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EFFECTIVE AVERAGE ACTION FOR GAUGE THEORIES AND EXACT EVOLUTION EQUATIONS

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Abstract:
We propose a new nonperturbative evolution equation for Yang-Mills theories. It describes the scale dependence of an effective action. The running of the nonabelian gauge coupling in arbitrary dimension is computed.

1 Introduction

The perturbative expansion for a nonabelian gauge theory like QCD is plagued by severe infra-red problems. This well-known fact is closely linked to confinement and to the running gauge coupling becoming strong near the confinement scale. The nonperturbative effects of QCD should be small for Green functions where all relevant momenta are large compared to the confinement scale or for similar infra-red insensitive quantities. In contrast, the properties of the vacuum and the low lying excitations are completely dominated by the nonperturbative infra-red physics. A similar problem appears in the electroweak theory at high temperature. The nonabelian gauge coupling becomes strong at a scale only somewhat below the temperature (typically around $\frac{1}{10}$ T) [1]. The infrared problems in perturbation theory are even more severe in the effective three dimensional theory relevant for high temperatures. Again, one has to deal with a genuine nonperturbative problem if one wants to describe the electroweak high temperature phase transition [2]. Successful nonperturbative studies of nonabelian gauge theories are available by lattice theories. Needless to say that a nonperturbative method formulated in continuous space and maintaining the symmetries of rotations and translations would be a valuable complement. Obviously such a method will have to control carefully the long distance or infra-red behaviour of the theory.

Recently such a method has been developed for scalar theories. It is based on the concept of the average action [3] which is for continuous space the analog of the block spin action [4]. One effectively integrates out all quantum fluctuations with squared momentum q^2 larger than k^2, where k^{-d} determines the volume over which the average of the fields is taken. The average action can be formulated as a functional integral over microscopic variables with a constraint which enforces the averages of the microscopic variables to equal new macroscopic fields. The average action is the effective action for these macroscopic "average fields". The vacuum properties are obtained in the limit $k \to 0$ where the "average volume" tends to infinity. The dependence of the effective average action Γ_k on the "average scale" k can be described by an exact nonperturbative evolution equation [5]. This new exact equation is very sensitive to the infra-red properties of the theory since it is similar in structure to a perturbative one-loop equation, with the important modification that the propagator is the full propagator rather than the classical propagator and vertices are the k-dependent 1PI Green functions derived from Γ_k[1]. In a wider sense the exact evolution equation is a differential version of the Schwinger-Dyson equation [7] with an additional infra-red cutoff given by k. The evolution equation is a partial differential equation for infinitely many variables comprising $t = \ln k$ and the fields $\varphi(x)$ (or their Fourier modes $\varphi(q)$)

$$\frac{\partial}{\partial t}\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\frac{\partial}{\partial t}R_k(\Gamma_k^{(2)}[\varphi] + R_k)^{-1}\right\}. \tag{1.1}$$

Here the trace is over internal indices and contains a d-dimensional momentum integration and $\Gamma_k^{(2)}$ is the second functional derivative with respect to the fields. The infra-red cutoff R_k has typically the form

$$R_k = \frac{q^2\exp(-q^2/k^2)}{1-\exp(-q^2/k^2)} \tag{1.2}$$

and provides for an effective mass-like term $\sim k^2$ in the effective inverse propagator for low momenta $q^2 \ll k^2$. This makes the momentum integral in (1.1) infrared finite even for theories with massless modes. On the other side the integrand in (1.1) decreases exponentially for $q^2 \gg k^2$ due to the exponential fall off of R_k. In consequence the integral in (1.1) is very well converging in the ultraviolet and no specification of the regularization of the theory is needed. (The role of the regularization is replaced in our approach by the initial values for Γ_k at some short distance scale $k = \Lambda$.) It can be proven [5] that Γ_k approaches the classical action for large k and becomes the generating functional for the 1PI Green function for $k \to 0$. A solution of the evolution equation therefore interpolates between the classical action (short-distance action) and the effective action.

The simple physical interpretation of Γ_k, together with the limiting behaviour for $k \to \infty$ and $k \to 0$, and the form of the evolution equation in close analogy to a perturbative one-loop equation permit simple approximate methods for a solution of (1.1). They consist in a truncation of the most general form of Γ_k to a few invariants which should contain the essential physics of the model. The remaining few ordinary coupled nonlinear differential equations for the k-dependent couplings can be solved either analytically or numerically. Green functions and vacuum properties can be reliably computed in a nonperturbative way. These features are in contrast to earlier versions of "exact renormalization group equations" [8], which have mainly been used

[1]The main features of this equation have been obtained earlier [3][6] as a renormalization group improved one-loop equation for the average action.

so far in order to gain beautiful insight in the general structure of renormalizable theories and not so much for practical computations.[2] A formal relation of our new exact evolution equation to the "exact renormalization group equation" [8] can be established [10], [9] by an appropriate Legendre transform.

So far the nonperturbative aspects of the evolution equation (1.1) have been successfully tested for scalar theories by describing the pattern of phase transitions in arbitrary dimensions [6] (including the Kosterlitz-Thouless phase transition [11]), and giving a reliable picture of the second-order phase transition of the four-dimensional theory at high temperature [12]. The critical exponents of the three-dimensional theory have been computed [13] within a precision of a few percent of previous highly accurate estimates [14]. In the large N approximation it is possible to establish explicitly [15] the relation between the exact evolution equation (1.1) and resummation techniques for graphs [16] (gap equations). Furthermore, this approach describes correctly [17] how the effective action becomes convex for $k \to 0$ in the phase with spontaneous symmetry breaking.

A generalization of the average action for abelian gauge theories has been developed [18],[1] within its representation as a functional integral. It has been used to derive evolution equations for the scalar potential of the abelian Higgs model in arbitrary dimension d, as well as for the scalar wave function renormalization [18]. The running of the abelian gauge coupling in arbitrary dimension has been computed in ref. [1]. Based on the fast running of the three-dimensional coupling we have suggested that the electroweak phase transition in the standard model can only be reliably described by nonperturbative methods.

In the present paper we further develop our approach in two directions. First we present a generalization for nonabelian gauge theories and secondly we derive an exact evolution equation in analogy to (1.1). We fully confirm the results in ref. [18], [1], which can now be based on more systematic grounds. For a first time we have obtained an exact nonperturbative evolution equation for nonabelian gauge theories formulated in continuous space and preserving gauge invariance. It can be used for practical computations. We demonstrate the power of our method by computing the running of the nonabelian gauge coupling in a pure SU(N) Yang-Mills theory in arbitrary dimension. In four dimensions the simple truncation used in this paper not only gives the correct one-loop β function for small gauge couplings.

[2]The difficulties for practical calculations are explained in some detail in ref. [9].

It also reproduces the two-loop coefficient with an accuracy of 7%. (More precise computations will be presented elsewhere [19]) The important implications of our three-dimensional results for the high temperature behaviour of nonabelian gauge theories are briefly discussed in the last section.

2 Gauge-invariant generating functionals

For a (Euclidean) gauge theory the generating functional of the connected Green functions is given by

$$\exp W[K;\bar{A}] = \int Da \ \det\left[\frac{\delta' G^z}{\delta'\theta^y}\right]$$

$$\times \exp -\{S[\bar{A}+a] + S_{\rm gf}[a;\bar{A}] - \int d^dx \ K^\mu_z a^z_\mu\} \tag{2.1}$$

$$S_{\rm gf}[a;\bar{A}] = \frac{1}{2\alpha}\int d^dx \ G^*_z G^z$$

Here we have written the gauge field $\mathcal{A}^z_\mu$ as a sum of a background field $\bar{A}^z_\mu$ and a fluctuation field a^z_μ

$$\mathcal{A}^z_\mu = \bar{A}^z_\mu + a^z_\mu \tag{2.2}$$

and we note that the sources K^μ_z are coupled to the fluctuation field a^z_μ. The action

$$S[\mathcal{A}] = S[\bar{A}+a] = \frac{1}{4}\int d^dx \mathcal{F}^{\mu\nu}_z \mathcal{F}^z_{\mu\nu}$$

with

$$\begin{aligned}
\mathcal{F}^z_{\mu\nu} &= \partial_\mu \mathcal{A}^z_\nu - \partial_\nu \mathcal{A}^z_\mu + \bar{g} f_{wy}{}^z \mathcal{A}^w_\mu \mathcal{A}^y_\nu \\
&= \bar{F}^z_{\mu\nu} + (D_\mu[\bar{A}])^z{}_y a^y_\nu - (D_\nu[\bar{A}])^z{}_y a^y_\mu + \bar{g} f_{wy}{}^z a^w_\mu a^y_\nu
\end{aligned}$$

where

$$\begin{aligned}
\bar{F}^z_{\mu\nu} &= \partial_\mu \bar{A}^z_\nu - \partial_\nu \bar{A}^z_\mu + \bar{g} f_{wy}{}^z \bar{A}^w_\mu \bar{A}^y_\nu \\
(D_\mu[\bar{A}])^z{}_y a^y_\nu &= \partial_\mu a^z_\nu - i\bar{g}\bar{A}^w_\mu (T_w)^z{}_y a^y_\nu
\end{aligned} \tag{2.3}$$

is invariant under the gauge transformation

$$\delta \mathcal{A}^z_\mu = \frac{1}{\bar{g}}(D_\mu[\mathcal{A}])^z{}_y \ \delta\theta^y \tag{2.4}$$

(Here w, y, z are adjoint group indices.) This transformation can be split such that the background field $\bar{A}$ transforms inhomogeneously whereas the fluctuation a_μ transforms homogeneously as a tensor in the adjoint representation:

$$\begin{aligned} \delta \bar{A}^z_\mu &= \frac{1}{g}(D_\mu[\bar{A}])^z{}_y \; \delta\theta^y \\ \delta a^z_\mu &= i\delta\theta^w (T_w)^z{}_y a^y_\mu \end{aligned} \tag{2.5}$$

Here T_w are the generators in the adjoint representation ($T_w^\dagger = T_w$) which are related to the real, totally antisymmetric structure constants in the usual way:

$$\begin{aligned} f_{wy}{}^z &= i(T_w)_y{}^z = -i(T_w)^z{}_y \\ [T_w, T_y] &= i f_{wy}{}^z T_z \end{aligned} \tag{2.6}$$

The last term in (2.1) is gauge invariant with respect to the transformation (2.5) provided the source K^μ_z transforms homogeneously as an adjoint tensor. The derivatives of W with respect to the source K yield the connected Green functions for the fluctuation field a_μ in presence of a background $\bar{A}_\mu$. From there the Green functions for $\mathcal{A}_\mu$ can be recovered by substituting $a_\mu = \mathcal{A}_\mu - \bar{A}_\mu$.

We work in the background gauge with gauge condition

$$G^z = (D^\mu[\bar{A}])^z{}_y a^y_\mu \tag{2.7}$$

The Faddeev-Popov determinant obtains from a gauge variation of G^z with $\bar{A}$ fixed:

$$\frac{\delta' G^z(x)}{\delta\theta^y(x')} = \frac{1}{g}(D^\mu[\bar{A}])^z{}_w (D_\mu[\bar{A}+a])^w{}_y \delta(x-x') \tag{2.8}$$

$$\det\left[\frac{\delta' G}{\delta\theta}\right] = \det\left[-D^\mu[\bar{A}]D_\mu[\bar{A}+a]\right] \tag{2.9}$$

More precisely, the variation δ' corresponds to the gauge transformation (2.4) acting only on the fluctuation field a_μ:

$$\begin{aligned} \delta' \bar{A}^z_\mu &= 0 \\ \delta' a^z_\mu &= \frac{1}{g}(D_\mu[\bar{A}+a])^z{}_y \delta\theta^y \end{aligned} \tag{2.10}$$

As usual, one can write the determinant (2.9) as a functional integral over anticommuting ghost fields $\xi^z(x)$ and $\bar{\xi}_z(x)$:

$$\begin{aligned} \det\left[\frac{\delta' G}{\delta\theta}\right] &= \int D\xi D\bar{\xi} \exp -S_{gh}[\xi,\bar{\xi},\bar{A},a] \\ S_{gh}[\xi,\bar{\xi},\bar{A},a] &= \int d^d x \; \bar{\xi}_z (-D^\mu[\bar{A}]D_\mu[\bar{A}+a])^z{}_y \xi^y \end{aligned} \tag{2.11}$$

On the other hand, G^z transforms as an adjoint tensor with respect to the split transformation (2.5). This implies that $S_{\rm gf}$ is invariant. Similarly, det $\left[\frac{\delta' G}{\delta\theta}\right]$ and S_{gh} are invariant with respect to the gauge transformation (2.5). (The ghost fields transform as adjoint tensors.) Using an invariant measure $\int Da$ for the functional integration over a^z_μ one concludes that

$$
\begin{aligned}
&W[K,\bar{A}] = \ln \int DaD\xi D\bar{\xi} \\
&\exp -\{S[\bar{A}+a] + S_{gf}[a;\bar{A}] + S_{gh}[\xi,\bar{\xi},\bar{A},a] - \int d^dx\ K^\mu_z a^z_\mu\} \qquad (2.12)
\end{aligned}
$$

is invariant under a standard gauge transformation of $\bar{A}$ (2.5) with K transforming simultaneously as a tensor in the adjoint representation.

