

Minimal U(1) gauge fields in two dimensions

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Abstract

Gribov copies of the vacuum in two dimensional compact U(1) lattice gauge models are constructed. On the basis of this a gauge fixing algorithm is developed, which finds the minimum of the sum of link field squares. Numerical experience in a two dimensional Higgs model with fixed length scalar field is reported and the extension to three and four dimensional U(1) and four dimensional SU(2) gauge theories is briefly discussed.

1 Introduction

Lattice gauge theory [1] is explicitly gauge invariant but gauge fixing is often desirable, for instance, to establish relation with perturbation theory. Therefore, the properties of the space of gauge orbits has been extensively studied both in continuum and lattice formulations [2, 3, 4] (for further references on the extensive literature see the review [5]).

On the lattice, gauge fixing to the Landau gauge can be formulated as a minimization problem: one has to find the minimum of the sum of link gauge field squares along gauge orbits. In most cases the absolute minimum is unique, but the appearance of a large number of secondary local minima leads to *Gribov copies* [2], which are difficult to sort out even in the simple case of compact U(1) gauge fields. (For recent work on gauge fixing in U(1) and further references see [6, 7, 8].)

In the present paper the problem of minimizing the gauge field by gauge transformation will be considered in a two dimensional Higgs model. The lattice action depending on the compact U(1) gauge field $U_{x\mu} = \exp(iA_\mu(x))$, ($\mu = 1, 2$) and, for simplicity, fixed length Higgs scalar field $\phi(x)$, $|\phi(x)| = 1$ has two parameters: the inverse gauge coupling squared $\beta = 1/g^2$ and the hopping parameter of the scalar field κ :

$$S = \beta \sum_x \sum_{\mu=1, \nu=2} [1 - \cos(F_{\mu\nu}(x))] - 2\kappa \sum_x \sum_{\mu=1}^2 \phi^*(x + \hat{\mu}) U_{x\mu} \phi(x) . \quad (1)$$

Here the lattice gauge field strength is defined for $\mu, \nu = 1, 2$ as

$$F_{\mu\nu}(x) = A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x) . \quad (2)$$

$\kappa = 0$ gives pure U(1) gauge theory without interaction with the scalar field. Real angular variables $-\pi < \theta_{x\mu} \leq \pi$ on the links $(x, x + \hat{\mu})$ can be introduced by

$$U_{x\mu} \equiv \exp(i\theta_{x\mu}) , \quad \theta_{x\mu} \equiv \pi t_{x\mu} = \text{cpt}(A_\mu(x)) \equiv A_\mu(x) - 2\pi \cdot \text{NINT}(A_\mu(x)/2\pi) , \quad (3)$$

where NINT() denotes nearest integer and the normalized angle $-1 < t_{x\mu} \leq 1$ as well as the function cpt() are introduced for later convenience.

The Lorentz gauge condition $\partial_\mu A_\mu(x) = 0$ can be latticezed [6, 7, 8] by looking for extrema of

$$f_1[U] \equiv \sum_x \sum_{\mu=1}^2 [1 - \cos \theta_{x\mu}] . \quad (4)$$

Another choice, instead of f_1 , is simply

$$f_2[U] \equiv \sum_x \sum_{\mu=1}^2 \theta_{x\mu}^2 . \quad (5)$$

A gauge transformation with parameters α_x acts on the link angles according to

$$\theta_{x\mu}^{(\alpha)} = \text{cpt}(\theta_{x\mu} + \alpha_x - \alpha_{x+\hat{\mu}}) . \quad (6)$$

Therefore the extremum condition for f_2 under gauge transformations is

$$0 = \sum_{\mu=1}^2 (\theta_{x\mu} - \theta_{x-\hat{\mu},\mu}) = \Delta_\mu^b \theta_{x\mu} , \quad (7)$$

with the backward lattice derivative Δ_μ^b . This is a simple discretization of the continuum Lorentz condition $\partial_\mu A_\mu(x) = 0$. For small fields in lattice units $|\theta_{x\mu}| \ll 1$, which dominate in perturbation theory, the minimization of $f_1[U]$ and $f_2[U]$ is equivalent. For large fields the choice is free, but f_2 is technically simpler, therefore in the present paper f_2 will be used.

