# PATH INTEGRATION AND SEPARATION OF VARIABLES IN SPACES OF CONSTANT CURVATURE IN TWO AND THREE DIMENSIONS

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**Abstract.** In this paper path integration in two- and three-dimensional spaces of constant curvature is discussed: i.e. the flat spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, the two- and three-dimensional sphere and the two- and three dimensional pseudosphere. The Laplace operator in these spaces admits separation of variables in various coordinate systems. In all these coordinate systems the path integral formulation will be stated, however in most of them an explicit solution in terms of the spectral expansion can be given only on a formal level. What can be stated in all cases, are the propagator and the corresponding Green function, respectively, depending on the invariant distance which is a coordinate independent quantity. This property gives rise to numerous identities connecting the corresponding path integral representations and propagators in various coordinate systems with each other.

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# 1. Introduction

The invention of the path integral by Feynman [31] is one of the major achievements of theoretical physics. In its now 50 years history it has become an indispensable tool in field theory, cosmology, molecular physics, condensed matter physics and string theory as well [42, 96].

Originally developed as a "space-time approach to non-relativistic quantum mechanics" [31] with the famous solution of the harmonic oscillator, it became soon of paramount importance in quantum electrodynamics (QED), especially in the development of the nowadays so-called "Feynman rules". It did not take long and Feynman succeeded in discussing problems not only in QED but also in the theory of superfluidity. [1] However, it took a considerably long time before it was generally accepted by most physicists as a powerful tool to analyse a physical system, giving non-perturbative global information, instead of only perturbative, respectively local information, as in an operator approach.

Eventually, a satisfying theory should be based on a field theory formulation, let it be the second quantization of the Schrödinger, respectively the Dirac equation, let it be a field theory path integral. However, the path integral is quite a formidable and difficult functional-analytic object. The very early field theory formulations by a path integral, first by Feynman and in the following by e.g. Matthews and Salam [35, 82] remained only on a formal level, however with well-described rules to extract the relevant information, say, for Feynman diagrams. Indeed, in field theory the path integral is cursed by several pathologies which cause people now and then to state that "the path integral does not exist".

What does exists, however, is the very originally Feynman path integral, i.e. Feynman's "space-time approach to non-relativistic quantum mechanics". Thanks to the work of many mathematicians and physicists as well, the theory of the Feynman path integral can be considered as quite comprehensively developed. Actually, the theory of the "Wiener-integral" [35] existed some twenty years right before Feynman published his ideas, and was developed in the theory of diffusion processes. The Wiener integral itself represents an "imaginary time" version of the "real time" Feynman path integral. This particular feature of the Wiener integral makes it a not too complicated and convenient tool in functional analysis, mostly because convergence properties are easily shown. These convergence properties are absent in the Feynman path integral and the emerging challenge attracted many mathematicians and mathematical physicists, c.f. the references given in [35]. Let us in addition mention Nelson [88] concerning the Feynman path integral in cartesian coordinates, DeWitt concerning curvilinear coordinates [18], Morette-DeWitt et al. [86] and Albeverio et al. (e.g. [2, 3]) who developed a theory of "pseudomeasures" appropriate to the interference of probability amplitudes in the path integral. This interference of probabilities, in particular in the lattice formulation of the path integral (see below in Section 2) leads to the very interpretation of the Feynman path integral. One encounters for finite lattice spacing,

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[1] For this kind of general overview we do not want to cite all the relevant references in detail, but would instead invite the interested reader to consult some of the textbooks on the Feynman path integral, in particular Feynman and Hibbs [32], respectively the collection of Schwinger [101] and the early review paper of Gel'fand and Yaglom [35].

i.e. finite $N$, a complex number $\Phi(\mathbf{q}_1, \ldots, \mathbf{q}_{N-1})$ which is a function of the variables $\mathbf{q}_j$ defining a $\mathbf{q}(t)$, the path integral can be interpreted as a "sum over all paths" or a "sum over all histories"

$$K(\mathbf{q}'', \mathbf{q}'; T) = \sum_{\substack{\text{over all paths} \\ \text{from } \mathbf{q}' \text{ to } \mathbf{q}''}} \Phi[\mathbf{q}(t)] = \sum_{\substack{\text{over all paths} \\ \text{from } \mathbf{q}' \text{ to } \mathbf{q}''}} e^{\mathrm{i}\, S[\mathbf{q}(t)]/\hbar} \quad . \qquad (1.1)$$

The path integral then gives a prescription how to compute the important quantity $\Phi$ for each path: "The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of $\hbar$). ... That is to say, the contribution $\Phi[\mathbf{q}(t)]$ from a given path $\mathbf{q}(t)$ is proportional to $\exp\left(\frac{\mathrm{i}}{\hbar} S[\mathbf{q}(t)]\right)$, where the *action* is the time integral of the classical Lagrangian taking along the path in question" [31]. All possible paths enter and interfere which each other in the convolution of the probability amplitudes. In fact, the nowhere differentiable paths span the continuum in the set of all paths, the differentiable ones are being a set of measure zero (the quantity $\Delta\mathbf{q}_j/\Delta t_j$ does not exist, whereas $(\Delta\mathbf{q}_j)^2/\Delta t_j$ does).

A more comprehensive discussion will be given in a forthcoming publication [58], including a table of exactly solvable path integrals, in our lecture notes [59], and alternatively, we refer to the several already existing textbooks on path integrals Albeverio et at. [2, 3], Dittrich and Reuter [19], Feynman and Hibbs [32], Glimm and Jaffe [40], Kleinert [73], Roepstorff [98], Schulman [100], Simon [103], and Wiegel [112]. For a short reference we refer to [57].

The subject of this paper will be the path integral formulations in (spatial) two and three-dimensional spaces of constant curvature. These are the most important needed in physics. It is often necessary to consider a given problem from various points of views, i.e. coordinate system realizations, say, and moreover find the relevant Fourier expansions needed for the harmonic analysis and for the transition from one coordinate system to another one, respectively. E.g. in the space $\Lambda^{(3)}$ (the three-dimensional pseudosphere) this task was undertaken by Vilenkin and Smorodinsky [111] for five coordinate systems on $\Lambda^{(3)}$. Historically, Lamé started the project of finding all trirectangular systems in Euclidean space which admit separation of variables of the corresponding Hamilton-Jacobi and Laplace equations. Further, the works of Darboux [16], Eisenhart [26] and Stepanov [105] were set out to solve this problem. The result is that there are four real coordinate systems in two dimensions and eleven real coordinates systems in three dimensions of the sought type (which e.g. can be found in Morse and Feshbach [87]). The corresponding problem for spaces of (non-vanishing) constant curvature, i.e. on spheres and pseudospheres was solved by Olevskiǐ [90], and a systematic approach for the D-dimensional generalization is due to Kalnins et al. [69, 70] and references therein. I will return to this classification in the next Section.

Some of the coordinate systems - cartesian, polar, parabolic - are very familar, others much less so. They are all obtained as degenerations of the confocal ellipsoidal coordinates (c.f. [84, 87] for such discussions in the three-dimensional Euclidean plane). It turns out that the pseudosphere $\Lambda^{(D-1)}$ for a fixed dimension D has the richest structure of all of them. This is not too surprising. The uniformization theorem for Riemann surfaces states that the fundamental domain of $\Lambda^{(2)}/\Gamma$, where $\Gamma$ is a discrete fixed-point free subgroups of the automorphisms on $\Lambda^{(2)}$, i.e. $\Gamma$ is a Fuchsian group, tesselates the entire hyperbolic plane. The Riemannian surfaces may have any genus

$g \geq 2$ and may be arbitrarily shaped according to the corresponding Teichmüller space. This rich structure makes $\Lambda^{(2)}$ interesting in the Polyakov approach to string theory, respectively in $1+1$-dimensional quantum gravity [42]: in the perturbative expansion of the Polyakov path integral one is left with a summation over all topologies of world sheets a string can sweep out, and an integral over the moduli space of Riemann surfaces. In contrast, the sphere $S^{(2)}$ can represent only a genus zero Riemann surface, i.e. a sphere.

The pseudosphere $\Lambda^{(3)}$ is on the one hand of particular importance, because the manifolds of constant $u^2$, $u$ being the four-velocity or the four-momentum, represent the physical domain of the variables for particles with real mass (on the mass shell). On the other, a similar harmonic analysis as in $\Lambda^{(2)}$ can be studied, c.f. [27, 107], giving a tessalation of $\Lambda^{(3)}$ in the form of three-manifolds. This structure makes $\Lambda^{(3)}$ interesting in $2+1$-dimensional quantum gravity.

I am now going to study and compute explicitly as far as possible, all the path integral formulations in spaces of constant curvature in two and three dimensions in terms of all possible coordinate systems allowing the complete separation of variables. Not all our path integral representations will be new. However, we put them all, i) into the context of exactly solvable examples of the path integral in a given space of constant curvature. ii) We establish in this way numerous identities in terms of a specific coordinate formulation of a path integral and its corresponding spectral expansion on the one hand, and its *explicitly known* form of the propagator and the Green function in terms of the invariant distance (norm) in the space expressed in these coordinates on the other. Spectral theory of functional analysis guarantees the equivalence of the identities. In the majority of the cases, especially for $\Lambda^{(3)}$, where no explicit path integral evaluations are possible, this equivalence sets out further progress in finding exactly solvable examples for the Feynman path integral by connecting the various examples with each other.