Next we define the generating functional for the 1PI Green functions by a Legendre transformation:

$$
\begin{aligned}
&\Gamma[\bar{a};\bar{A}] = -W[K;\bar{A}] + \int d^dx K^\mu_z \bar{a}^z_\mu \\
&\bar{a}^z_\mu = \frac{\delta W}{\delta K^\mu_z}; \quad K^\mu_z = \frac{\delta \Gamma}{\delta \bar{a}^z_\mu} \qquad (2.13)
\end{aligned}
$$

Here the variations are performed at fixed background field $\bar{A}$. The field $\bar{a}^z_\mu$ is the expectation value of the fluctuation a^z_μ in presence of the background $\bar{A}$. It transforms as an adjoint tensor and $\Gamma[\bar{a};\bar{A}]$ is invariant under a simultaneous gauge transformation (2.5) for $\bar{A}_\mu$ and the homogeneous transformation of $\bar{a}_\mu$. We may also use the gauge field variable

$$
A^z_\mu = \bar{A}^z_\mu + \bar{a}^z_\mu \qquad (2.14)
$$

which has the usual inhomogeneous transformation properties under gauge transformations. The effective action

$$
\Gamma[A,\bar{A}] \equiv \Gamma[A-\bar{A};\bar{A}] \qquad (2.15)
$$

depends in this formulation on two gauge fields A and $\bar{A}$ and is gauge-invariant under simultaneous transformations of both fields. One may define $\Gamma[A,\bar{A}]$ implicitly by the functional integral

$$
\begin{aligned}
&\exp -\Gamma[A,\bar{A}] = \int Da \det\left[-D^\mu[\bar{A}]D_\mu[\bar{A}+a]\right] \\
&\exp -\Big\{S[\bar{A}+a] + \frac{1}{2\alpha}\int d^dx \left(D^\mu[\bar{A}]a_\mu\right)^2 - \int d^dx \frac{\delta\Gamma[A,\bar{A}]}{\delta A_\mu}(a_\mu + \bar{A}_\mu - A_\mu)\Big\} \\
&= \int D\mathcal{A} \det\left[-D^\mu[\bar{A}]D_\mu[\mathcal{A}]\right] \exp -\Big\{S[\mathcal{A}] + \frac{1}{2\alpha}\int d^dx \left[D^\mu[\bar{A}](\mathcal{A}_\mu - \bar{A}_\mu)\right]^2 \\
&- \int d^dx \frac{\delta\Gamma[A,\bar{A}]}{\delta A_\mu}(\mathcal{A}_\mu - A_\mu)\Big\} \qquad (2.16)
\end{aligned}
$$

(Appropriate summations over internal indices are always implied.) It is easy to recognize in (2.16) the standard definition [20] for the effective action $\Gamma[A]_{|\alpha,\bar{A}}$ with gauge fixing $G = D^\mu[\bar{A}](\mathcal{A}_\mu - \bar{A}_\mu)$. This effective action depends on α and $\bar{A}$ only through the gauge fixing. Gauge symmetry implies that all physical observables computed from $\Gamma[A]_{|\alpha,\bar{A}}$ must be independent of α and $\bar{A}$. This clarifies the role of the second argument of $\Gamma[A,\bar{A}]$:

$$\Gamma[A,\bar{A}] = \Gamma[A]_{|\alpha,\bar{A}} \tag{2.17}$$

In particular, one may choose the background field $\bar{A}$ in the gauge fixing to coincide with A

$$\Gamma[A] \equiv \Gamma[A]_{|\alpha,A} = \Gamma[A,A] = \Gamma[\bar{a}=0;\bar{A}=A] \tag{2.18}$$

Then $\Gamma[A]$ is a gauge-invariant functional of only one gauge field A_μ.

In general, we may parametrize the dependence of $\Gamma[A,\bar{A}]$ on the choice of the background field by

$$\kappa_z^\mu[A] = \frac{\delta\Gamma[A,\bar{A}]}{\delta\bar{A}_\mu^z}{}_{|\bar{A}=A} \tag{2.19}$$

where the partial functional derivative on the r.h.s. should be taken at fixed A. We then obtain our final functional integral representation for the gauge-invariant effective action of eq. (2.18):

$$\begin{aligned}\exp -\Gamma[A] &= \int Da\,\det(-D^\mu[A]D_\mu[A+a])\times\\ \exp &-\Big\{S[A+a]+\frac{1}{2\alpha}\int d^dx\,(D^\mu[A]a_\mu)^2-\int d^dx\left(\frac{\delta\Gamma}{\delta A_\mu}-\kappa^\mu\right)a_\mu\Big\}\end{aligned} \tag{2.20}$$

The functional derivative of $\Gamma[A]$ is related to the source K by

$$\frac{\delta\Gamma[A]}{\delta A_\mu} = K^\mu + \kappa^\mu[A] \tag{2.21}$$

We shall return to the functional $\kappa^\mu[A]$ later on.

3 Effective action and gauge bosons

In the previous section we have recapitulated the gauge-invariant effective actions $\Gamma[A,\bar{A}]$ and $\Gamma[A]=\Gamma[A,A]$. From there we want to extract the properties of particles as photons, W-bosons or gluons with high momenta. In case of the photon for an abelian gauge theory the procedure is straightforward. We start with $\Gamma[\bar{a};\bar{A}]$ (2.13) and split the fluctuation field $\bar{a}_\mu$ into transversal and longitudinal components

$$\bar{a}_\mu = \bar{a}_{T\mu} + \bar{a}_{L\mu} \tag{3.1}$$

$$\bar{a}_{L\mu} = P_\mu^\nu \bar{a}_\nu, \ \bar{a}_{T\mu} = (\delta_\mu^\nu - P_\mu^\nu)\bar{a}_\nu \tag{3.2}$$

$$P_\mu^\nu = \frac{\partial_\mu \partial^\nu}{\partial^2} \tag{3.3}$$

Here P_μ^ν is the projector on the longitudinal component

$$P_\mu^\rho P_\rho^\nu = P_\mu^\nu \tag{3.4}$$

which obeys

$$P_\mu^\nu \partial_\nu = \partial_\mu, \quad \partial^\mu P_\mu^\nu = \partial^\nu \tag{3.5}$$

This guarantees that the gauge transformation (2.10) acts only on the longitudinal photons which are considered as gauge degrees of freedom

$$\begin{aligned} \delta' \bar{a}_{L\mu} &= \frac{1}{g}\partial_\mu \delta\theta \\ \delta' \bar{a}_{T\mu} &= 0 \end{aligned} \tag{3.6}$$

The transversal fluctuations are unaffected by the gauge transformation and are identified with the photon[3]. The field equations (2.13) for a vanishing source $K = 0$ read (2.13)

$$\frac{\delta \Gamma[\bar{a};\bar{A}]}{\delta \bar{a}_\mu} = 0 \tag{3.7}$$

and have typically a solution

$$A_\mu = \bar{A}_\mu + \bar{a}_\mu = 0 \tag{3.8}$$

Since the background field $\bar{A}$ can be chosen arbitrarily, one may conveniently employ $\bar{A}_\mu = 0$ in this case. The photon propagator

$$\begin{aligned} &< a_{T\mu}(x) a_{T\nu}(y) > - < a_{T\mu}(x) >< a_{T\nu}(y) > \\ &= (2\pi)^{-d} \int d^d q G_{\mu\nu}(q) \exp -iq_\mu(x^\mu - y^\mu) \end{aligned} \tag{3.9}$$

obtains as the inverse of the second functional derivative of $\Gamma[\bar{a};\bar{A}]$

$$\frac{\delta^2 \Gamma[\bar{a};\bar{A}]}{\delta \bar{a}_{T\mu}(q) \delta \bar{a}_{T\nu}(-q')} = G_{\mu\nu}^{-1}(q)\delta(q-q') \tag{3.10}$$

[3]For massless photons $\bar{a}_{T\mu}$ contains only $d-2$ physical degrees of freedom.

(Here $\bar{a}_{T\mu}(q)$ are the Fourier components $\bar{a}_{T\mu}(q) = (2\pi)^{-\frac{d}{2}} \int d^d x \exp(iq_\mu x^\mu)\bar{a}_{T\mu}(x)$ and (3.10) is evaluated for the solution (3.7) corresponding to a minimum of the (Euclidean) effective action $\Gamma[\bar{a};\bar{A}]$ for fixed $\bar{A}$, which is assumed to be translation-invariant.) The higher functional derivatives of $\Gamma[\bar{a};\bar{A}]$ with respect to $\bar{a}_{T\mu}$ yield the 1PI photon vertices. (We observe that our approach is not limited to the free abelian gauge theory. We may include matter fields or start with an action $S[A]$ containing photon-photon interactions, e.g. a term $\sim (F_{\mu\nu}F^{\mu\nu})^2$. The latter can be imaginated as a result of integrating out matter degrees of freedom.)

In order to describe the propagation and interactions of nonabelian gauge bosons in terms of the effective action, we first need a generalization of the separation into gauge degrees of freedom and "physical" degrees of freedom generalizing (3.1)-(3.5). We generalize the concept of longitudinal and transversal fluctuations in a gauge-covariant way and define[4]

$$\begin{aligned} \bar{a}^z_{L\mu} &= P^{z\nu}_{\mu y}\bar{a}^y_\nu \\ \bar{a}^z_\mu &= \bar{a}^z_{T\mu} + \bar{a}^z_{L\mu} \end{aligned} \tag{3.11}$$

The generalized projector $P^{z\nu}_{\mu y}$ is now also a matrix in internal space and depends on the gauge field $A_\mu = \bar{A}_\mu + \bar{a}_\mu$. We propose

$$P^{z\nu}_{\mu y}[A] = (D_\mu[A]D^{-2}[A]D^\nu[A])^z{}_y \tag{3.12}$$

with $D^{-2}[A] = (D_\rho[A]D^\rho[A])^{-1}$. In a matrix notation in internal space the projector obeys

$$P_\mu{}^\nu[A]D_\nu[A] = D_\mu[A] \tag{3.13}$$

$$D^\mu[A]P_\mu{}^\nu[A] = D^\nu[A] \tag{3.14}$$

$$P_\mu{}^\rho P_\rho{}^\nu = P_\mu{}^\nu \tag{3.15}$$

The transverse fluctuations obey therefore the constraint

$$(D^\mu[A])^z{}_y\bar{a}^y_{T\mu} = 0 \tag{3.16}$$

Using the identity

$$(D^\mu[\bar{A}+\bar{a}])^z{}_y\bar{a}^y_\mu = (D_\mu[\bar{A}])^z{}_y\bar{a}^y_\mu \tag{3.17}$$

[4]The notion of "physical" degrees of freedom employed here should not be confused with the usual concept of physical states in the sense of, say, BRST-cohomology classes. Clearly $a^\mu_T(x)$ considered as a field operator does not create physical states from the vacuum.

we can equivalently write the constraint in the form

$$(D^\mu[\bar{A}])^z{}_y \bar{a}^y_{T\mu} = \bar{g}\bar{a}_T^{w\mu}\bar{a}^y_{L\mu}f_{wy}{}^z \tag{3.18}$$