2 Gribov copies of the vacuum

In what follows we always assume periodic boundary conditions on the L^2 lattice. The Gribov copies of the vacuum satisfy, besides the extremum condition (7), the requirement that every plaquette and Polyakov line winding around the torus gives a group element 1 in U(1). That is:

$$\theta_{x1} + \theta_{x+\hat{1},2} - \theta_{x2} - \theta_{x+\hat{2},1} = 2\pi n_{x12} , \quad (8)$$

and for $\mu = 1, 2$

$$\sum_{k=0}^{L-1} \theta_{x+k\hat{\mu},\mu} = 2\pi l_{x\mu} . \quad (9)$$

Here $n_{x\mu\nu}$ and $l_{x\mu}$ are integers. Due to $|\theta_{x\mu}| \leq \pi$ the allowed values are $-2 \leq n_{x\mu\nu} \leq 2$ and $-L/2 \leq l_{x\mu} \leq L/2$. In the real (identically zero: $\theta_{x\mu} \equiv 0$) vacuum we obviously have $n_{x12} = l_{x1} = l_{x2} = 0$. Solutions with some non-zero $n_{x\mu\nu}$ and $l_{x\mu}$ give Gribov copies.

Plaquettes with non-zero values of n_{x12} play an important rôle in the dynamics of compact U(1) lattice gauge fields [9, 10]. Such plaquettes are said to carry a *Dirac string*, or a *gauge field kink*. In the present paper we shall call a plaquette with $n_{x12} = 1$ a *kink*, with $n_{x12} = -1$ an *antikink*, with $n_{x12} = 2$ a *double kink*, and with $n_{x12} = -2$ a *double antikink*.

Let us define in the general case, for a gauge configuration which is not necessarily gauge equivalent to the vacuum, the plaquette variables

$$\begin{aligned}\Theta_{x\mu\nu} &\equiv \theta_{x\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x\nu} - \theta_{x+\hat{\nu},\mu} , \\ \theta_{x\mu\nu} &\equiv \text{cpt}\Theta_{x\mu\nu} \equiv \pi t_{x\mu\nu} , \quad n_{x\mu\nu} \equiv \frac{1}{2\pi} (\Theta_{x\mu\nu} - \theta_{x\mu\nu}) .\end{aligned}\tag{10}$$

Then the integer valued topological charge Q of the configuration (see ref. [11] and references therein) is given by

$$Q = \frac{1}{2\pi} \sum_x \theta_{x12} = \frac{1}{2} \sum_x t_{x12} = - \sum_x n_{x12} .\tag{11}$$

Since in the vacuum $Q = \theta_{x12} = 0$, the sum of the integers n_{x12} on the right hand side of (8) has to vanish for every vacuum gauge configuration.

Note that n_{x12} is not gauge invariant. Namely, a gauge transformation where the argument of $\text{cpt}()$ in eq. (6) on some link is greater in absolute value than π adds a kink-antikink pair to the two plaquettes containing the link. At the same time the value of the Polyakov line going through the link jumps by $\pm 2\pi$, therefore the integer $l_{x\mu}$ on the right hand side of (9) jumps by ± 1 . Let us call such a link *overtransformed*. Since for independent gauge transformations one can restrict the parameters to $-\pi < \alpha_x \leq \pi$, starting from the identically zero vacuum configuration, one can overtransform every link at most once. Therefore, a set of overtransformed links with signs uniquely determines the values of the integers (n_{x12} , l_{x1} , l_{x2}) in eqs. (8), (9). If the linear system of equations (7), (8), (9) has a solution then we obtain a Gribov copy of the vacuum, provided that the solution is an admissible link angle configuration ($-1 < t_{x1}, t_{x2} \leq 1$) and the extremum of f_2 is a minimum.

The stability of the extrema can be easily seen, because the second derivative matrix D_{yx} defined by

$$D_{yx} \equiv \frac{\partial^2 f_2[U^{(\alpha)}]}{\partial \alpha_y \partial \alpha_x}\tag{12}$$

is equal to

$$D_{yx} = 4\delta_{yx} - \sum_{\mu=1}^2 [\delta_{y,x+\hat{\mu}} + \delta_{y,x-\hat{\mu}}] .\tag{13}$$

As it can be easily proven, this matrix has a single zero eigenvalue corresponding to $\alpha_x = \text{const.}$, which is a global gauge transformation. All the other eigenvalues are positive, therefore the vacuum configurations satisfying (7) are all stable minima.