Whereas in this paper we focus on coordinate systems and their corresponding path integral representations, one can also look on the matter from a group theoretical point of view [10]: For flat space we have the Euclidean group, for the sphere we have $S^{(D-1)} \cong \mathrm{SO}(D)/\mathrm{SO}(D-1)$, in particular $S^{(3)} \cong \mathrm{SO}(4)/\mathrm{SO}(3)$ and $S^{(2)} \cong \mathrm{SO}(3)/\mathrm{SO}(2)$ and for the pseudosphere $\Lambda^{(D-1)} \cong \mathrm{SO}_0(D-1,1)/\mathrm{SO}(D-1)$.

In order to look at such a path integral formulation we consider the Lagrangian $\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) - V(\mathbf{x})$ ($\mathbf{x} \in \mathbb{R}^{p+q}$) as formulated, say, in a not-necessarily positive definite space with signature

$$(g_{ab}) = \mathrm{diag}\left(\underbrace{+1, \ldots, +1}_{p \ times}, \underbrace{-1, \ldots, -1}_{q \ times}\right) . \tag{1.2}$$

One introduces polar coordinates

$$x_\nu = r e_\nu(\theta_1, \ldots, \theta_{p+q-1}) , \qquad \nu = 1, \ldots, p+q , \tag{1.3}$$

where the $\mathbf{e}$ are unit vectors in some suitable chosen (timelike, spacelike or lightlike) set [10]. One then expresses the Lagrangian in terms of these polar coordinates and seeks for an expansion of the quantity $\mathrm{e}^{z(\mathbf{e}_1 \cdot \mathbf{e}_2)}$ expressed in terms of group elements $g_1, g_2$. If this is possible one can re-express the path integration of the coordinates $\mathbf{x}$ into a path

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integration over group elements $g$ yielding [10]

$$\int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\mathcal{L}(\mathbf{x},\dot{\mathbf{x}})dt\right] \mapsto \int dE_\lambda d_\lambda \sum_{mn} \hat{f}_{mn}^\lambda D_{mn}^\lambda(g'^{-1}g'') \tag{1.4}$$

$$= \int dE_\lambda d_\lambda \sum_{mn} \hat{f}_{mn}^\lambda \sum_k D_{m,k}^{\lambda *}(g') D_{m,k}^\lambda(g'') \; . \tag{1.5}$$

Here $\hat{f}_{mn}^\lambda$ is defined via the Fourier transformation

$$f(g) = \int dE_\lambda d_\lambda \sum_{mn} \hat{f}_{mn}^\lambda D_{mn}^\lambda(g) \; , \qquad \hat{f}_{mn}^\lambda = \int_G f(g) D_{mn}^{\lambda *}(g^{-1}) dg \; , \tag{1.6}$$

and $dg$ is the invariant group (Haar) measure. $\int dE_\lambda$ stands for a Lebesque-Stieltjes integral to include discrete ($\int dE_\lambda \to \sum_\lambda$) as well as continuous representations. $\int dE_\lambda$ is to be taken over the complete set $\{\lambda\}$ of class one representations. $d_\lambda$ denotes (in the compact case) the dimension of the representation and we take

$$d_\lambda \int_G D_{mn}^\lambda(g) D_{m'n'}^{\lambda' *}(g) dg = \delta(\lambda,\lambda') \delta_{m,m'} \delta_{n,n'} \tag{1.7}$$

as a definition for $d_\lambda$. $\delta(\lambda,\lambda')$ can denote a Kronecker delta, respectively, a $\delta$-function, depending whether the quantity $\lambda$ is a discrete or continuous parameter. We have furthermore used the group (composition) law

$$D_{mn}^\lambda(g_a^{-1}g_b) = \sum_k D_{kn}^{\lambda *}(g_a) D_{km}^\lambda(g_b) \; . \tag{1.8}$$

Choosing a basis $\{\mathbf{b}\}$ in the relevant Hilbert space fixes the matrix elements $D_{mn}^\lambda$ through $D_{mn}^\lambda = (D^\lambda(g)b_m, b_n)$ of the representation $D^\lambda(g)$ of the group. In particular the $D_{0m}^\lambda$ are called associated spherical harmonics, and the $D_{00}^\lambda$ are the zonal harmonics. These spherical functions are eigen-functions of the corresponding Laplace-Beltrami operator on a, say, homogeneous space, and the entire Hilbert space is spanned by a complete set of associated spherical functions $D_{0m}^\lambda$ [38, 39, 110].

"All what remains" is to look at a convenient representation, i.e. a coordinates system. One may say, that this is the very subject of this paper.

In our case of the free motion things are not too difficult, and the group path integral technique can be fruitfully exploited. This kind of approach is always possible in cases where a model has a known underlying dynamical group structure (e.g. [65]). Here the most famous example is the hydrogen atom with its O(4) symmetry. Actually, this property enabled Duru and Kleinert to apply the so-called Kustaanheimo-Stiefel transformation to the path integral problem of the hydrogen atom [23, 24]. This dynamical group structure is also important in order to discuss the so-called "super-integrable" potentials [30] (where the Coulomb potential and the harmonic oscillator in $\mathbb{R}^3$ are but two examples), and of course their path integral representations; we return shortly to this topic in the discussion in Section 6.

The further content will be as follows: In the second Section we outline some basic information concerning the path integral in curved spaces. We sketch the relevant (transformation) techniques along the lines as presented in our earlier work [43, 49, 55]. Next we sketch the classification of separating variables for Riemannian spaces of constant curvature. Here some basic information as found in Kalnins [69] is given. This includes the development of separability of variables in the language of path integrals.

Sections 4 and 5 then deal with the enumeration of the path integral representations, spectral expansions, and path integral evaluations in two and three dimensions, respectively. These two Sections represent the principal part of the paper and because we also list some already known results, some parts will have a review character.

The last Section contains a summary and a discussion of the results. In the three appendices some additional information is given as needed in Sections 4 and 5: In Appendix 1 some basic path integrals are cited, in Appendix 2 a particular dispersion relation is discussed, and in Appendix 3 the path integral solution on $\Lambda^{(D-1)}$ in a particular coordinate system is given.

## 2. Formulation of the Path Integral

In order to set up our notation we proceed in the canonical way for path integrals on curved spaces (DeWitt [18], D'Olivio and Torres [20], Feynman [31], Gervais and Jevicki [36], [43, 55], McLaughlin and Schulman [80], Mayes and Dowker [83], Mizrahi [85], and Omote [89]). In the following $\mathbf{x}$ denotes a D-dimensional cartesian coordinate, $\mathbf{q}$ a D-dimensional arbitrary coordinate, and $x, y, z$ etc. one-dimensional coordinates. A quantity $\underline{o}$ denotes an operator. We start by considering the classical Lagrangian corresponding to the line element $ds^2 = g_{ab}dq^a dq^b$ of the classical motion in some $D$-dimensional Riemannian space

$$\mathcal{L}_{Cl}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\left(\frac{ds}{dt}\right)^2 - V(\mathbf{q}) = \frac{m}{2}g_{ab}\dot{q}^a\dot{q}^b - V(\mathbf{q}). \qquad (2.1)$$

The quantum Hamiltonian is *constructed* by means of the Laplace-Beltrami operator

$$\underline{H} = -\frac{\hbar^2}{2m}\Delta_{LB} + V(\mathbf{q}) = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_a}g^{ab}\sqrt{g}\frac{\partial}{\partial q_b} + V(\mathbf{q}) \qquad (2.2)$$

as a *definition* of the quantum theory on a curved space [94]. Here $g = \det(g_{ab})$ and $(g^{ab}) = (g_{ab})^{-1}$. The scalar product for wave-functions on the manifold reads $(f, g) = \int d\mathbf{q}\sqrt{g}\,f^*(\mathbf{q})g(\mathbf{q})$, and the momentum operators which are hermitean with respect to this scalar product are given by

$$p_a = \frac{\hbar}{i}\left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2}\right)\,, \qquad \Gamma_a = \frac{\partial \ln\sqrt{g}}{\partial q^a}\,. \qquad (2.3)$$

In terms of the momentum operators (2.3) we can rewrite $H$ by using the Weyl-ordering prescription ([55, 85], $W = W$eyl):

$$\underline{H}(\mathbf{p}, \mathbf{q}) = \frac{1}{8m}(\underline{g}^{ab}\,\underline{p}_a\underline{p}_b + 2\underline{p}_a\,\underline{g}^{ab}\,\underline{p}_b + \underline{p}_a\underline{p}_b\,\underline{g}^{ab}) + V(\mathbf{q}) + \Delta V_W(\mathbf{q})\,. \qquad (2.4)$$

Here a well-defined quantum correction appears which is given by [55, 85, 89]:

$$\Delta V_W = \frac{\hbar^2}{8m}(g^{ab}\Gamma^d_{ac}\Gamma^c_{bd} - R) = \frac{1}{8m}\left[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + g^{ab}_{\ \ ,ab}\right] \qquad (2.5)$$