With respect to the gauge transformation (2.5) the longitudinal and transversal components transform separately as tensors. From

$$\begin{aligned}\delta\bar{a}^z_{T\mu} &= (\bar{a}^z_\mu + \delta\bar{a}^z_\mu) - P^{z\nu}_{\mu y}[A+\delta A](\bar{a}^y_\nu + \delta\bar{a}^y_\nu)\\ &\quad -(\bar{a}^z_\mu - P^{z\nu}_{\mu y}[A]\bar{a}^y_\nu)\\ &= \delta\bar{a}^z_\mu - P^{z\nu}_{\mu y}[A]\delta\bar{a}^y_\nu - \delta P^{z\nu}_{\mu y}[A]\bar{a}^y_\nu \end{aligned}\tag{3.19}$$

$$\delta\bar{a}^z_{L\mu} = P^{z\nu}_{\mu y}[A]\delta\bar{a}^y_\nu + \delta P^{z\nu}_{\mu y}[A]\bar{a}^y_\nu \tag{3.20}$$

and using the transformation of P as an adjoint tensor (with $\delta\theta = \delta\theta^z T_z$)

$$\delta P_\mu{}^\nu[A] = P_\mu{}^\nu[A+\delta A] - P_\mu{}^\nu[A] = -i[P_\mu{}^\nu, \delta\theta] \tag{3.21}$$

we obtain

$$\begin{aligned}\delta\bar{a}^z_{T\mu} &= i\delta\theta^w(T_w)^z{}_y\bar{a}^y_{T\mu}\\ \delta\bar{a}^z_{L\mu} &= i\delta\theta^w(T_w)^z{}_y\bar{a}^y_{L\mu}\end{aligned}\tag{3.22}$$

Here we have used the analogue of (2.5) for $\delta\bar{a}_\mu$ and $\delta A_\mu = \delta\bar{A}_\mu + \delta\bar{a}_\mu$. The transformation law (3.22) also applies to the transversal and longitudinal component of the quantum fluctuation $a_{T\mu}, a_{L\mu}$ defined similar to (3.11). We emphasize that the simple transformation property (3.22) holds only for the transformation (2.5) and not for the transformation with fixed background field (2.10). We could also attempt to define $A_{T\mu}$ and $A_{L\mu}$ in analogy to (3.11). These fields have, however, no simple transformation properties since the symmetry (2.5) mixes transverse and longitudinal components. On the other hand, the homogeneous transformation of $\bar{a}_{T\mu}$ and $\bar{a}_{L\mu}$ would also obtain if we replace $P[A]$ by $P[\bar{A}]$ in the definition (3.11), since $P[\bar{A}]$ transforms according to (3.21).

The split into transversal and longitudinal degrees of freedom becomes useful for a study of small fluctuations around a background field. Let us consider the standard covariant kinetic term $\sim (F_{\mu\nu})^2$ in second order in $\bar{a}_\mu$:

$$I_1 = \frac{1}{4}\int d^dx F_z^{\mu\nu}F^z_{\mu\nu} = \frac{1}{4}\int d^dx \bar{F}_z^{\mu\nu}\bar{F}^z_{\mu\nu} + I_1^{(1)} + I_1^{(2)} + 0(\bar{a}^3) \tag{3.23}$$

$$I_1^{(1)} = \int d^dx(\bar{F}^{\mu\nu};_\nu)_z\bar{a}^z_\mu \tag{3.24}$$

$$I_1^{(2)} = \frac{1}{2} \int d^d x \bar{a}_y^\nu \left\{ (\mathcal{D}_T)^{y\mu}_{\nu z} - (\mathcal{D}_L)^{y\mu}_{\nu z} \right\} \bar{a}^z_\mu \tag{3.25}$$

Here $\bar{F}_{\mu\nu}$ and $F_{\mu\nu}$ are the field strengths formed with $\bar{A}_\mu$ and $A_\mu = \bar{A}_\mu + \bar{a}_\mu$, respectively. The kinetic operators $\mathcal{D}_T$ and $\mathcal{D}_L$ are matrices with both internal and Lorentz indices and read

$$\begin{aligned} (\mathcal{D}_T)^{y\mu}_{\nu z} &= -(D^2)^y{}_z \delta^\mu_\nu + 2i\bar{g}(T_w)^y{}_z \bar{F}_\nu^{w\mu} \\ (\mathcal{D}_L)^{y\mu}_{\nu z} &= -(D_\nu D^\mu)^y{}_z \\ (D^2)^y{}_z &= (D^\rho D_\rho)^y{}_z \end{aligned} \tag{3.26}$$

It is sometimes convenient to use a notation in terms of matrices with internal indices only where

$$\begin{aligned} \bar{A}_\mu &= \bar{A}^z_\mu T_z \\ D_\mu &= \partial_\mu - i\bar{g}\bar{A}_\mu \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} (\mathcal{D}_T)_\nu{}^\mu &= -D_\rho D^\rho \delta^\mu_\nu + 2i\bar{g}\bar{F}_\nu{}^\mu \\ (\mathcal{D}_L)_\nu{}^\mu &= -D_\nu D^\mu \end{aligned} \tag{3.28}$$

We can now employ the identities ($D^2 = D^\rho D_\rho$, $F_{\mu\nu;\rho} = T_z (D_\rho)^z{}_y F^y_{\mu\nu}$, etc.)

$$[D_\mu, D_\nu] = -i\bar{g}\bar{F}_{\mu\nu} \tag{3.29}$$

$$[D_\rho, \bar{F}_\mu{}^\rho] = \bar{F}_\mu{}^\rho{}_{;\rho} \tag{3.30}$$

$$[D^2, D_\mu] = -i\bar{g}(D^\rho \bar{F}_{\rho\mu} + \bar{F}_{\rho\mu} D^\rho) \tag{3.31}$$

to establish

$$(\mathcal{D}_L)_\mu{}^\rho P_\rho{}^\nu[\bar{A}] = P_\mu{}^\rho[\bar{A}](\mathcal{D}_L)_\rho{}^\nu = (\mathcal{D}_L)_\mu{}^\nu \tag{3.32}$$

$$(\mathcal{D}_T)_\mu{}^\rho P_\rho{}^\nu[\bar{A}] = -D_\mu D^\nu - i\bar{g}\bar{F}_\mu{}^\rho{}_{;\rho} D^{-2} D^\nu \tag{3.33}$$

$$P_\mu{}^\rho[\bar{A}](\mathcal{D}_T)_\rho{}^\nu = -D_\mu D^\nu - i\bar{g} D_\mu D^{-2} \bar{F}^{\nu\rho}{}_{;\rho} \tag{3.34}$$

In particular we consider solutions of the Yang-Mills (or Maxwell) equation

$$\bar{F}_{\mu\nu;}{}^\nu = 0 \tag{3.35}$$

In this case $I_1^{(1)}$ vanishes and the operators $\mathcal{D}_T$ and $\mathcal{D}_L$ commute

$$\mathcal{D}_T \mathcal{D}_L = \mathcal{D}_L \mathcal{D}_T = \mathcal{D}_L^2 \tag{3.36}$$

The projector now obeys

$$P[\bar{A}] = \mathcal{D}_T^{-1}\mathcal{D}_L \tag{3.37}$$

and we can perform the split into transversal and longitudinal components in $I_1^{(2)}$ replacing $P_\mu{}^\nu[\bar{A}+\bar{a}]$ by $P_\mu{}^\nu[\bar{A}]$. Then the longitudinal component $\bar{a}_{L\mu}$ does not contribute in quadratic order to I_1

$$I_1^{(2)} = \frac{1}{2}\int d^dx \bar{a}_T^\mu (\mathcal{D}_T)_\mu{}^\nu \bar{a}_{T\nu} \tag{3.38}$$

At least for fields satisfying (3.35) this suggests to identify the transversal fluctuation $\bar{a}_{T\mu}$ with the physical gauge bosons and to consider the longitudinal components $\bar{a}_{L\mu}$ as gauge degrees of freedom. These considerations are important in the following sections where we construct a scale-dependent effective action. We shall use the eigenvalues with respect to the operator $\mathcal{D}_T$ instead of momentum squared in order to generalize the concept of "integrating out the high momentum modes" in a gauge-invariant way.

4 The effective average action

The functional integral (2.20) is our starting point for the generalization to a scale-dependent effective action - the effective average action $\Gamma_k[A]$. Whereas in (2.20) quantum fluctuations with arbitrary momenta have to be included, the scale-dependent effective action $\Gamma_k[A]$ should involve only an integration over modes with momenta larger than some infrared cutoff k. A variation of k describes then the successive integration of fluctuations corresponding to different length scales with the aim to recover $\Gamma[A]$ in the limit $k \to 0$. We propose

$$\Gamma_k[A] = -\ln\int Da \;\; \det\left[-D^\mu[A]D_\mu[A+a]\right] \;\; E_k[A] \tag{4.1}$$

$$\exp - \left\{ S[A+a] + \frac{1}{2\alpha}\int d^dx\,(D^\mu[A]a_\mu)^2 + \Delta_k S_G[A,a] - \int d^dx \left(\frac{\delta\Gamma_k}{\delta A_\mu} - \kappa^\mu\right) a_\mu \right\}$$

The additional term

$$\Delta_k S_G[a;A] = \frac{1}{2}\int d^dx \;\, a_y^\nu (R_k(\mathcal{D}_\mathrm{T}))_{\nu z}^{y\mu} a_\mu^z$$

$$+\frac{1}{2}(\frac{1}{\alpha_k}-1)\int d^dx \;\left(D^\nu[A]a_\nu\right)_y (\mathcal{D}_\mathrm{S}^{-1}R_k(\mathcal{D}_\mathrm{S}))^y{}_z \left(D^\mu[A]a_\mu\right)^z \tag{4.2}$$

provides for an infrared cutoff for the fluctuations a_μ. Here $R_k(\mathcal{D}_\mathrm{T})$ and $R_k(\mathcal{D}_\mathrm{S})$ are matrix-valued functions of the matrices (3.26) with $\bar{A}$ replaced by A

$$(\mathcal{D}_\mathrm{S}[A])^y{}_z = -(D_\mu[A]D^\mu[A])^y{}_z$$

$$(\mathcal{D}_{\rm T}[A])^{y\mu}_{\nu z} = (\mathcal{D}_{\rm S}[A])^y{}_z\delta^\mu_\nu + 2i\bar{g}(T_w)^y{}_z F^{w\mu}_\nu[A] \tag{4.3}$$

(For the abelian theory one has $\mathcal{D}_{\rm S} = -\partial_\mu\partial^\mu$, $\mathcal{D}_{\rm T}{}^\mu{}_\nu = -\partial_\rho\partial^\rho\delta^\mu_\nu$.) The effective classical propagator for the functional integration of a_μ in (4.1) is obtained by combining the quadratic pieces in S and $\Delta_k S$. Its inverse reads, for $\alpha = 1$ and $S = I_1$,

$$\bar{P}(\mathcal{D}_{\rm T}) = \mathcal{D}_{\rm T} + R_k(\mathcal{D}_{\rm T}) \tag{4.4}$$

with a suitable modification for $\alpha \neq 1$. We will use an analytic function $R_k(x)$ with the property

$$\lim_{k\to 0} R_k(x) = 0 \tag{4.5}$$

in order to recover the effective action $\Gamma[A]$ for $k \to 0$. A convenient choice is

$$R_k(x) = Z_k\ x\frac{\exp(-x/k^2)}{1-\exp(-x/k^2)} \tag{4.6}$$

so that

$$\lim_{x\to 0} R_k(x) = Z_k k^2 \tag{4.7}$$

with Z_k an appropriate wave function renormalization constant. Our choice assures that for a generalized momentum squared $x \gg k^2$ all effects of the presence of an infrared cutoff are exponentially suppressed. More generally, the detailed form of R_k specifies the details of how the fluctuations with eigenvalues of $\mathcal{D}_{\rm T}$ or $\mathcal{D}_{\rm S}$ larger than k^2 are integrated out in the computation of Γ_k. Our particular choice (4.2) contains two separate pieces - the first involving $\mathcal{D}_T$ acts as an infrared cutoff for the transverse fluctuations whereas the second involving D_S deals with the longitudinal fluctuations or gauge degrees of freedom (cf. sect. 3).

The quantity

$$E_k[A] = \det\left[1 + \mathcal{D}_{\rm S}^{-1}[A]R_k(\mathcal{D}_{\rm S}[A])\right] \tag{4.8}$$

provides an infrared cutoff for the ghost fields of the nonabelian theory. (It reduces to an irrelevant constant for the abelian theory.) Combined with the Faddeev-Popov determinant it leads to an effective inverse ghost propagator $\bar{P}(\mathcal{D}_{\rm S}[A]) \equiv \mathcal{D}_{\rm S} + R_k(\mathcal{D}_{\rm S})$. With the choice (4.6) we observe the limits

$$\lim_{k\to\infty} E_k = \det(\mathcal{D}_{\rm S}[A]/Z_k k^2)^{-1} \tag{4.9}$$

$$\lim_{k\to 0} E_k = 1 \tag{4.10}$$

where the second limit (4.10) applies for modes with nonzero eigenvalue of $\mathcal{D}_{\rm S}[A]$. (For $\mathcal{D}_{\rm S}[A] \ll k^2, k \to 0$, the dominant part of E_k is given by the r.h.s. of (4.9).)

