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Resummations in Hot Scalar Electrodynamics

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ABSTRACT

The gauge-boson sector of perturbative scalar electrodynamics is investigated in detail as a testing ground for resummation methods in hot gauge theories. It also serves as a simple non-trivial reference system for the non-Abelian gluon plasma. The complete next-to-leading order contributions to the polarization tensor are obtained within the resummation scheme of Braaten and Pisarski. The simpler scheme proposed recently by Arnold and Espinosa is shown to apply to static quantities only, whereas Braaten-Pisarski resummation turns out to need modification for collective phenomena close to the light-cone. Finally, a recently proposed resummation of quasi-particle damping contributions is assessed critically.

1. Introduction

Perturbative thermal gauge field theories at ultra-relativistic temperatures [1] are hoped to be applicable for the description of phenomena associated with the envisaged formation of a quark-gluon plasma in high-energy ion collisions and for the physics of the early universe. In recent years, a lot of work has been put into developing a systematic perturbation scheme, which has turned out to require a resummation of the usual series of loop diagrams.

In non-Abelian gauge theories it was found by Linde [2] that there exists a barrier raised by infrared divergences associated with unscreened static magnetic fields, but even before this barrier is hit, conventional perturbation theory breaks down for certain mass scales much smaller than the temperature. In hot quantum chromodynamics (QCD), which is the fundamental theory relevant for the hypothetical quark-gluon plasma, it is for example the spectrum of quasi-particles whose perturbative treatment requires a resummed perturbation theory. After some failed attempts to calculate dissipative properties of gluonic plasma excitations [3], such a resummation scheme was developed finally by Braaten and Pisarski [4]. Also in spontaneously broken theories, there has been a flurry of activity centered around the issue of determining corrections to the effective potentials of scalar fields which describe the cosmological phase transition from the high-temperature symmetric to the low-temperature broken phase. These developments, which have been pioneered in the early seventies [5], are summarized e.g. in Ref. [6].

The need for a resummation of the ordinary loop expansion is not peculiar to non-Abelian theories, but can be studied also in the Abelian case. One of the simplest models is scalar electrodynamics. As Higgs model, it has already been used extensively to investigate the nature of the phase transition at finite temperature, e.g. in Ref. [7]. There the scalar sector is of central interest. In this paper we shall concern ourselves with the sector of the gauge fields, and we shall use hot scalar electrodynamics to study *en miniature* the issues that have been raised in the case of hot QCD. Scalar electrodynamics appears to be a potentially interesting toy model for purely gluonic QCD, since it involves self-interacting bosons and a gauge-field sector. In hot QCD, the resummed perturbation theory of Ref. [4] has been employed to derive a consistent, gauge-independent damping constant for the lowest-energy plasmon mode [8] as well as for energetic quasi-particles [9], and also some next-to-leading order results: corrections to the plasma frequency [10]

and to Debye screening [11]. In the much simpler case of hot scalar electrodynamics, we shall be able to give a fairly complete picture for the photonic quasi-particle spectrum.

In doing so, we shall largely follow the resummation program of Braaten and Pisarski [4]. After presenting the leading-order results in Sect. 2, this scheme is reviewed for the simple case of scalar electrodynamics in Sect. 3. In Sect. 4 we first specialize to the static limit and obtain next-to-leading order results for Debye screening and magnetic permeability. Here the simpler resummation scheme of Ref. [6], which has originally been put forward for the resummation of the static effective potential, agrees with full resummation. In Sect. 5, the limit of long-wavelength oscillations is considered and the next-to-leading order result for the plasma frequency is found. In this case, a naive application of the resummation scheme of Arnold and Espinosa is shown to fail, because nonstatic modes can no longer be neglected. We also show to what extent classical considerations can explain the value of the plasma frequency obtained before. In Sect. 6 we exhibit the complete next-to-leading order results for the polarization tensor, deriving the corrected spectrum of propagating photonic quasi-particles. Here we find a qualitative change in the case of the longitudinal plasmons, which turn out to exist only for a finite range of frequencies or momenta. The canonical resummation according to Braaten and Pisarski actually breaks down in this example due to singularities at the light-cone, but can be amended so that this upper bound for the frequencies of longitudinal plasmons can be calculated accurately. A similar result is shown to hold in the case of QCD. Finally, in Sect. 7 we consider the additional resummation of scalar damping contributions, critically assessing a recent proposal in Ref. [12]. Sect. 8 contains our conclusions.

2. Leading-order results

Scalar electrodynamics with a scalar potential leading to spontaneous symmetry breaking has been extensively used as a toy model to study symmetry restoration at high temperature and the corresponding phase transition which occurs when the temperature is lowered [7]. In this paper we shall focus our attention to the gauge field sector of this model rather than the effective potential of the scalars. For this we shall for simplicity consider charged scalar particles without self-interactions. At sufficiently high temperature, the bare mass of the scalar particles can be neglected, so our starting point is the

Lagrangian [13]

$$\mathcal{L} = (D_\mu \varphi)^* D^\mu \varphi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 \quad , \quad (2.1)$$

with the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ and the Abelian field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The signature of the Minkowski metric $g_{\mu\nu}$ is $+- - -$. In (2.1) we have included a gauge breaking term corresponding to general covariant gauges; the ghost term is omitted since it decouples from the rest.

We shall mainly use the Matsubara formalism, where Green's functions are first defined for discrete imaginary frequencies proportional to $2\pi iT$. After the evaluation of all frequency sums, they are finally extended to real external frequencies by an appropriate analytic continuation. In the first chapters we shall have to consider exclusively the continuation of two-point functions, for which we choose retarded boundary conditions prescribing $Q_0 = 2\pi inT \rightarrow \omega + i\varepsilon$ [1, 14].

The theory (2.1) has two propagators and two vertices. Writing a four-momentum as $Q = (i\omega_n, \vec{q})$, $\omega_n = 2\pi nT$, the bare scalar propagator is $S^0 = -1/Q^2$, and the bare photon propagator reads $G_{\mu\nu}^0 = g_{\mu\nu}/Q^2 + (\alpha - 1)D_{\mu\nu}/Q^2$. Here D is the fourth matrix of the following basis for symmetric Lorentz tensors,

$$A = g - B - D \quad , \quad B = \frac{V \circ V}{V^2} \quad , \quad C = \frac{Q \circ V + V \circ Q}{\sqrt{2} Q^2 q} \quad , \quad D = \frac{Q \circ Q}{Q^2} \quad , \quad (2.2)$$

where $V \equiv Q^2 U - (UQ)Q$ is a projector to longitudinal fields built from the rest-frame velocity of the heat bath U , which in the following we choose as $U = (1, \vec{0})$. A and B are transverse with respect to Q_μ ; A is transverse also with respect to the three-momentum \vec{q} . The translation into the notation of the textbook of Kapusta [1] is given by $A = -P_T$ and $B = -P_L$. The vertex which couples a scalar line (Q ingoing, P outgoing) with one photon is $-\epsilon(Q + P)^\mu$, and that connecting to two photon lines is $2e^2 g^{\mu\nu}$ with the Lorentz indices carried by the two photons.

At one-loop level, the propagators are given by Dyson's equation

$$G = G^0 + G^0 \Pi G \quad \text{and} \quad S = S^0 + S^0 \Xi S \quad , \quad (2.3)$$

where Π and Ξ are the self energies of photon and scalar, respectively. The corresponding Feynman diagrams are depicted in Fig. 1.

The bare 1-loop expression for the photon self-energy reads

$$\Pi^{\mu\nu}(Q) = \frac{3}{2} m^2 g^{\mu\nu} + e^2 \sum \frac{4K^\mu K^\nu - Q^\mu Q^\nu}{K^2(K - Q)^2} \quad \text{with} \quad m \equiv \frac{\epsilon T}{3} \quad , \quad (2.4)$$

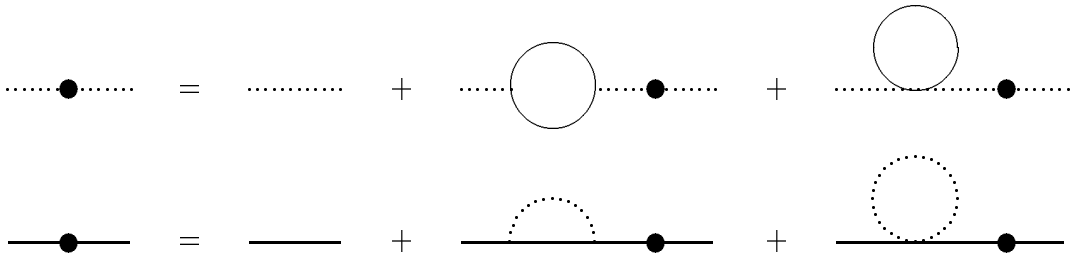


Figure 1: Propagators dressed by hard thermal loops. Dotted lines represent photons, and solid lines stand for scalar particles.

where the blank sum symbol stands for $(2\pi)^{-3} \int d^3k T \sum_n$. We shall always use K as loop-momentum variable. In contrast to the self-energy of gluons in QCD, (2.4) is gauge parameter independent, and also somewhat simpler. $\Pi^{\mu\nu}$ is transverse, $Q_\mu \Pi^{\mu\nu} = 0$ which in Abelian theories holds for any linear gauge choice (in non-Abelian theories the finite-temperature gluon self-energy is, in general, non-transverse even in covariant gauges). It therefore decomposes into

$$\Pi^{\mu\nu} = A^{\mu\nu} \Pi_t + B^{\mu\nu} \Pi_\ell \quad (2.5)$$

with $\Pi_t = \text{Tr} A\Pi/2$ and $\Pi_\ell = \text{Tr} B\Pi$.

In the high-temperature limit, the leading order contributions are given by

$$\Pi_\ell = -\frac{Q^2}{q^2} \Pi_{00} = 3m^2 \left(1 - \frac{Q_0^2}{q^2}\right) \left(1 - \frac{Q_0}{2q} \ln \frac{Q_0 + q}{Q_0 - q}\right), \quad (2.6)$$

$$\Pi_t = \frac{1}{2} (3m^2 - \Pi_\ell). \quad (2.7)$$

For the comparison with subleading results to be discussed later on, in Fig. 2 the real part of Π_ℓ is displayed as a function of real $Q_0 = \omega$ and q (according to (2.7) Π_t is simply given by the inverted picture, appropriately relabelled). Notice in particular the non-analytic behavior at the origin, where the value of Π_ℓ (and also of Π_t) depends on whether it is approached from space-like, time-like, or light-like directions.

The scalar self-energy on the other hand is given by

$$\Xi = -\mu^2 - e^2 Q^2 \sum \left(\frac{3 - \alpha}{K^2(K - Q)^2} + \frac{2(\alpha - 1)KQ}{K^4(K - Q)^2} \right) \quad \text{with} \quad \mu \equiv \frac{eT}{2}. \quad (2.8)$$

(Including φ^4 -self-interactions by $\mathcal{L} \rightarrow \mathcal{L} - \frac{\lambda}{4}(\varphi^* \varphi)^2$ would only change $\mu^2 \rightarrow e^2 T^2/4 + \lambda T^2/12$.) In contrast to the photon self-energy, Ξ does depend on the gauge parameter α , but the only contribution $\propto T^2$ is the first term, which is again gauge-independent.

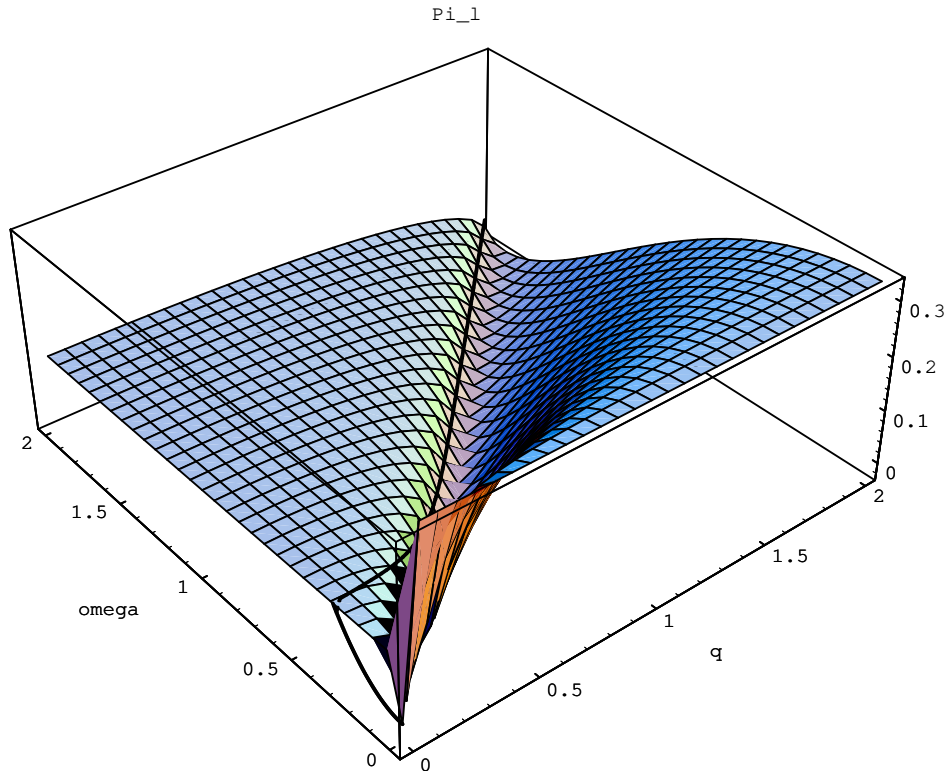


Figure 2: The real part of $\Pi_\ell(\omega, q)$ in units of $e^2 T^2$. Also given is the intersection with the surface $\omega^2 - q^2$ which gives the dispersion curve for longitudinal plasmons.

Including only the leading order contributions $\propto e^2 T^2$, the scalar propagator according to (2.3) is simply

$$S = \frac{-1}{Q^2 - \mu^2} \quad , \quad (2.9)$$

with a constant (thermal) mass term. The photon propagator, however, has a richer structure, and reads in the tensorial basis (2.2)

$$G^{\mu\nu}(Q) = A^{\mu\nu} \frac{1}{Q^2 - \Pi_t(Q)} + B^{\mu\nu} \frac{1}{Q^2 - \Pi_\ell(Q)} + D^{\mu\nu} \frac{\alpha}{Q^2} \quad , \quad (2.10)$$

with Π_t and Π_ℓ at leading order as given by (2.6, 2.7).

The ultra-relativistic photon propagator (2.10) has been first derived within classical kinetic theory by Silin [15], and within field theory by Fradkin [16]. Exactly the same expressions were later found in high-temperature QCD [17, 18], the only difference being a replacement $e^2 \rightarrow g^2(N + N_f/2)$ for $SU(N)$ with N_f flavors. There are now two physical structure functions in the gauge-field propagator. The one associated with the spatially transverse tensor A corresponds to transverse photons (or gluons), whereas the

one associated with B describes a new collective mode, the so-called plasmon mode. The poles of these structure functions correspond to the normal modes of the gauge field sector of the ultra-relativistic plasma. For real frequencies and momenta, these give the spectrum of propagating photonic quasi-particles. It starts at the minimal (plasma) frequency $\omega(q=0) = \epsilon T/3 \equiv m$. For larger frequencies the two branches differ. At $q \rightarrow \infty$ the spatially transverse branch approaches the form $\sqrt{q^2 + m_\infty^2}$ with an asymptotic thermal mass $m_\infty = \epsilon T/\sqrt{6}$, whereas the longitudinal branch approaches the light-cone exponentially. This longitudinal branch is also shown in Fig. 2. It disappears from the spectrum eventually because with increasing momentum the residue of the corresponding pole in (2.10) vanishes exponentially. Perturbing the plasma with frequencies $\omega < m$ does not give rise to propagating quasi-particles, but results in dynamical screening, both in the spatially transverse and in the longitudinal mode. In the static limit, there is screening only in the longitudinal sector corresponding to Debye screening of longitudinal electric fields with screening mass $m_{el.} = \epsilon T/\sqrt{3}$, whereas static magnetic fields remain unscreened.

The main theme of this work will be to determine this spectrum of the photonic quasi-particles and the screening properties beyond the level of the leading temperature contributions.

3. Resummation of hard thermal loops

In perturbative quantum field theories it sometimes happens that higher-order loop diagrams are not suppressed by correspondingly high powers of the coupling constants, and that therefore the perturbation series has to be reorganized in order to actually be perturbative. In thermal field theories this is a ubiquitous phenomenon. In particular the zero modes of massless Bose particles are sensitive to the appearance of thermal masses. Repeated self-energy insertions cause higher-order diagrams to become more and more infra-red singular, but summing them up according to Fig. 1 the resulting dressed propagators are well-behaved. In the case of the thermodynamic potential, this particular resummation is known as “ring resummation” and dates back to the work of Gell-Mann and Brueckner [19].