The corresponding *Lagrangian path integral* reads ($MP = M$id-$P$oint):

$$K(\mathbf{q}'',\mathbf{q}';T) = [g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4}\int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\sqrt{g}\,\mathcal{D}_{MP}\mathbf{q}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathcal{L}_{eff}(\mathbf{q},\dot{\mathbf{q}})dt\right]$$

$$\equiv [g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4}\lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{\frac{ND}{2}}\left(\prod_{j=1}^{N-1}\int d\mathbf{q}_j\right)\prod_{j=1}^{N}\sqrt{g(\bar{\mathbf{q}}_j)}$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\left[\frac{m}{2\epsilon}g_{ab}(\bar{\mathbf{q}}_j)\Delta q_j^a\Delta q_j^b - \epsilon V(\bar{\mathbf{q}}_j) - \epsilon\Delta V_W(\bar{\mathbf{q}}_j)\right]\right\}\quad.$$
$$(2.6)$$

Here we have used the abbreviations $\epsilon = (t'' - t')/N \equiv T/N$, $\Delta\mathbf{q}_j = \mathbf{q}_j - \mathbf{q}_{j-1}$, $\bar{\mathbf{q}}_j = \frac{1}{2}(\mathbf{q}_j + \mathbf{q}_{j-1})$ for $\mathbf{q}_j = \mathbf{q}(t'+j\epsilon)$ ($t_j = t' + \epsilon j$, $j = 0,\dots,N$) and we interpret the limit $N\to\infty$ as equivalent to $\epsilon\to 0$, $T$ fixed. The lattice representation can be obtained by exploiting the composition law of the time-evolution operator $\underline{\mathrm{U}} = \exp(-\mathrm{i}\,\underline{\mathrm{H}}T/\hbar)$, respectively its semi-group property. The Weyl-ordering prescription is the most discussed ordering prescription in the literature.

In an alternative approach the metric tensor is rewritten as a product according to $g_{ab} = h_{ac}h_{cb}$ [43]. Then we obtain for the Hamiltonian (2.2)

$$\underline{\mathrm{H}} = -\frac{\hbar^2}{2m}\Delta_{LB} + V(\mathbf{q}) = \frac{1}{2m}\underline{\mathrm{h}}^{ac}\underline{\mathrm{p}}_a\underline{\mathrm{p}}_b\underline{\mathrm{h}}^{cb} + \Delta V_{PF}(\mathbf{q}) + V(\mathbf{q}) \qquad (2.7)$$

and for the path integral (PF - $P$roduct-$F$orm)

$$K(\mathbf{q}'',\mathbf{q}';T)$$

$$= \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\sqrt{g}\,\mathcal{D}_{PF}\mathbf{q}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}h_{ac}(\mathbf{q})h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V(\mathbf{q})\right]dt\right\}$$

$$\equiv \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int d\mathbf{q}_j\sqrt{g(\mathbf{q}_j)}$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{bc}(\mathbf{q}_j)h_{ac}(\mathbf{q}_{j-1})\Delta q_j^a\Delta q_j^b - \epsilon V(\mathbf{q}_j) - \epsilon\Delta V_{PF}(\mathbf{q}_j)\right]\right\}\quad.$$
$$(2.8)$$

$\Delta V_{PF}$ denotes the well-defined quantum potential

$$\Delta V_{PF} = \frac{\hbar^2}{8m}\left[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_b)_{,b} + g^{ab}_{\ \ ,ab}\right] + \frac{\hbar^2}{8m}\left(2h^{ac}h^{bc}_{\ \ ,ab} - h^{ac}_{\ \ ,a}h^{bc}_{\ \ ,b} - h^{ac}_{\ \ ,b}h^{bc}_{\ \ ,a}\right)$$
$$(2.9)$$

arising from the specific lattice formulation for the path integral, respectively the ordering prescription for position and momentum operators in the quantum Hamiltonian. We only use the lattice formulation of (2.8) in this paper unless otherwise (and explicitly) stated.

Indispensable tools in path integral techniques are transformation rules. In order to avoid cumbersome notation, we restrict ourselves to the one-dimensional case. For the general case we refer to DeWitt [18], Fischer, Leschke and Müller [33], Gervais and Jevicki [36], [50, 53, 55, 57], Junker [67], Kleinert [72, 73], Pak and Sökmen [91], Steiner [104] and Storchak [106], and references therein. We consider the one-dimensional path integral

$$K(x'', x'; T) = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2}\dot{x}^2 - V(x)\right) dt\right] \qquad (2.10)$$

and perform the coordinate transformation $x = F(q)$. Implementing this transformation, one has to keep all terms of $O(\epsilon)$ in (2.10). Expanding about midpoints, the result is

$$K\big(F(q''), F(q'); T\big) = \big[F'(q'')F'(q')\big]^{-1/2} \lim_{N \to \infty} \left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{1/2} \prod_{j=1}^{N-1} \int dq_j \cdot \prod_{j=1}^{N} F'(\bar{q}_j)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar} \sum_{j=1}^{N} \left[\frac{m}{2\epsilon} F'^2(\bar{q}_j)(\Delta q_j)^2 - \epsilon V(F(\bar{q}_j)) - \frac{\epsilon\hbar^2}{8m} \frac{F''^2(\bar{q}_j)}{F'^4(\bar{q}_j)}\right]\right\} \quad . \quad (2.11)$$

It is obvious that the path integral representation (2.11) is not completely satisfactory. Whereas the transformed potential $V(F(q))$ may have a convenient form when expressed in the new coordinate $q$, the kinetic term $\frac{m}{2}F'^2\dot{q}^2$ is in general nasty. Here the so-called "time-transformation" comes into play which leads in combination with the "space-transformation" already carried out to general "space-time transformations" in path integrals. The time-transformation is implemented [23, 24, 104, 106] by introducing a new "pseudo-time" $s''$ by means of $s'' = \int_{t'}^{t''} ds/F'^2(q(s))$. A rigorous lattice derivation is far from being trivial and has been discussed by many authors. Recent attempts to put it on a sound footing can be found in Refs. [14, 33, 113]. A convenient way to derive the corresponding transformation formulæ uses the energy dependent Green's function $G(E)$ of the kernel $K(T)$ defined by

$$G(x'', x'; E) = \left\langle q'' \left| \frac{1}{\mathbb{H} - E - \mathrm{i}\,\epsilon} \right| q' \right\rangle = \frac{\mathrm{i}}{\hbar} \int_0^\infty dT\, \mathrm{e}^{\mathrm{i}(E+\mathrm{i}\,\epsilon)T/\hbar} K(x'', x'; T) \quad . \quad (2.12)$$

where a small positive imaginary part ($\epsilon > 0$) has been added to the energy $E$. (Usually we not explicitly write the $\mathrm{i}\,\epsilon$, but will tacitly assume that the various expressions are regularized according to this rule). For the path integral (2.10) one obtains the following transformation formulæ

$$K(x'', x'; T) = \int_{\mathbb{R}} \frac{dE}{2\pi\,\mathrm{i}} \mathrm{e}^{-\mathrm{i}\,ET/\hbar} G(q'', q'; E) \quad , \qquad (2.13)$$

$$G(q'', q'; E) = \frac{\mathrm{i}}{\hbar} \big[F'(q'')F'(q')\big]^{1/2} \int_0^\infty ds'' \widehat{K}(q'', q'; s'') \quad , \qquad (2.14)$$

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with the transformed path integral $\widehat{K}(s'')$ given by

$$\widehat{K}(q'', q'; s'') = \lim_{N \to \infty} \left( \frac{m}{2\pi \,\mathrm{i}\, \epsilon \hbar} \right)^{1/2} \prod_{j=1}^{N-1} \int dq_j$$

$$\times \exp \left\{ \frac{\mathrm{i}}{\hbar} \sum_{j=1}^{N} \left[ \frac{m}{2\epsilon} (\Delta q_j)^2 - \epsilon F'^2(\bar{q}_j) \Big( V(F(\bar{q}_j)) - E \Big) \right. \right.$$

$$\left. \left. - \frac{\epsilon \hbar^2}{8m} \left( 3 \frac{F''^2(\bar{q}_j)}{F'^2(\bar{q}_j)} - 2 \frac{F'''(\bar{q}_j)}{F'(\bar{q}_j)} \right) \right] \right\} \ . \quad (2.15)$$

Further refinements are possible and general formulæ of practical interest and importance can be derived. Let us note that also an explicitly time-dependent "space-time transformation" $x = F(q, t)$ can be formulated similarly to the formulæ (2.13-2.15), c.f. Refs. [53, 58, 59, 73, 93, 106].