More generally, the determinant E_k cancels the contribution from the Faddeev-Popov determinant (for $a = 0$) for all modes with eigenvalues of $\mathcal{D}_S$ much smaller than k^2. This is required since the contribution of gauge degrees of freedom with $\mathcal{D}_S \ll k^2$ is suppressed by the infrared cutoff contained in $\Delta_k S_G$ (4.2) and the Faddeev-Popov determinant serves exactly the purpose of removing unphysical contributions from fluctuations of gauge degrees of freedom. Finally, the quantity κ^μ in eq. (4.1) is given by the k-dependent analog to (2.21). It will be specified in sect. 6.

Using the limits (4.5) and (4.10), we find that the effective average action $\Gamma_k[A]$ reduces to the effective action $\Gamma[A]$ of eq. (2.20) in the limit $k \to 0$:

$$\lim_{k\to 0} \Gamma_k[A] = \Gamma[A] \tag{4.11}$$

On the other hand we observe that for $k \to \infty$ the functional integral (4.1) becomes Gaussian since the action becomes dominated by the divergent piece $\Delta_k S_G$ which is purely quadratic in a_μ. In this limit $\Gamma_k[A]$ is given by its classical approximation (putting $a_\mu = 0$ in (4.1) instead of functional integration). Together with (4.9), this implies that $\Gamma_k[A]$ becomes identical with the classical action $S[A]$ in the limit $k \to \infty$:

$$\lim_{k\to \infty} \Gamma_k[A] = S[A] \tag{4.12}$$

The effective average action Γ_k therefore interpolates between the classical action S and the effective action Γ, corresponding to a successive integration of quantum fluctuations with different length scales.

In the following sections we will motivate the definition (4.1) of the effective average action $\Gamma_k[A]$ in more detail. We generalize our construction to include matter fields. For convenience we discuss scalars, but fermions pose no particular problem [21]. This includes chiral fermions provided they belong to an anomaly-free representation such that their functional measure can be defined in a gauge-invariant way. We will show that the dependence of the effective average action on the scale k can be described by an exact evolution equation whose structure is similar to a renormalization-group improved one-loop equation.

5 Exact evolution equation with background gauge field

Let us start by defining a k-dependent generating functional

$$\begin{aligned}\exp W_k[J,K;\bar{A}] &= E_k[\bar{A}]\int D\chi Da \exp -\{\hat{S}[\chi,a,\bar{A}]+\Delta_k S_{\rm S}[\chi;\bar{A}]+\Delta_k S_G[a;\bar{A}]\\ &- \int d^dx (J_a(x)\chi^a(x)+K^\mu_z(x)a^z_\mu(x))\}\end{aligned} \tag{5.1}$$

for scalar fields χ and gauge fields $\mathcal{A}$ with J and K the corresponding sources. Here $\hat{S}$ obtains from the gauge-invariant classical action $S[\chi,A]$ by shifting $\mathcal{A}=\bar{A}+a$ and contains gauge-fixing terms (see sect. 2)

$$\hat{S}[\chi,a,\bar{A}]=S[\chi,\bar{A}+a]+S_{gf}[a;\bar{A}]-\ln\det(-D^\mu[\bar{A}]D_\mu[\bar{A}+a]) \tag{5.2}$$

As discussed in the last section, W_k is gauge-invariant if $\Delta_k S_{\rm S}$ and $\Delta_k S_G$ are invariant under the transformation (2.5). (The scalar field transforms in the appropriate representation. We note that $\int D\chi Da$ is defined in presence of the background field $\bar{A}$ and must be invariant only under the combined transformation of χ, a and $\bar{A}$. Gauge invariance of W_k relates to the simultaneous homogeneous transformation of the sources J and K and inhomogeneous transformation of the background field $\bar{A}$.) The gauge-invariant effective infrared cutoff reads

$$\begin{aligned}\Delta_k S_{\rm S} &= \frac{1}{2}\int d^dx\ \chi_a(x)(R_k(\mathcal{D}_{\rm S}[\bar{A}]))^a{}_b\chi^b(x)\\ \Delta_k S_{\rm G} &= \frac{1}{2}\int d^dx\ a^\nu_y(R_k(\mathcal{D}_{\rm T}[\bar{A}]))^{y\mu}_{\nu z}a^z_\mu\\ &+\frac{1}{2}(\frac{1}{\alpha_k}-1)\int d^dx\ (D^\nu[\bar{A}]a_\nu)_y(\mathcal{D}_{\rm S}^{-1}[\bar{A}]R_k(\mathcal{D}_{\rm S}[\bar{A}]))^y{}_z(D^\mu[\bar{A}]a_\mu)^z\end{aligned} \tag{5.3}$$

with $D_\mu[\bar{A}]$ in the appropriate representation for the real scalar field χ. The determinant E_k is given by (4.8) in case of a nonabelian theory[5]. We note that $\Delta_k S=\Delta_k S_{\rm S}+\Delta_k S_G$ is quadratic in a_μ and χ and therefore only contributes to the propagators but not to the vertices in the functional integral (5.1). (The same

[5] In principle, one could choose different cutoff functions R_k^S and R_k^G for scalars and gauge fields and also introduce an arbitrary additional term for the gauge degrees of freedom $\tilde{\Delta}S_G=\frac{1}{2\alpha}\int d^dx(D_\mu a^\mu)\tilde{R}_k(D_\nu a^\nu)$. We will not exploit this additional freedom here except for the use of different wave function renormalization constants Z_k for the various fields. It is also easy to employ a formulation for complex scalars. Then $\Delta_k S_{\rm S}$ reads $\int d^dx\chi^*_a(R_k)^a_b\chi^b$ and $J_a\chi^a$ should be replaced by $J^*_a\chi^a+J^a\chi^*_a$ in (5.1).

holds for the ghost integration if E_k is included in the ghost Lagrangian. We will not explicitly work with ghost fields in this paper and rather keep E_k as a given functional of the background field $\bar{A}$.) For small eigenvalues of $\mathcal{D}_{\rm S}[\bar{A}]$ or $\mathcal{D}_{\rm T}[\bar{A}]$ the term $\Delta_k S$ provides a "masslike" term $\sim k^2$ for both the scalars and the gauge fields and therefore acts as an infrared cutoff. On the other side, the integration of modes with eigenvalues much larger than k^2 is only affected by exponentially small corrections. In the limit $k \to 0$ the generating functional W_k becomes the usual generating functional W for the connected Green functions.

We next define k-dependent expectation values

$$\varphi^a(x) = \frac{\delta W_k}{\delta J_a(x)} = \langle \chi^a(x) \rangle_{\bar{A}} \tag{5.4}$$

$$\bar{a}^z_\mu(x) = \frac{\delta W_k}{\delta K^\mu_z(x)} = \langle a^z_\mu(x) \rangle_{\bar{A}} \tag{5.5}$$

Here the functional derivatives of W_k are taken with a fixed background field $\bar{A}$ and we have indicated that the expectation values are taken in presence of the background field. Both φ and $\bar{a}$ are functionals of the sources J and K and depend in addition on $\bar{A}$ and k. If we denote the original gauge field by $\mathcal{A} = \bar{A} + a$ the Green functions of χ and $\mathcal{A}$ in presence of the quadratic constraint term $\Delta_k S$ are directly related to the derivatives of W_k for $J = K = 0$. In particular one has for the vacuum expectation values without sources

$$\langle \mathcal{A}^z_\mu \rangle_0 = \bar{A}^z_\mu + \bar{a}^z_\mu[\bar{A}]_{|J,K=0} \tag{5.6}$$

$$\langle \chi^a \rangle_0 = \varphi^a[\bar{A}]_{|J,K=0} \tag{5.7}$$

In the limit $k \to 0$ they reduce to the usual expectation values.

The Legendre transform $\tilde{\Gamma}_k$ of W_k obeys

$$\tilde{\Gamma}_k[\varphi, \bar{a}; \bar{A}] + W_k[J, K; \bar{A}] = \int d^d x (J_a(x)\varphi^a(x) + K^\mu_z(x)\bar{a}^z_\mu(x)) \tag{5.8}$$

with

$$\frac{\delta \tilde{\Gamma}_k}{\delta \varphi^a(x)} = J_a(x), \quad \frac{\delta \tilde{\Gamma}_k}{\delta \bar{a}^z_\mu(x)} = K^\mu_z(x) \tag{5.9}$$

Here all functional derivatives are taken at fixed background field $\bar{A}$. We combine $\varphi^a(x)$ and $\bar{a}^z_\mu(x)$ into a vector ψ and similarly $J_a(x)$ and $K^\mu_z(x)$ into j:

$$\begin{aligned} \psi^i(x) &= (\varphi^a(x),\ \bar{a}^z_\mu(x)) \\ j_i(x) &= (J^a(x),\ K^z_\mu(x)) \end{aligned} \tag{5.10}$$

The second functional derivatives of W_k and $\tilde{\Gamma}_k$ (at fixed $\bar{A}$) are then related by the well-known identity

$$G_k\ \tilde{\Gamma}_k^{(2)} = 1 \tag{5.11}$$

where $G_k^{(2)}$ and $\tilde{\Gamma}_k^{(2)}$ are considered as matrices with group indices (including Lorentz indices) and coordinate or (covariant) momentum labels:

$$(\tilde{\Gamma}_k^{(2)})_{ij}(x,y) = \frac{\delta^2 \tilde{\Gamma}_k}{\delta\psi^i(x)\delta\psi^j(y)} \tag{5.12}$$

$$(G_k)^{ij}(x,y) = \frac{\delta^2 W_k}{\delta j_i(x)\delta j_j(y)} \tag{5.13}$$

The identity (5.11) is valid for all $\varphi, \bar{a}$ where $\tilde{\Gamma}_k^{(2)}$ exists and is invertible. We restrict our discussion to this region in field space.

We next define

$$\Gamma_k[\psi;\bar{A}] = \tilde{\Gamma}_k[\psi;\bar{A}] - \Delta_k S[\psi;\bar{A}] \tag{5.14}$$

where $\Delta_k S$ of eq. (3.3) reads in an obvious matrix notation

$$\Delta_k S = \frac{1}{2}\int d^dx d^dy\ \psi_i(x) R_k[\bar{A}]^i{}_j(x,y)\psi^j(y) \equiv \frac{1}{2}\psi^T R_k \psi \tag{5.15}$$

¿From (5.8) and (5.1) one finds for the scale dependence of $\tilde{\Gamma}_k$

$$\begin{aligned}
&\frac{\partial}{\partial t}\tilde{\Gamma}_{k\ |\psi,\bar{A}} \equiv k\frac{\partial}{\partial k}\tilde{\Gamma}_{k|\psi,\bar{A}} = -\frac{\partial}{\partial t}W_{k|j,\bar{A}} = \\
&= \frac{\partial}{\partial t} < \Delta_k S[\sigma;\bar{A}] >_{\bar{A}} - \mathcal{E}_k = \frac{1}{2}\langle \mathrm{Tr}\{\sigma \otimes \left(\frac{\partial}{\partial t}R_k\right)\sigma\}\rangle_{\bar{A}} - \mathcal{E}_k \\
&= \frac{1}{2}\mathrm{Tr}\{\left(\frac{\partial}{\partial t}R_k\right)(G + \psi\otimes\psi)\} - \mathcal{E}_k
\end{aligned} \tag{5.16}$$

Here we defined $t \equiv \ln k, \sigma \equiv (\chi, a)$ and

$$\mathcal{E}_k = \frac{\partial}{\partial t}\ln\ E_k$$

Inserting

$$\frac{\partial}{\partial t}\Delta_k S[\psi,\bar{A}] = \frac{1}{2}\mathrm{Tr}\{\left(\frac{\partial}{\partial t}R_k\right)(\psi\otimes\psi)\} \tag{5.17}$$

yields

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_k[\psi;\bar{A}] &= \frac{1}{2}\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k\right)G_k\right\} - \mathcal{E}_k \\
&= \frac{1}{2}\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k\right)\left(\tilde{\Gamma}_k^{(2)}\right)^{-1}\right\} - \mathcal{E}_k \\
&= \frac{1}{2}\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k\right)\left(\Gamma_k^{(2)} + R_k\right)^{-1}\right\} - \mathcal{E}_k
\end{aligned} \tag{5.18}$$

This equation expresses the scale dependence of Γ_k in terms of the second functional derivative $\Gamma_k^{(2)}$. We observe that $\mathcal{E}_k$ can be written in the form

$$\mathcal{E}_k = \mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k(\mathcal{D}_{\mathrm{S}}[\bar{A}])\right)(\mathcal{D}_{\mathrm{S}}[\bar{A}] + R_k(\mathcal{D}_{\mathrm{S}}[\bar{A}]))^{-1}\right\} \tag{5.19}$$

where Tr symbolizes a trace in the adjoint representation (as well as over coordinate or momentum labels). The two terms on the r.h.s. of (5.18) have therefore a similar structure.