Concerning $|t_{x1}, t_{x2}| \leq 1$ at the candidate Gribov copies, to obtain a general statement does not seem easy. Numerical experience shows that in most cases with $|n_{x12}| = 1$ the values of the normalized link angles t_{x1}, t_{x2} are in the allowed range. As an example, a kink at $(x_1, x_2) = (1, 3)$ and antikink at $(5, 3)$ on 8^2 lattice looks like (14), (15) where, respectively, the normalized link angles $t_{\mu=1}$ and $t_{\mu=2}$ are given. The sum of link angle squares in this configuration is $f_2 = (1011/238)\pi^2$. Similar results on larger lattices are too voluminous to be published here in detail. Just as an example, a kink-antikink pair at $(8, 16), (24, 16)$ on 32^2 gives $f_2 = (21.7141 \dots / 3.60523 \dots)\pi^2$.

$$t_1 = \begin{bmatrix} \frac{23}{1904} & \frac{79}{1904} & \frac{159}{1904} & \frac{215}{1904} & -\frac{215}{1904} & -\frac{159}{1904} & -\frac{79}{1904} & -\frac{23}{1904} \\ -\frac{9}{476} & \frac{67}{952} & \frac{171}{952} & \frac{229}{476} & -\frac{229}{476} & -\frac{171}{952} & -\frac{67}{952} & -\frac{9}{476} \\ \frac{23}{1904} & \frac{79}{1904} & \frac{159}{1904} & \frac{215}{1904} & -\frac{215}{1904} & -\frac{159}{1904} & -\frac{79}{1904} & -\frac{23}{1904} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{23}{1904} & -\frac{79}{1904} & -\frac{159}{1904} & -\frac{215}{1904} & \frac{215}{1904} & \frac{159}{1904} & \frac{79}{1904} & \frac{23}{1904} \\ -\frac{9}{476} & -\frac{67}{952} & -\frac{171}{952} & -\frac{229}{476} & \frac{229}{476} & \frac{171}{952} & \frac{67}{952} & \frac{9}{476} \\ -\frac{23}{1904} & -\frac{79}{1904} & -\frac{159}{1904} & -\frac{215}{1904} & \frac{215}{1904} & \frac{159}{1904} & \frac{79}{1904} & \frac{23}{1904} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{x_1=0\dots 7, x_2=0\dots 7} \quad (14)$$

$$t_2 = \begin{bmatrix} \frac{19}{272} & \frac{27}{952} & -\frac{15}{272} & -\frac{20}{119} & -\frac{15}{272} & \frac{27}{952} & \frac{19}{272} & \frac{39}{476} \\ \frac{27}{272} & \frac{67}{952} & -\frac{7}{272} & -\frac{375}{952} & -\frac{7}{272} & \frac{67}{952} & \frac{27}{272} & \frac{101}{952} \\ \frac{41}{272} & \frac{171}{952} & \frac{75}{272} & \frac{613}{952} & \frac{75}{272} & \frac{171}{952} & \frac{41}{272} & \frac{137}{952} \\ \frac{49}{272} & \frac{211}{952} & \frac{83}{272} & \frac{199}{476} & \frac{83}{272} & \frac{211}{952} & \frac{49}{272} & \frac{20}{119} \\ \frac{49}{272} & \frac{211}{952} & \frac{83}{272} & \frac{199}{476} & \frac{83}{272} & \frac{211}{952} & \frac{49}{272} & \frac{20}{119} \\ \frac{41}{272} & \frac{171}{952} & \frac{75}{272} & \frac{613}{952} & \frac{75}{272} & \frac{171}{952} & \frac{41}{272} & \frac{137}{952} \\ \frac{27}{272} & \frac{67}{952} & -\frac{7}{272} & -\frac{375}{952} & -\frac{7}{272} & \frac{67}{952} & \frac{27}{272} & \frac{101}{952} \\ \frac{19}{272} & \frac{27}{952} & -\frac{15}{272} & -\frac{20}{119} & -\frac{15}{272} & \frac{27}{952} & \frac{19}{272} & \frac{39}{476} \end{bmatrix}_{x_1=0\dots 7, x_2=0\dots 7} \quad (15)$$

An exceptional situation is, when the kink and antikink are touching each other in a common overtransformed link. In this case the solution of the linear equations give for the overtransformed link a value outside the allowed range. Therefore, the minimum is on the boundary: one has to set the overtransformed link to $t = \pm 1$ and solve the equations without the ones in (7) for x at the two endpoints of this link. It turns out that the remaining equations have a unique solution within the allowed range and on an 8^2 lattice one obtains $f_2 = (128/63)\pi^2$.