Finally we consider a pure time transformation in a path integral. We consider

$$G(\mathbf{q}'', \mathbf{q}'; E) = \sqrt{f(\mathbf{q}')f(\mathbf{q}'')} \frac{\mathrm{i}}{\hbar} \int_0^\infty ds'' \left\langle \mathbf{q}'' \left| \exp \left( -\mathrm{i}\, s'' \sqrt{f} \,(\underline{\mathrm{H}} - E) \sqrt{f} / \hbar \right) \right| \mathbf{q}' \right\rangle \ , \quad (2.16)$$

which corresponds to the introduction of the "pseudo-time" $s'' = \int_{t'}^{t''} ds/f(\mathbf{q}(s))$ and we assume that the Hamiltonian $H$ is product ordered. Then

$$G(\mathbf{q}'', \mathbf{q}'; E) = \frac{\mathrm{i}}{\hbar} (f' f'')^{\frac{1}{2}(1-D/2)} \int_0^\infty \widetilde{K}(\mathbf{q}'', \mathbf{q}'; s'') \, ds'' \quad (2.17)$$

with the path integral

$$\widetilde{K}(\mathbf{q}'', \mathbf{q}'; s'') = \int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \sqrt{\tilde{g}} \, \mathcal{D}_{PF} \mathbf{q}(t)$$

$$\times \exp \left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[ \frac{m}{2} \tilde{h}_{ac} \tilde{h}_{cb} \dot{q}^a \dot{q}^b - f \Big( V(\mathbf{q}) + \Delta V_{PF}(\mathbf{q}) - E \Big) \right] ds \right\} \ . \quad (2.18)$$

Here are $\tilde{h}_{ac} = h_{ac}/\sqrt{f}$, $\sqrt{\tilde{g}} = \det(\tilde{h}_{ac})$ and (2.18) is of the canonical product form.

The third ingredient in our calculations will be the technique of separation of variables in path integrals [44]. Let us sketch the most important features of this technique. We assume that a potential problem $V(\mathbf{x})$ has an exact solution according to

$$\int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp \left[ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{\mathbf{x}}^2 - V(\mathbf{x}) \right) dt \right] = \int dE_\lambda \, e^{-\mathrm{i}\, E_\lambda T/\hbar} \, \Psi_\lambda^*(\mathbf{x}') \Psi_\lambda(\mathbf{x}'') \ , \quad (2.19)$$

and $\mathbf{x}$ is a set of variables of dimension $d$. Now we consider the path integral

$$K(\mathbf{z}'', \mathbf{z}', \mathbf{x}'', \mathbf{x}'; T) = \int_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}(t'')=\mathbf{z}''} f^d(\mathbf{z}) G(\mathbf{z}) \mathcal{D}\mathbf{z}(t) \int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left((\mathbf{g}\cdot\dot{\mathbf{z}})^2 + f^2\dot{\mathbf{x}}^2 - \left(\frac{V(\mathbf{x})}{f^2(\mathbf{z})} + V(\mathbf{z}) + \Delta\tilde{V}(\mathbf{z})\right)\right)\right]dt\right\}\;. \tag{2.20}$$

Here, $\mathbf{z}$ denotes a $d'$-dimensional coordinate with $d + d' = D$, $g_i$ and $f$ the elements of the $D$-dimensional metric tensor $g_{ab} = \mathrm{diag}[g_1^2,\ldots,g_{d'}^2, f^2,\ldots, f^2]$, $\Delta\tilde{V}$ the quantum potential of (2.9), and $\det(g_{ab}) = f^{2d}\prod_i g_i^2 \equiv f^{2d}G(z)$. As shown in Ref. [44] by performing a time-transformation (see Duru and Kleinert [24] and Kleinert [73]) forth and back in the path integral (2.20) we can separate the $\mathbf{x}$ from the $\mathbf{z}$ variables yielding

$$K(\mathbf{z}'',\mathbf{z}',\mathbf{x}'',\mathbf{x}';T) = [f(\mathbf{z}')f(\mathbf{z}'')]^{-d/2}\int dE_\lambda \Psi_\lambda^*(\mathbf{x}')\Psi_\lambda(\mathbf{x}'')$$

$$\times \int \sqrt{G(\mathbf{z})}\,\mathcal{D}\mathbf{z}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\mathbf{g}\cdot\dot{\mathbf{z}})^2 - V(\mathbf{z}) - \Delta\tilde{V}(\mathbf{z}) - \frac{E_\lambda}{f^2(\mathbf{z})}\right]dt\right\}\;. \tag{2.21}$$

Of course, also $\mathbf{x}$-depended metric terms can be included (with their corresponding quantum potentials) in the separated $\mathbf{x}$-path integration without changing the general feature.

## 3. Separable Coordinate Systems on Spaces of Constant Curvature

### 3.1. The Coordinates Systems

A systematic approach for the D-dimensional on the problem on the classification of separable coordinate systems in spaces of constant curvature (positive, zero, negative) is due to Kalnins et al. [69, 70]. By the notion "separable coordinate system" we man any coordinate system which separate the classical Hamilton-Jacobi equations, respectively the Schrödinger equation. The classification goes at follows [69]:

*3.1.1. The Sphere $S^{(D-1)}$.* We denote the coordinates on the sphere $S^{(D-1)}$ by the vector $\mathbf{s} = (s_0,\ldots,s_{D-1})$. The basic building blocks of separable coordinates systems on $S^{(D-1)}$ are the (D-1)-sphere *elliptic* coordinates

$$s_j^2 = \frac{\prod_{i=1}^{D-1}(\rho_i - e_j)}{\prod_{j\neq i}(e_i - e_j)}\;,\quad (j = 0,\ldots,D-1)\;,\quad \sum_{j=0}^{D-1}s_j^2 = 1\;, \tag{3.1}$$

corresponding to a metric

$$ds^2 = -\frac{1}{4k}\sum_{i=1}^{D-1}\frac{1}{P_D(\rho_i)}\left[\prod_{j\neq i}(\rho_i - \rho_j)\right](d\rho_i)^2\;,\qquad P_D(\rho) = \prod_{i=0}^{D}(\rho - e_i) \tag{3.2}$$

($k > 0$ curvature). In order to find the possible explicit coordinate systems one must pay attention to the requirements that, (i) the metric must be positive definite, (ii) the variables $\{\rho\}_{i=1}^{D-1}$ should vary in such a way that they correspond to a coordinates patch which is compact. There is a unique solution to these requirements given by

$$e_0 < \rho_1 < e_1 < \cdots < e_{D-1} < \rho_D < e_D\;. \tag{3.3}$$

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*3.1.2. The Euclidean Space $E^{(D)}$.* In D-dimensional Euclidean space we have first the coordinate system corresponding to the D-sphere elliptic (3.2)

$$x_j^2 = c^2 \frac{\prod_{i=1}^{D}(\rho_i - e_j)}{\prod_{j \neq i}(e_i - e_j)} \ , \qquad (j = 1, \ldots, D) \tag{3.4}$$

($c^2$ constant). In addition there is a second class of coordinate systems, namely the parabolic coordinates

$$\begin{aligned}
x_1^2 &= \frac{c}{2}(\rho_1 + \cdots + \rho_D + e_1 + \cdots + e_{D-1}) \ , \\
x_j^2 &= -c^2 \frac{\prod_{i=1}^{D}(\rho_i - e_j)}{\prod_{j \neq i}(e_i - e_j)} \ , \quad (j = 2, \ldots, D) \ .
\end{aligned} \tag{3.5}$$

*3.1.3. The Pseudo-Sphere $\Lambda^{(D-1)}$.* On the pseudosphere $\Lambda^{(D-1)}$ the complexity increases considerably. One starts by considering the line element

$$ds^2 = -\frac{1}{4k} \sum_{i=1}^{D-1} \frac{1}{P_D(\rho_i)} \left[ \prod_{j \neq i}(\rho_i - \rho_j) \right] (d\rho_i)^2 \tag{3.6}$$

($k < 0$ curvature), and one must require that $ds^2 > 0$. It turns out that there are four classes of solutions determined by the character of the solutions of the characteristic equation $P_D(\rho) = 0$.