In the case of the spectrum of quasi-particle excitations, the necessity of a resummed

perturbation theory has become apparent in gauge theories by the failure of the unimproved perturbation theory to give gauge-independent results for the damping constants of thermal quasi-particle excitations [3]. A systematic perturbation theory was developed most notably by Braaten and Pisarski [4]. These authors have shown that Green's functions involving soft external momenta (i.e. of the same order of magnitude as the plasma frequency) require the resummation of all so-called hard thermal loops (HTL's) [20, 4]. The latter are the contributions from one-loop diagrams that are dominated by hard loop momenta $\sim T$. In renormalizable theories they go like T^2 for $T \rightarrow \infty$. They can compensate for powers of the coupling as follows. Consider in particular the dressed propagators as given by Fig. 1. In the diagrams on the right-hand-side of Fig. 1 the self-energy part contributes a factor $\sim \epsilon^2 T^2$. If the bare propagator attached to it carries momentum $\sim \epsilon T$, it brings in a factor $\epsilon^{-2} T^{-2}$ and the contribution involving the self-energy insertion is of relative order 1. Only for larger external momentum the perturbative nature of Fig. 1 is restored, and loop-corrections are suppressed by factors of ϵ . Consequently, contributions from loop integrals (in any higher order) that probe these soft scales are not perturbatively stable, unless (at least) these hard thermal loops are resummed. The hard thermal loops themselves are stable, because they are sensitive to hard scales only.

In non-Abelian gauge theories, the same situation occurs for vertex functions, because there the one-loop vertices receive also contributions $\sim T^2$, such that HTL vertices are of the same order of magnitude as bare ones for soft external momenta. Therefore, they have to be included in the resummation as well.

In scalar electrodynamics, there are no HTL vertices. Superficial power counting [4] rules out (as in QCD) all candidates for HTL vertices, which contain a bare 4-vertex or different kinds of lines in the loop. Moreover, scalar loops with an odd number of external lines are easily shown to vanish, because they change sign by reversing momentum flows. For the remaining scalar loops with an even number n of external photons with momenta Q_j one arrives at

$$\epsilon^n \sum \frac{K_{\mu_1} K_{\mu_2} \cdots K_{\mu_n}}{(K + Q_1)^2 (K + Q_1 + Q_2)^2 \cdots (K - Q_n)^2 K^2} + \text{perm. of } Q_1, \dots, Q_{n-1} \quad (3.1)$$

with $Q_1 + Q_2 + \dots + Q_n = 0$. The power-counting arguments of Ref. [4] would estimate that (3.1) goes like T^2 for $T \rightarrow \infty$, so that with $Q_j \sim \epsilon T$ (3.1) would become of the order $\epsilon^{n-2} |Q|^{4-n}$, the same order of magnitude as bare vertices are on dimensional grounds. However, for $n = 4$ it has been shown in Ref. [20] that the leading-order contributions in (3.1) in fact cancel, and in Ref. [21] it was proved by induction that this holds for all

$n > 4$ as well.

In non-Abelian gauge theories with or without fermions, the HTL vertices are essential to guarantee the gauge invariance of the generating functional of all HTL's. An explicit effective action S_{eff} has been constructed by Taylor and Wong [22] by starting from its bilinear part $S_{\text{eff}}^{(2)}$ as determined by the HTL self-energy diagrams and determining higher orders in the fields by gauge invariance. A nice representation of this can be found in Ref. [23]. In scalar electrodynamics it turns out that $S_{\text{eff}}^{(2)}$ is already gauge invariant so that $S_{\text{eff}} = S_{\text{eff}}^{(2)}$, and the uniqueness of this construction, which is presented in some detail in App. A, explains the absence of HTL vertices in the Abelian case.

For soft momenta, the HTL's are of the same order as the tree-level terms, so together they represent the 'zeroth order' of the high-temperature limit of the theory. Thus, the effective action, which combines them into a single formula, represents the first term of the high-temperature asymptotics of this theory. In its manifest gauge invariant form, it reads

$$S_{\text{eff}} = S + \delta S = \int^{\beta} \mathcal{L}_{\text{eff}} \quad , \quad \mathcal{L}_{\text{eff}} = \mathcal{L} - \mu^2 \varphi^* \varphi + \frac{3}{4} m^2 \int_{\Omega} Y^{\rho} F_{\rho\mu} \frac{1}{(Y\partial)^2} F^{\mu\lambda} Y_{\lambda} \quad , \quad (3.2)$$

where $\int^{\beta} \equiv \int_0^{\beta} d\tau \int d^3r$ and $Y^{\mu} \equiv (1, \vec{e})$ with $\vec{e}^2 = 1$. The angular integral \int_{Ω} over the directions of \vec{e} is normalized to one: $\int_{\Omega} 1 = 1$, $\int_{\Omega} Y = U$.

When adopting (3.2) as the adequate starting point for a perturbative treatment, one must however take care of not to change the underlying theory itself. One is therefore led to rewrite S as $S = S_{\text{eff}} - \delta S$, where $-\delta S$ generates counter-terms that subtract at higher loop orders what has been added in at lower ones.

The first contributions beyond the new zeroth order are contained in 1-loop diagrams, but now with resummed internal lines. For the photon self-energy this resummation amounts to insert the massive scalar propagators (2.9) into the diagrams of Fig.1. The general form of the photon Green's function is still given by (2.10) by virtue of gauge invariance, i.e. transversality.

The resummed 1PI expressions $\Pi_{\ell} = \text{Tr} B\Pi$ and $\Pi_t = \text{Tr} A\Pi/2$ derive from

$$\Pi^{\mu\nu} = -2e^2 g^{\mu\nu} \sum \Delta + 4e^2 \sum K^{\mu} K^{\nu} \Delta^{-} \Delta \quad , \quad \Delta \equiv \frac{1}{K^2 - \mu^2} \quad , \quad \Delta^{-} \equiv \frac{1}{(K - Q)^2 - \mu^2} \quad , \quad (3.3)$$

and are given more explicitly by

$$\Pi_\ell = -2e^2 \sum \Delta + 4e^2 \sum \Delta^- \Delta \left[\bar{k}^2 - \frac{(\bar{k} \bar{q})^2}{q^2} + K^2 - \frac{(KQ)^2}{Q^2} \right] , \quad (3.4)$$

$$\Pi_t = \frac{1}{2} (\Pi_g - \Pi_\ell) , \quad (3.5)$$

where Π_t is determined to the slightly simpler auxiliary quantity

$$\Pi_g \equiv \text{Tr } g\Pi = -4e^2 \sum \Delta + e^2(4\mu^2 - Q^2) \sum \Delta^- \Delta . \quad (3.6)$$

In (3.3) a term involving $Q^\mu Q^\nu$ has been dropped, because it does not contribute at the subleading order under study.

An alternative version of Π_ℓ is

$$\Pi_\ell = e^2(4\mu^2 - Q^2) \sum \Delta^- \Delta + 4e^2 \sum \Delta^- \Delta \left[k^2 - \frac{(\bar{k} \bar{q})^2}{q^2} \right] . \quad (3.7)$$

Special elements of $\Pi^{\mu\nu}$ may be read off from $\Pi^{\mu\nu} = A^{\mu\nu} \Pi_t + B^{\mu\nu} \Pi_\ell$. In particular,

$$\Pi_{00} = -\frac{q^2}{Q^2} \Pi_\ell \quad , \quad \Pi_{ii} = -2\Pi_t - \frac{Q_0^2}{Q^2} \Pi_\ell = \Pi_{00} - \Pi_g . \quad (3.8)$$

4. Static screening at next-to-leading order

In the static limit the spatially-transverse and spatially-longitudinal structure functions in the photon propagator, as given by (3.8), simplify to

$$\Pi_t(0, q) = -\frac{1}{2} \Pi_{ii}(0, q) \quad , \quad \Pi_\ell(0, q) = \Pi_{00}(0, q) . \quad (4.1)$$

Before resummation, the leading and subleading terms in the high-temperature expansion of the static one-loop self-energy (2.4) read

$$\Pi_{00}(0, q) = \frac{e^2 T^2}{3} + \frac{e^2}{24\pi^2} q^2 \ln \frac{\sigma}{T} + \dots = 3m^2 (1 + O(\epsilon^2)) \quad (4.2)$$

and

$$\Pi_{ii}(0, q) = -\frac{1}{8}e^2 qT - \frac{e^2}{12\pi^2} q^2 \ln \frac{\sigma}{T} + \dots = -\frac{3}{8} m q e (1 + O(\epsilon)) \quad , \quad (4.3)$$

where here and in what follows we count orders in ϵ always for soft momenta $\sim m \propto eT$. Notice further that the $T = 0$ part has been assumed to be renormalized at the scale σ , which contributes a term proportional to $\ln \sigma^2/Q^2$. This has been combined with a similar term proportional to $\ln Q^2/T^2$ from the temperature-dependent parts.

The leading contribution in Π_{00} is the familiar electric (Debye) screening mass $m_{\text{el}}^2 = 3m^2$, while there is no screening mass of this order of magnitude in Π_{ii} , i.e. no magnetic mass. At relative order ϵ there is no contribution in Π_{00} , but one in Π_{ii} which when taken seriously spells trouble. Including the latter in the static transverse propagator would give

$$\Delta_t(0, q) = \frac{-1}{q^2 - e^2 qT/16} \quad , \quad (4.4)$$

which has a space-like pole at $q = e^2 T/16$. A similar behaviour is found also in QCD, sometimes called the Landau ghost of thermal QCD.

However, as we have argued in the previous sections, the subleading terms in (4.2) and (4.3) are not accurate for $q \lesssim eT$ and require resummation of the hard thermal loops. Only the terms proportional to $e^2 \ln(\sigma/T)$ are to be trusted as they are determined by the ultraviolet part of the loop integrals.

4. A RESUMMATION

$\Pi_{00}(0, q)$ upon resummation of the hard thermal loop, which is just the thermal scalar mass μ , reads

$$\Pi_{00}(0, q) = e^2 \sum \left\{ 4K_0^2 \Delta \Delta^- - 2\Delta \right\} \quad , \quad (4.5)$$

see (3.4), with Δ and Δ^- as defined in (3.3). Evaluating the sum over the Matsubara frequencies by means of a contour integral leads to

$$\Pi_{00}(0, q) = \frac{e^2}{\pi^2} \int_0^\infty dk k^2 \frac{n(\sqrt{k^2 + \mu^2})}{\sqrt{k^2 + \mu^2}} \left[1 + \frac{k^2 + \mu^2}{kq} \ln \left| \frac{2k + q}{2k - q} \right| \right] \quad . \quad (4.6)$$

After separating off the T^2 -contribution, we can write

$$\Pi_{00}(0, q) = \frac{e^2 T^2}{3} - \frac{e^2 T}{\pi^2} \int_0^\infty dk \left\{ \frac{\mu^2}{k^2 + \mu^2} + 1 - \frac{k}{q} \ln \left| \frac{2k + q}{2k - q} \right| \right\} + O(e^2 q^2 \ln(T)) \quad . \quad (4.7)$$

This can be evaluated by integration by parts, which reveals that the q -dependence of the contribution at order T is completely spurious:

$$\Pi_{00}(0, q) = \frac{\epsilon^2 T^2}{3} - \frac{\epsilon^2 T \mu}{2\pi} + O(\epsilon^2 q^2 \ln(T)) = 3m^2 \left(1 - \frac{3}{4\pi} \epsilon + O(\epsilon^2) \right) . \quad (4.8)$$

The static Π_{00} at next-to-leading order is a negative constant, resulting in a decrease of the classical value of the electric screening mass. In QCD the corresponding calculation has been performed recently by one of the present authors in Ref. [11]. There the next-to-leading order correction to $\Pi_{00}(0, q)$ turns out to be a nontrivial function of q , which is such that it diverges logarithmically for $q^2 \rightarrow -m_{\text{el.}}^2$, where $\Pi_{00}(0, q)$ defines the correction term to the Debye mass. Assuming that this divergence is cut-off by a magnetic mass, which is expected to arise non-perturbatively at order $g^2 T$ (g being the QCD coupling constant), leads to $\delta m_{\text{el.}}^2 = O(g \ln(1/g) m^2)$ such that $\delta m_{\text{el.}} > 0$ for small g . Evidently, in this quantity there is a qualitative difference between an Abelian and a non-Abelian gauge field plasma beyond leading order.

In a similar manner, using (3.5) and (3.6), one can evaluate the resummed expression for $\Pi_{ii}(0, q)$,

$$\Pi_{ii}(0, q) = \epsilon^2 \sum \left\{ (4k^2 - q^2) \Delta \Delta^- + 6\Delta \right\} . \quad (4.9)$$

Its high-temperature limit is found to be

$$\Pi_{ii}(0, q) = \frac{\epsilon \mu}{2\pi} \left[2\mu - \frac{q^2 + 4\mu^2}{q} \arctan \left(\frac{q}{2\mu} \right) \right] + O(\epsilon^2 \mu^2) , \quad (4.10)$$

which is to be compared with (4.3). The latter is only accurate for $q \gg eT$, and in this regime it indeed coincides with the resummed result (4.10), since $\arctan(q/(2\mu)) \rightarrow \pi/2$ for $q/\mu \rightarrow \infty$.

For $q \lesssim eT$, the resummed result deviates considerably from the bare one. In particular for $q \rightarrow 0$, the former approaches zero like q^2 rather than q , so that there is no longer any unphysical pole at space-like momentum. The vanishing of $\Pi_{ii}(0, q \rightarrow 0)$ also implies that there is no magnetic mass squared of the order of $e^3 T^2$. In fact, in the present Abelian case it can be shown rigorously that there is no magnetic screening mass [16, 17]. In QCD [17] the situation is again quite different. Resummation changes the unphysical pole at space-like momenta, but does not remove it. However, since this pole arises at momentum scale $g^2 T$, this just points to the relevance of a magnetic mass in the non-Abelian case.

For finite $q \sim eT$, $\Pi_{ii}(0, q)$ is always negative, which implies that the magnetic permeability

$$\left(\frac{1}{\mu} - 1\right)_{\text{static}} = -\frac{1}{2q^2} \Pi_{ii}(0, q) \quad (4.11)$$

is positive, decreasing monotonously from $e/(12\pi)$ at $q = 0$ to zero for $q \rightarrow \infty$. Hence, the hot scalar plasma is weakly diamagnetic at distances $\gtrsim 1/(eT)$.

4.B STATIC RESUMMATION

Up to now we have strictly followed the resummation scheme outlined by Braaten and Pisarski for hot gauge theories (a detailed exposition in the simpler scalar case can be found in Ref. [24]). However, the above calculations can be somewhat simplified by the following observations due to Arnold and Espinosa [6] made in the context of a resummed perturbation theory for the finite temperature effective potential in gauge theories. In the Matsubara formalism, K^2 has Euclidean signature, $-K^2 = (2\pi nT)^2 + k^2$, so that the momentum K_μ can be soft only with $n = 0$. Accordingly it is sufficient to dress only these static modes:

$$\Delta_{n=0}(k) = \frac{-1}{k^2 + \Pi(0, k)} \quad , \quad (4.12)$$

whereas the correction terms to the nonstatic propagators in

$$\Delta_{n \neq 0}(k) = \frac{-1}{(2\pi nT)^2 + k^2} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \Pi^m(2\pi inT, k)}{[(2\pi nT)^2 + k^2]^m} \right] \quad (4.13)$$

are down by powers of g^{2m} when $\Pi \sim g^2 T^2$.

A systematic resummation scheme based on this splitting has been put forward in Ref. [6], where the decisive simplifications for gauge theories are due to the fact that the hard thermal loop $\Pi(0, k)$ is a constant mass term. In its lowest order version, it just coincides with the well-known ring resummation introduced by Gell-Mann and Brueckner [19].

The limitations of this restricted resummation scheme become apparent when one considers nonstatic Green's functions. Because of energy conservation at vertices, external frequencies are also fixed to Matsubara frequencies, and because of the special treatment of the static sector, it is quite impossible to perform an analytic continuation to nonzero soft external frequencies in the end. However, this scheme should be sufficient for the evaluation of corrections to static quantities like the effective potential [6] or screening

masses [11], and to Green's functions with hard external frequencies, e.g. damping of energetic particles [9].

Let us exemplify this simplified resummation method by recalculating the next-to-leading order screening mass in hot scalar electrodynamics. Separating static from non-static modes in $\Pi_{00}(0, k)$ gives

$$\begin{aligned} \Pi_{00}(0, q) &= e^2 \sum \left\{ 4K_0^2 \Delta \Delta^- - 2\Delta \right\} \\ &= e^2 T \sigma^{2\varepsilon} \int \frac{d^{3-2\varepsilon} k}{(2\pi)^{3-2\varepsilon}} \frac{2}{k^2 + \mu^2} \\ &\quad + 2e^2 T \sigma^{2\varepsilon} \sum_{n=1}^{\infty} \int \frac{d^{3-2\varepsilon} k}{(2\pi)^{3-2\varepsilon}} \left\{ -4[(k-q)^2 + \mu^2] \Delta_n \Delta_n^- + 6\Delta_n \right\} . \end{aligned} \quad (4.14)$$

Here we have employed dimensional regularization in order to render the splitted expressions well-defined. From (4.14) it is apparent that indeed only the $n = 0$ contributions are capable of producing a relative order ε through $e^2 T / \mu$ — the other, nonstatic contributions can be expanded out in powers of μ^2 and will not give something nonanalytic in μ^2 , neither i.e., in ε^2 .