We finally combine $\bar{a}$ with the background field $\bar{A}$ into the gauge field

$$A_\mu^z = \bar{A}_\mu^z + \bar{a}_\mu^z \tag{5.20}$$

The gauge field A has the standard inhomogeneous transformation properties. Replacing the variable $\bar{a}$ by A the effective action Γ_k becomes now a functional of φ, A and $\bar{A}$

$$\Gamma_k = \Gamma_k[\varphi, A, \bar{A}] \tag{5.21}$$

By construction $\Gamma_k[\varphi, A, \bar{A}]$ is manifestly gauge-invariant under the simultaneous transformations of φ, A and $\bar{A}$. It obeys

$$\begin{aligned}&\frac{\partial}{\partial t}\Gamma_k[\varphi, A, \bar{A}] \\ &= \frac{1}{2}\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k[\bar{A}]\right)\left(\Gamma_k^{(2)}[\varphi, A, \bar{A}] + R_k[\bar{A}]\right)^{-1}\right\} - \mathcal{E}_k[\bar{A}]\end{aligned} \tag{5.22}$$

where

$$\Gamma_k^{(2)}[\varphi, A, \bar{A}] = \Gamma_k^{(2)}[\varphi, \bar{a}; \bar{A}]_{|\bar{a}=A-\bar{A}} \tag{5.23}$$

is the matrix for the second functional derivative of $\Gamma_k[\varphi, A, \bar{A}]$ with respect to φ and A at fixed $\bar{A}$.

6 The role of the background field

At this point Γ_k is a functional of two gauge fields, A and $\bar{A}$, which are considered as independent variables. It is tempting to identify A and $\bar{A}$ such that the effective action only depends on one gauge field. Before doing so we have, however, to worry about the fact that the second functional derivative $\Gamma_k^{(2)}$ in (5.22) is taken at fixed $\bar{A}$ and does not coincide with the second functional derivative of $\bar{\Gamma}_k[\varphi, A] \equiv$

$\Gamma_k[\varphi, A, A]$.[6] Indeed one has

$$\frac{\delta}{\delta A}\bar{\Gamma}_k[\varphi, A] = \frac{\delta}{\delta A}\Gamma_k[\varphi, A, A] = \frac{\delta}{\delta A}\Gamma_k[\varphi, A, \bar{A}]_{|\bar{A}=A} + \frac{\delta}{\delta \bar{A}}\Gamma_k[\varphi, A, \bar{A}]_{|\bar{A}=A} \tag{6.1}$$

and similarly for the second functional derivative. In order to understand the role of this difference we need some insight in the general dependence of $\Gamma_k[\varphi, A, \bar{A}]$ on the background field $\bar{A}$.

Let us start by taking the limit $k \to \infty$. We can use the definitions (5.8), (5.9), and (5.14) to derive a functional integral representation of Γ_k,

$$\begin{aligned}
\exp -\Gamma_k[\varphi, \bar{a}; \bar{A}] \;=\; & \int D\chi Da \exp -\{S[\chi, \bar{A}+a] \\
& +S_{gf}[a; \bar{A}] - \ln\det(-D^\mu[\bar{A}]D_\mu[\bar{A}+a]) - \ln E_k[\bar{A}] \\
& +\Delta_k S_{\rm S}[\chi; \bar{A}] - \Delta_k S_{\rm S}[\varphi; \bar{A}] + \Delta_k S_G[a; \bar{A}] - \Delta_k S_G[\bar{a}; \bar{A}] \\
& -\int d^d x [J_a(\chi^a - \varphi^a) + K_z^\mu(a_\mu^z - \bar{a}_\mu^z)]\}
\end{aligned} \tag{6.2}$$

with

$$\begin{aligned}
J_a &= \frac{\delta \Gamma_k}{\delta \varphi^a} + \frac{\delta \Delta_k S_{\rm S}[\varphi; \bar{A}]}{\delta \varphi^a} \\
K_z^\mu &= \frac{\delta \Gamma_k}{\delta \bar{a}_\mu^z} + \frac{\delta \Delta_k S_G[\bar{a}; \bar{A}]}{\delta \bar{a}_\mu^z}
\end{aligned} \tag{6.3}$$

Using shifted integration variables

$$\begin{aligned}
\chi' &= \chi - \varphi \\
a' &= a - \bar{a} = a + \bar{A} - A
\end{aligned} \tag{6.4}$$

eq. (6.2) yields

$$\begin{aligned}
\exp -\Gamma_k[\varphi, A, \bar{A}] \;=\; & \int D\chi' Da' \exp -\{S[\varphi + \chi', A + a'] \\
& +S_{gf}[A - \bar{A} + a'; \bar{A}] - \ln\det(-D^\mu[\bar{A}]D_\mu[A + a']) - \ln E_k[\bar{A}] \\
& +\Delta_k S_{\rm S}[\chi'; \bar{A}] + \Delta_k S_G[a'; \bar{A}] \\
& -\int d^d x [\frac{\delta \Gamma_k}{\delta \varphi^a}\chi'^a + \frac{\delta \Gamma_k}{\delta \bar{a}_\mu^z} a'^z_\mu]\}
\end{aligned} \tag{6.5}$$

In the limit $k \to \infty$ the function R_k approaches a constant and one has

$$\begin{aligned}
&\lim_{k\to\infty} R_k = Z_k k^2 \\
&\lim_{k\to\infty} \exp -\{\Delta_k S_{\rm S}[\chi'; \bar{A}] + \Delta_k S_G[a'; \bar{A}]\} \propto \delta[\chi']\delta[a']
\end{aligned} \tag{6.6}$$

[6] We put in this section a bar on $\bar{\Gamma}[\varphi, A]$ in order to facilitate the reading.

One obtains (up to an irrelevant constant)

$$\lim_{k\to\infty}\Gamma_k[\varphi,A,\bar{A}] = S[\varphi,A]$$
$$+S_{gf}[A-\bar{A};\bar{A}] - \ln\det(D^\mu[\bar{A}]D_\mu[A]D^{-2}[\bar{A}]) \tag{6.7}$$

Therefore $\Gamma_{k\to\infty}$ corresponds to the classical action with gauge fixing. For $k\to\infty$ the dependence of Γ_k on $\bar{A}$ arises only from the gauge fixing terms (including the determinant in the last term). We also observe

$$\lim_{k\to\infty}\Gamma_k[\varphi,A,A] = S[\varphi,A] \tag{6.8}$$

The above discussion suggests to write for arbitrary k

$$\Gamma_k[\varphi,A,\bar{A}] = \bar{\Gamma}_k[\varphi,A] + \Gamma_k^{\text{gauge}}[\varphi,A,\bar{A}] \tag{6.9}$$

Here Γ_k^{gauge} is a generalized gauge fixing which vanishes for $\bar{A}=A$, and

$$\bar{\Gamma}_k[\varphi,A] \equiv \Gamma_k[\varphi,A,A] \tag{6.10}$$

The second functional derivative at fixed $\bar{A}$ is then a sum of the second functional derivative of $\bar{\Gamma}_k$ and a contribution from the generalized gauge fixing term:

$$\Gamma_k^{(2)}[\varphi,A,\bar{A}] = \bar{\Gamma}_k^{(2)}[\varphi,A] + \Gamma_k^{\text{gauge}(2)}[\varphi,A,\bar{A}]_{|\bar{A}} \tag{6.11}$$

(We have indicated that the second functional derivative in the second term should be performed for fixed $\bar{A}$.) Since Γ_k^{gauge} vanishes for $A=\bar{A}$ we may expand for small $A-\bar{A}=\bar{a}$

$$\Gamma_k^{\text{gauge}} = \int d^dx\left\{-\kappa_z^\mu[\varphi,\bar{A}]\bar{a}_\mu^z + \frac{1}{2}\bar{a}_\nu^y\bar{Q}_{yz}^{\nu\mu}\bar{a}_\mu^z + ...\right\} \tag{6.12}$$

with

$$\kappa_z^\mu = -\frac{\delta\Gamma_k^{\text{gauge}}[\varphi,A,\bar{A}]}{\delta A_\mu^z}_{|\bar{A}=A} = \frac{\delta\Gamma_k[\varphi,A,\bar{A}]}{\delta\bar{A}_\mu^z}_{|\bar{A}=A} \tag{6.13}$$

$$\bar{Q}_{yz}^{\nu\mu} = \frac{\delta^2\Gamma_k^{\text{gauge}}[\varphi,A,\bar{A}]}{\delta A_\nu^y\delta A_\mu^z}_{|\bar{A}=A} \tag{6.14}$$

In (6.13) we have used the identities

$$\begin{aligned}\frac{\delta\bar{\Gamma}_k}{\delta A} &= \frac{\delta\Gamma_k[A,\bar{A}]}{\delta A}_{|\bar{A}=A} + \frac{\delta\Gamma_k[A,\bar{A}]}{\delta\bar{A}}_{|\bar{A}=A}\\ K &= \frac{\delta\Gamma_k[A,\bar{A}]}{\delta A}_{|\bar{A}=A} = \frac{\delta\bar{\Gamma}_k}{\delta A} + \frac{\delta\Gamma_k^{\text{gauge}}}{\delta A}_{|\bar{A}=A}\\ &= \frac{\delta\bar{\Gamma}_k}{\delta A} - \frac{\delta\Gamma_k}{\delta\bar{A}}_{|\bar{A}=A} = \frac{\delta\bar{\Gamma}_k}{\delta A} - \kappa\end{aligned} \tag{6.15}$$

These relations further motivate the definition of κ^μ in eq. (2.21). For $\bar{A} = A$ the (A, A) component of $\Gamma_k^{\rm gauge(2)}$ coincides with $\bar{Q}$ whereas the (A, φ) component is given by $\delta\kappa/\delta\varphi$:

$$\begin{aligned} \frac{\delta^2\Gamma_k^{\rm gauge}}{\delta\varphi^a\delta A_\mu^z}_{|\bar{A}=A} &= -\frac{\delta\kappa_z^\mu[\varphi, A]}{\delta\varphi^a} \\ \frac{\delta^2\Gamma_k^{\rm gauge}}{\delta\varphi^a\delta\varphi^b}_{|\bar{A}=A} &= 0 \end{aligned} \tag{6.16}$$

For $k \to \infty$ one finds from (6.7)

$$\begin{aligned} \frac{\delta\kappa}{\delta\varphi} &= 0 \\ \bar{Q} &= \bar{Q}_{\rm gf} + \bar{Q}_{det} \\ (\bar{Q}_{\rm gf})_{yz}^{\nu\mu} &= -\frac{1}{\alpha}(D^\nu[\bar{A}]D^\mu[\bar{A}])_{yz} \\ (\bar{Q}_{\rm det})_{yz}^{\nu\mu} &= -\frac{\delta^2}{\delta\bar{a}_\nu^y\delta\bar{a}_\mu^z}{\rm Tr}\ln(-D^\rho[\bar{A}]D_\rho[\bar{A}+\bar{a}])_{|\bar{a}=0} \end{aligned} \tag{6.17}$$

The role of $\Gamma_k^{\rm gauge(2)}$ in the evolution equation (5.22) becomes obvious now: Since $\bar{\Gamma}_k[\varphi, A]$ is gauge-invariant, its second functional derivative $\bar{\Gamma}_k^{(2)}$ must have zero eigenvalues in the direction of the gauge modes. In absence of $\bar{Q}$ the momentum integrals implied by the trace in (5.22) would not be ultraviolet finite. (Note that $\lim_{x\to\infty}\frac{\partial}{\partial t}\ln R_k(x) = \frac{\partial}{\partial t}\ln Z_k + 2\frac{x}{k^2}$ diverges for large x!) This disease corresponds to the usual fact that no well-behaving propagator can be derived from a gauge-invariant action. The additional term $\bar{Q}$ cures this problem. The inverse propagator $\Gamma_k^{(2)} = \bar{\Gamma}_k^{(2)} + \bar{Q}$ is now well behaved for large eigenvalues of $x = \mathcal{D}_{\rm T}$ or $x = \mathcal{D}_{\rm S}$ since it corresponds to a gauge-fixed propagator. More precisely, all eigenvalues of $\Gamma_k^{(2)}$ are strictly positive for large x. For high eigenvalues the integral ${\rm Tr}\frac{\partial}{\partial t}R_k(\Gamma^{(2)})^{-1}$ is then exponentially suppressed in the ultraviolet due to the exponential decay of R_k. We conclude that the momentum integrals in (5.22) are both ultraviolet and infrared finite for $k \neq 0$. The evolution equation (5.22) is therefore well defined and does not need a specification of the regularization of the theory.