*3.1.3.A.* The first class is characterised by $e_i \neq e_j$ for $i, j = 0, \ldots, D - 1$. If $D - 1 = n = 2p + 1$ is odd then

$$\ldots \rho_{i-2} < e_{i-2} < \rho_{i-1} < e_{i-1} < e_i < e_{i+1} < \rho_i < e_{i+2} < \cdots < e_{2p+2} < e_{2p+1} \ , \tag{3.7}$$

($i = 0, \ldots, p$) with the convention that $e_j, \rho_j = 0$ for $j$ a non-positive integer which give $p + 1$ distinct possibilities. Using $E_i^{(j)} = e_{i+j+1}$ ($i = 1, \ldots, 2p + 2$, $j = 1, \ldots, p + 1$) and $e_r = e_s$ for $r = s \mod (n + 1)$, the coordinates on $\Lambda^{(D-1)}$ are written in the following way

$$u_0^2 = \frac{\prod_{i=1}^{n}(\rho_i - E_1^{(j)})}{\prod_{k \neq 1}(E_k^{(j)} - E_1^{(j)})} \ , \qquad u_l^2 = \frac{\prod_{i=1}^{n}(\rho_i - E_{l+1}^{(j)})}{\prod_{k \neq l+1}(E_k^{(j)} - E_{l+1}^{(j)})} \ . \tag{3.8}$$

Similarly if $D - 1 = n = 2p + 2$ is even ($i = 0, \ldots, p$)

$$\ldots \rho_{i-2} < e_{i-2} < \rho_{i-1} < e_{i-1} < e_i < e_{i+1} < \rho_i < \cdots < e_{2p+1} < e_{2p} \ . \tag{3.9}$$

*3.1.3.B.* The second class is characterised by the fact that there can be two complex conjugate zeros of $P_D(\rho) = 0$ denoted by $e_1 = \alpha + i\beta$, $e_2 = \alpha - i\beta$ ($\alpha, \beta \in \mathbb{R}$), respectively. Together with the convention $e_{i+1} \equiv f_{i-1}$ for all other $e_j$ there is the one possibility

$$\rho_1 < f_1 < \rho_2 < f_2 < \cdots < \rho_{n-1} < f_{n-1} < \rho_n \ . \tag{3.10}$$

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A suitable choice of coordinates is $(j = 2, \ldots, n)$

$$(u_0 + \mathrm{i}\, u_1)^2 = \frac{\mathrm{i}}{\beta} \frac{\prod_{i=1}^{n}(\rho_i - \alpha - \mathrm{i}\,\beta)}{\prod_{i=1}^{n-1}(f_i - \alpha - \mathrm{i}\,\beta)} \quad , \quad u_j^2 = \frac{-\prod_{i=1}^{n}(\rho_i - f_{j-1})}{[(\alpha - f_{j-1})^2 + \beta^2]\prod_{i \neq j-1}(f_i - f_{j-1})} \quad .$$

(3.11)

*3.1.3.C.* In the third class we have the two-fold root $e_1 = e_2 = a$, say. Let us denote $G_j^{(i)} = g_{j+1}$ $(j = 1, \ldots, n-1, i = 0, \ldots, p)$, where $e_j = g_{j-2}$ $(j = 3, \ldots, n+1)$ with $g_k \neq g_l$ if $k \neq l$ and $g_k \neq a$ for any $k$, $g_r = g_s$ for $r = s \mod (n+1)$, $n = 2p+1$ for $n$ odd, and $n = 2p$ for $n$ even, respectively. This case divides into two families with coordinates varying in the ranges $(i = 0, \ldots, p)$

$$\ldots \rho_{i-1} < g_{i-1} < \rho_i < g_i < a < \rho_{i+1} < g_{i+1} < \ldots g_{n-1} < \rho_n \ , \qquad (3.12a)$$

$$\ldots \rho_{i-1} < g_{i-1} < \rho_i < g_i < \rho_{i+1} < a < g_{i+1} < \ldots g_{n-1} < \rho_n \ , \qquad (3.12b)$$

and in either case of $n$ there are $p + 1$ distinguishable cases to consider. A suitable choice of coordinates is

$$\left. \begin{aligned} (u_0 - u_1)^2 &= \epsilon \frac{\prod_{i=1}^{n}(\rho_i - a)}{\prod_{k=1}^{n-1}(G_j^{(i)} - a)} \ , \\ (u_0^2 - u_1^2) &= \frac{\partial}{\partial a} \frac{\prod_{i=1}^{n}(\rho_i - a)}{\prod_{k=1}^{n-1}(G_j^{(i)} - a)} \ , \\ u_j^2 &= -\frac{\prod_{i=1}^{n}(\rho_i - G_{j-1}^{(i)})}{(a - G_{j-1}^{(i)})^2 \prod_{l \neq j-1}(G_l^{(i)} - G_{j-1}^{(i)})} \ , \end{aligned} \right\} \qquad (3.13)$$

$(j = 2, \ldots, n)$ and $\epsilon = +1$ in (3.12a), $\epsilon = -1$ in (3.12b).

*3.1.3.D.* The forth case is characterised by $e_1 = e_2 = e_3 = b$. We set $e_j = h_{j-3}$ $(j = 4, \ldots, n+1)$ with $h_k \neq k_l$ for $k \neq l$ and $h_k \neq b$ for any $k$. Then

$$\cdots < \rho_{i-1} < h_{i-1} < \rho_i < \rho_{i+1} < b < \rho_{i+2} < h_{i+1} < \cdots < \rho_{n-1} < h_{n-2} < \rho_n \ , \quad (3.14)$$

$(i = 0, \ldots, p)$ and there are $p + 1$ distinct cases. A suitable choice of coordinates is

$$\left. \begin{aligned} (u_0 - u_1)^2 &= -\frac{\prod_{i=1}^{n}(\rho_i - b)}{\prod_{k=1}^{n-2}(H_j^{(i)} - b)} \ , \\ 2u_2(u_0 - u_1) &= -\frac{\partial}{\partial b} \frac{\prod_{i=1}^{n}(\rho_i - b)}{\prod_{k=1}^{n-2}(H_j^{(i)} - b)} \ , \\ (u_0^2 - u_1^2 - u_2^2) &= -\frac{1}{2} \frac{\partial^2}{\partial b^2} \frac{\prod_{i=1}^{n}(\rho_i - b)}{\prod_{k=1}^{n-2}(H_j^{(i)} - b)} \\ u_j^2 &= -\frac{\prod_{i=1}^{n}(\rho_i - H_{j-2}^{(i)})}{\prod_{l \neq j-2}(H_l^{(i)} - H_{j-2}^{(i)})(b - H_{j-2}^{(i)})^3} \ , \end{aligned} \right\} \qquad (3.15)$$

$(j = 3, \ldots, n)$ and $H_j^{(i)} = h_{j+1}$ $(j = 1, \ldots, n-2, i = 0, \ldots, p)$, and $h_r = h_s \mod (n-2)$.

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For the lowest two case we obtain the following small table of distinct coordinate systems

|  | $D = 2$ | $D = 3$ |
|---|---|---|
| $S^{(D-1)}$ | 2 | 6 |
| $E^{(D)}$ | 4 | 11 |
| $\Lambda^{(D-1)}$ | 9 | 34 |

There seems to be no obvious closed recursions for D a natural number [77].

### 3.2. Separability of Variables in the Path Integral

Separation of variables is always of great importance in many calculations. The separation formula (2.21) provides us already with a prescription in the case where we know a separating coordinate system. However, the situation is often somewhat more complicated and one has to look first for the coordinate systems which separate the relevant partial differential equations, i.e. the Hamiltonian, and, more important from our point of view, the path integral. In order to develop such a theory we consider according to [87] the Lagrangian $\mathcal{L} = \frac{m}{2} \sum_{i=1}^{D} h_i^2 \dot{x}_i^2$ and the Laplacian $\Delta_{LB}$, respectively, in the following way (where only orthogonal coordinate systems are taken into account)

$$
\Delta_{LB} = \sum_{i=1}^{D} \frac{1}{\prod_{j=1}^{D} h_j(\{\xi\})} \frac{\partial}{\partial \xi_i} \left( \frac{\prod_{k=1}^{D} h_k(\{\xi\})}{h_i^2(\{\xi\})} \frac{\partial}{\partial \xi_i} \right)
$$
$$
=: \sum_{i=1}^{D} \frac{1}{\prod_{j=1}^{D} h_j(\{\xi\})} \frac{\partial}{\partial \xi_i} \left( g_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_D) f(\xi_i) \frac{\partial}{\partial \xi_i} \right) \ ,
$$
(3.16)

where $\{\xi\}$ denotes the set of variables $(\xi_1, \ldots, \xi_D)$, and the existence of the functions $f_i, g_i$ is necessary for the separation [69, 87]. Note that the factorizing in terms of the $h_i$ is perfectly adopted to the product form prescription. We introduce the Stäckel-determinant [69, 87, 90]

$$
S(\{\xi\}) = \det(\Phi_{ij}) = \prod_{i=1}^{D} \frac{h_i(\{\xi\})}{f_i(\xi_i)} \ , \quad M_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_D) = \frac{\partial S}{\partial \Phi_{i1}} = \frac{S(\{\xi\})}{h_i^2(\{\xi\})} \ ,
$$
(3.17)

and abbreviate $\Gamma_i = f_i'/f_i$. Then

$$
g_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_D) = M_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_D) \prod_{\substack{j=1 \\ i \neq j}}^{D} f_j(\xi_j) \ ,
$$
(3.18)

which fixes the functions $g_i$. Introducing the (new) momentum operators

$$
P_{\xi_i} = \frac{\hbar}{i} \left( \frac{\partial}{\partial \xi_i} + \frac{1}{2} \frac{f_i'}{f_i} \right) \ ,
$$
(3.19)