The next-to-leading order term is thus contained in

$$\begin{aligned} \delta \Pi_{00}(0, q) &= 2e^2 T \sigma^{2\varepsilon} \int \frac{d^{3-2\varepsilon} k}{(2\pi)^{3-2\varepsilon}} \frac{1}{k^2 + \mu^2} + O(\varepsilon^2 q^2 \ln T) \\ &= 2e^2 T \mu \frac{\Gamma(-\frac{1}{2} + \varepsilon)}{(4\pi)^{3/2}} \left(\frac{4\pi \sigma^2}{\mu^2} \right)^\varepsilon + O(\varepsilon^2 q^2 \ln T) \\ &= -\frac{e^2 T \mu}{2\pi} + O(\varepsilon e^2 T \mu) + O(\varepsilon^2 q^2 \ln T) , \end{aligned} \quad (4.15)$$

where σ is the mass scale introduced by dimensional regularization. The limit $\varepsilon \rightarrow 0$ is regular so that there arise no UV singularities proportional to T , and the result coincides with (4.8). Actually, in the evaluation of $\delta \Pi$, the use of dimensional regularization could have been avoided by subtracting off the unresummed zero-mode contribution, i.e.

$$\delta \Pi_{00}(0, q) = 2e^2 T \int \frac{d^3 k}{(2\pi)^3} \left[\frac{1}{k^2 + \mu^2} - \frac{1}{k^2} \right] . \quad (4.16)$$

Compared to the full calculation in the preceding subsection, the q -independence of the next-to-leading order result is now manifest, which greatly simplifies its evaluation. The simplifications are less conspicuous in the case of $\delta \Pi_{ii}(0, q)$, which can be computed in an analogous manner, readily reproducing (4.10).

5. Plasma frequency at next-to-leading order

In this section we shall restrict ourselves to another limiting case, the one of long wavelengths, $q \rightarrow 0$, which again simplifies all calculations considerably.

In this limit $\Pi_{\mu\nu}$ degenerates and contains only one independent structure function, because $\Pi_{00}(Q_0, q \rightarrow 0) = 0$ due to transversality and $\Pi_{ij}(Q_0, q \rightarrow 0) \propto \delta_{ij}$ if $\Pi_{\mu\nu}$ is to remain regular. Since $\delta_{ij}A^{ij} = \delta_{ij}B^{ij}$, this entails that the longitudinal and the transverse polarization function coincide, $\Pi_t(Q_0, 0) = \Pi_\ell(Q_0, 0)$. In other words, without a wave vector there is no way to tell longitudinal photonic quasi-particles from transverse ones.

At $q = 0$, the unimproved one-loop result for $T \gg Q_0 \sim m$ reads

$$\Pi_{t,\ell}(Q_0, 0) = \frac{\epsilon^2 T^2}{9} - \frac{\epsilon^2 T}{12\pi} i Q_0 - \frac{\epsilon^2}{24\pi^2} Q_0^2 \ln \frac{\sigma}{T} + O(\epsilon^2 Q_0^2 T^0) = m^2 - \frac{\epsilon}{4\pi} i Q_0 m + O(\epsilon^2 m^2) . \quad (5.1)$$

The mass m , often identified as *the* plasmon mass (although there is no longer any momentum independent notion of mass) determines the plasma frequency, below which the medium cannot sustain free oscillations. At relative order ϵ , (5.1) implies a non-zero damping constant

$$\gamma = -\frac{1}{2m} \text{Im} \Pi_{t,\ell}(Q_0 = m, 0) = \frac{\epsilon^2 T}{24\pi} = \frac{\epsilon}{8\pi} m . \quad (5.2)$$

However, we shall presently show that the subleading term is not stable under resummation. In particular, the corrected γ will turn out to vanish at relative order ϵ .

5.A RESUMMATION

Proceeding as in the previous section we find for the resummed polarization functions

$$\begin{aligned} \Pi_t(Q_0, 0) &= \Pi_\ell(Q_0, 0) = \epsilon^2 \sum \left\{ -\frac{4}{3} k^2 \Delta \Delta^- - 2\Delta \right\} \\ &= \frac{\epsilon^2}{3\pi^2} \int_0^\infty k^2 dk \frac{n(\sqrt{k^2 + \mu^2})}{\sqrt{k^2 + \mu^2}} \left(2 + \frac{\mu^2 - (Q_0/2)^2}{k^2 + \mu^2 - (Q_0/2)^2} \right) . \end{aligned} \quad (5.3)$$

Because of $\vec{q} = 0$, the angular integration is trivial this time, and after separating off the constant T^2 -contribution, the $O(T)$ -part is easily evaluated, yielding

$$\Pi_t(Q_0, 0) = \Pi_\ell(Q_0, 0) = \frac{\epsilon^2 T^2}{9} + \frac{\epsilon^2 T}{2\pi} \left\{ -\mu - \frac{4}{3Q_0^2} \left([\mu^2 - (Q_0/2)^2]^{\frac{3}{2}} - \mu^3 \right) \right\} . \quad (5.4)$$

At $Q_0 = m$ this determines the correction to the plasma frequency,

$$m^2 + \delta m^2 = \frac{e^2 T^2}{9} \left(1 - \frac{8\sqrt{2} - 9}{2\pi} e \right) \approx \frac{e^2 T^2}{9} (1 - 0.37e) , \quad (5.5)$$

as well as the $O(e^2 T)$ -contribution to the plasmon damping in the long-wavelength limit. However, (5.3) is real at $Q_0 = m$, so that the damping constant vanishes at this order of magnitude. The bare result (5.2) is wrong because by not taking into account the thermal masses of the scalar particles a plasmon seems capable of decaying into those. In view of the long story of the plasmon damping puzzle in hot QCD [3] let us make the trivial remark that the manifest gauge independence of the bare result (5.2) did not prevent it from being incomplete. The nonvanishing result for the resummed QCD damping constant by the way arises from the possibility of Landau damping, which is absent for thermal scalars. Thus in scalar QED an imaginary part to the polarization tensor appears only when pair decay becomes possible. Indeed, for $Q_0 > 2\mu$ (5.4) does become complex, but $m < 2\mu$.

In pure QCD at high temperature the next-to-leading contribution to the plasma frequency has recently been calculated by one of the present authors in Ref. [10], yielding $(\delta m^2/m^2)_{\text{QCD}} \approx -0.18\sqrt{g^2 N}$. Let us see how far this latter result can be understood by the above result on scalar QED. From the leading order terms it is clear that e^2 corresponds to $g^2 N$, so we might try to apply (5.4) by inserting the plasmon mass in place of the thermal mass of the scalars. This would give $(\delta m^2/m^2) \approx -0.028\sqrt{g^2 N}$, which is over a factor of 6 short of the actual result. Hence, the correction to the QCD plasma frequency is much larger than what might be expected from just the appearance of thermal masses in the loop integrals.

5.B STATIC RESUMMATION

In the preceding chapter we have seen that the next-to-leading order results on static Green's functions could be obtained also in a simplified scheme that resums only static modes. In the imaginary-time formalism, nonstatic modes are automatically hard so that their hard-thermal-loop corrections can be treated as perturbations. This scheme is particularly advantageous in gauge theories, where it simplifies tremendously the calculation of static quantities and also of Green's functions with hard external frequencies. However, dynamical Green's functions with soft external frequencies are quite intractable because they require analytic continuation from the imaginary Matsubara frequencies which are

either zero or hard. The separation into static and nonstatic contributions renders this analytic continuation to general soft external frequencies rather impossible.

One might perhaps nevertheless expect that also in the nonstatic situation it is the resummation of internal static modes that gives the next-to-leading order correction which is nonanalytic in the coupling constant, and perform the analytic continuation on just their contribution. This is wrong, as we shall show in the present example of $\Pi_{t,l}(Q_0, 0)$.

Taking only the $n = 0$ -contribution of the sum in (5.3) and continuing at once to soft $Q_0 \neq 0$ would give

$$\delta\Pi_{t,\ell}(Q_0, 0) = -\frac{1}{3}e^2T \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{4k^2}{(k^2 + \mu^2)(k^2 + \mu^2 - Q_0^2)} - \frac{6}{k^2 + \mu^2} \right\} , \quad (5.6)$$

where dimensional regularization is understood (see above). This is readily evaluated, yielding

$$\delta\Pi_{t,\ell}(Q_0, 0)\Big|_{\text{static contr.}} = -\frac{e^2T}{6\pi} \left\{ \frac{2}{Q_0} \left[\frac{\mu}{Q_0} - \sqrt{\frac{\mu^2}{Q_0^2} - 1} \right] (Q_0^2 - \mu^2) + \mu \right\} . \quad (5.7)$$

It obviously disagrees with (5.4). It has a wrong analytic structure in that the imaginary part sets in already at $Q_0 \geq \mu$ rather than 2μ , and of course does not reproduce the actual real contribution either. For instance, it would predict the correction to the plasma frequency squared to be

$$m^2 + \delta m^2\Big|_{\text{static contr.}} = \frac{e^2T^2}{9} \left(1 - \frac{5\sqrt{5} - 9}{8\pi} e \right) \approx \frac{e^2T^2}{9} (1 - 0.09e) , \quad (5.8)$$

which underestimates the next-to-leading order term by more than a factor of 4.

Evidently, the nonstatic modes may not be neglected and even give the largest contribution in this example. (In Ref. [25], the importance of the nonstatic modes was also noticed in the case of the gluonic plasmon damping.) The failure of the above reasoning may be traced to the premature analytic continuation (a collection of similar pitfalls with analytic continuation at finite temperature can be found in [26]). In our case this analytic continuation was not possible because the polarization function in the imaginary-time formalism was given only at one point in the complex plane of *soft* energies, $Q_0 = 0$, which of course cannot be continued unambiguously into a function over nonzero (soft) frequencies.

5.C CLASSICAL PHYSICS

We close this chapter by trying to understand the results on the plasma frequency in the familiar intuitive terms of classical physics rather than through full-fledged quantum (field) theory. Looking at the hot scalar plasma as a simple system of particles interacting through Coulomb forces explains fully the leading order result $\omega^2 = e^2 T^2/9$, as we shall see, and also part of the next-to-leading order result.

First, to count the states of scalar particles, imagine a box of volume V inside the hot plasma and apply periodic boundary conditions. Then

$$N = \frac{V}{(2\pi)^3} \int d^3k n \left(\sqrt{\mu^2 + k^2} \right) \quad (5.9)$$

is the number of positive scalar particles in V as well as that of the negative ones. We shall consider again massless particles; the mass μ in (5.9) is to make room for a dynamically acquired self-energy. Admittedly, the Bose distribution function n stems from quantum physics, but this is the only exterior element we shall employ.

If a longitudinal electric plane wave $\vec{E} = E_0 \vec{e}_1 \sin(qx - \omega t)$ with an infinitesimal amplitude E_0 is somehow activated in the medium, then Maxwell's equations tell us that there is no magnetic field associated with it. They reduce to

$$E_0 q \cos(qx - \omega t) = \rho \quad , \quad j_1 = E_0 \omega \cos(qx - \omega t) \quad . \quad (5.10)$$

In the long-wavelength limit $q \rightarrow 0$ the charge density ρ vanishes. For the first component j_1 of the current density we may write $j_1 = 2e(N/V)v_1$, the factor of 2 coming from the oppositely charged particles ($-e$) moving with the opposite velocity $-v_1$. By whatever Newtonian dynamics the Maxwell equations (5.10) are accompanied, it will lead to a factor λ of proportionality between $\partial_t \vec{v}$ and the force to the particles. Combining this, we obtain the plasma frequency ω :

$$\partial_t \vec{v} = \lambda e \vec{E} \quad \Rightarrow \quad \omega^2 = \lambda 2e^2 \frac{N}{V} \quad . \quad (5.11)$$

Guessing the inverse mass λ to be simply $1/T$, and determining N/V from (5.9) as $\approx T^3/8.2$ leads to $\omega^2 \approx e^2 T^2/4.1$ – which is wrong. The simple resolution to this puzzle is that one has to take the relativistic version $\partial_t \vec{k} = e \vec{E}$ of Newton's equation and identify the velocity \vec{v} with a mean value:

$$v_1 = \frac{1}{N} \frac{V}{(2\pi)^3} \int d^3k f(\vec{k}, t) \frac{k_1}{\sqrt{\mu^2 + k^2}} \quad , \quad (5.12)$$

where f is the distribution function (which in equilibrium is the Bose function) and $k_1/\sqrt{\mu^2 + k^2}$ is nothing but the relativistic velocity-momentum relation. Particles when accelerated carry their probability with them, so

$$\partial_t f = -\nabla_k f \partial_t \bar{k} = -eE_1 \partial_{k_1} n \left(\sqrt{\mu^2 + k^2} \right) + O(E^2) . \quad (5.13)$$

Using this in (5.12) and (5.11) we end up with

$$\lambda = -\frac{1}{3} \frac{V}{N} \frac{1}{2\pi^2} \int dk k^2 \frac{k^2}{\mu^2 + k^2} n' \left(\sqrt{\mu^2 + k^2} \right) , \quad (5.14)$$

which when taken at $\mu = 0$ gives $\lambda = (V/N)T^2/18$. Thus we obtain indeed $\omega^2 = e^2 T^2/9$, where one factor $1/3$ came from averaging over the directions of \bar{k} and the other from the integration in (5.14).

Trying now to go beyond leading order, we could take into account the thermal mass $\mu = eT/2$ acquired by the scalar particles. From formula (5.14) we then obtain for the 'classical' next-to-leading-order plasma frequency

$$\omega_{\text{classical}}^2 = \frac{e^2}{3\pi^2} \int dk k^2 \frac{n \left(\sqrt{k^2 + \mu^2} \right)}{\sqrt{k^2 + \mu^2}} \left(2 + \frac{\mu^2}{k^2 + \mu^2} \right) , \quad (5.15)$$

which is to be compared with the true result (5.3). The only difference is the missing back-reaction in the form of corrections involving $Q_0 = \omega_{\text{classical}}$ in the integrand. In (5.3) these terms give rise to an imaginary part when $Q_0 > 2\mu$, which obviously corresponds to pair creation. This cannot be captured in purely classical terms, of course.

Despite the missing terms, (5.15) is extremely close to the correct result (5.5), to wit

$$m^2 + \delta m^2|_{\text{classical}} = \frac{e^2 T^2}{9} \left(1 - \frac{9}{8\pi} e \right) \approx \frac{e^2 T^2}{9} (1 - 0.36e) . \quad (5.16)$$

6. The complete plasmon spectrum at next-to-leading order

Because of the simplicity of the resummed scalar propagator determining the two polarisation functions Π_ℓ and Π_t up to and including the next-to-leading order terms,

these can in fact be calculated analytically for the entire ω - q -plane. In contrast to the QCD counterparts, the spectral density of the dressed scalar propagator $\Delta = 1/(K^2 - \mu^2)$ has no cut contribution:

$$\Delta(K) = \int dx \frac{1}{K_0 - x} \rho(x, k) \quad , \quad \rho(x, k) = \frac{1}{2\bar{\omega}} [\Delta(x - \bar{\omega}) - \Delta(x + \bar{\omega})] \quad , \quad (6.1)$$

where $\bar{\omega} \equiv \sqrt{\mu^2 + k^2}$. Π_ℓ and Π_t as given in (3.5), (3.6) and (3.7) are determined by the two sums (and integrals)

$$\sum \Delta^- \Delta \quad \text{and} \quad \sum \Delta^- \Delta \left[k^2 - \frac{(\bar{k} \bar{q})^2}{q^2} \right] \quad . \quad (6.2)$$

Here, the upper index on Δ^- refers to the shift $K_0 \rightarrow Q_0 - K_0$ as well as to $\bar{k} \rightarrow \bar{q} - \bar{k}$. Hence, the propagator Δ^- has the spectral representation (6.1) with K_0 shifted and with the frequency $\bar{\omega}_- \equiv \sqrt{\mu^2 + (\bar{k} - \bar{q})^2}$ in place of $\bar{\omega}$. Equivalently, we may view it as 'another propagator' taken at $Q_0 - K_0$ and with a spectral density ρ_- as described. At this point we recall two formulae derived earlier, (6.6) and (6.8) in Ref. [10]. They concern the real and imaginary parts, respectively, of expressions like (6.2). The two are related by the dispersion relation

$$\Re \sum \Delta^- \Delta f(\bar{k}) = \int dt \frac{1}{t - \omega} \frac{1}{\pi} \left[\Im \sum \Delta^- \Delta f(\bar{k}) \right]_{\omega=t} \quad . \quad (6.3)$$

We may therefore concentrate on

$$\Im \sum \Delta^- \Delta f(\bar{k}) = \pi \omega T \left(\frac{1}{2\pi} \right)^3 \int d^3 k f(\bar{k}) \int dx \frac{1}{x(x - \omega)} \rho(x, \bar{k}) \rho_-(x - \omega, \bar{k}) \quad , \quad (6.4)$$

where on the right-hand-side only the leading temperature-dependent part resulting from $n(x) \approx T/x$ has been included, assuming that the integrations are restricted to soft arguments of the Bose function either automatically or after suitable subtractions.