For finite values of k the generalized gauge fixing contribution $\Gamma_k^{\rm gauge}$ will in general not remain of the simple form (6.7). We may nevertheless try some relatively simple expansion

$$\begin{aligned} \Gamma_k^{\rm gauge} &= \frac{1}{2}\int d^dx\Big\{\alpha^{-1}(k)\left(D^\mu[\bar{A}]\bar{a}_\mu\right)^2 \\ &\quad +\mu_g^2(k)\bar{a}_\mu^z\bar{a}_z^\mu - Z_g(k)(D_\mu[\bar{A}]\bar{a}_\nu)(D^\mu[\bar{A}]\bar{a}^\nu) \\ &\quad +l_g(k)F^{\mu\nu}{}_{;\nu}[\bar{A}]\bar{a}_\mu + \ldots\Big\} \end{aligned} \tag{6.18}$$

where α^{-1}, μ_g^2, Z_g, and l_g may be functions of invariants constructed from φ and $\bar{A}$. The k-dependence of $\Gamma_k^{\rm gauge}$ can directly be read off from (5.22) with $\bar{a} = A - \bar{A}$

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_k^{\rm gauge}[\varphi, A, \bar{A}] &\equiv \frac{\partial}{\partial t}\Gamma_k^{\rm gauge}[\varphi, \bar{a}; \bar{A}] \\ &= \frac{1}{2}\mathrm{Tr}\Big\{\frac{\partial}{\partial t}R_k[\bar{A}](\Gamma_k^{(2)}[\varphi, \bar{a}; \bar{A}] + R_k[\bar{A}])^{-1} \\ &- \frac{\partial}{\partial t}R_k[\bar{A}+\bar{a}](\Gamma_k^{(2)}[\varphi, 0; \bar{A}+\bar{a}] + R_k[\bar{A}+\bar{a}])^{-1}\Big\} \quad (6.19)\end{aligned}$$

Corresponding evolution equations for functional derivatives of $\Gamma_k^{\rm gauge}$ with respect to $\bar{a}$ (or A) obtain by performing the corresponding derivative on the r.h.s. of (6.19).

To conclude this section we emphasize that at this point we are not guaranteed that the field equations, propagators and also the vertices can be extracted from $\bar{\Gamma}_k[\varphi, A]$ alone. In general, one needs in addition the dependence of $\Gamma_k^{\rm gauge}[\varphi, A, \bar{A}]$ on the field $\bar{a} = A - \bar{A}$. This field transforms as a Lorentz vector and as an adjoint tensor with respect to gauge transformations. The additional field permits many new invariants. Terms linear in $\bar{a}$ (as for example the last term in (6.18)) give a contribution to the relation between the sources and the first functional derivatives of $\bar{\Gamma}_k[\varphi, A]$, see eqs. (2.21) and (6.3). They therefore modify the field equations which obtain for $J = K = 0$. Similarly, the propagators are given by the second functional derivative of W_k with respect to the sources. As a consequence the inverse propagators correspond to $\Gamma_k^{(2)}$ $(\varphi, A, \bar{A})$ rather than to the second functional derivatives of $\bar{\Gamma}_k[\varphi, A]$. We observe that these complications are not particular to the effective average action for $k > 0$. They are also present for the effective action $\Gamma[A]$ of eq. (2.20). This is the price one has to pay for the definition of a gauge-invariant effective action involving only one gauge field A_μ. Only the quantity $\Gamma[A, \bar{A}]$ is simply related to field equations, propagators, and vertices by functional differentiation with respect to A for fixed $\bar{A}$. In a sense, the use of the functional with only one gauge field becomes useful only if the contributions from $\Gamma^{\rm gauge}$ can be neglected or are quantitatively understood. The discussion in sect. 3 suggests indeed that often all relevant properties of the physical gauge bosons are encoded in $\Gamma[A]$. A proof under what circumstances this is the case would clearly be of great value.

As a comment we point out that the additional freedom related to the new field $\bar{a}_\mu$ in $\Gamma_k^{\rm gauge}$ should not only be considered as a negative complication. The new invariants appearing in $\Gamma^{\rm gauge}$, as for example the mass term $\sim \mu_g^2$ in (6.18), may sometimes play a role for the nonabelian theory even in the limit $k \to 0$. In a pure nonabelian gauge theory a mass term $\mu_g^2(k = 0) > 0$ would provide an infrared cutoff and therefore set the scale for the theory. If it exists, it must therefore be closely related to the confinement scale. It is difficult to construct a gauge-invariant infrared cutoff in terms of one gauge field only, and such a term is typically nonlocal. In terms of two gauge fields A and $\bar{A}$ local effective infrared cutoffs as simple as a mass term are allowed, and the description in terms of $\Gamma_k[A, \bar{A}]$ may actually become much simpler than the one in terms of $\bar{\Gamma}_k[A]$. It is interesting in this respect that nonperturbative mass terms in the propagators of a gauge fixed theory have been discussed earlier. An example is the treatment of the "transverse gauge boson mass" in gauge theories at nonvanishing temperature by the approximate solution of Schwinger-Dyson equations [16].

We finally remark that despite of appearance the many new possible invariants in $\Gamma_k^{\text{gauge}}[\varphi, A, \bar{A}]$ do not contain additional freedom. In fact, the dependence of Γ_k on the background field $\bar{A}$ is fixed by identities which are spelled out in detail in the appendix. These identities contain the information about the gauge invariance of the original theory which will manifest itself in the form of Ward identities in the limit $k \to 0$. A detailed discussion of these relations for the case of an abelian gauge theory can be found in ref. [22].

7 Running gauge coupling in arbitrary dimensions

The exact evolution equation (5.22) for the scale dependence of the effective average action is a nonlinear differential equation for a function of infinitely many variables. There seems to be little hope for finding a general solution. The successful use of this equation therefore depends crucially on the existence of an appropriate approximation scheme. This will consist in a truncation of the infinitely many invariants characterizing Γ_k to a finite number. The truncation has to reflect the dominant aspects of the particular model or physical situation.

In this section we demonstrate the practical use of our equation by computing approximately the running of the nonabelian gauge coupling in arbitrary dimension d. For $d = 4$ this should reproduce for small couplings the well-known perturbative β-function. The result for $d = 3$ is an important ingredient for an understanding of nonabelian gauge theories at high temperature [1].

We consider the pure Yang-Mills theory with gauge group SU(N). From (5.22) with (5.23) and (5.19) we obtain in this case

$$\begin{aligned} \frac{\partial}{\partial t}\Gamma_k[A,\bar{A}] &= \frac{1}{2}\mathrm{Tr}\left\{\frac{\partial}{\partial t}R_k\left(\frac{\delta^2}{\delta A^2}\Gamma_k[A,\bar{A}]+R_k\right)^{-1}\right\} \\ &\quad -\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k(\mathcal{D}_\mathrm{S})\right)(\mathcal{D}_\mathrm{S}+R_k(\mathcal{D}_\mathrm{S}))^{-1}\right\} \end{aligned} \tag{7.1}$$

Both traces on the r.h.s of (7.1) are over coordinate and (adjoint) group labels; the first one involves an additional trace over Lorentz indices. The infrared cutoff R_k depends on the background field $\bar{A}$ and is given by (4.6), with $Z_k = Z_{Fk}$ (see below) for the first term (4.2) and $Z_k = 1$ for the second term (4.8).

In order to approximate the solution $\Gamma_k[A,\bar{A}]$ of (7.1) by a functional with at most second derivatives, we make the ansatz

$$\Gamma_k[A,\bar{A}] = \int d^dx\left\{\frac{1}{4}Z_{Fk}F^z_{\mu\nu}(A)F^{\mu\nu}_z(A)+\frac{Z_{Fk}}{2\alpha_k}[D_\mu[\bar{A}](A^\mu-\bar{A}^\mu)]^2\right\} \tag{7.2}$$

We want to determine the running of Z_{Fk} from the flow equation. Referring back to the decomposition $\Gamma_k[A,\bar{A}] = \Gamma_k[A]+\Gamma_k^{\mathrm{gauge}}[A,\bar{A}]$ of eq. (6.9), we see that the first term of the r.h.s. of (7.2) corresponds to $\Gamma_k[A]$ and the second one to $\Gamma_k^{\mathrm{gauge}}[A,\bar{A}]$. The truncation (7.2) leads to

$$\frac{\delta^2\Gamma_k[A,\bar{A}]}{\delta A^\nu_y(x)\delta A^\mu_z(x')} = Z_{Fk}\left\{\mathcal{D}_\mathrm{T}[A]_{\nu\mu}+D_\nu[A]D_\mu[A]-\frac{1}{\alpha_k}D_\nu[\bar{A}]D_\mu[\bar{A}]\right\}^{yz}\delta(x-x') \tag{7.3}$$

In the following we neglect the running of α_k and restrict our discussion to $\alpha_k = 1$. According to (4.2) the infrared cutoff R_k in the first trace in (7.1) depends then only on $\mathcal{D}_\mathrm{T}$. Furthermore, (7.3) yields

$$\frac{\delta^2}{\delta A^2}\Gamma_k[A,\bar{A}]\Big|_{\bar{A}=A} = Z_{Fk}\mathcal{D}_\mathrm{T}(A) \tag{7.4}$$

Hence the evolution equation reads for $\bar{A}=A$:

$$\begin{aligned} \frac{\partial}{\partial t}\Gamma_k[A,A] &= \frac{\partial Z_{Fk}}{\partial t}\int d^dx\frac{1}{4}F_{\mu\nu}{}^zF^{\mu\nu}{}_z \\ &= \frac{1}{2}\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k(\mathcal{D}_\mathrm{T})\right)(Z_{Fk}\mathcal{D}_\mathrm{T}+R_k(\mathcal{D}_\mathrm{T}))^{-1}\right\} \\ &\quad -\mathrm{Tr}\left\{\left(\frac{\partial}{\partial t}R_k(\mathcal{D}_\mathrm{S})\right)(\mathcal{D}_\mathrm{S}+R_k(\mathcal{D}_\mathrm{S}))^{-1}\right\} \end{aligned} \tag{7.5}$$

Here $\mathcal{D}_\mathrm{T}$ and $\mathcal{D}_\mathrm{S}$ depend on A_μ now. In order to determine $\partial Z_{Fk}/\partial t$ it is sufficient to evaluate the traces for the simplest possible field configuration $A^z_\mu(x)$ which gives a nonzero value to the invariant $F_{\mu\nu}{}^{z}F^{\mu\nu}{}_{z}$. We choose a covariantly constant color-magnetic field of the form

$$A_\mu^z(x) = n^z A_\mu(x) \tag{7.6}$$

whereby n^z is a constant unit vector in color space, and A_μ is an abelian vector potential whose field strength amounts to a constant magnetic field of strength B. Therefore we get $F_{\mu\nu}{}^{z}F^{\mu\nu}{}_{z} = 2B^2$ and we have to evaluate the traces in (7.5) in linear order in B^2. For this background the spectra of $\mathcal{D}_{\rm T}$ and $\mathcal{D}_{\rm S}$ can be found by elementary methods [23]. Taking the degeneracy of the eigenvalues correctly into account, one finds for the trace of any function $H(\mathcal{D}_{\rm T})$

$$\begin{aligned}\mathrm{Tr}[H(\mathcal{D}_{\rm T})] &= \Omega \sum_{\ell=1}^{N^2-1} \frac{\bar{g}\nu_\ell B}{2\pi} \sum_{n=0}^{\infty} \int \frac{d^{d-2}q}{(2\pi)^{d-2}} \\ &\cdot\{(d-2)H(q^2+(2n+1)\bar{g}\nu_\ell B) \\ &+H(q^2+(2n+3)\bar{g}\nu_\ell B) \\ &+H(q^2+(2n-1)\bar{g}\nu_\ell B)\}\end{aligned} \tag{7.7}$$