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we write the Legendre transformed Hamiltonian [54] as follows

$$0 = H - E = -\frac{\hbar^2}{2m}\Delta_{LB} - E$$

$$= -\frac{\hbar^2}{2m}\sum_{i=1}^{D}\frac{1}{\prod_{j=1}^{D}h_j}\frac{\partial}{\partial\xi_i}\left(\frac{\prod_{k=1}^{D}k_k}{h_i^2}\frac{\partial}{\partial\xi_i}\right) - E = -\frac{\hbar^2}{2m}\frac{1}{S}\sum_{i=1}^{D}\left[\frac{1}{f_i}\frac{\partial}{\partial\xi_i}\left(f_i\frac{\partial}{\partial\xi_i}\right)\right] - E$$

$$= -\frac{\hbar^2}{2m}\frac{1}{S}\sum_{i=1}^{D}M_i\left(\frac{\partial^2}{\partial\xi_i^2} + \Gamma_i\frac{\partial}{\partial\xi_i}\right) - E$$

$$= \frac{1}{S}\left[\frac{1}{2m}\sum_{i=1}^{D}M_i P_i^2 - ES + \frac{\hbar^2}{8m}\sum_{i=1}^{D}M_i\left(\Gamma_i^2 + 2\Gamma_i'\right)\right] \quad . \tag{3.20}$$

Therefore we obtain according to the general theory the following identity in the path integral by means of the space-time transformation technique

$$\prod_{i=1}^{D}\int_{\xi_i(t')=\xi_i'}^{\xi_i(t'')=\xi_i''}h_i\mathcal{D}\xi_i(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\sum_{i=1}^{D}h_i^2\dot\xi_i^2 - \Delta V_{PF}(\{\xi\})\right]dt\right\}$$

$$= \prod_{i=1}^{D}\int_{\xi_i(t')=\xi_i'}^{\xi_i(t'')=\xi_i''}\sqrt{\frac{S}{M_i}}\mathcal{D}\xi_i(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}S\sum_{i=1}^{D}\frac{\dot\xi_i^2}{M_i} - \Delta V_{PF}(\{\xi\})\right]dt\right\}$$

$$= (S'S'')^{\frac{1}{2}(1-D/2)}\int_{\mathrm{I\!R}}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty ds''\prod_{i=1}^{D}\int_{\xi_i(0)=\xi_i'}^{\xi_i(s'')=\xi_i''}M_i^{-1/2}\mathcal{D}\xi_i(s)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\sum_{i=1}^{D}\frac{\dot\xi_i^2}{M_i} + ES - \frac{\hbar^2}{8m}\sum_{i=1}^{D}M_i\left(\Gamma_i^2 + 2\Gamma_i'\right)\right]ds\right\} \quad . \tag{3.21}$$

## 4. Separation of Variables in Two Dimensions

The subject of the next two Sections will be the path integral formulation in the three two-dimensional spaces of constant curvature, the flat space $E^{(2)}$, the sphere $S^{(2)}$ and the pseudosphere $\Lambda^{(2)}$, respectively the path integral formulation in the three three-dimensional spaces of constant curvature, the flat space $E^{(3)}$, the sphere $S^{(3)}$, and the pseudosphere $\Lambda^{(3)}$. For notation, in the two-dimensional case we denote by $\mathbf{x} = (x, y) = (x_1, x_2)$ the flat space coordinates, by $\mathbf{s} = (s_0, s_1, s_2)$ coordinates on the sphere, and by $\mathbf{u} = (u_0, u_1, u_2)$ coordinates on the pseudosphere, and in the three-dimensional case $\mathbf{x} = (x, y, z)$ the flat space coordinates and $\mathbf{s}$ and $\mathbf{u}$ in obvious generalization from the two-dimensional case. In each case, we first state the general form of the propagator and the Green function in $D = 2$ and $D = 3$, respectively, in terms of the invariant distance (norm) $d_{E^{(D)}}(\mathbf{q}'', \mathbf{q}') = \mathbf{x}' \cdot \mathbf{x}''$ in $E^{(D)}$, $\cos\psi_{S^{(D-1)}}(\mathbf{q}'', \mathbf{q}') = \mathbf{s}' \cdot \mathbf{s}''/R^2$ in $S^{(D-1)}$, and $\cosh d_{\Lambda^{(D-1)}}(\mathbf{q}'', \mathbf{q}') = \mathbf{u}' \cdot \mathbf{u}''/R^2$ in

$\Lambda^{(D-1)}$. The sphere $S^{(D-1)}$ is described by the constraint $\mathbf{s}^2 = R^2$, and $\Lambda^{(D-1)}$ by $\mathbf{u}^2 = u_0^2 - \sum_{i=1}^{D-1} u_i^2 = R^2$ which fixes the signature of the metric in $\mathbf{u}$-space. The metric is determined via the classical Lagrangians, say,

$$\mathcal{L}_{Cl}^{E^{(D)}}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m}{2}\dot{\mathbf{x}}^2 \ , \qquad \mathcal{L}_{Cl}^{S^{(D-1)}}(\mathbf{s}, \dot{\mathbf{s}}) = \frac{m}{2}\dot{\mathbf{s}}^2 \ , \qquad \mathcal{L}_{Cl}^{\Lambda^{(D-1)}}(\mathbf{u}, \dot{\mathbf{u}}) = -\frac{m}{2}\dot{\mathbf{u}}^2 \ . \quad (4.1)$$

We put $R^2 = 1$, since $R$ is just a scaling factor of the systems. Consequently, the Laplacians are determined by

$$\Delta_{E^{(D)}} = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2} \ , \qquad \Delta_{S^{(D-1)}} = \sum_{i=0}^{D-1} \frac{\partial^2}{\partial s_i^2} \ , \qquad \Delta_{\Lambda^{(D-1)}} = \frac{\partial^2}{\partial u_0^2} - \sum_{i=1}^{D-1} \frac{\partial^2}{\partial u_i^2} \ . \quad (4.2)$$

We would like to point out that the various path integral evaluations are straightforward by applying the basic path integral solutions of Appendix 1. For completeness we present in this Section some examples somewhat more comprehensive to help understanding; in the next Section it is then sufficient just to take reference to the former ones. It is obvious that in two dimensions, the more easy path integral solutions are already known and the emphasize of this Section will be more on the collection of the various identities, instead on the path integral evaluations. On the other, where the theory of special functions is not well developed, such path integral evaluations cannot be explicitly done, therefore only an indirect reasoning is possible. Especially in these cases we state the general form of the propagator, respectively the Green function. We do not discus coordinate systems which are equivalent with a presented one and which can be obtained by a simple reparameterization of the variables. Compare e.g. the two different definitions of the parabolic coordinates [30, 87] in $E^{(2)}, E^{(3)}$, and of the equidistant coordinates [45, 90, 111] for $\Lambda^{(2)}$, respectively.

### 4.1. The Flat Space $E^{(2)}$

4.1.1. General Form of the Propagator and the Green Function. First of all we discuss the general form of the propagator and the Green function in the two-dimensional flat space $E^{(2)}$, which are easiest calculated in cartesian coordinates (see below). One finds that in $E^{(2)}$ they are four different coordinate systems [87]. In terms of the norm $d_{E^{(2)}}(\mathbf{q}'', \mathbf{q}')$ (where $\mathbf{q}$ denotes any coordinate system) they are given by

$$K^{E^{(2)}}\left(d_{E^{(2)}}(\mathbf{q}'', \mathbf{q}'); T\right) = \frac{m}{2\pi \mathrm{i}\hbar T} \exp\left[\frac{\mathrm{i}\,m}{2\hbar T}d_{E^{(2)}}^2(\mathbf{q}'', \mathbf{q}')\right] \ , \qquad (4.3)$$

$$G^{E^{(2)}}\left(d_{E^{(2)}}(\mathbf{q}'', \mathbf{q}'); E\right) = \frac{m}{\pi\hbar^2} K_0\left(\frac{d_{E^{(2)}}(\mathbf{q}'', \mathbf{q}')}{\hbar}\sqrt{-2mE}\right) \ . \qquad (4.4)$$

The norm in $E^{(2)}$ is given by (for the definition of the coordinates see below)

Cartesian Coordinates:
$$d_{E^{(2)}}^2(\mathbf{q}'', \mathbf{q}') = |\mathbf{x}'' - \mathbf{x}'|^2 \ , \qquad\qquad\qquad\qquad (4.5a)$$

Cylindrical Coordinates:

$$= r'^2 + r''^2 - 2r'r'' \cos(\phi'' - \phi') \ , \tag{4.5b}$$

Parabolic Coordinates:

$$= \frac{1}{4}\Big[(\eta''^2 + \xi''^2)^2 + (\eta'^2 + \xi'^2)^2 - 2(\eta'^2 - \xi'^2)(\eta''^2 - \xi''^2) - 8\eta'\eta''\xi'\xi''\Big] \ , \tag{4.5c}$$

Elliptic Coordinates:

$$= d^2\big(\cosh\mu' \cosh\mu'' \cos\nu' \cos\nu'' + \sinh\mu' \sinh\mu'' \sin\nu' \sin\nu''\big) \ . \tag{4.5d}$$

4.1.2. *Cartesian Coordinates.* We consider the usual cartesian coordinates $(x, y) = \mathbf{x} \in \mathrm{I\!R}^2$. The metric is given by $(g_{ab}) = \mathbb{1}_2$ and for the momentum operators we have $p_x = -\mathrm{i}\,\hbar\partial_x$, $p_y = -\mathrm{i}\,\hbar\partial_y$. Therefore:

$$-\frac{\hbar^2}{2m}\Delta_{E^{(2)}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{2m}(p_x^2 + p_y^2) \ . \tag{4.6}$$

The path integral formulation is well-known and given by [31, 32]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2\,dt\right) = \frac{m}{2\pi\,\mathrm{i}\,\hbar T}\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}|\mathbf{x}'' - \mathbf{x}'|^2\right) \tag{4.7}$$

$$= \int_{\mathrm{I\!R}^2}\frac{d\mathbf{p}}{4\pi^2}\exp\left[-\frac{\mathrm{i}\,\hbar T}{2m}\mathbf{p}^2 + \mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}'' - \mathbf{x}')\right] \ . \tag{4.8}$$

Note the formulation via a path integration over the Euclidean group [11].