The details of first evaluating the simpler imaginary part (6.4) and then using (6.3) are given in the Appendix B. The results may be put together as follows. The prefix δ again indicates that the known leading hard-thermal-loop contributions have been subtracted. Three regions in the ω - q -plane are to be distinguished: $\omega^2 < q^2$ (region I), $q^2 < \omega^2 < 4\mu^2 + q^2$ (region II), and $4\mu^2 + q^2 < \omega^2$ (region III). Then:

$$\delta \Pi_\ell = \frac{e^2 T}{8\pi} \frac{\omega^2 - q^2}{q^2} \left(4\mu + 2i\mathcal{E} - \frac{\omega^2}{q} [\mathcal{R} + i\mathcal{J}] \right) \quad , \quad (6.5)$$

$$\delta\Pi_g = \frac{e^2 T}{8\pi} \left(-8\mu + \frac{4\mu^2 + q^2 - \omega^2}{q} [\mathcal{R} + i\mathcal{J}] \right) , \quad (6.6)$$

$$\begin{aligned} \delta\Pi_t &= \frac{1}{2} (\delta\Pi_g - \delta\Pi_\ell) \\ &= \frac{e^2 T}{16\pi} \left(-4\mu \frac{\omega^2 + q^2}{q^2} - \frac{\omega^2 - q^2}{q^2} 2i\mathcal{E} + \left[4\mu^2 + \frac{(\omega^2 - q^2)^2}{q^2} \right] \frac{1}{q} [\mathcal{R} + i\mathcal{J}] \right) , \end{aligned} \quad (6.7)$$

where

$$\mathcal{R} \equiv \begin{cases} \arctan\left(\frac{\Omega + q}{2\mu}\right) - \arctan\left(\frac{\Omega - q}{2\mu}\right) & \text{in I and III} \\ 2 \arctan\left(\frac{q}{2\mu + |\Omega|}\right) & \text{in II} \end{cases} \quad (6.8)$$

$$\mathcal{J} \equiv \begin{cases} \ln \left| \frac{\Omega q + \omega^2}{\Omega q - \omega^2} \right| & \text{in I and III} \\ 0 & \text{in II} \end{cases} \quad (6.9)$$

$$\mathcal{E} \equiv \begin{cases} \Omega & \text{in I and III} \\ i|\Omega| & \text{in II} \end{cases} \quad (6.10)$$

with Ω , if real, the positive square root of

$$\Omega^2 = \omega^2 \frac{\omega^2 - q^2 - 4\mu^2}{\omega^2 - q^2} . \quad (6.11)$$

As one learns in Appendix B, the combination $\mathcal{R} + i\mathcal{J}$ can be cast into the compact form

$$\mathcal{R} + i\mathcal{J} = i \ln \left(\frac{2\mu - i\mathcal{E} - iq}{2\mu - i\mathcal{E} + iq} \right) . \quad (6.12)$$

The static limit of these results obviously reproduces the ones derived before, (4.8) and (4.10), whereas a bit of calculation is required to verify that the long-wavelength limit $q \rightarrow 0$ indeed gives $\delta\Pi_\ell(Q_0, 0) = \delta\Pi_t(Q_0, 0)$ and coincides with the result derived in (5.3).

Notice that $\delta\Pi_\ell$ and $\delta\Pi_t$ are purely real in region II. In region I, where the hard-thermal-loop contribution has an imaginary part corresponding to Landau damping, there is now a correction term of relative order ϵ , whereas the imaginary part appearing in region III is the leading term resulting from the possibility of the decay of virtual photonic plasma excitations into pairs of scalar quasi-particles.

The real part of $\delta\Pi_\ell$ and $\delta\Pi_t$ is displayed in figs. 3 and 4, respectively. For a comparison to the leading-order result see Fig. 2. In Sect. 5 we have seen that the effect

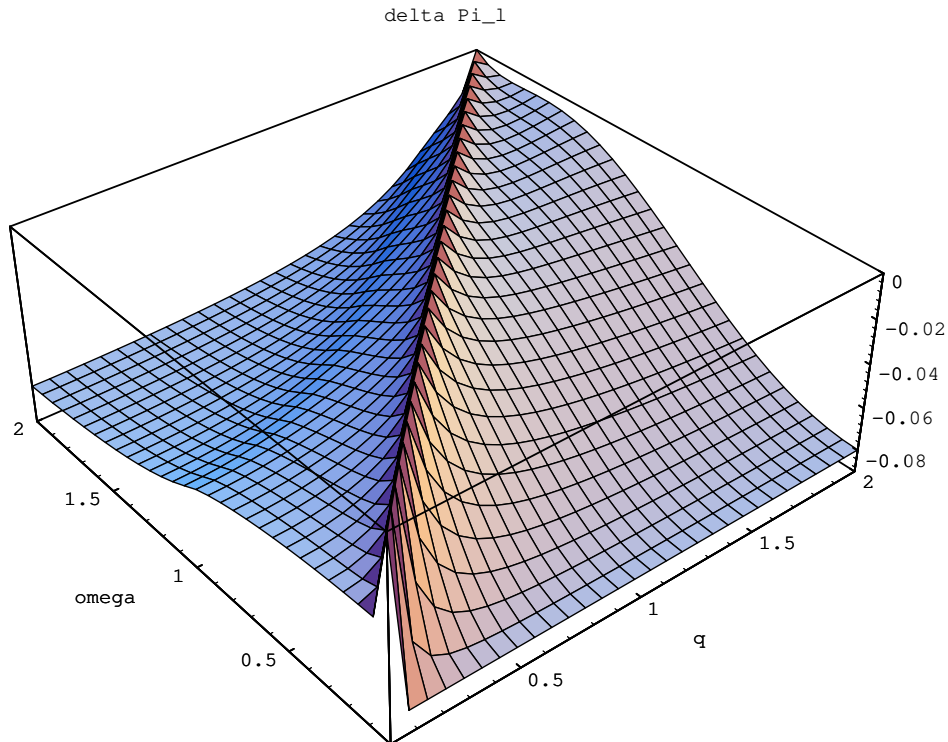


Figure 3: The real part of $\delta\Pi_\ell$ in units of $\epsilon(\epsilon T)^2$. ω and q are in units of ϵT .

of the next-to-leading order contributions is to lower the plasma frequency according to (5.5). We shall now study the whole spectrum of propagating photonic quasi-particles, i.e. $\bar{q} \neq 0$.

In the case of transverse photonic excitations, it turns out that for increasing q the ratio of the corrected frequency $\omega_t(q)$ to the lowest-order one decreases somewhat. For $q \gg \epsilon T$ the effective thermal mass becomes momentum-independent, and the dispersion curve goes over into a perfect mass hyperboloid with asymptotic mass

$$m_\infty^2 + \delta m_\infty^2 = \frac{\epsilon^2 T^2}{6} \left(1 - \frac{3\epsilon}{2\pi}\right) . \quad (6.13)$$

We now turn to the the more interesting longitudinal plasmons, which are collective modes without analogues at zero temperature. As we have seen in chapter 2, their leading-order dispersion curve approaches the light-cone exponentially with increasing q . At the same time the residue of the corresponding pole in the propagator vanishes exponentially [27], quickly rendering them unimportant for larger values of q .

Including the next-to-leading order terms now, we expand everything around the

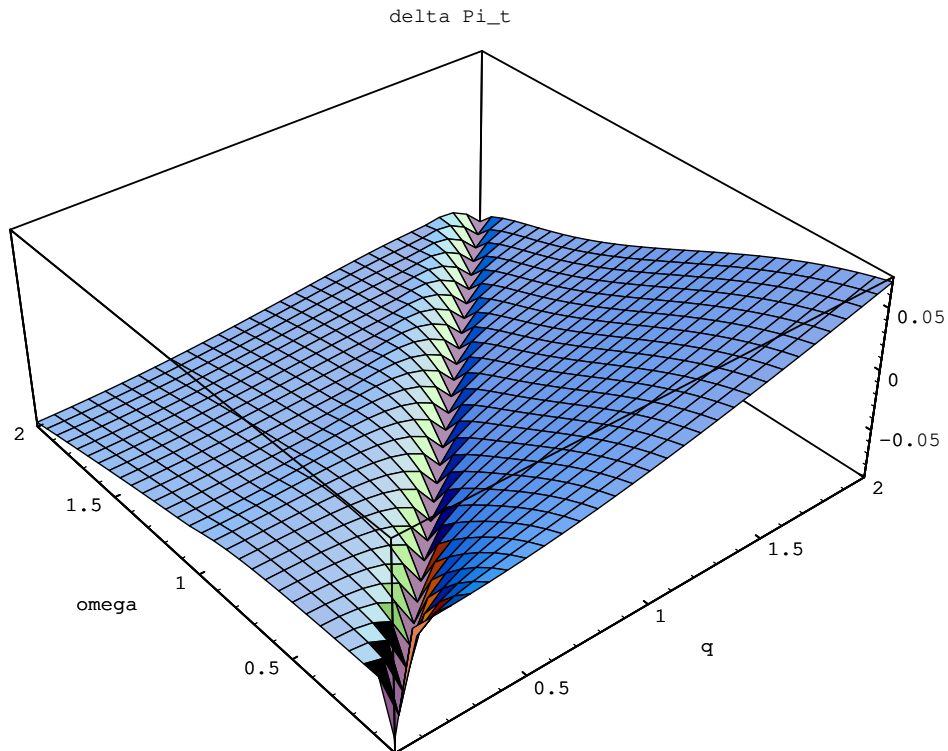


Figure 4: The real part of $\delta\Pi_t$ in the same units as used in Fig. 3.

leading-order result $\omega_0(q)^2$ and rewrite the condition $\omega^2 = q^2 + \Pi_\ell(\omega, q)$ for fixed q as

$$\omega^2 = \omega_0^2 + \delta\Pi_\ell(\omega_0, q) + (\omega^2 - \omega_0^2) \partial_{\omega_0^2} \Pi_\ell^{\text{hard}}(\omega_0, q) + O(\epsilon^2 m^2) \quad (6.14)$$

which gives

$$\delta\omega^2 \equiv \omega^2 - \omega_0^2 = \frac{\delta\Pi_\ell(\omega_0, q)}{1 - \partial_{\omega_0^2} \Pi_\ell^{\text{hard}}(\omega_0, q)} + O(\epsilon^2 m^2) . \quad (6.15)$$

In order to work consistently at relative order ϵ , we drop all contributions that are of higher order. With Π_ℓ^{hard} given by (2.6), (6.15) can be rewritten as

$$\omega^2 = q^2 + \Pi_\ell^{\text{hard}}(\omega_0, q) + \frac{2\omega_0^2}{3m^2 + q^2 - \omega_0^2} \delta\Pi_\ell(\omega_0, q) . \quad (6.16)$$

The resulting dispersion curves for $\epsilon = 0.3, 1$, and 2 are shown in Fig. 5. It is seen that the corrected curves are not just a slightly down-scaled version of the leading-order result. The dispersion curves now hit the light-cone at a finite value of q , above which there is no longer any solution corresponding to propagating plasmons, since for $\omega < q$, Π_ℓ has a large imaginary part $\sim \epsilon^2 T^2$, which prevents the existence of weakly damped excitations.

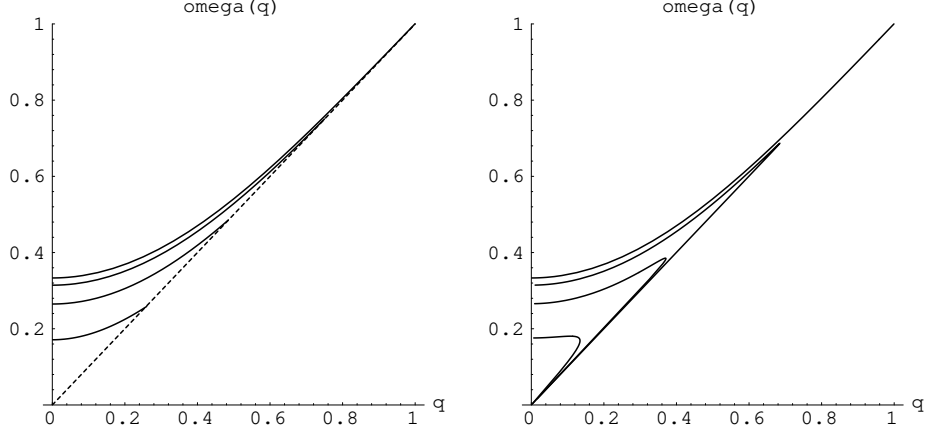


Figure 5: Leading and next-to-leading order dispersion curves of the longitudinal plasmons. The upper curve is the leading-order result; below it are the corrected ones for $\epsilon = 0.3, 1, \text{ and } 2$, respectively. The version on the left consistently discards contributions to $\delta\omega^2(q)$ that are beyond relative order ϵ ; the one on the right gives the location of the poles in the propagator when all next-to-leading order contributions to the self-energy are kept.

The residue of the plasmon pole can be defined in a gauge-independent manner by projecting the propagator $G_{\mu\nu}(Q)$ onto conserved currents J with $Q^\mu J_\mu = 0$ and restricting to spatially longitudinal $J_\ell^i = \frac{q^i q^j}{q^2} J^j$. From the component of the propagator associated with J_ℓ^2 in

$$J^\mu G_{\mu\nu}(Q) J^\nu = -\frac{Q^2}{Q_0^2} \bar{J}_\ell^2 \Delta_\ell + \dots, \quad (6.17)$$

we extract the residue as

$$Z_\ell = \lim_{Q_0 \rightarrow \omega(q)} \frac{Q^2}{Q_0^2} \frac{Q_0^2 - \omega^2(q)}{Q^2 - \Pi_\ell}. \quad (6.18)$$

Other definitions, differing in normalization, are possible [27], but the one given here avoids unnecessary kinematical singularities. Abbreviating $\Phi \equiv \omega_0^2 \Pi_\ell(\omega_0, q)/(\omega_0^2 - q^2)$, (6.18) gives

$$Z_\ell(q) = Z_\ell^{\text{hard}} \left[1 + Z_\ell^{\text{hard}} \left(\partial_{\omega_0^2}(\delta\Phi) + \delta\omega^2 \partial_{\omega_0^2}^2 \Phi_\ell^{\text{hard}} \right) \right], \quad (6.19)$$

where $Z_\ell^{\text{hard}} = 1 - \partial_{\omega_0^2} \Phi^{\text{hard}}$.

In Fig. 6 the leading-order result and the corrected one are compared for the same values of ϵ as in Fig. 5. For both, the residue decreases rapidly for increasing q , with the one-loop-resummed result being smaller than the leading-order one for all $q > 0$ and arriving at zero at a finite value of q .

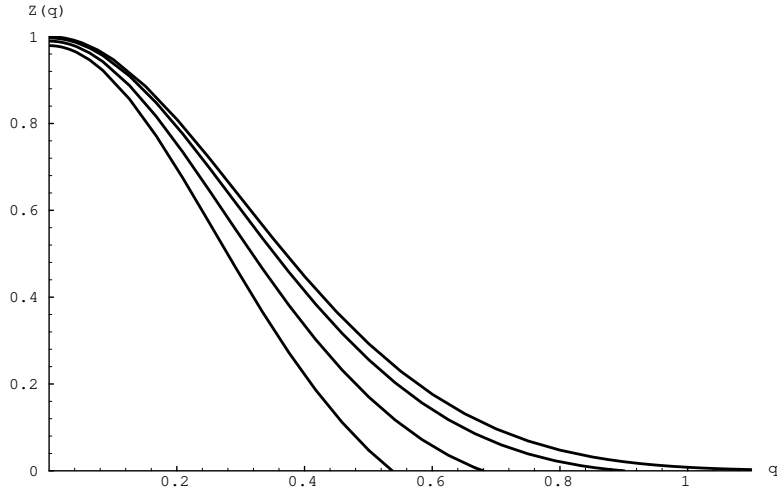


Figure 6: The residues $Z_\ell(q)$ of the plasmon poles associated with the dispersion curves of Fig. 5.

Both results indicate that there is now only a finite range of q and ω in which longitudinal plasmons can exist. However, the very fact that the corrected result changes qualitatively signals that a break-down of the *resummed* perturbation theory is occurring. This qualitative change is possible only by $\delta\Pi_\ell/Q^2$ becoming greater than $\Pi_\ell^{\text{hard}}/Q^2$, although the former is down by one power of ϵ . The reason is that $\Pi_\ell^{\text{hard}}/Q^2$ is logarithmically divergent for $Q^2 \rightarrow 0$, but $\delta\Pi_\ell/Q^2$ diverges even stronger, like $1/\sqrt{Q^2}$ (see (6.5) and (6.10)). Hence, $\sqrt{Q^2}/q \sim \epsilon$ eventually (over-)compensates for the factor of ϵ in $\delta\Pi_\ell$. In fact, had we solved $Q^2 = \Pi_\ell^{\text{hard}} + \delta\Pi_\ell$ for a small finite value of ϵ without expanding $\omega(q) = \omega_0(q) + \epsilon\omega_1(q) + \dots$ as we have done above, the result, shown in the right half of Fig. 5, would have been that the dispersion curve follows the result $\omega_0(q) + \epsilon\omega_1(q)$ closely until Q^2/q^2 becomes comparable to ϵ , after which it bends back sharply, running down hard by the light-cone back to $\omega = q = 0$. If the higher-order corrections to Π_ℓ/Q^2 keep being more and more singular for $Q^2 \rightarrow 0$, this kinky behavior may change from order to order. It is clear that at any finite order one could trust the result only up to a certain small distance from the light-cone.

In fact, this break-down of our resummed perturbation theory is not actually due to higher-order diagrams, but is caused by the break-down of the high-temperature expansion of $\Pi_{\mu\nu}$ at $Q^2 = 0$. Since $\Pi_\ell/Q^2 \equiv -\Pi_{00}/q^2$ and the internal lines in $\Pi_{\mu\nu}$ have been dressed by their thermal masses, a kinematical singularity at $Q^2 = 0$ should not be there at all.