The numbers $\nu_\ell, \ell = 1 \ldots N^2-1$, are the eigenvalues of the matrix $n^z T_z$ with the generators T^z in the adjoint representation and $\Omega \equiv \int d^dx$. Here the $(d-2)$-component vector q^μ is the momentum in the directions which are not affected by the "background field". Writing $q^2 \equiv x$ we get

$$\begin{aligned}\mathrm{Tr}[H(\mathcal{D}_{\rm T})] &= \frac{v_{d-2}}{\pi}\Omega \sum_{\ell=1}^{N^2-1} \bar{g}\nu_\ell B \sum_{n=0}^{\infty} \int_0^\infty dx\ x^{\frac{d}{2}-2} \\ &\cdot\{(d-2)H(x+(2n+1)\bar{g}\nu_\ell B) \\ &+H(x+(2n+3)\bar{g}\nu_\ell B) \\ &+H(x+(2n-1)\bar{g}\nu_\ell B)\}\end{aligned} \tag{7.8}$$

with $v_d \equiv \left[2^{d+1}\pi^{d/2}\Gamma(d/2)\right]^{-1}$. For our purposes it is sufficient to extract from (7.8) the term which is quadratic in B. Using the Euler-McLaurin formula

$$\sum_{n=0}^{\infty} H(x+(2n+1)\bar{g}\nu_\ell B) = (2\bar{g}\nu_\ell B)^{-1}\int_0^\infty dy H(x+y) + \frac{1}{12}\bar{g}\nu_\ell B \frac{d}{dx}H(x) + 0(B^3) \tag{7.9}$$

it is easy to show that for $\lim_{x\to\infty}\ dH/dx = 0$ the quadratic piece is given by

$$\mathrm{Tr}\left[H(\mathcal{D}_{\rm T})\right]_{quad} = -2N(1-\frac{d}{24})\frac{v_{d-2}}{\pi}(\bar{g}B)^2\Omega \int_0^\infty dx\ x^{\frac{d}{2}-2}\frac{d}{dx}H(x). \tag{7.10}$$

In the last step we used $\sum_{\ell=1}^{N^2-1}\nu_\ell\nu_\ell = N$ which follows from $n^2=1$ and $\mathrm{Tr}(T_yT_z) = N\delta_{yz}$. This formula holds for $d>2$ whereas for $d=2$ the momentum integration in

(7.7) is absent and (7.10) is replaced by

$$\mathrm{Tr}\left[H(\mathcal{D}_{\mathrm{T}})\right]_{quad} = -\frac{11}{12}\frac{N}{\pi}(\bar{g}B)^2\Omega\frac{dH}{dx}\bigg|_{x=0}. \tag{7.11}$$

Inserting (7.10), and a similar formula [1] for $\mathcal{D}_{\mathrm{S}}$ in eq. (7.5) we find for $d > 2$

$$\begin{aligned}\frac{\partial}{\partial t}Z_{Fk} &= -2N\left(1-\frac{d}{24}\right)\frac{v_{d-2}}{\pi}\bar{g}^2\int_0^\infty dx\ x^{\frac{d}{2}-2}\frac{d}{dx}\frac{\partial_t R_k(x)}{Z_{Fk}x+R_k(x)}\\ &-\frac{1}{6}N\frac{v_{d-2}}{\pi}\bar{g}^2\int_0^\infty dx x^{\frac{d}{2}-2}\frac{d}{dx}\frac{\partial_t R_k(x)}{x+R_k(x)} \equiv \bar{g}^2 b_d k^{d-4} \equiv -\eta_F Z_{Fk}\end{aligned} \tag{7.12}$$

(The second contribution is due to the trace containing $\mathcal{D}_{\mathrm{S}}$ with $Z_k = 1$ in $R_k(x)$.)

Introducing the dimensionless, renormalized gauge coupling [1]

$$g^2(k) = k^{d-4}Z_{Fk}^{-1}\bar{g}^2 \tag{7.13}$$

the associated beta function reads

$$\beta_{g^2} \equiv \frac{\partial}{\partial t}g^2(k) = (d-4)g^2 + \eta_F g^2 = (d-4)g^2 - b_d g^4. \tag{7.14}$$

For $d = 4$ the result for the running of $g^2(k)$ becomes universal, i.e. b_4 is independent of the precise form of the cut-off function $R_k(x)$ - only its behaviour for $x \to 0$ enters in (7.12). One obtains, with $\lim_{x\to 0} R_k = Z_{Fk}k^2$ for the first term in (7.12) and $\lim_{x\to 0}R_k = k^2$ for the second term

$$b_4 = \frac{N}{24\pi^2}(11 - 5\eta_F) \tag{7.15}$$

In lowest order in g^2 we can neglect η_F on the r.h.s. of (7.15) and obtain the standard perturbative one-loop β-function. More generally, one finds for η_F the equation

$$\eta_F = -g^2 b_d(\eta_F) \tag{7.16}$$

which has, for $d = 4$, the solution

$$\eta_F = -\frac{11N}{24\pi^2}g^2\left(1 - \frac{5N}{24\pi^2}g^2\right)^{-1} \tag{7.17}$$

The resulting β-function can be expanded for small g^2

$$\begin{aligned}\beta_{g^2} &= -\frac{11N}{24\pi^2}g^4\left(1-\frac{5N}{24\pi^2}g^2\right)^{-1}\\ &= -\frac{22N}{3}\frac{g^4}{16\pi^2} - \frac{220}{9}N^2\frac{g^6}{(16\pi^2)^2} - \ldots\end{aligned} \tag{7.18}$$

Comparing with the standard perturbative two-loop expression

$$\beta_{g^2}^{(2)} = -\frac{22N}{3}\frac{g^4}{16\pi^2} - \frac{204}{9}N^2\frac{g^6}{(16\pi^2)^2} \tag{7.19}$$

we find a surprisingly good agreement even for the two-loop coefficient. The missing 7 % in the coefficient of the g^6-term in β_{g^2} should be due to our truncations.

In arbitrary dimension d we define as in ref. [1] the constants l_{NA}^d by

$$b_d = \frac{44}{3}Nv_d l_{NA}^d - \frac{20}{3}Nv_d l_{NA\eta}^d \eta_F \tag{7.20}$$

with $l_{NA}^4 = 1, l_{NA\eta}^4 = 1$. With

$$P(x) = x + Z_k^{-1}R_k(x) = \frac{x}{1 - f_k^2(x)} = \frac{x}{1 - \exp -(x/k^2)} \tag{7.21}$$

they read for $d > 2$

$$\begin{aligned} l_{NA}^d &= -\frac{1}{88}(26-d)(d-2)k^{4-d}\int_0^\infty dx\ x^{\frac{d}{2}-2}\frac{d}{dx}\frac{d}{dt}\ln P \\ &= \frac{(26-d)(d-2)}{44}n_1^{d-4} \end{aligned} \tag{7.22}$$

$$\begin{aligned} l_{NA\eta}^d &= -\frac{1}{40}(24-d)(d-2)k^{4-d}\int_0^\infty dx\ x^{\frac{d}{2}-2}\frac{d}{dx}\frac{P-x}{P} \\ &= \frac{(24-d)(d-2)}{40}l_1^{d-2} \end{aligned} \tag{7.23}$$

It is remarkable [24] that the β-function for the coupling $g_R^2 = g^2k^{4-d}$ vanishes precisely in the critical string dimension $d = 26$[7] . The integrals appearing in (7.22), (7.23) are discussed in ref. [13]. One has

$$l_1^{d-2} = \Gamma\left(\frac{d-2}{2}\right) \tag{7.24}$$

$$\begin{aligned} n_1^{d-4} &= -\frac{1}{2}N_{1,0}^{d-4}(0) = -\frac{1}{2}k^{4-d}\int_0^\infty dx\ x^{\frac{d}{2}-2}\frac{\partial}{\partial t}\frac{dP/dx}{P} \\ &= -\int_0^\infty dy\ y^{\frac{d}{2}-2}e^{-y}(1-y-e^{-y})(1-e^{-y})^{-2} > 0 \end{aligned} \tag{7.25}$$

The evolution equation for the running dimensionless renormalized gauge coupling g in arbitrary dimension

$$\frac{\partial g^2}{\partial t} = \beta_{g^2} = (d-4)g^2 - \frac{44N}{3}v_d l_{NA}^d g^4\left(1 - \frac{20N}{3}v_d l_{NA\eta}^d g^2\right)^{-1} \tag{7.26}$$

[7] In ref. [25] this result was related to the $\alpha' \to 0$-limit of one-loop string amplitudes.

has the general solution (for $d \neq 4$)

$$\frac{g^2(k)}{(1+a_2 g^2(k))^\gamma} = C\left(\frac{k}{k_0}\right)^{d-4} \tag{7.27}$$

with

$$\begin{aligned} a_1 &= \frac{44 N v_d l^d_{NA}}{3(4-d)} \\ a_2 &= a_1 - \frac{20N}{3} v_d l^d_{NA\eta} \\ \gamma &= a_1/a_2 \end{aligned} \tag{7.28}$$

and

$$C = \frac{g^2(k_0)}{(1+a_2 g^2(k_0))^\gamma} \tag{7.29}$$

The nonabelian Yang-Mills theory is asymptotically free for $d \leq 4$ with a "confinement scale" $\Lambda^{(d)}_{conf}$, where β_{g^2} diverges

$$\Lambda^{(d)}_{conf} = \left(\frac{C a_1^\gamma}{(a_1-a_2)^{\gamma-1}}\right)^{\frac{1}{4-d}} k_0 \tag{7.30}$$

At this scale our truncation gives no quantitatively reliable results any more since η_F diverges and the choice $Z_k = Z_{Fk}$ in R_k becomes inconvenient. Indeed, Z_{Fk} may vanish for some scale $k_{cf} > 0$, whereas Z_k should always remain strictly positive. A possible smoother definition in the region of rapidly varying Z_{Fk} could be $Z_\Lambda = Z_{F\Lambda}$ for $k = \Lambda$, and $\partial_t Z_k = -\eta_F(1+\eta_F^2)^{-1} Z_k$ for $k < \Lambda$. This modification does not influence the one and two loop β-function. It guarantees, however, that Z_k remains always strictly positive. Now the β function does not diverge for any finite value of g^2 and the confinement scale can always be associated with the scale where g^2 diverges or Z_{Fk} vanishes. This scale is slightly lower than (7.30). The "one-loop" confinement scale [1] obtains from (7.30) for $l^d_{NA\eta} \to 0,\ a_2 \to a_1,\ \gamma \to 1$

$$\Lambda^{(d)}_{conf} = \left[\frac{44 N v_d l^d_{NA}}{3(4-d)} g^2(k_0)\left(1 + \frac{44 N v_d l^d_{NA}}{3(4-d)} g^2(k_0)\right)^{-1}\right]^{\frac{1}{4-d}} k_0 \tag{7.31}$$

and corresponds as usual to a diverging gauge coupling. We observe that $\Lambda^{(d)}_{conf}$ (7.30) is always higher than the "one-loop" result (7.31) (for given k_0 and $g^2(k_0)$). We therefore consider the scale (7.31) as a lower bound for the confinement scale. For $4 < d < 24$ the β function (7.26) has an ultraviolet stable fixpoint separating the confinement phase for strong coupling (with a confinement scale given by the analog of (7.30) for negative a_1 and a_2) from the infrared free weak coupling phase. We note that there is no confinement phase for $d > 26$.

8 Discussion

We propose in this paper a new nonperturbative evolution equation for gauge theories. It is valid for Yang-Mills theories and simplifies for abelian gauge theories. (The latter will be dealt with in detail in a related publication [22].) Our evolution equation (5.22) is an exact differential equation relating the scale dependence of the effective average action to its second functional derivative with respect to the fields. A solution of this equation amounts to a complete solution for all 1PI Green functions of the theory. Obviously, an exact solution must be very complicated and is in most cases very unlikely to be found. It is therefore crucial to find an appropriate truncation of the most general form of the effective average action to a few invariants which should contain all important physics. This will allow one to reduce the exact evolution equation to a system of a few ordinary coupled nonlinear differential equation. An approximate solution, either by analytical or numerical methods, becomes then feasible.