4.1.3. *Cylindrical Coordinates.* We consider two-dimensional polar coordinates

$$\begin{aligned} x &= r\cos\phi \ , & r &> 0 \ , \\ y &= r\sin\phi \ , & 0 &\leq \phi < 2\pi \ . \end{aligned} \tag{4.9}$$

The metric is given by $(g_{ab}) = \mathrm{diag}(1, r^2)$, and the momentum operators have the form

$$p_r = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right) \ , \qquad p_\phi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\phi} \ . \tag{4.10}$$

This gives for the Hamiltonian

$$-\frac{\hbar^2}{2m}\Delta_{E^{(2)}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right) = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\phi^2\right) - \frac{\hbar^2}{8mr^2} \ . \tag{4.11}$$

We therefore obtain the path integral identity [4, 25, 54, 92, 104]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} r\mathcal{D}r(t) \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{\hbar^2}{8mr^2}\right]dt\right\}$$

$$= (r'r'')^{-1/2} \sum_{l \in \mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - \hbar^2\frac{l^2-\frac14}{2mr^2}\right)dt\right]$$

$$= (r'r'')^{-1/2} \sum_{l \in \mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_l[r^2] \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''} \dot{r}^2\,dt\right)$$

$$= \frac{m}{2\pi\,\mathrm{i}\,\hbar T} \exp\left[\frac{\mathrm{i}\,m}{2\hbar T}(r'^2 + r''^2)\right] \sum_{l \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')} I_l\left(\frac{mr'r''}{\mathrm{i}\,\hbar T}\right) \tag{4.12}$$

$$= \frac{m}{2\pi\,\mathrm{i}\,\hbar T} \exp\left[\frac{\mathrm{i}\,m}{2\hbar T}\left(r'^2 + r''^2 - 2r'r''\cos(\phi''-\phi')\right)\right] \tag{4.13}$$

$$= \sum_{l \in \mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi} \int_0^\infty p\,dp\, J_l(pr') J_l(pr'') \,\mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m} \quad . \tag{4.14}$$

**4.1.4. Elliptic Coordinates.** We consider the coordinate system

$$\begin{aligned}
x &= d\cosh\mu\cos\nu \ , & \mu &> 0 \ , \\
y &= d\sinh\mu\sin\nu \ , & -\pi &< \nu \le \pi \ .
\end{aligned} \tag{4.15}$$

The metric is $(g_{ab}) = d^2(\sinh^2\mu + \sin^2\nu)\,\mathbb{1}_2$ and we obtain for the momentum operators

$$p_\mu = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu} + \frac{\sinh\mu\cosh\mu}{\sinh^2\mu+\sin^2\nu}\right) \ , \qquad p_\nu = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\nu} + \frac{\sin\nu\cos\nu}{\sinh^2\mu+\sin^2\nu}\right) \ . \tag{4.16}$$

Consequently we have for the Hamiltonian

$$\begin{aligned}
-\frac{\hbar^2}{2m}\Delta_{E^{(2)}} &= -\frac{\hbar^2}{2md^2(\sinh^2\mu+\sin^2\nu)}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right) \\
&= \frac{1}{2md^2}(\sinh^2\mu+\sin^2\nu)^{-1/2}(p_\mu^2 + p_\nu^2)(\sinh^2\mu+\sin^2\nu)^{-1/2} \quad .
\end{aligned} \tag{4.17}$$

The path integral *construction* is straightforward, however no explicit path *integration* is possible. Actually, an expansion into the corresponding wave-functions in the coordinates $\mu$ and $\nu$ yields the Mathieu functions $\mathrm{me}_\nu(\eta, h^2)$ and $\mathrm{Me}_\nu^{(1)}(\xi, h^2)$ ($h^2 = mEd^2/2\hbar^2$), respectively, as eigen-function of the Hamiltonian, a specific class of higher transcendental functions [84]. However, because we know on the one side the eigen-functions of the Hamiltonian in terms of these functions [84], and on the other the kernel in $E^{(2)}$ in terms of the invariant distance $d_{E^{(2)}}$ we can state the following path integral identity (note the implemented time-transformation)

$$\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) d^2(\sinh^2\mu+\sin^2\nu)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}d^2\int_{t'}^{t''}(\sinh^2\mu+\sin^2\nu)(\dot\mu^2+\dot\nu^2)dt\right]$$

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$$
= \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} \, e^{-iET/\hbar} \int_0^\infty ds'' \int_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)
$$

$$
\times \exp\left\{ \frac{i}{\hbar} \int_0^{s''} \left[ \frac{m}{2}(\dot{\mu}^2 + \dot{\nu}^2) + Ed^2(\sinh^2\mu + \sin^2\nu) \right] ds \right\} \tag{4.18}
$$

$$
= \frac{1}{2\pi} \sum_{\nu \in \Lambda} \int_0^\infty p\,dp \; e^{-i\hbar p^2 T/2m}
$$

$$
\times \mathrm{me}_\nu^*(\eta', \tfrac{d^2 p^2}{4}) \, \mathrm{me}_\nu(\eta'', \tfrac{d^2 p^2}{4}) \, \mathrm{Me}_\nu^{(1)\,*}(\xi', \tfrac{d^2 p^2}{4}) \, \mathrm{Me}_\nu^{(1)}(\xi'', \tfrac{d^2 p^2}{4}) \tag{4.19}
$$

$$
= \frac{m}{2\pi i \hbar T} \exp\left[ \frac{im}{2\hbar T} d_{E^{(2)}}^2(\mathbf{q}'', \mathbf{q}') \right] \;, \tag{4.20}
$$

and $d_{E^{(2)}}(\mathbf{q}'', \mathbf{q}')$ must be taken in elliptic coordinates. The functions $\mathrm{me}_\nu(\eta, h^2)$ and $\mathrm{Me}_\nu^{(1)}(\xi, h^2)$, are mutually determined through the separation parameter $\lambda = \lambda_\nu(h^2)$ yielding a countable set of numbers $\nu \in \Lambda$. In particular, the functions $\mathrm{Me}_\nu^{(1)}(z, h^2)$ yield in the limit $h^2 \to 0$ the Bessel functions $J_\nu$, i.e. $\mathrm{Me}_\nu^{(1)}(z/h, h^2) \simeq J_\nu(z)$ $(h \to 0)$, and the functions $\mathrm{me}_\nu(z, h^2) \propto e^{i\nu z}$ $(h^2 \to 0)$, therefore obeying the correct boundary-conditions of our problem [84].

*4.1.5. Parabolic Coordinates.* We consider the coordinate system

$$
x = \xi\eta \;, \qquad y = \tfrac{1}{2}(\eta^2 - \xi^2) \;, \qquad \xi \in \mathbb{R}, \eta > 0 \tag{4.21}
$$

(alternatively $\xi > 0$, $\eta \in \mathbb{R}$ [84]). We have $(g_{ab}) = (\xi^2 + \eta^2)\,\mathbb{1}_2$, and consequently for the momentum operators

$$
p_\xi = \frac{\hbar}{i} \left( \frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2 + \eta^2} \right) \;, \qquad p_\eta = \frac{\hbar}{i} \left( \frac{\partial}{\partial\eta} + \frac{\eta}{\xi^2 + \eta^2} \right) \;. \tag{4.22}
$$

This gives for the Hamiltonian

$$
-\frac{\hbar^2}{2m}\Delta_{E^{(2)}} = -\frac{\hbar^2}{2m(\xi^2+\eta^2)} \left( \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} \right) = \frac{1}{2m}(\xi^2+\eta^2)^{-1/2}(p_\xi^2 + p_\eta^2)(\xi^2+\eta^2)^{-1/2} \;. \tag{4.23}
$$

Note $\Delta V_{PF} = 0$ which is a peculiarity of two dimensions with metric $\propto \mathbb{1}$. These coordinates have been used in literature to discuss the two-dimensional "Coulomb-problem", c.f. [24, 52, 64]. The transformation $(x,y) \mapsto (\xi,\eta)$ actually is the two-dimensional realization of the Kustaanheimo-Stiefel transformation. The arising path integral in these coordinates must be treated by a time-transformation due to the metric terms in the Lagrangian. The transformation in its continuous and lattice implementation, respectively, has the form

$$
s(t) = \int_{t'}^t \frac{d\sigma}{\xi^2(\sigma) + \eta^2(\sigma)} \;, \qquad \epsilon = (\widehat{\xi_j^2 + \eta_j^2})\Delta s_j \;, \tag{4.24}
$$

and decouples the $\xi$- and $\eta$-path integration giving two harmonic oscillator path integrals in $\xi$ and $\eta$ with frequency $\omega = \sqrt{-2E/m}$, respectively. Because $E > 0$ in our case,