Indeed, as shown in Appendix C, performing the high-temperature expansion directly at the light-cone $Q^2 = 0$, i.e. evaluating $\Pi_{\mu\nu}(Q_0 = q, q)$, the leading and next-to-leading terms are found to be

$$\lim_{Q_0 \rightarrow q} \frac{\Pi_\ell^{\text{resum.}}}{Q^2} = \frac{e^2 T^2}{3q^2} \left[\ln \frac{2T}{\mu} + \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} \right] - \frac{e^2 T \mu}{2\pi q^2} + O(e^2 q^2 T^0) \quad (6.20)$$

with γ_E being Euler's constant and ζ the Riemann zeta function. The first term on the right-hand-side is $\sim \ln(1/e)$, so the original logarithmic singularity evidently got cut off at $\sqrt{Q^2}/(\epsilon T) \sim e$ by the resummation of $\mu \sim \epsilon T$. The equation $1 = \Pi_\ell/Q^2$ with $Q^2 \rightarrow 0$ therefore has the solution

$$q_{\text{crit.}}^2/(\epsilon T)^2 = \frac{1}{3} \left[\ln \frac{4}{e} + \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} \right] - \frac{e}{4\pi} = \frac{1}{3} \ln \frac{2.094 \dots}{e} - \frac{e}{4\pi} . \quad (6.21)$$

Because higher-order diagrams are not singular at $Q^2 = 0$, either, the result (6.21) is stable up to and including order e . The fact that it is non-analytic in e makes it clear that it could not be obtained by organizing the resummed perturbation theory in powers of e . On the other hand, the strictly perturbative next-to-leading order result, depicted in the left half of Fig. 5, becomes inaccurate only close to the light-cone. Indeed, the ‘‘perturbative’’ result for $q_{\text{crit.}} \approx 0.77\epsilon T$ for $e = 0.3$ is not far off from the one of Eq. (6.21), which gives $q_{\text{crit.}} \approx 0.79\epsilon T$. The alternative result in the right half of Fig. 5, however, where $\delta\omega^2$ was not truncated at relative order e , but the entire next-to-leading order result for $\delta\Pi_\ell$ was kept to define a new full propagator, turns out to be qualitatively wrong.

The fact that the resummed result for Π_ℓ/Q^2 is no longer singular at the light-cone also implies that the residue of the plasmons does not completely vanish there. This means that there are longitudinal massless photonic excitations with a fixed value of $q = \omega$. For larger values of ω and q , Q^2 becomes negative, i.e. space-like, and Landau damping sets in. At the level of hard thermal loops, the imaginary part $\sim e^2 T^2$ switches on discontinuously, $\Im m \Pi_\ell(Q^2)/Q^2 = \theta(-Q^2) \frac{3\pi}{2} m^2 \omega / q^3$. Because the logarithmic singularity in the hard thermal loop is removed, this discontinuity is smoothed out. As we show in Appendix C, for $\omega < q$ and $q^2 - \omega^2 \ll e^2 q^2$, an extra factor $\exp(-e\sqrt{\frac{q}{8(q-\omega)}})$ arises — the imaginary part sets in with all of its derivatives vanishing. There is now a finite range in the space-like region with small damping. Consequently, the plasmons are removed from the spectrum only for $(q - \omega)/(\epsilon T) \gtrsim e^2$.

This phenomenon of a finite range of q and ω , where longitudinal plasmons can exist, was predicted in the case of QCD in [28, 29]. The simplicity of scalar electrodynamics

has allowed us to study it in full detail. In the case of non-ultrarelativistic QED (i.e. with electron mass $m > T$), a finite range for the longitudinal plasmons has been found already in 1961 by Tsytovich [30].

The simple step beyond the resummed perturbation that was necessary in the above calculation of q_{crit} might perhaps shed some light on how similar failures with Braaten-Pisarski resummation could be overcome. In Eq. (6.20) we have witnessed the hard thermal loop itself becoming subject to change under resummation of the hard thermal loops close to its singularities. So there are cases where the resummed perturbation theory cannot be organized in excess powers of e . A perhaps similar failure was encountered in a recent attempt to calculate the production rate of real non-thermal photons in a QCD plasma by Braaten-Pisarski resummation [31].

We close this chapter by suggesting that such kinematical singularities in the Braaten-Pisarski scheme for QCD can be taken care of by extending resummation to the hard thermal loops themselves in the same way as we have done above for scalar electrodynamics. At hard loop momenta it is in fact not necessary to use the full complicated form of resummed propagators and vertices in QCD for deriving the modification of the hard thermal loops near the kinematical singularities they otherwise give rise to. The latter are caused by the masslessness of the propagators in the bare hard thermal loop, so it will be the true quasi-particle poles (rather than cut contributions) that are important. At large loop momenta $\gg gT$, and only there, these are given by momentum-independent masses (cp. Eq. (6.13)), whereas the additional collective modes have exponentially vanishing residues at leading order. Only at relative order $O(g)$ the form of the dressed propagators and vertices at soft momenta become relevant, whereas the leading terms are determined by simple massive loop integrals. The QCD result analogous to the part $\propto T^2$ in (6.20) has the same form for the bosonic (gluon) contributions with $e^2 \rightarrow g^2 N$, $\mu \rightarrow m_\infty$. In the case of fermionic (quark) contributions a similar formula holds, obtained by replacing $\gamma_E \rightarrow \gamma_E - \ln 2$ and taking now the asymptotic value of the thermal quark mass. With N_f flavors, and $N_+ \equiv N + N_f/2$, this yields at leading order

$$q_{\text{crit}}^2 = 3m^2 \left[\frac{N}{N_+} \ln \frac{2\sqrt{6}}{g\sqrt{N_+}} + \frac{N_f}{2N_+} \ln \frac{8}{g\sqrt{C_F}} + \frac{1}{2} - \gamma_E + \frac{\zeta'(2)}{\zeta(2)} \right], \quad (6.22)$$

where $C_F = (N^2 - 1)/(2N)$. (This result differs somewhat from the one of Ref. [28], but this seems to be due simply to an incorrect evaluation of the integrals given therein, with which we otherwise agree.) We intend to cover the case of QCD more fully in a future publication. Let us just note here that q_{crit}^2 , as given by (6.22) goes down to zero when

$g = 1.48\dots$ (for $N = 3, N_f = 0$) or $g = 1.31\dots$ (for $N = 3, N_f = 2$). Clearly, next-to-leading results are becoming decisive here to determine the fate of the QCD plasmons.

7. Resummation of scalar damping

Up to now we have studied exclusively the effects of the thermal mass μ acquired by the scalar fields, i.e. of the leading term of the self-energy of the scalars. A calculation of the next-to-leading order terms in this self-energy would already involve the more complicated thermal photon propagator, making it necessary to resort to numerical integrations. However, the limiting case of large scalar momenta turns out to be accessible by purely analytical means.

For external scalar momentum Q with $Q_0 \sim T$, the next-to-leading order terms can be most easily derived by using the static ring resummation (see Sect. 4B). For this we only need to resum the zero-mode propagators

$$S(0, k) = \frac{1}{k^2 + \mu^2} \quad , \quad G(0, k) = \frac{U \circ U - g}{k^2} + (\alpha - 1) \frac{\widetilde{K} \circ \widetilde{K}}{k^4} - \frac{U \circ U}{k^2 + m_{el}^2} \quad (7.1)$$

of scalar and gauge fields, respectively, with the four-vector $\widetilde{K} \equiv K - (KU)U$ and $m_{el}^2 = e^2 T^2/3$, see (2.9),(2.10) and (2.2).

The correction term to the scalar self-energy, for large $Q_0, q \sim T$, is then given by

$$\delta\Xi = \frac{e^2 T}{(2\pi)^3} \int d^3 k \left[g^{\mu\nu} G_{\mu\nu}(K) - \frac{(2Q - K)^\mu (2Q - K)^\nu}{(K - Q)^2} G_{\mu\nu}(K) - \frac{(Q + K)^\mu (Q + K)^\nu}{K^2 - \mu^2} G_{\mu\nu}^0(Q - K) - (\mu, m_{el.} \rightarrow 0) \right]_{K_0=0} \quad , \quad (7.2)$$

where, while the first term clearly corresponds to the tadpole diagram, the loop contributes twice, once with the hard momentum Q running through the scalar line (second term) and once with Q in the photon line (third term). Only the propagator that carries the soft momentum K requires resummation. On mass-shell, $Q^2 = \mu^2$, the real part of (7.2) gives the next-to-leading order term in the thermal mass of energetic scalar particles, yielding (after some tedious integrations)

$$\delta\mu^2 = -\Re \delta\Xi \Big|_{Q^2=\mu^2} = -\frac{e^2 T}{2\pi} \left(m_{el.} + \frac{1}{2}\mu \right) = -\frac{1}{2\pi} \left(\frac{4}{\sqrt{3}} + 1 \right) e\mu^2 \quad . \quad (7.3)$$

The imaginary part, which determines the damping rate ζ of energetic scalar particles defined by

$$\Im m \delta \Xi \Big|_{Q^2=\mu^2} \equiv 2Q_0 \zeta \quad , \quad (7.4)$$

however leads to an infrared-divergent expression, which is in fact virtually identical to what is obtained in QED or QCD [9]

$$\begin{aligned} \zeta &= \frac{e^2 T}{4\pi} \Im m \int_{-1}^1 dz \int_{\lambda}^{\infty} dk \frac{k}{z+k/(2q) - i\varepsilon} \left[\frac{1}{k^2} - \frac{1}{k^2 + m_{el.}^2} - (1-\alpha) \frac{z^2}{k^2} \right] \\ &= \frac{e^2 T}{4\pi} \left[\ln \frac{m_{el.}}{\lambda} + O(\lambda^0) \right] \quad . \end{aligned} \quad (7.5)$$

There is a logarithmic singularity caused by the absence of screening for static transverse electromagnetic fields. In non-Abelian theories the infrared singularity in (7.5) could perhaps be removed by the appearance of a magnetic screening mass, but in the Abelian case $\Pi_{ii}(0, k) \propto k^2$. In Ref. [29, 32, 33] it was suggested that this infrared divergence is instead cut off by the damping of the energetic particles itself, entailing $\lambda \sim e^2 T$ and

$$\zeta = \frac{e^2 T}{4\pi} \ln \frac{1}{e} \quad . \quad (7.6)$$

However, this cut-off depends on having the external momentum slightly off the assumed complex pole [34] and this in turn gives rise to gauge dependences at the order $e^2 T \lambda^0$. There is still an on-going discussion on how to remedy this situation and to become able to go beyond the leading logarithmic term of (7.6), see [35]. In particular it seems likely that at least in Abelian theories the propagators of moving quasi-particles do not have simple poles on the unphysical sheet but rather branch singularities.

At any rate, in order to go beyond the approximations leading to (7.5), one evidently should include self-consistently physics at scales below eT . Taking the result of (7.6) for granted, one has $\zeta \sim \mu e \ln(1/e) \gg \delta\mu$, so seemingly damping effects are the next important corrections to be included in an improved resummation scheme. The consequences of this have been analysed in Ref. [32] in the case of QED, and recently also in scalar electrodynamics in Ref. [12]. Here we shall follow the lines of Ref. [32] and correct what has been presented in Ref. [12].

If one tries to improve the propagator for scalars at large momenta by including the unexpectedly large damping (7.6), one finds that in contrast to the thermal mass a small but finite damping constant may even change the leading hard thermal loop result for small momenta. It does so, however, by violating gauge invariance [32]. In particular one

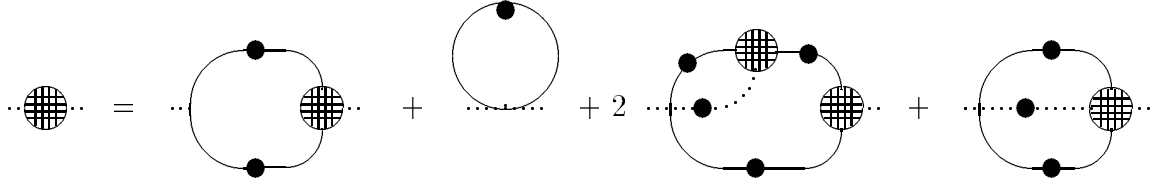


Figure 7: Schwinger-Dyson equation for the photon self-energy. The lines with a black bullet are the resummed propagators of figure 1.

obtains

$$\Pi_{00}(Q_0, 0) = \frac{2i\zeta}{Q_0 + 2i\zeta} \frac{e^2 T^2}{3}, \quad (7.7)$$

which is inconsistent with transversality of the self-energy. This seems to have gone unnoticed in a recent paper on the analytic properties of finite-temperature scalar electrodynamics Ref. [40], where it was claimed that inclusion of damping would restore analyticity of the zero-momentum limit of thermal Green's functions.

In order to include a finite damping self-consistently one obviously has to consider vertex corrections. Lacking an effective action, which would furnish corrected Feynman rules and compensating (“thermal”) counterterms, one may analyse the possible contributions from vertex functions through the coupled Schwinger-Dyson equations they have to satisfy. The Schwinger-Dyson equation for the photon self-energy is depicted in Fig. 7. It involves the fully dressed scalar propagator as well as bare and dressed vertices. In Ref. [32, 12] it has been argued that in the infrared the dominant corrections coming from the vertices are given by ladder diagrams and that their leading terms can be obtained by solving the Ward identities of vertex diagrams in terms of the self-energies and discarding possible transverse contributions.

In order to avoid the intricacies of analytic continuation, in particular that of vertex diagrams, we shall like Ref. [32, 12] work in the real-time (Schwinger-Keldysh) formalism [36]. In this formalism perturbation theory is based on a time contour that runs along the real axis and back, which leads to a doubling of the fields corresponding to the part of the contour they are living on, and to a contour-ordered propagator that is a matrix with respect to this bisection. If $a, b = 1, 2$ denotes the field type, a scalar propagator is now given by the matrix

$$-i\Delta(x-y) = \langle T_c \phi_a(x) \phi_b(y) \rangle = \begin{pmatrix} \langle T \phi(x) \phi(y) \rangle & \langle \phi(y) \phi(x) \rangle \\ \langle \phi(x) \phi(y) \rangle & \langle \bar{T} \phi(x) \phi(y) \rangle \end{pmatrix}, \quad (7.8)$$

where the contour-ordering T_c is broken up into time-ordering T for type-1 fields and anti-time-ordering \bar{T} for type-2. Mixed propagators correspond always to a fixed order of the operators because type-2 fields are always further down the contour than type-1.

Analytic continuation from the imaginary-time formalism eventually leads to a collection of Green's functions which depend on the prescriptions chosen for each external line. In the real-time formalism, these are given by the various linear combinations of the components of the corresponding matrix quantities [36, 37]. The retarded and advanced propagators are given by

$$\Delta_R(P) = \Delta_{11}(P) - \Delta_{12}(P) \ , \quad \Delta_A(P) = \Delta_{11}(P) - \Delta_{21}(P) \ . \quad (7.9)$$

The symmetric combination, defined by

$$\Delta_P(P) = \Delta_{11}(P) + \Delta_{22}(P) \equiv \Delta_{12}(P) + \Delta_{21}(P) \ , \quad (7.10)$$

is, in thermal equilibrium, related to the former,

$$\Delta_P(P) = (1 + 2n(P_0))(\Delta_R(P) - \Delta_A(P)) \ . \quad (7.11)$$

Whereas at tree level, vertices are exclusively connecting either type-1 or type-2 fields, there is a large variety of full vertex functions which differ in their analytic properties. In the first diagram on the right-hand-side of Fig. 7 we have a full vertex function connecting an incoming scalar, a photon, and an outgoing scalar. Defining retarded vertices with respect to these three lines in turn we have

$$\Gamma_{R(i)} = \sum_{a,b} \Gamma_{1ab} \ , \quad \Gamma_{R(\gamma)} = \sum_{a,b} \Gamma_{a1b} \ , \quad \Gamma_{R(o)} = \sum_{a,b} \Gamma_{ab1} \ , \quad (7.12)$$

where the indices are in the order of incoming scalar (i), photon, and outgoing scalar (o). There are various symmetric combinations, of which we shall need only

$$\Gamma_P = \sum_{a,b} \Gamma_{aba} \quad (7.13)$$

which is the one that singles out the photon line.

With these definitions the retarded piece of the first diagram on the right-hand-side of Fig. 7 is given by

$$\begin{aligned} \Pi_{\mu\nu}^{(1)R}(Q) &\equiv \Pi_{\mu\nu}^{(1)11}(Q) + \Pi_{\mu\nu}^{(1)12}(Q) \\ &= \frac{e}{2} \int \frac{d^4 K}{(2\pi)^4} (2K + Q)_\mu \left\{ \Delta_R(Q + K) \Gamma_\nu^{R(i)} [\Delta_P(K) + \Delta_R(K)] \right. \\ &\quad + [\Delta_P(Q + K) + \Delta_A(Q + K)] \Gamma_\nu^{R(o)} \Delta_A(K) \\ &\quad \left. + \Delta_R(Q + K) \Gamma_\nu^P \Delta_A(K) \right\} \ . \end{aligned} \quad (7.14)$$

The second diagram involves only the bare 4-vertex and therefore differs from the usual tadpole (or sea-gull) diagram only in that the full propagator has to be taken. In terms of the real-time quantities it is simply given by

$$\Pi_{\mu\nu}^{(2)R}(Q) = i\epsilon^2 g_{\mu\nu} \int \frac{d^4 K}{(2\pi)^4} [\Delta_P(K) + \Delta_R(K) + \Delta_A(K)] . \quad (7.15)$$

The other diagrams are of higher (explicit) power in ϵ and we shall not need them in the following discussion.