In this paper we have demonstrated the practical use of our equation by computing the running of the gauge coupling of a nonabelian theory in arbitrary dimension. Even though we use a very simple truncation we reproduce in four dimensions correctly the one-loop β-function and also the coefficient of the two-loop term with an accuracy of about 7 %. This seems rather remarkable. The running of the gauge coupling in three dimensions is crucial in order to understand gauge theories at high temperatures. This applies to QCD as well as to the electroweak theory in the symmetric phase or near the phase transition. The problem is notoriously difficult to tackle by other methods, mainly due to the strong infrared divergences appearing in standard perturbation theory for the one-loop quantum correction to the gauge coupling. We compute explicitly the β-function and the runnning of the coupling in a three-dimensional theory. Using the formulae of sect. 7 with

$$\begin{aligned} n_1^{-1} &= 1.2942 \\ l_{NA}^3 &= 0.6765 \\ l_{NA\eta}^3 &= 0.9305 \end{aligned} \tag{8.1}$$

we can compute the three-dimensional confinement scale where the gauge coupling becomes strong. The scale where the effective three-dimensional running starts is proportional to the temperature T and we use here

$$k_0 = 2\pi T \tag{8.2}$$

(Previous work on scalar theories at high temperature suggests a more accurate value of $k_0 \approx 5T$[12].) Using for the initial value of the three-dimensional gauge coupling

$$g^2(k_0) = 4\pi\alpha(k_0) \cdot \frac{T}{k_0} \tag{8.3}$$

with $\alpha(k_0)$ the appropriate four-dimensional fine structure constant we find for the ratio between the three-dimensional confinement scale $\Lambda^{(3)}_{conf}$ and temperature in the electroweak theory $\left(\alpha_w(k_0) = \frac{1}{30}\right)$

$$\frac{\Lambda^{(3)}_{conf}}{T} = 0.227 \ (0.104) \tag{8.4}$$

Here the second value in brackets corresponds to (7.31), whereas the first is given by (7.30). Three-dimensional confinement is a crucial feature for the electroweak phase transition [1]. We observe that $\Lambda^{(3)}_{conf}$ is larger than the estimated value of the transversal gauge boson mass $M_t \approx \frac{4}{3}\alpha_w T$ [16] by about a factor of 2-5. This puts severe questions on the applicability of high temperature perturbation theory in the symmetric phase or near the phase transition.

For QCD the three-dimensional confinement scale obtains from (7.31) as

$$\Lambda^{(3)}_{conf} = \frac{10.4 T\alpha_s(2\pi T)}{(1 + 0.28\alpha_s(2\pi T))^{\frac{8}{3}}} \tag{8.5}$$

to be compared with the "one-loop" result (7.30)

$$\Lambda^{(3)}_{conf} = \frac{4.74 T\alpha_s(2\pi T)}{(1 + 0.75\alpha_s(2\pi T))} \tag{8.6}$$

Using the usual expression for $\alpha_s(2\pi T)$ in terms of the (one-loop) four-dimensional confinement scale $\Lambda^{(4)}_{conf}$

$$\alpha_s(2\pi T) = \frac{2\pi}{11 \ln(2\pi T/\Lambda^{(4)}_{conf})} \tag{8.7}$$

we can express $\Lambda^{(3)}_{conf}$ in terms of $\Lambda^{(4)}_{conf}$ and T. There is a characteristic temperature T_{ch} where the three-dimensional confinement scale becomes larger than the (one-loop) four-dimensional confinement scale. We find for the "one-loop" expression (7.30)

$$\frac{2\pi T_{ch}}{\Lambda^{(4)}_{conf}} = \frac{3\pi}{9.48}\left(1 + \frac{22}{3\pi}\ln\left(\frac{2\pi T_{ch}}{\Lambda^{(4)}_{conf}}\right)\right) \tag{8.8}$$

which gives T_{ch} close to $\Lambda^{(4)}_{conf}/2\pi$. For $T < T_{ch}$ the three-dimensional running would start below $\Lambda^{(4)}_{conf}$ where the description of physics in terms of gluonic degrees of freedom is not meaningful.[8] For temperature larger than $\Lambda^{(4)}_{conf}$ and even possibly as low as about $\Lambda^{(4)}_{conf}/2\pi$ the three-dimensional running of the gauge coupling at non-vanishing temperature affects the nonperturbative mass scale of QCD. We expect that all spacelike correlation functions and magnetic condensates change quantitatively for $T > T_{ch}$. Their characteristic mass scale at high temperatures is given by $\Lambda^{(3)}_{conf} \sim T$ rather than by the scale $\Lambda^{(4)}_{conf}$ relevant for low temperatures. This does not necessarily imply an important qualitative change in the spacelike correlation functions at $T \approx T_{ch}$. The qualitative similarity between three-dimensional confinement and four-dimensional confinement rather suggests a smooth change in these correlations.

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Appendix: Background field dependence

In this section we study more quantitatively the dependence of $\Gamma_k[\varphi, A, \bar{A}]$ on the background field $\bar{A}$. We are interested in the quantity $\kappa[A,\bar{A}]$, which generalizes (6.13) for $\bar{A} \neq A$:

$$\begin{aligned}\kappa[A,\bar{A}] \;&\equiv\; \frac{\delta\Gamma_k[\varphi, A, \bar{A}]}{\delta\bar{A}}_{|A} = \frac{\delta\Gamma_k[\varphi,\bar{a};\bar{A}]}{\delta\bar{a}}_{|\bar{A}}\frac{\delta\bar{a}}{\delta\bar{A}}_{|A}\\ &+\; \frac{\delta\Gamma_k[\varphi,\bar{a};\bar{A}]}{\delta\bar{A}}_{|\bar{a}}\end{aligned} \tag{A.1}$$

For convenience we combine the gauge-fixing terms, Faddeev-Popov determinants and the infrared cutoff

$$\begin{aligned}\hat{\Delta}S[\chi, a;\bar{A}] \;&\equiv\; \Delta_k S_{\rm S} + \Delta_k S_G\\ &-\frac{1}{2\alpha}\int d^dx\; a_z^\nu (D_\nu[\bar{A}]D^\mu[\bar{A}])^z{}_y a^y_\mu\\ &-\ln\det(-D^\mu[\bar{A}]D_\mu[\bar{A}+a]) - \ln E_k[\bar{A}]\end{aligned} \tag{A.2}$$

The dependence on the background field arises through the dependence of $\hat{\Delta}S$ on $\bar{A}$. Our starting point are the definitions (5.8) and (5.14) and the functional integral

[8] Our description in terms of T_{ch} is only a rough approximation. To account for the temperature effects more precisely one should use the formalism of ref. [12] instead of an abrupt change from a four-dimensional to a three-dimensional description. The $T \neq 0$ confinement scale $\Lambda^{(T)}_{conf}$ is always larger than $\Lambda^{(4)}_{conf}$. For $T \ll T_{ch}$ the difference is small, whereas for $T \gg T_{ch}$ one has $\Lambda^{(T)}_{conf} \approx \Lambda^{(3)}_{conf}$.

representation (5.1) for W_k. Inserting (5.9) for the first term in (A.1) and observing (5.8), (5.14)

$$\frac{\delta \Delta_k S[\varphi, \bar{a}; \bar{A}]}{\delta \bar{a}} = R_k[\bar{A}]\bar{a} \tag{A.3}$$

$$\frac{\delta \tilde{\Gamma}_k}{\delta \bar{A}}_{|\varphi,\bar{a}} = -\frac{\delta W_k}{\delta \bar{A}}_{|J,K} \tag{A.4}$$

one finds

$$\frac{\delta \Gamma_k}{\delta \bar{A}}_{|A} = -K + R_k[\bar{A}]\bar{a} - \frac{\delta \Delta_k S[\varphi, \bar{a}; \bar{A}]}{\delta \bar{A}} - \frac{\delta W_k}{\delta \bar{A}}_{|J,K} \tag{A.5}$$

In order to evaluate $\delta W_k / \delta \bar{A}$ we define

$$\begin{aligned} &\exp W_k[J, K; \bar{A}, \hat{a}] = \exp W_k[J, K; \bar{A} + \hat{a}] \\ &= \int D\chi D a \exp -\{S[\chi, \bar{A} + \hat{a} + a] + \hat{\Delta} S[\chi, a; \bar{A} + \hat{a}] - J\chi - Ka\} \end{aligned} \tag{A.6}$$

so that

$$\frac{\delta W_k}{\delta \bar{A}}_{|J,K,\bar{A}} = \frac{\delta W_k[J, K; \bar{A}, \hat{a}]}{\delta \hat{a}}_{|J,K,\bar{A}} (\hat{a} = 0) \tag{A.7}$$

We shift the variable a,

$$a' = \hat{a} + a \tag{A.8}$$

and obtain

$$\begin{aligned} \frac{\delta W_k}{\delta \bar{A}}_{|J,K} &= W_k^{-1}[J, K; \bar{A}] \int D\chi D a' \exp -\{S[\chi, \bar{A} + a'] - J\chi - Ka'\} \\ &\quad \frac{\delta}{\delta \hat{a}} \exp -\{\hat{\Delta} S[\chi, a' - \hat{a}; \bar{A} + \hat{a}] + K\hat{a}\}_{|\hat{a}=0} \\ &= -K - \langle \frac{\delta}{\delta \hat{a}} \hat{\Delta} S[\chi, a - \hat{a}; \bar{A} + \hat{a}]_{|\hat{a}=0} \rangle \end{aligned} \tag{A.9}$$

The contributions $\Delta_k S$ and S_{gf} are quadratic in a and χ, and one finds

$$\langle \frac{\delta}{\delta \hat{a}} \Delta_k S_{|\hat{a}=0} \rangle = -R_k[\bar{A}]\ \bar{a} + \frac{1}{2} \langle \sigma \frac{\delta R_k}{\delta \bar{A}} \sigma \rangle \tag{A.10}$$

$$\begin{aligned} \langle \frac{\delta}{\delta \hat{a}^z_\mu} S_{gf\ |\hat{a}=0} \rangle &= \frac{1}{\alpha} (D^\mu[\bar{A}] D^\rho[\bar{A}])_{zw} \bar{a}^w_\rho \\ &\quad - \frac{1}{2\alpha} \langle a^\nu_y \frac{\delta}{\delta \bar{A}^z_\mu} (D_\nu[\bar{A}] D^\rho[\bar{A}])^y{}_w a^w_\rho \rangle \end{aligned} \tag{A.11}$$

Here we have used the notation of sect. 3, $\sigma = (\chi, a)$, and we employ for simplicity the same notation R_k for matrices acting on gauge fields and scalars or acting on gauge fields alone, with an obvious meaning for $\delta R_k / \delta \bar{A}$. On the other hand, the contributions from the determinants have a more complicated functional dependence on a. They are best expressed in terms of ghost correlators $\langle\bar{\xi}\xi\rangle$, cf. (2.11). Combining these results, the dependence of Γ_k on the background field $\bar{A}$ is described by

$$\begin{aligned}\frac{\delta\Gamma_k}{\delta\bar{A}^z_\mu} &= \frac{1}{2}\mathrm{Tr}\left\{\frac{\delta R_k[\bar{A}]}{\delta\bar{A}^z_\mu}\left(\Gamma^{(2)}_k + R_k\right)^{-1}\right\} \\ &\quad + \langle\frac{\delta}{\delta\hat{a}^z_\mu}(S_{\mathrm{gf}} + S_{\mathrm{det}})_{|\hat{a}=0}\rangle \end{aligned} \tag{A.12}$$

with

$$\begin{aligned} S_{det}[a,\bar{A},\hat{a}] &= -\ln\det(-D^\mu[\bar{A}+\hat{a}]D_\mu[\bar{A}+a]) \\ &\quad -\ln E_k[\bar{A}+\hat{a}] \end{aligned} \tag{A.13}$$

We observe the similarity of the first term in (A.12) with (5.22). The second term simplifies considerably for an abelian gauge theory where it reads $\frac{1}{\alpha}\partial^\mu\partial^\rho\bar{a}_\rho$. A few consequences of this simple form are discussed in detail in ref. [22].

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