In case of a plaquette with double kink the only possibility is to have $t_{x1} = t_{x+\hat{1},2} = -t_{x2} = -t_{x+\hat{2},1} = \pm 1$. Therefore, these values have to be fixed before solving the linear system of equations. Correspondingly, one has to omit the equations in (7) for points on plaquettes with double kinks. The remaining system of equations usually does not have a unique solution, therefore one has to look for the minimum of f_2 in the remaining few variables. For instance, for a double kink pair at the same positions as in (14), (15) one has to fix a single remaining variable. The solution gives $f_2 = (1328940/76409)\pi^2$, a value substantially higher than for a normal kink pair. This is one of the reasons why in numerical simulations Gribov copies with double kinks usually are not important.

3 Gauge fixing algorithm

Introducing a small gauge coupling will not qualitatively change the Gribov copies. Therefore one can characterize them by the kink configuration, as in the case of the vacuum. The system of linear equations in (7), (8), (9) changes, because on the right hand side of (8) also the non-

integer part θ_{x12} of the total plaquette angle Θ_{x12} appears. Similarly, in (9) also the non-integer part of the Polyakov lines have to be included.

On smallish lattices one can then determine the values of f_2 at every local minimum and choose the one with the lowest value. However, the number of local minima is increasing exponentially with the number of lattice points, therefore this is not practicable for larger lattices. A possibility is to visit a large number of local minima in a random walk, and keep always the lowest one. Such an algorithm can consist of the following steps:

1. *Gauge cooling;*
2. *Removing kink pairs;*
3. *Changing kink configuration.*

In the first step one can apply an iterative algorithm to take the gauge configuration to a local minimum. One can sweep systematically on sites and minimize f_2 locally, or choose links randomly and try random gauge transformations at both ends and accept the change if f_2 gets smaller.

In the second step one looks for the kink distribution on the configuration and tries to remove all kink-antikink pairs. A possibility is to choose the closest pairs and perform the opposite of the gauge transformation which would create this pair from the zero vacuum (see (14), (15)). In praxis, however, it is simpler to work with idealized kinks, since the kinks on finite lattices and at finite distances from each other are somewhat influencing each other. An idealized kink is shown in fig. 1. An idealized kink-antikink pair together with the gauge transformation creating it from the vacuum is given in fig. 2. One can see that on the horizontal and vertical straight sections between the kink pair there is a string of overtransformed links, which connects the kink and antikink. After removing all kink pairs one has still to transform the Polyakov loops in such a way that the integer numbers $l_{x\mu}$ on the right hand side of eq. (9) on average vanish in both directions (x_1 and x_2 direction). This can be achieved by introducing an appropriate number of strings of overtransformed links which are closed by the periodic boundary conditions. After performing this transformations, on a typical non-vacuum configuration at sufficiently small gauge coupling, the configuration comes close to a local minimum. At zero gauge coupling and for topological charge zero one would create in such a way the identically zero vacuum configuration. For configurations with non-zero topological charge the result is a smooth configuration with "instantons". Of course, applying step 1. again one can take the configuration at any time arbitrarily close to the nearby local minimum.

The purpose of step 2. is to create a good starting configuration for the final step, which is a random walk on local minima in order to find with high probability the absolute minimum. This is necessary, because on non-vacuum configurations the different kink positions give different f_2 values and one has to find the optimal positions. The moving of kinks and antikinks is done by gauge transformations corresponding to the desired moves in a zero vacuum background (see the example in fig. 2). It is also possible that a deeper minimum is achieved by adding again a few kink-antikink pairs. This can be taken into account during the random walk by allowing for kink-antikink pair creation with some small probability. Annihilation of kink-antikink pairs occurs if during the random moves a kink hits an antikink or vice versa.

The strategy of the random walk can be chosen differently. A good way is to look for a deeper minimum in some given finite number of steps. If it is found than move the configuration to the new minimum, otherwise start again another sequence of moves from the known deepest

Table 1: Comparison of deepest minima in sectors with different numbers of kink-antikink pairs N_p . The value of f_2 is given. The lattice size is 64^2 .