We wish to consider a resummation of both thermal masses and the “anomalously” [32] large damping of the scalars. We therefore begin by adding the damping to the retarded and advanced scalar self-energies according to

$$-\Xi_R(P) = \mu^2 - 2iP_0\zeta , \quad -\Xi_A(P) = \mu^2 + 2iP_0\zeta = -\Xi_R^* = -\Xi_R(-P) . \quad (7.16)$$

The momentum dependence of these additions spoil the Abelian Ward identities and this leads to a violation, proportional to ζ , of the generally valid transversality of the photon self-energy already at the order of $\epsilon^2 T^2$, i.e. at the level of hard thermal loops [32]. In Ref. [32] it has been shown however that higher-order vertex diagrams start to contribute because with the above modification of the propagators there are now contributions proportional to ζ from would-be pinch singularities which are now cut off by the small but finite damping. It turned out that the dominant vertex contributions just correspond to solving the Ward identities according to

$$\begin{aligned} \Gamma_\mu^{R(i)}(K+Q, K) &= -\left(\Gamma_\mu^{R(o)}(K+Q, K)\right)^* \\ &= -ie(2K+Q)_\mu \left(1 + \frac{1}{2KQ+Q^2} [\Xi_R(K+Q) - \Xi_R(K)]\right) \\ &= -ie(2K+Q)_\mu \left[1 + \frac{2i\zeta Q_0}{2KQ+Q^2}\right] , \end{aligned} \quad (7.17)$$

$$\begin{aligned} \Gamma_\mu^P(K+Q, K) &= -ie(2K+Q)_\mu \frac{1}{2KQ+Q^2} [\Xi_P(K+Q) - \Xi_P(K)] \\ &= \frac{-ie(2K+Q)_\mu}{2KQ+Q^2} (4i\zeta) \left[Q_0 \right. \\ &\quad \left. + 2(K_0+Q_0)n(K_0+Q_0) - 2K_0n(K_0)\right] . \end{aligned} \quad (7.18)$$

(In the case of QED [32] only Γ^P needed correction because there the damping of the electrons arises from a constant contribution to the self-energy.)

Inserting the corrected vertices into (7.14) and keeping only terms involving the distribution function n , one finds for the dressed one-loop contribution to the retarded photon self-energy

$$\begin{aligned}
\Pi_{\mu\nu}^R(Q) &= -2ie^2 \int \frac{d^4K}{(2\pi)^4} (2K+Q)_\mu (2K+Q)_\nu \left\{ \right. \\
&\quad \Delta^R(K+Q) \left[1 + \frac{2i\zeta Q_0}{2KQ+Q^2} \right] n(K_0) \left(\Delta^R(K) - \Delta^A(K) \right) \\
&\quad \left. + 2i\zeta \frac{(K_0+Q_0)n(K_0+Q_0) - K_0n(K_0)}{2KQ+Q^2} \Delta^R(K+Q)\Delta^A(K) \right\} \\
&\quad + 2ie^2 g_{\mu\nu} \int \frac{d^4K}{(2\pi)^4} n(K_0) \left(\Delta^R(K) - \Delta^A(K) \right) . \tag{7.19}
\end{aligned}$$

By a change of variables, this can be brought into a form that can be more easily compared with the corresponding expressions given in Ref. [12],

$$\begin{aligned}
\Pi_{\mu\nu}^R(Q) &= 2ie^2 \int \frac{d^4K}{(2\pi)^4} (2K+Q)_\mu (2K+Q)_\nu n(K_0) \left\{ \right. \\
&\quad \Delta^R(K+Q)\Delta^R(K) \frac{4i\zeta K_0}{2KQ+Q^2} \\
&\quad \left. - \Delta^R(K+Q) \left(\Delta^R(K) - \Delta^A(K) \right) \left[1 + 2i\zeta \frac{2K_0+Q_0}{2KQ+Q^2} \right] \right\} \\
&\quad + 2ie^2 g_{\mu\nu} \int \frac{d^4K}{(2\pi)^4} n(K_0) \left(\Delta^R(K) - \Delta^A(K) \right) . \tag{7.20}
\end{aligned}$$

again up to temperature-independent contributions. This does not completely agree with Eq. (25) of [12]: there the first term in the first integrand was omitted, and in the second integral a dressed 4-vertex was used, contrary to what the Schwinger-Dyson equation prescribes. Moreover, it was assumed in Ref. [12] that the dominant contributions from the vertices would be the correction terms, because they involve a factor $1/(2KQ+Q^2)$, so the tree-level vertices were deleted. Further, $(2K+Q)_\mu/(2KQ+Q^2)$ was simplified to K_μ/KQ . Neither of these truncations is justified, however, as a complete calculation readily shows. The part of (7.14) which involves only the tree vertices does produce the same leading temperature results as without resummation of the scalar damping plus terms proportional to ζ , and the vertex corrections that are proportional to ζ stay so when the integrals are evaluated. The replacement $(2K+Q)_\mu/(2KQ+Q^2) \rightarrow K_\mu/KQ$ on the other hand changes the contributions coming from the vertex corrections, but even after discarding the pure tree-level contributions the leading temperature contributions

are not restored. For instance, the Debye mass Π_{00} comes out minus half the correct value. In Ref. [12], the missing factor of 2 was compensated by taking the first diagram in the right-hand-side of Fig. 7 twice; the minus sign seems to have been simply lost.

Another thing that goes wrong with the expressions put forward in Ref. [12] is in fact gauge invariance. As a special case of the transversality of the photon self-energy one has to have $\Pi_{00}(Q_0, 0) \equiv 0$. The expressions given in Ref. [12] violate this requirement already in the leading terms of the high-temperature expansion, both with and without the unjustified simplifications performed therein: $\Pi_{00}(Q_0, 0)$ in each case turns out to be proportional to $ie^2T^2\zeta/Q_0$.

Coming back to the complete results (7.19,7.20), they can be evaluated by closing the contour in the K_0 -integration. In the upper half plane, there are poles from the advanced Green function $G^A(K)$ and an infinite set of poles from the Bose distribution function n . There are also potential poles from the denominators of the vertex corrections, which are ambiguous because the Ward identities do not determine the prescription for these. We shall therefore consider only such cases where these denominators cannot give rise to poles on algebraic grounds. This holds generally for $Q_0 = 0$ and with $Q_0 \neq 0$ at least for $\Pi_{00}(Q_0, 0)$, which is of special interest as a non-zero result indicates a loss of gauge invariance.

In the form (7.20) there are nonvanishing contributions from the poles of the Bose distribution function. For example, with $\bar{q} = 0$, $Q_0 \ll T$, they contribute at leading order in T

$$\begin{aligned}
& -16e^2T\zeta \sum_{n=1}^{\infty} \int \frac{d^{(3-\varepsilon)}k}{(2\pi)^{(3-\varepsilon)}} \frac{(2\pi inT\delta_{\mu}^0 + k_i\delta_{\mu}^i)(2\pi inT\delta_{\nu}^0 + k_j\delta_{\nu}^j)}{iQ_0((2\pi nT)^2 + k^2)^2} \\
& = \frac{ie^2\zeta T^2}{3Q_0} \left(\delta_{\mu}^0\delta_{\nu}^0 + \delta_{ij}\delta_{\mu}^i\delta_{\nu}^j \right) + O(\varepsilon) \ , \tag{7.21}
\end{aligned}$$

which is due to the first term in the first integral of (7.20). The omission of this contribution is what led to the (unnoticed) loss of gauge invariance in Ref. [12]. In the seemingly more complicated form (7.19), however, it turns out that in all the considered cases the contributions from the poles of the distribution functions which would be either proportional to T^2 or T exactly cancel upon integration over spatial momenta; only the poles of $G^A(K)$ contribute by closing the contour around the upper half plane.

At leading order T^2 , the correction terms to the vertices in Eq. (7.19) are found to exactly cancel the result (7.7) which indicated that dressing of only the propagators

violates gauge invariance. However, contrary to what has been claimed both in Ref. [32] and [12], the one-loop resummed expressions are not completely independent of ζ . There are still contributions $\propto T$ in (7.19) that are dependent on ζ . Checking gauge invariance first, we find for the leading term that violates transversality of the photon self-energy

$$\Pi_{00}(Q_0, 0) = -\frac{i e^2 \zeta^3 T}{\pi \mu (Q_0 + 2i\zeta)} . \quad (7.22)$$

For $Q_0 \lesssim \zeta$, this is of order $e^5 \ln^3(1/e) T^2$; for larger Q_0 it is of even higher order. (A similar result is obtained also in the case of QED.)

Inspecting now potential contributions to $\Pi_{\mu\nu}(0, q)$, we find that the only new corrections are of precisely the order where gauge invariance is being lost,

$$\Pi_{00}(0, q \rightarrow 0) = \frac{e^2 T^2}{3} - \frac{e^2 T \mu}{2\pi} - \frac{e^2 T \zeta^2}{4\pi \mu} , \quad (7.23)$$

$$\Pi_{ii}(0, q \rightarrow 0) = \frac{3e^2 T \zeta^2}{8\pi \mu} . \quad (7.24)$$

The latter result, which if correct would correspond to a magnetic screening mass, is by the way just another expression of the fact that gauge invariance has been lost — in scalar electrodynamics one can easily prove the absence of a magnetic screening mass to all orders of perturbation theory [16, 38].

The form of the above results in fact shows why the resummation of scalar damping is bound to fail eventually. The contributions involving ζ are obviously dominated by the infrared region of integration, which is effectively cut off by the thermal mass μ . The damping of the scalar fields on the other hand has its simple constant form certainly only for momenta $\gg \epsilon T$, so precisely where interesting effects might show up, this resummation is clearly insufficient.

In view of recent attempts to go beyond the Braaten-Pisarski resummation scheme by a simple resummation of damping effects [39], we close by emphasizing again that a resummation of damping only in the propagators proves to be incorrect and insufficient already in the Abelian case. Vertex corrections are of vital importance to keep gauge invariance, and they tend to just undo the resummation of damping in the propagators. Moreover, in the infrared regime where a resummation of damping might eventually become important, the full-fledged self-energy and vertex functions will be relevant, which can hardly be approximated by a constant damping term.

8. Summary and conclusions

The simplicity of hot scalar electrodynamics has allowed us to determine the complete next-to-leading order corrections to the spectrum of photonic quasi-particles. For the most part, this was a straightforward application of the resummation scheme of Braaten and Pisarski developed for QCD, with the bonus of being able to do all calculations analytically, so that we did not depend on numerical approximations. As we have shown in detail in the Appendix, these calculations can be most efficiently done by evaluating first the simpler imaginary parts and exploiting dispersion relations to obtain the complete expressions.

Against these results we tested the simplified resummation scheme of Ref. [6], which is a systematic extension of the old ring resummation prescription of dressing zero modes only, and found that it works only for static quantities (for which it has been put forward originally), whereas non-static Green's functions turn out to receive next-to-leading-order corrections also from non-static modes. In the case of static Green's functions, on the other hand, the resummation scheme of Ref. [6] constitutes a computational simplification, which is of course more pronounced in non-Abelian applications.

The next-to-leading order results for the plasmon spectrum constituted mostly small corrections to the ones derived from a bare one-loop calculation, but on two occasions they led to a qualitative change.

First, the bare one-loop result gave a nonvanishing result for plasmon damping of the order of $e^2T \sim em$. Like in QCD, this result turned out to be inaccurate, although unlike the bare one-loop gluonic plasmon damping, it was positive and gauge-independent. The resummed result revealed that the photonic plasmon damping is in fact zero at relative order e , because the thermal scalars are too massive to be produced by pair creation and also because at this order there is no Landau damping contribution from scalar quasi-particles. The obvious moral is that gauge-independence is only a necessary criterion for a correct result, not a sufficient one.

Second, the next-to-leading order results for the longitudinal plasmons turned out to break down very close to the light-cone. At leading order, the longitudinal plasmon branch of the dispersion curves approaches the light-cone exponentially. Very close to the light-cone, the next-to-leading order corrections derived strictly along the lines of

the Braaten-Pisarski resummation scheme begin to dominate over the leading-order contributions, so this scheme ceases to be actually perturbative. The reason for this is a singularity, at the light-cone, in the hard thermal loops that are being resummed. A more persistent resummation removes this singularity and allowed us to calculate the true next-to-leading order dispersion curve, which hits the light-cone at a finite value of momentum $\sim \epsilon T \ln^{1/2}(1/\epsilon)$, and continues even a short distance below the light-cone before Landau damping prohibits the existence of weakly damped excitations. We have argued that the same phenomenon occurs in QCD, as has been done before in Ref. [28], whose corresponding result we believe to have corrected. The encountered break-down of the canonical Braaten-Pisarski scheme and its redress could be relevant also with respect to another recently observed break-down of this scheme in Ref. [31], which also is due to an irregular behavior of the hard thermal loops at the light-cone.

Finally, we have investigated another modification of the Braaten-Pisarski scheme, which attempts the next step by resumming also damping contributions from internal lines. In accord with Ref. [32], where similar issues have been studied in the case of fermionic QED, we found that there are no contributions to the photonic polarization tensor from such a further resummation, if and only if the vertices are corrected in addition to the propagators. We disagree with the findings of Ref. [12], however, where this sort of resummation was examined for scalar electrodynamics. In contrast to Ref. [12], we found that the corrections to the vertices do not dominate over tree contributions, an assumption which led Ref. [12] to postulate a new diagrammatic framework. Instead, both contribute at the same order of magnitude and give correct results with the conventional set of diagrams as prescribed by the Schwinger-Dyson equations. Further, we observed that the assumption of a constant damping term can be sufficient only for leading-order results, and that it leads to a violation of gauge invariance at higher orders even with vertices corrected such that they satisfy the Ward identities. However, without correcting the vertices, gauge invariance is lost already at lowest order. This seems to have gone unnoticed in a recent paper on analyticity properties of scalar electrodynamics, Ref. [40]. It also casts doubt on recent attempts to improve on perturbative results by an ad hoc resummation of damping contributions in QCD [39] only in propagators.

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Appendix A

In this Appendix, which supplements Sect. 3, we briefly go through the construction of the effective action summarizing the hard thermal loops along the lines of Refs. [22, 41, 42] restricted to the case of scalar electrodynamics. As was first pointed out in [22], this effective action is completely determined by the bilinear part of the generating functional $S_{\text{eff}}^{(2)}$ that is proportional to $\epsilon^2 T^2$, i.e. by the HTL self-energies.

To construct $S_{\text{eff}}^{(2)}$, or rather its nontrivial part $\delta S^{(2)}$, we sandwich $-\mu^2$ with scalar fields and $\frac{1}{2}II^{\mu\nu}$ with photon fields. With the leading term of (2.4), if rewritten in terms of Y , [20] one obtains

$$\delta S^{(2)} = -\mu^2 \int^\beta \varphi^* \varphi + \frac{3}{2} m^2 \int^\beta \ell'' \quad , \quad \ell'' = \int_\Omega \left(A_0 A_0 + (\partial_0 A Y) \frac{1}{d} A Y \right) \quad (\text{A.1})$$

$$\text{with} \quad A Y = A_0 - \bar{e} \bar{A} \quad \text{and} \quad d \equiv Y_\mu \partial_\mu = \partial_0 + \bar{e} \nabla \quad . \quad (\text{A.2})$$

Now consider the gauge variations $\delta_\omega A_0 = \partial_0 \omega$ and $\delta_\omega A Y = d\omega$. This gives

$$\delta_\omega \int^\beta \ell'' = \int^\beta \int_\Omega \left(2A_0 \partial_0 \omega + (\partial_0 d\omega) \frac{1}{d} A Y + \omega \partial_0 A Y \right) = 0 \quad , \quad (\text{A.3})$$

i.e. gauge invariance of $S_{\text{eff}}^{(2)}$. The vanishing of (A.3) is due to $\int_\Omega \bar{e} = 0$ in the first term (which allows us to rewrite it as $2A Y \partial_0 \omega$) and partial integrations in the two others. Therefore, gauge invariance does not require the existence of HTL vertices. Moreover, since all HTL's are uniquely determined by $S_{\text{eff}}^{(2)}$ [22], this indeed proves the absence of HTL vertices (cf. Sect. 3).

While the generating functional thus obtained, $S_{\text{eff}} = S_{\text{eff}}^{(2)}$, is gauge invariant, the corresponding Lagrangian density is not. A manifestly gauge invariant (mgi) density ℓ' can be obtained by adding to ℓ'' a suitable total derivative $\partial_\mu J^\mu$. It is not hard to construct this current as $J^\mu = -\int_\Omega Y^\mu A_0 \frac{1}{d} A Y$. Thus:

$$\ell' = \ell'' + \partial_\mu J^\mu = \int_\Omega (\partial_0 A Y - d A_0) \frac{1}{d} A Y \quad . \quad (\text{A.4})$$

(A.4) is easily checked to be mgi indeed: $\delta_\omega \ell' = 0$. But in this check $\int_\Omega \bar{e} = 0$ has to be exploited, and in fact one can do better. The density ℓ' can be expressed entirely by mgi objects, which are the fields $\vec{E} = -\nabla A_0 - \partial_0 \bar{A}$ and $\vec{B} = \nabla \times \bar{A}$ or combinations of them as e.g.

$$f_\mu \equiv Y^\nu F_{\nu\mu} = d A_\mu - \partial_\mu A Y \quad , \quad \text{i.e.} \quad (\text{A.5})$$

$$f_0 = -\bar{e}\bar{E} \quad \text{and} \quad \{f^k\} = \bar{f} = -\bar{E} - \bar{e} \times \bar{B} \quad . \quad (\text{A.6})$$

The mgi action in the QCD case was constructed independently by Frenkel and Taylor [41] and by Braaten and Pisarski [42], where the latter just guessed the only possible form in accord with known facts and principles. Ref. [41] also gives a proof of its equivalence with the non-mgi version. Here we read off the mgi action for scalar electrodynamics from eq. (20) of [42] by taking its abelian equivalent:

$$\ell = -\frac{1}{2} \int_{\Omega} f_{\mu} \frac{1}{d^2} f^{\mu} \quad . \quad (\text{A.7})$$

Note that the trace over generators has turned into a factor $1/2$. The two versions ℓ' and ℓ , both mgi, could differ by a total derivative which is mgi too. However, there is no such difference, as we show next. This is in fact already covered by the proof for the non-Abelian case [41], but in the much simpler Abelian case, we can do a straightforward calculation which starts with ' ℓ' ' and results in ' ℓ '.

In a first step we try to eliminate the potentials from the angular integral ℓ' , (A.4), in favour of the fields \bar{E} , \bar{B} . This can be done by using the first two of the following five identities:

$$\frac{\bar{e}}{d} = X\nabla \quad , \quad \frac{\bar{e} \circ \bar{e}}{d} = -X\partial_0 \left(1 + \frac{3}{2}Z\right) - \frac{1}{2d}Z \quad , \quad (\text{A.8})$$

$$\frac{\bar{e}}{d^2} = \frac{1}{d}X\nabla \quad , \quad \frac{1}{d^2} = \frac{1}{\partial^2} \quad , \quad (\text{A.9})$$

$$\frac{\bar{e} \circ \bar{e}}{d^2} = X(2 + 3Z) + \frac{1}{d^2}(1 + Z) \quad , \quad (\text{A.10})$$

where

$$X \equiv \frac{1}{\Delta} \left(1 - \frac{\partial_0}{d}\right) \quad \text{and} \quad Z \equiv \frac{1}{\Delta} \nabla \circ \nabla - 1 = \frac{1}{\Delta} \nabla \times (\nabla \times \dots) \quad . \quad (\text{A.11})$$

(A.8) to (A.10) are replacements allowed under the angular average \int_{Ω} if no further dependence on \bar{e} occurs. Using (A.8) and, in the last step, $Z\bar{A} = (1/\Delta)\nabla \times \bar{B}$ one obtains

$$\begin{aligned} \ell' &= \int_{\Omega} \left(\bar{E} \frac{\bar{e}}{d} A_0 - \bar{E} \frac{\bar{e} \circ \bar{e}}{d} \bar{A} \right) \\ &= \int_{\Omega} \left(\bar{E} X\nabla A_0 + \bar{E} X\partial_0 \left(1 + \frac{3}{2}Z\right) \bar{A} + \bar{E} \frac{1}{2d} Z \bar{A} \right) \\ &= \int_{\Omega} \bar{E} \left(-X\bar{E} + \left(\frac{3}{2}X\partial_0 + \frac{1}{2d}\right) \frac{1}{\Delta} \nabla \times \bar{B} \right) \quad . \end{aligned} \quad (\text{A.12})$$

We get rid of ∂_0 through the homogeneous Maxwell equation $\partial_0 \bar{B} = -\nabla \times \bar{E}$, i.e. $\partial_0(1/\Delta)\nabla \times \bar{B} = -Z\bar{E}$. Now (A.10), used in backward direction, and (A.11) lead to

$$\ell' = \frac{1}{2} \int_{\Omega} \left(\bar{E} \frac{1}{d^2} \bar{E} - \bar{e} \bar{E} \frac{1}{d^2} \bar{e} \bar{E} + \bar{E} \frac{\bar{e}}{d^2} \times \bar{B} \right) . \quad (\text{A.13})$$

All three terms are contained also in the desired result ℓ . But two terms are missing. So, in the last step, we exploit that

$$\left(\bar{e} \times \bar{B} \right) \frac{1}{d^2} \left(\bar{e} \times \bar{B} \right) + \left(\bar{e} \times \bar{B} \right) \frac{1}{d^2} \bar{E} = 0 . \quad (\text{A.14})$$

Adding this under \int_{Ω} in (A.13) the calculation ends up with

$$\ell' = -\frac{1}{2} \int_{\Omega} \left[\bar{e} \bar{E} \frac{1}{d^2} \bar{e} \bar{E} - \left(\bar{E} + \bar{e} \times \bar{B} \right) \frac{1}{d^2} \left(\bar{E} + \bar{e} \times \bar{B} \right) \right] = \ell , \quad (\text{A.15})$$

see (A.7) with (A.6).

To verify (A.14), we use (A.9) and (A.10), note that $Z\bar{B} = -\bar{B}$ and eliminate \bar{E} by means of the homogeneous Maxwell equation:

$$\begin{aligned} \bar{B} \frac{1}{d^2} \bar{B} - \bar{B} \bar{e} \frac{1}{d^2} \bar{e} \bar{B} - \bar{B} \frac{\bar{e}}{d^2} \times \bar{E} &= \bar{B} \frac{1}{d^2} \bar{B} + \bar{B} X \bar{B} - \bar{B} X \frac{1}{d} \nabla \times \bar{E} \\ &= \bar{B} \frac{1}{\Delta} \left(1 - \frac{\partial^2}{d^2} \right) \bar{B} = 0 , \end{aligned} \quad (\text{A.16})$$

where (A.9) explains the last step. (A.1) with $\ell'' \rightarrow \ell' = \ell$ and (A.7) constitute the result (3.2) given in the main text.

Appendix B

Here the next-to-leading order contributions to the polarization functions Π_{ℓ} and Π_t as given by Eqs. (3.4) and (3.5) are evaluated for all ω and q . The strategy for doing this was already outlined at the beginning of section 6. We start with the imaginary parts of the two sums (6.2) and apply formula (6.4). For convenience we write

$$\Sigma_i \equiv \sum \Delta^- \Delta f_i \quad (i = 1, 2) \quad \text{with} \quad f_1 = 1 \quad \text{and} \quad f_2 = k^2 - \frac{(\bar{k} \bar{q})^2}{q^2} . \quad (\text{B.1})$$

The two integrals in (6.4) allow for the substitutions $x \rightarrow \omega - x$ and/or $\bar{k} \rightarrow \bar{q} - \bar{k}$. Doing both transformations the integrand remains unchanged, since the f_i are invariants under this transformation and the density ρ is an odd function of x . Thus, the prefactor ω may be taken inside the integral as $\omega = x + (\omega - x) \rightarrow 2x$:

$$\Im m \Sigma_i = 2\pi T \left(\frac{1}{2\pi} \right)^3 \int d^3 k f_i \int dx \frac{1}{x - \omega} \rho_-(x, \bar{k}) \rho(x - \omega, \bar{k}) . \quad (\text{B.2})$$

We insert the spectral densities of (6.1). After some trivial manipulations with the delta functions (with the aim to remove $\bar{\omega}_-$ from the prefactors) we obtain

$$\begin{aligned} \Im m \Sigma_i &= \frac{T}{4\pi} \int_0^\infty dk k^2 \int_{-1}^1 du f_i \frac{1}{\bar{\omega}^2} \text{sgn}(\omega - \bar{\omega}) \delta(2kqu + q^2 - \omega^2 + 2\omega\bar{\omega}) - (\omega \rightarrow -\omega) \\ &= \frac{T}{8\pi q} \int_0^\infty dk \frac{k}{\bar{\omega}^2} f_i \text{sgn}(\omega - \bar{\omega}) \Theta - (\omega \rightarrow -\omega) , \end{aligned} \quad (\text{B.3})$$

where, while f_1 will always equal 1, the function f_2 has turned into

$$f_2 = k^2 - \frac{1}{4q^2} \left[q^2 - \omega^2 + 2\omega\bar{\omega} \right]^2 . \quad (\text{B.4})$$

Θ stands for the step function $\Theta = \theta(2kq - |\omega^2 - 2\omega\bar{\omega} - q^2|)$. Next we change to the integration variable $x = \sqrt{\mu^2 + k^2} = \bar{\omega}$ and manipulate the step function:

$$\Theta = \theta \left(\left[q^2 - \omega^2 \right] \left[\left(x - \frac{\omega}{2} \right)^2 - \frac{q^2}{4\omega^2} \Omega^2 \right] \right) , \quad (\text{B.5})$$

where $\Omega^2 \equiv \omega^2(\omega^2 - q^2 - 4\mu^2)/(\omega^2 - q^2)$ as given in (6.11) in the main text. Obviously (and notwithstanding the remaining integration), there is no imaginary part in region II, i.e. in $q^2 < \omega^2 < 4\mu^2 + q^2$. For a convenient formulation in the other two regions we introduce

$$x_1 = \frac{\omega}{2} - \frac{q}{2\omega} \Omega \quad \text{and} \quad x_2 = \frac{\omega}{2} + \frac{q}{2\omega} \Omega , \quad (\text{B.6})$$

so that

$$\Theta = \begin{cases} \theta \left([x - x_1][x_2 - x] \right) & \text{in III} \\ 0 & \text{in II} \\ \theta \left([x - x_1][x - x_2] \right) & \text{in I} . \end{cases} \quad (\text{B.7})$$

The function f_2 now reads

$$f_2 = \frac{\omega^2 - q^2}{q^2} \left[\omega x - x^2 - \frac{4\mu^2 q^2 + (\omega^2 - q^2)^2}{4(\omega^2 - q^2)} \right] , \quad (\text{B.8})$$

and the expression for $\Im m \Sigma_i$ so far obtained is

$$\Im m \Sigma_i = \frac{T}{8\pi q} \int_\mu^\infty dx \frac{1}{x} f_i \text{sgn}(\omega - x) \Theta - (\omega \rightarrow -\omega) . \quad (\text{B.9})$$

Clearly, the integrals (B.9) can be evaluated analytically, once the range of the parameters is specified. Consider region III and assume $\omega > 0$. Then $\mu < x_1 < x_2 < \omega$. Hence, $\text{sgn}(\omega - x) = +1$ and the term $-(\omega \rightarrow -\omega)$ vanishes. Thus, in region III:

$$\delta \Im m \Sigma_i = \frac{T}{8\pi q} \begin{cases} \mathcal{J} & i = 1 \\ \frac{\omega^2 - q^2}{q^2} \left(\frac{q\Omega}{2} - \frac{4\mu^2 q^2 + (\omega^2 - q^2)^2}{4(\omega^2 - q^2)} \mathcal{J} \right) & i = 2 \end{cases} \quad (\text{B.10})$$

with $\mathcal{J} = \ln |(\omega^2 + q\Omega)/(\omega^2 - q\Omega)|$ as given in (6.9) in the main text. Of course, in region III neither the prefix δ nor the absolute value in the argument of the logarithm are necessary. See however below.

In region I and for $\omega > 0$ we have $x_1 < \mu < x_2$ and $\text{sgn}(\omega - x) = -1$. Hence, in the first term of (B.9), the x -integration runs from x_2 to ∞ . The second term, i.e. $-(\omega \rightarrow -\omega)$, needs one more position: $\bar{x}_1 = -\frac{\omega}{2} + \frac{q}{2\omega}\Omega$. Note that $0 < \mu < \bar{x}_1 < x_2$. Also, f_2 might be split into its even and odd part with respect to ω . In region I and for positive ω we obtain

$$\Im m \Sigma_2 = \frac{T}{8\pi q} \left(\int_{\bar{x}_1}^{x_2} dx \frac{1}{x} f_2^{\text{even}} - \frac{\omega^2 - q^2}{q^2} \omega \left[\int_{\bar{x}_1}^{\infty} dx + \int_{x_2}^{\infty} dx \right] \right) . \quad (\text{B.11})$$

Obviously, this expression needs subtraction of the hard leading-order imaginary part, which is in fact non-zero in region I. This amounts to subtracting $2 \int_0^{\infty} dx$ from the two diverging integrals in (B.11). The result agrees precisely with (B.10), but the prefix δ and the absolute value in \mathcal{J} are no longer superfluous. For $i = 1$ there is only the first integral in (B.11). For an immediate check, whether the above partial results are correctly used in the main text, the reader may take the imaginary parts of (3.6), (3.7), insert (B.10) and compare with (6.6) and (6.5).

We turn to the real part. It has to be determined from the dispersion relation (6.3). But instead of directly treating the corresponding integrals (a torture) there is a pleasant way by guessing and using analytical properties. First of all, we put a δ in front of both sides of (6.3) to indicate subtraction of leading-order terms. The integration over t runs from $-\infty$ to $-\sqrt{4\mu^2 + q^2}$, from $-q$ to q and from $\sqrt{4\mu^2 + q^2}$ to ∞ . This fact may be dealt with by including the step function

$$\Theta' \equiv \theta \left(\left[q^2 - t^2 \right] \left[4\mu^2 + q^2 - t^2 \right] \right) . \quad (\text{B.12})$$

Let us start with the case $i = 1$, i.e. with $f = 1$ in (6.3). The imaginary part is an odd function of t . To see this explicitly, and by using (B.10), (6.9), (6.11), we write (6.3) as

$$\delta \Re e \Sigma_1 = \frac{T}{8\pi^2 q} \int dt \frac{1}{t - \omega} L(t) \Theta' , \quad (\text{B.13})$$

with

$$L(t) \equiv \ln \left| \frac{t^2 + qtW(t)}{t^2 - qtW(t)} \right| \quad \text{and} \quad W(t) \equiv \sqrt{\frac{4\mu^2 + q^2 - t^2}{q^2 - t^2}} . \quad (\text{B.14})$$

Principal values are understood whenever a real variable runs over a pole. In order to get rid of the absolute value signs in (B.14) we rewrite L as

$$L(t) = \frac{1}{2} \ln \left(\frac{4\mu^2 + (tW + q)^2}{4\mu^2 + (tW - q)^2} \right) . \quad (\text{B.15})$$

Next we tackle with the step function in (B.13). Note that $\Re e W(t)$ vanishes automatically in the unwanted region $q^2 < t^2 < 4\mu^2 + q^2$. The same holds true for $\Re e g(t)$ with

$$g(z) = \ln \left(\frac{2\mu - izW(z) - iq}{2\mu - izW(z) + iq} \right) . \quad (\text{B.16})$$

Moreover, in the other regions $\Re e g(t)$ agrees with (B.15). Thus, in all regions

$$L(t) \Theta' = \Re e g(t) \equiv g_1 . \quad (\text{B.17})$$

At this point we recall that $\Re e \Sigma_i$ and $\Im m \Sigma_i$ are originally defined by approaching the real axis from the upper half complex plane (UHP). Note that, with ε a positive infinitesimal, $W(t + i\varepsilon)$ turns into $+i|W(t)|$ when entering region II from larger as well as from lower t . Both the functions $W(z)$ and $g(z)$ have cuts on the real axis, but they are analytic in the UHP. Since, in addition, $g(z)$ behaves as $1/z$ at $z \rightarrow \infty$, we may state

$$\frac{1}{2\pi i} \int_{\mathcal{C}} dz' \frac{1}{z' - z} g(z') = g(z) , \quad (\text{B.18})$$

where \mathcal{C} surrounds the UHP counterclockwise. Through $z \rightarrow t + i\varepsilon$ and $g(t + i\varepsilon) \equiv g_1 + ig_2$, (B.18) tells us the common dispersion relations

$$g_2(t) = -\frac{1}{\pi} \int dt' \frac{1}{t' - t} g_1(t') \quad , \quad g_1(t) = \frac{1}{\pi} \int dt' \frac{1}{t' - t} g_2(t') . \quad (\text{B.19})$$

Combining the left of these equations with (B.13), (B.17) and (B.16), we have

$$\Re e \Sigma_1 = -\frac{T}{8\pi q} g_2(\omega + i\varepsilon) = \frac{T}{8\pi q} \mathcal{R} \quad (\text{B.20})$$

with \mathcal{R} given by (6.8). The procedure is unique. One can now verify $\delta \Pi_g$ as given in (6.6).

In treating the case $i = 2$ we follow the same lines of reasoning. $\delta \Re e \Sigma_2$ is given by the integral (B.13) but with $L(t)$ replaced by the function

$$M(t) = \frac{t^2 - q^2}{2q} t W(t) - \frac{4\mu^2 q^2 + (t^2 - q^2)^2}{4q^2} L(t) . \quad (\text{B.21})$$

To restore convergence of (B.13) we subtract (and add) on both sides the same expression taken at $\mu = 0$, i.e. we split

$$\delta \Re \epsilon \Sigma_2 = \delta_\mu \Re \epsilon \Sigma_2 + \delta \Re \epsilon \sum \Delta_0^- \Delta_0 f_2 \quad (\text{B.22})$$

with $\delta_\mu \Re \epsilon \Sigma_2 \equiv \Re \epsilon \Sigma_2 - \Re \epsilon \Sigma_2^{\mu=0}$, and treat the second term of (B.22) at the end. This decomposition was used recently in the QCD case (§4 of [10]) with the terms called 'one-loop soft' and 'one-loop hard', respectively. We observe that $\delta_\mu M(t) \Theta'$ is the real part $\Re g(\omega + i\varepsilon)$ of the following complex function:

$$\begin{aligned} g(z) = & (z^2 - q^2) \frac{z}{2q} [W(z) - 1] - \frac{4\mu^2 q^2 + (z^2 - q^2)^2}{4q^2} \ln \left(\frac{2\mu - izW(z) - iq}{2\mu - izW(z) + iq} \right) \\ & + \frac{(z^2 - q^2)^2}{4q^2} \ln \left(\frac{z + q + i\varepsilon}{z - q + i\varepsilon} \right) - i(z^2 - q^2) \frac{\mu}{q} , \end{aligned} \quad (\text{B.23})$$

where the last term, which obviously is not fixed by g_1 , has been determined from the requirement $g(z) \sim 1/z$ at $z \rightarrow \infty$. The relations (B.18) and (B.19) hold true again, and thus

$$\delta_\mu \Re \epsilon \Sigma_2 = \frac{T}{8\pi^2 q} \int dt \frac{1}{t - \omega} g_1(t) = -\frac{T}{8\pi q} g_2(\omega + i\varepsilon) . \quad (\text{B.24})$$

Note that $\omega W(\omega + i\varepsilon)$ is equal to the quantity \mathcal{E} , (6.10), used in the main text. With $g_2(\omega + i\varepsilon)$ taken from (B.23) we end up with

$$\delta_\mu \Re \epsilon \Sigma_2 = \frac{T}{8\pi q} \left[\frac{\omega^2 - q^2}{2q} (2\mu - \Im m \mathcal{E}) - \frac{4\mu^2 q^2 + (\omega^2 - q^2)^2}{4q^2} \mathcal{R} + \frac{(\omega^2 - q^2)^2}{4q^2} \pi \Theta_I \right] , \quad (\text{B.25})$$

where $\Theta_I = \theta(q^2 - \omega^2)$ restricts the last term to region I.