β	κ	Q	$N_p = 0$	$N_p = 1$	$N_p = 2$	$N_p = 3$
8.0	0.0	5	<i>367.2</i>	<i>372.2</i>	383.3	398.9
8.0	0.1	4	<i>420.7</i>	<i>403.5</i>	389.3	
8.0	0.2	7	<i>353.3</i>	362.8	380.1	
8.0	0.3	7	<i>337.4</i>	355.1	368.3	
8.0	0.4	-3	<i>316.3</i>	327.6	345.7	
8.0	0.5	-4	337.4	339.6	351.3	
8.0	0.6	0	<i>294.7</i>	<i>278.0</i>	275.3	299.0
8.0	0.7	-2	<i>263.2</i>	280.3	305.3	
8.0	0.8	-1	<i>241.0</i>	262.2	285.4	

minimum. Another possibility is to do a sequence of single steps and to allow with some small and decreasing probability also increasing values of f_2 at the visited local minima ("annealing").

I collected some numerical experience with this algorithm in the two dimensional Higgs model defined by the lattice action (1) on 64^2 lattice at $\beta = 2$ and $\beta = 8$ for $0 \leq \kappa \leq 0.8$. At $\beta = 8$ where the plaquette expectation value is about $\simeq 0.94$, on most configurations taken from a Metropolis-overrelaxation updating the absolute minimum could be found in many different subsequent random searches. A representative sample of f_2 values at the local minima with different numbers of additional kink-antikink pairs are contained in table 1. The uniquely determined minima are given in *italic*. The roman entries indicate that there are several local minima with close values or the minimum is not easy to find. Three additional pairs typically give substantially higher minima, which are usually not interesting and were therefore not precisely determined.

In general, the question has to be answered what is to do if there are two or several almost equally (or even exactly) deep minima. A correct prescription for gauge fixing in this case is to take the average over these degenerate absolute minima.

At the stronger gauge coupling $\beta = 2$ with a plaquette expectation value $\simeq 0.7$ there are about 5-10 additional kink-antikink pairs at the best found minimum. The positions of these many kinks and antikinks are usually not sharply defined, there are many local minima with very similar values of f_2 . Nevertheless the algorithm still works fine, the obtained minima are about a factor 3 smaller after the third step than after a standard gauge cooling in the first step. Due to the large number of kinks, however, the numerical work is increased substantially compared to $\beta = 8$.

4 Generalizations

As the numerical experience shows, the gauge fixing algorithm based on the knowledge of the vacuum Gribov copies works well as long as the gauge coupling is not very strong. The decent range can be characterized by the plaquette expectation value being above $\simeq 0.6$, where the topological charge of the gauge configuration is reasonably well defined [10].

The generalization of this algorithm to higher (three or four) dimensional $U(1)$ gauge theories

is possible. Step 1. and 3. remain essentially the same. In step 2. one can remove the superfluous kink-antikink pairs first in parallel two dimensional subspaces and then remove the remaining closed strings of kinks between these "slices", first in three and then in four dimensions.

The close analogy between the topological features in two dimensional $U(1)$ and four dimensional $SU(2)$ gauge theories is very helpful for a generalization to four dimensional $SU(2)$. The rôle of the gauge field kinks in the vacuum is played there by vacuum gauge configurations on 2^4 hypercubes which contribute by a non-zero integer to the representation of the topological charge in terms of the "sections" [12, 13]. Nevertheless, the numerical task seems considerably more difficult because of the non-linearity of the problem. I hope to return to this in a subsequent publication.

References

- [1] K.G. Wilson, Phys. Rev. D10 (1974) 2445.
- [2] V.N. Gribov, Nucl. Phys. B139 (1978) 1.
- [3] D. Zwanziger, Nucl. Phys. B209 (1982) 336; Nucl. Phys. B378 (1992) 525.
- [4] P. van Baal, Nucl. Phys. B369 (1992) 259.
- [5] M. Salmhofer, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 81.
- [6] A. Hulsebos, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 539.
- [7] V.G. Bornyakov, V.K. Mitrjushkin, M. Müller-Preussker, F. Pahl, Phys. Lett. B317 (1993) 596.
- [8] M.I. Polikarpov, K. Yee, M.A. Zubkov, preprint LSU-431-93.
- [9] T.A. DeGrand, D. Toussaint, Phys. Rev. D22 (1980) 2478.
- [10] I. Montvay, DESY preprints 93-134, 93-164 (1993); to appear in *Proc. of the 17th Johns Hopkins Workshop*, Budapest, July 1993; *Proc. of the 2nd IMACS Conference*, St. Louis, October, 1993; *Proc. of the Lattice '93 Conference*, Dallas, October, 1993.
- [11] M. Göckeler, A.S. Kronfeld, G. Schierholz, U.-J. Wiese, Nucl. Phys. B404 (1993) 839.
- [12] M. Lüscher, Comm. Math. Phys. 85 (1982) 39.
- [13] A.S. Kronfeld, M.L. Laursen, G. Schierholz, U.-J. Wiese, Nucl. Phys. B292 (1987) 330; M. Göckeler, A.S. Kronfeld, M.L. Laursen, G. Schierholz, U.-J. Wiese, Nucl. Phys. B292 (1987) 349.