It remains to study the second term of (B.22). Its imaginary part may be read off from (B.10),

$$\delta \Im m \sum \Delta_0^- \Delta_0 f_2 = \frac{T}{8\pi q} \frac{\omega^2 - q^2}{4q^2} \left[2q\omega - (\omega^2 - q^2) \ln \left| \frac{\omega + q}{\omega - q} \right| \right] , \quad (\text{B.26})$$

and the appropriate, in the UHP analytic function is

$$g(z) = (z^2 - q^2) \frac{z}{2q} - \frac{(z^2 - q^2)^2}{4q^2} \ln \left(\frac{z + q - i\varepsilon}{z - q - i\varepsilon} \right) - \frac{1}{3} qz . \quad (\text{B.27})$$

Again the last term (though in conflict with (B.26)) has been added by hand to make $g(z)$ convergent at large z . In reality convergence is restored by the Bose function (which one could reintroduce in (B.9) by $1/x \rightarrow n(x)/T$). However, such details (if real on

the real axis) do not influence the result, as we shall take the imaginary part. In fact, $\Im m g(\omega + i\varepsilon) = g_2 = -(\omega^2 - q^2) \pi \Theta_I$. Consequently, when the second term of (B.22) is included, the last term in (B.25) simply drops out.

The results (6.5) to (6.7) in the main text, which we have also derived by a much more laborious direct calculation, are now readily verified.

Appendix C

In this Appendix, the ratio $\Re e \Pi_\ell / Q^2$ is calculated at the light cone $\omega = q$, in order to verify the statement (6.20) in the main text. Here the decomposition of the Braaten-Pisarski scheme into soft and hard loop momenta, where only the former need resummation, fails. Instead, the resummed version (3.7) of the longitudinal polarization Π_ℓ must be used throughout even though the leading contribution will be seen to arise from hard internal momenta. Also, the Bose function cannot be expanded.

The Bose function may be reintroduced by the replacement $T/x \rightarrow n(x)$ in the formulas of the preceding Appendix B. The latest opportunity for doing so is at (B.9):

$$\Im m \Sigma_i = \frac{1}{8\pi q} \int_\mu^\infty dx n(x) [h_2(\omega) - h_2(-\omega)] \quad (\text{C.1})$$

$$\text{with } h_2(\omega) = f_i(\omega) \operatorname{sgn}(\omega - x) \Theta \quad (\text{C.2})$$

and $f_1 = 1$. f_2 and Θ are given by (B.8) and (B.5), respectively. The two sums Σ_i , which constitute Π_ℓ , are defined in (B.1). If we are able to grasp the analytical function $h(z)$, which has $\Im m h(\omega + i\varepsilon) = h_2(\omega)$ and is convergent at large z , we may proceed as in Appendix B, but this time inside the real x -integration. Indeed, we find

$$h(z) = f_i(z) \frac{1}{\pi} \ln \left(\frac{z^2 - q^2 - 2xz - 2kq}{z^2 - q^2 - 2xz + 2kq} \right) - \frac{k}{\pi q} (z^2 - q^2 - 2xz) \delta_{i,2} \quad , \quad (\text{C.3})$$

where $k = \sqrt{x^2 - \mu^2}$. Only in the case $i = 2$ there is a term to be subtracted for convergence. Thus, with (B.19) and $h_1(\omega) \equiv \Re e h(\omega + i\varepsilon)$ we obtain the real parts

$$\Re e \Sigma_i = \frac{1}{8\pi q} \int_\mu^\infty dx n(x) [h_1(\omega) + h_1(-\omega)] \quad . \quad (\text{C.4})$$

One can now form the linear combination (3.7) of the above real parts, with all details filled in. The result is conveniently written down with the integration variable k instead of $x = \sqrt{\mu^2 + k^2} \equiv \bar{\omega}$:

$$\begin{aligned} \Re e \Pi_\ell = & (\omega^2 - q^2) \frac{e^2}{8\pi^2 q^3} \int_0^\infty dk \frac{k}{\bar{\omega}} n(\bar{\omega}) \left[-(2\bar{\omega} - \omega)^2 \ln \left(\frac{\omega^2 - q^2 - 2\bar{\omega}\omega - 2kq}{\omega^2 - q^2 - 2\bar{\omega}\omega + 2kq} \right) \right. \\ & \left. -(2\bar{\omega} + \omega)^2 \ln \left(\frac{\omega^2 - q^2 + 2\bar{\omega}\omega - 2kq}{\omega^2 - q^2 + 2\bar{\omega}\omega + 2kq} \right) - 8kq \right] . \end{aligned} \quad (\text{C.5})$$

The expression (C.5) is the appropriate place for approaching the light cone. After dividing both sides by $\omega^2 - q^2 = Q^2$, (C.5) leads to

$$\lim_{\omega \rightarrow q} \frac{\Re e \Pi_\ell}{Q^2} = \frac{e^2}{\pi^2 q^2} J \quad \text{with} \quad J \equiv \int_0^\infty dk \frac{k}{\bar{\omega}} n(\bar{\omega}) \left[-k + \bar{\omega} \ln \left(\frac{\bar{\omega} + k}{\bar{\omega} - k} \right) \right] . \quad (\text{C.6})$$

To study the high temperature expansion of this integral J , we subtract and add the large- k limit of the integrand and write

$$J = J_0 + \delta J \quad \text{with} \quad J_0 = \int_0^\infty dk k n(k) \left[-1 + 2 \ln \left(\frac{2k}{\mu} \right) \right] . \quad (\text{C.7})$$

In the small difference term δJ one can use $n(x) \approx T/x$, so that it is readily evaluated to be $\delta J = T\mu\pi/2$, which leads to the contribution $e^2 T\mu/2\pi q^2$ to $\Re e \Pi_\ell/Q^2$. The leading term J_0 may be given the form

$$J_0 = T^2 \left[-\frac{\pi^2}{6} + \frac{\pi^2}{3} \ln \left(\frac{2T}{\mu} \right) + 2 \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\infty du \frac{u^{1+\varepsilon} - u}{e^u - 1} \right] , \quad (\text{C.8})$$

where the last term brings in derivatives of the Riemann zeta and Gamma functions [43]. Equation (6.20) is thus obtained. Since accurate values of derivatives of the zeta function are rarely found in tables, we note that $\frac{\zeta'(2)}{\zeta(2)} = -0.569\,960\,993\dots$

We now turn to the imaginary part of the resummed polarization Π_ℓ at the level of (C.5). It derives from (C.1), (C.2) directly and is strictly zero in region II. For $\omega^2 < 4\mu^2 + q^2$, i.e. in the two regions I and II which have the light cone as the common border, we find

$$\Im m \Pi_\ell(\omega + i\varepsilon, q) = -Q^2 \theta(q^2 - \omega^2) \frac{e^2}{2\pi q^3} \int_{x_0}^\infty dx x^2 \left[n \left(x - \frac{\omega}{2} \right) - n \left(x + \frac{\omega}{2} \right) \right] . \quad (\text{C.9})$$

Here, the effect of resummation is hidden in the lower endpoint x_0 of the integration interval:

$$x_0 = \frac{1}{2} q \sqrt{\frac{4\mu^2 + q^2 - \omega^2}{q^2 - \omega^2}} . \quad (\text{C.10})$$

Clearly, due to $\mu \neq 0$, x_0 comes down from infinity as q becomes larger than ω , giving a smooth onset of the imaginary part in region I. For $\omega < q$ and $q - \omega \ll \epsilon^2 q$,

$$\Im m \Pi_\ell(\omega + i\varepsilon, q) = \frac{\epsilon^4 T^2}{8\pi} \exp\left(-\epsilon \sqrt{\frac{q}{8(q-\omega)}}\right), \quad (\text{C.11})$$

so that the imaginary part sets in infinitely smoothly in fact. Alarming, (C.11) has the wrong sign to give rise to weak damping. As it stands, it would give *anti-damping* as soon as the light-cone is traversed into region I. The simple resolution of this startling puzzle is that, in region I, $\Im m \Pi_\ell(\omega - i\varepsilon, q) = -\Im m \Pi_\ell(\omega + i\varepsilon, q)$. In region II, where the next-to-leading order dispersion curve is still undamped, higher-order corrections will generate positive damping. The dispersion curve, when it is followed towards and through the light-cone, crosses into region I with $Q_0 = \omega(q) - i\gamma(q)$, so that only the Riemann sheet where $\Im m \Pi_\ell < 0$ can be accessed, which corresponds to Landau *damping* instead of anti-damping. Consequently, for $(q - \omega)/(\epsilon T) \ll \epsilon^2$ one has weakly damped plasmon excitations with phase velocities just below 1, whereas further down in region I these excitations quickly become overdamped.

Appendix D

In this Appendix we add some details to the calculations presented in Chap. 7.

First, as a consistency check we rederive Eqs. (7.19,7.20) from an equivalent Schwinger-Dyson equation, where the diagrams of Fig. 7 are flipped so that the bare vertices are on the right side. Since $\Pi_{\mu\nu}^{(R)}$ is an asymmetric combination of the real-time components, this leads to a different starting point for $\Pi_{\mu\nu}^{(1)R}$. In place of Eq. (7.14) we obtain

$$\begin{aligned} \Pi_{\mu\nu}^{(1)R}(Q) &\equiv \Pi_{\mu\nu}^{(1)11}(Q) + \Pi_{\mu\nu}^{(1)12}(Q) \\ &= \frac{e}{2} \int \frac{d^4 K}{(2\pi)^4} \left\{ \left[\Gamma_\mu^{R(i)} + \Gamma_\mu^{P(o)} \right] \Delta_R(Q+K) \Delta_R(K) \right. \\ &\quad + \left[\Gamma_\mu^{R(o)} + \Gamma_\mu^{P(i)} \right] \Delta_A(Q+K) \Delta_A(K) \\ &\quad \left. + \Gamma_\mu^{R(\gamma)} \left[\Delta_R(Q+K) \Delta_P(K) + \Delta_P(Q+K) \Delta_A(K) \right] \right\} (2K+Q)_\nu, \quad (\text{D.1}) \end{aligned}$$

where we have additionally introduced

$$\Gamma^{P(i)} = \sum_{a,b} \Gamma^{baa} \quad , \quad \Gamma^{P(o)} = \sum_{a,b} \Gamma^{aab} \quad . \quad (\text{D.2})$$

The Ward identities that allowed us to solve $\Gamma^{R(i)}$, $\Gamma^{R(o)}$, and $\Gamma^{(P)}$ in terms of the corresponding self-energies can be derived by subjecting the type-1 and type-2 field to the same gauge transformations [32]. This would not determine the vertices appearing above. Assuming, however, that the theory is separately invariant under type-1 and type-2 gauge transformations relates all components of the vertices to the corresponding self-energy components. In analogy to (7.17) we are then led to

$$\Gamma_{\mu}^{R(\gamma)}(K, K+Q) = -i\epsilon(2K+Q)_{\mu} \left(1 - \frac{1}{2KQ+Q^2} [\Xi_A(K) - \Xi_R(K+Q)] \right) \quad , \quad (\text{D.3})$$

$$\Gamma_{\mu}^{P(i)}(K, K+Q) = -i\epsilon(2K+Q)_{\mu} \frac{1}{2KQ+Q^2} \Xi_P(K+Q) \quad , \quad (\text{D.4})$$

$$\Gamma_{\mu}^{P(o)}(K, K+Q) = i\epsilon(2K+Q)_{\mu} \frac{1}{2KQ+Q^2} \Xi_P(K) \quad . \quad (\text{D.5})$$

Inserting these into (D.1) and keeping only terms involving the Bose distribution function exactly reproduces (7.20).

In the following we shall concentrate on $\Pi_{\mu\nu}^R$ in the form (7.19), since there the contributions from the poles of the Bose distribution functions that are proportional to T^2 or to T cancel upon integration of the 3-momenta. Closing the contour around the upper half plane, only the poles of $G_A(K)$ contribute, which are located at $K_0 = i\zeta \pm \omega_k$ with $\omega_k = \sqrt{k^2 + \mu^2 - \zeta^2}$.

We consider first $\Pi_{00}(Q_0, 0)$, which when nonvanishing signals the loss of gauge invariance. Always omitting temperature-independent contributions, Eq. (7.19) leads to

$$\begin{aligned} \Pi_{00}(Q_0, 0) = & -\frac{e^2}{Q_0 + 2i\zeta} \int \frac{dk k^2}{\pi^2} \left\{ n(\omega_k + i\zeta) - n(\omega_k - i\zeta) \right. \\ & - \frac{i\zeta}{Q_0\omega_k} (\omega_k + i\zeta + Q_0) [n(\omega_k + i\zeta + Q_0) - n(\omega_k + i\zeta)] \\ & \left. + \frac{i\zeta}{Q_0\omega_k} (\omega_k - i\zeta - Q_0) [n(\omega_k - i\zeta - Q_0) - n(\omega_k - i\zeta)] \right\} \quad . \quad (\text{D.6}) \end{aligned}$$

Taylor-expanding the distribution functions and keeping only terms up to ζ^3 , we have

$$\Pi_{00}(Q_0, 0) = -\frac{e^2 i\zeta}{Q_0 + 2i\zeta} \int \frac{dk k^2}{\pi^2} \left\{ 2n'(\omega_k) - \zeta^2 n'''(\omega_k) \right\}$$

$$\begin{aligned}
& - \frac{\omega_k + i\zeta + Q_0}{\omega_k} \sum_{m=1}^{\infty} \left[n^{(m)} + i\zeta n^{(m+1)} - \frac{\zeta^2}{2} n^{(m+2)} \right] \frac{Q_0^{m-1}}{m!} \\
& - \frac{\omega_k - i\zeta - Q_0}{\omega_k} \sum_{m=1}^{\infty} \left[n^{(m)} - i\zeta n^{(m+1)} - \frac{\zeta^2}{2} n^{(m+2)} \right] \frac{(-Q_0)^{m-1}}{m!} \} \quad (\text{D.7})
\end{aligned}$$

The integrals associated with higher powers of ζ are increasingly infrared-dominated, so that the terms neglected are down by powers of $\zeta/\mu \sim e \ln(1/e)$. Collecting the various powers in Q_0 , we find for $Q_0 \ll T$

$$\begin{aligned}
H_{00}(Q_0, 0) &= \frac{4i\zeta^3 e^2 T}{Q_0 + 2i\zeta} \int \frac{dk k^2}{\pi^2} \sum_{m=1}^{\infty} \left\{ \right. \\
& \quad \left. - \frac{n Q_0^{2m-2}}{(k^2 + \mu^2)^{m+1}} + \frac{(n+1) Q_0^{2m}}{(k^2 + \mu^2)^{m+2}} + O(\zeta/\mu) \right\} . \quad (\text{D.8})
\end{aligned}$$

This sum is telescoping — all terms but the first cancel each other, and (7.22) is readily obtained.

Considering next $H_{\mu\nu}(0, q)$, we have

$$\begin{aligned}
H_{00}(0, q) &= \frac{e^2}{\pi^2} \Re e \int dk k^2 \frac{n(\omega_k + i\zeta)}{\omega_k} \left[1 \right. \\
& \quad \left. + \frac{k^2 + \mu^2 - 2\zeta^2 + 2i\zeta\omega_k}{kq} \ln \frac{2kq - 4i\zeta\omega_k + q^2 + 4\zeta^2}{-2kq - 4i\zeta\omega_k + q^2 + 4\zeta^2} \right] \quad (\text{D.9})
\end{aligned}$$

$$\begin{aligned}
H_{ii}(0, q) &= - \frac{e^2}{\pi^2} \Re e \int dk k^2 \frac{n(\omega_k + i\zeta)}{\omega_k} \left[1 \right. \\
& \quad \left. - \frac{4k^2 + 8i\zeta\omega_k - q^2 - 8\zeta^2}{4kq} \ln \frac{2kq - 4i\zeta\omega_k + q^2 + 4\zeta^2}{-2kq - 4i\zeta\omega_k + q^2 + 4\zeta^2} \right] . \quad (\text{D.10})
\end{aligned}$$

In the limit $q \rightarrow 0$ this simplifies to

$$\begin{aligned}
H_{00}(0, q \rightarrow 0) &= \frac{e^2}{\pi^2} \Re e \int dk k^2 \frac{in(\omega_k + i\zeta)}{\zeta} \\
&= \frac{e^2}{\pi^2} \int dk k^2 \left(-n'(\sqrt{k^2 + \mu^2}) + \frac{\zeta^2}{6} n'''(\sqrt{k^2 + \mu^2}) + \dots \right) \quad (\text{D.11})
\end{aligned}$$

$$H_{ii}(0, q \rightarrow 0) = - \frac{e^2}{\pi^2} \int dk k^2 \left(\frac{\zeta^2}{3} \frac{k^4}{k^2 + \mu^2} n'''(\sqrt{k^2 + \mu^2}) + \dots \right) , \quad (\text{D.12})$$

which yields Eqs. (7.23).

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