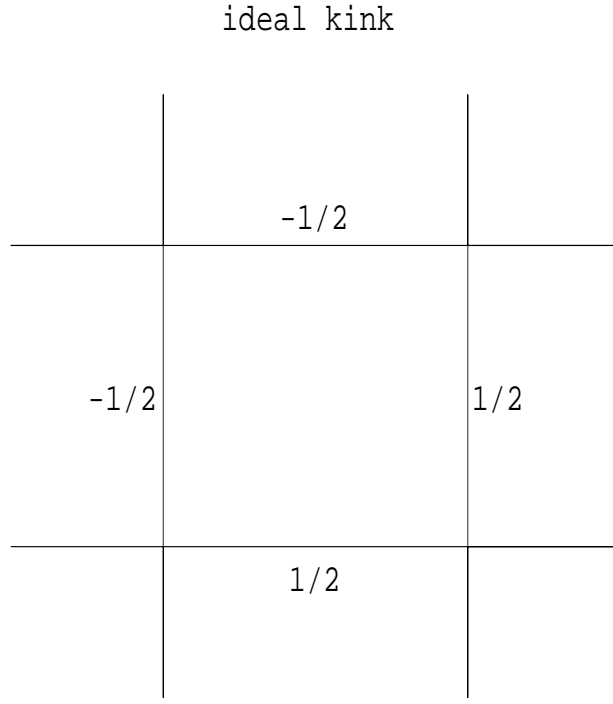


Figure 1: An idealized kink: the normalized link angles $t_{x\mu}$ are given.

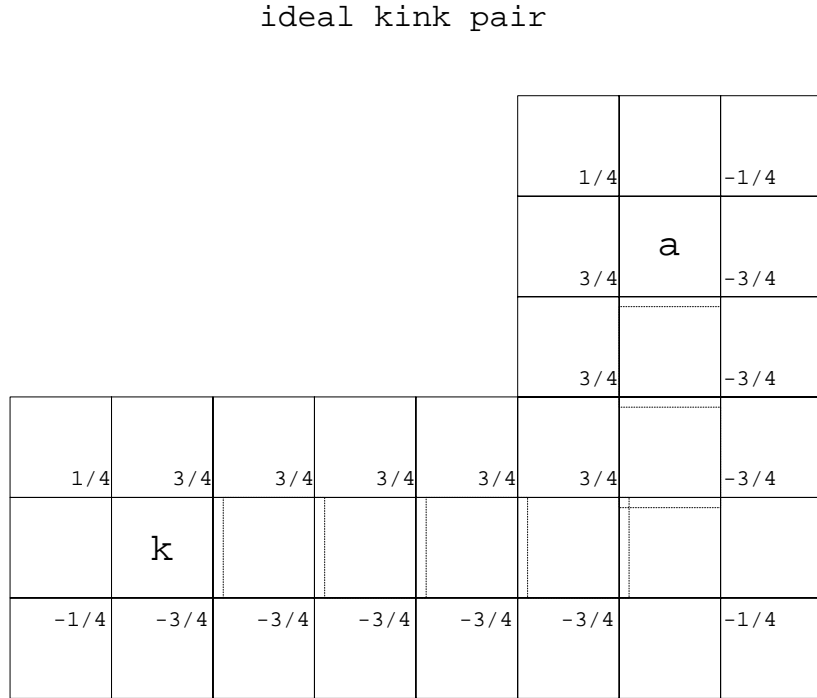


Figure 2: An example of an idealized kink-antikink pair: kink on plaquette k and antikink on plaquette a . The normalized gauge transformation angles $a_x \equiv \alpha_x/\pi$ are given, which create this pair from the identically zero vacuum. The "overtransformed" links are marked by double lines.