$\omega$ is purely imaginary and we obtain two repelling oscillators. With the help of the harmonic oscillator path integral and the relations in Appendix 2 we obtain the path integral identity

$$
\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)(\xi^2+\eta^2) \exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\xi^2+\eta^2)(\dot{\xi}^2+\dot{\eta}^2)dt\right]
$$

$$
= \int_{\mathbb{R}} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}\,ET/\hbar} \int_0^\infty ds'' \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s) \int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(t)
$$

$$
\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot{\xi}^2+\dot{\eta}^2)+E(\xi^2+\eta^2)\right]ds\right\}
$$

$$
= \int_{\mathbb{R}} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}\,ET/\hbar} \int_0^\infty ds'' \frac{m\omega}{\pi\,\mathrm{i}\,\hbar\sin\omega s''}
$$

$$
\times \exp\left[\frac{\mathrm{i}\,m\omega}{2\hbar\sin\omega s''}(\xi'^2+\xi''^2+\eta'^2+\eta''^2)\cos\omega s''\right]\cosh\left(\frac{m\omega(\xi'\xi''+\eta'\eta'')}{\mathrm{i}\,\hbar\sin\omega s''}\right)
$$

$$
(4.25)
$$

$$
= \sum_{e,o}\int_{\mathbb{R}} d\zeta \int_{\mathbb{R}} dp\, \mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m}\,\Psi_{p,\zeta}^{(e,o)\,*}(\xi',\eta')\Psi_{p,\zeta}^{(e,o)}(\xi'',\eta'')\ , \qquad (4.26)
$$

and $\sum_{e,o}$ denotes the summation over even and odd states respectively; the functions $\Psi_{p,\zeta}^{(e,o)}(\xi,\eta)$ are given by

$$
\Psi_{p,\zeta}^{(e,o)}(\xi,\eta) = \frac{\mathrm{e}^{\pi/2\,ap}}{\sqrt{2}4\pi^2}
$$

$$
\times \begin{pmatrix}\left|\Gamma(\frac{1}{4}-\frac{\mathrm{i}\,\zeta}{2p})\right|^2 E_{-\frac{1}{2}+\mathrm{i}\,\zeta/p}^{(0)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2p}\,\xi)E_{-\frac{1}{2}-\mathrm{i}\,\zeta/p}^{(0)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2p}\,\eta) \\ \left|\Gamma(\frac{3}{4}-\frac{\mathrm{i}\,\zeta}{2p})\right|^2 E_{-\frac{1}{2}+\mathrm{i}\,\zeta/p}^{(1)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2p}\,\xi)E_{-\frac{1}{2}-\mathrm{i}\,\zeta/p}^{(1)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2p}\,\eta)\end{pmatrix}\ , \qquad (4.27)
$$

which are $\delta$-normalized according to [79]

$$
\int_0^\infty d\eta \int_{\mathbb{R}} d\xi(\xi^2+\eta^2)\Psi_{p',\zeta'}^{(e,o)\,*}(\xi,\eta)\Psi_{p,\zeta}^{(e,o)}(\xi,\eta) = \delta(p'-p)\delta(\zeta'-\zeta)\ . \qquad (4.28)
$$

Note that in the evaluation of the path integral one has to take into account that by using the harmonic oscillator solution for the $\xi$- and $\eta$-variable, respectively, one actually uses a double covering of the original $(x,y)\equiv(x_1,x_2)\in\mathbb{R}^2$-plane, i.e. $\mathbf{u}\equiv(\xi,\eta)\in\mathbb{R}^2$. Furthermore we have taken into account that our mapping is of the "square-root" type which gives rise to a sign ambiguity. "Thus, if one considers all paths in the complex $z = x+\mathrm{i}\,y$-plane from $z'$ to $z''$, they will be mapped into two different classes of paths in the $\mathbf{u}$-plane: Those which go from $\mathbf{u}'$ to $\mathbf{u}''$ and those going from $\mathbf{u}'$ to $-\mathbf{u}''$. In the cut complex $z$-plane for the function $|\mathbf{u}| = \sqrt{|z|}$ these are the paths passing an even or odd number of times through the square root from $|z| = 0$ and $|z| = -\infty$. We may choose

the $\mathbf{u}'$ corresponding to the initial $z'$ to lie on the first sheet (i.e. in the right half $\mathbf{u}$-plane). The final $\mathbf{u}''$ can be in the right as well as the left half-plane and all paths on the $z$-plane go over into paths from $\mathbf{u}'$ to $\mathbf{u}''$ and those from $\mathbf{u}'$ to $-\mathbf{u}''$ " [24]. Thus the two contributions arise in (4.25). The last line of (4.26) is then best obtained by considering the Coulomb problem $-q_1 q_2/r$ in $\mathrm{I\!R}^2$, applying (A.2.2), respectively (A.2.4), performing a momentum variable transformation $(p_\xi, p_\eta) \rightarrow \left( \frac{1}{2p}(\frac{1}{a} + \zeta), \frac{1}{2p}(\frac{1}{a} - \zeta) \right)$ ($a = \hbar^2/m q_1 q_2$ is the Bohr radius) with the new variables $(p, \zeta)$ and finally setting $q_1 q_2 = 0$, i.e. $a = \infty$.

### 4.2. The Sphere $S^{(2)}$

*4.2.1. General Form of the Propagator and the Green Function.* The sphere $S^{(2)}$ is the first non-trivial space of constant positive curvature. There exist two coordinate systems which admit separation of variables on $S^{(2)}$: polar coordinates and sphero-conical coordinates. The invariant distance $\cos \psi_{S^{(2)}} = \mathbf{s}' \cdot \mathbf{s}''$ in these coordinates is given by (for the definition of the coordinates see below)

Spherical Coordinates:
$$\cos \psi_{S^{(2)}}(\mathbf{q}'', \mathbf{q}') = \cos \theta'' \cos \theta' + \sin \theta'' \sin \theta' \cos(\phi'' - \phi') \ , \tag{4.29a}$$
Sphero-Conical Coordinates:
$$= \operatorname{sn} \mu'' \operatorname{sn} \mu' \operatorname{dn} \nu'' \operatorname{dn} \nu' + \operatorname{cn} \mu'' \operatorname{cn} \mu' \operatorname{cn} \nu'' \operatorname{cn} \nu' + \operatorname{dn} \mu'' \operatorname{dn} \mu' \operatorname{sn} \nu'' \operatorname{sn} \nu' \ . \tag{4.29b}$$

The propagator and the Green function on $S^{(2)}$ are best calculated in terms of polar coordinates. One obtains

$$K^{S^{(2)}} \left( S^{(2)}(\mathbf{q}'', \mathbf{q}'); T \right) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l \left( \cos \psi_{S^{(2)}}(\mathbf{q}'', \mathbf{q}') \right) \exp \left[ -\frac{\mathrm{i}\hbar T}{2m} l(l+1) \right] \ , \tag{4.30}$$

$$G^{S^{(2)}} \left( \psi_{S^{(2)}}(\mathbf{q}'', \mathbf{q}'); E \right) = \frac{m}{2\hbar^2} \frac{P_{-\frac{1}{2} + \sqrt{2mE/\hbar^2 + 1/4}} \left( -\cos \psi_{S^{(2)}}(\mathbf{q}'', \mathbf{q}') \right)}{\sin \left[ \pi \left( \frac{1}{2} - \sqrt{2mE/\hbar^2 + \frac{1}{4}} \right) \right]} \ . \tag{4.31}$$

*4.2.2. Spherical Coordinates.* We consider the polar coordinates

$$\left. \begin{aligned} s_0 &= \sin \theta \cos \phi \ , & 0 < \theta < \pi \ , \\ s_1 &= \cos \theta \ , & 0 \le \phi < 2\pi \ , \\ s_2 &= \sin \theta \sin \phi \ . & \end{aligned} \right\} \tag{4.32}$$

These are the usual two-dimensional polar coordinates on the sphere. The metric tensor is $(g_{ab}) = \operatorname{diag}(1, \sin^2 \theta)$, and the momentum operators have the form

$$p_\theta = \frac{\hbar}{\mathrm{i}} \left( \frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta \right) \ , \qquad p_\phi = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial \phi} \ . \tag{4.33}$$

For the Hamiltonian we obtain

$$-\frac{\hbar^2}{2m}\Delta_{S^{(2)}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$= \frac{1}{2m}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\phi^2\right) - \frac{\hbar^2}{8m}\left(1 + \frac{1}{\sin^2\theta}\right) \ . \qquad (4.34)$$

The corresponding path integral is well-known [10, 55, 92] and we have the identity

$$\int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \sin\theta\mathcal{D}\theta(t) \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot\theta^2 + r^2\sin^2\theta\dot\phi^2\right) + \frac{\hbar^2}{8m}\left(1 + \frac{1}{\sin^2\theta}\right)\right]dt\right\}$$

$$= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l\big(\cos\psi_{S^{(2)}}(\mathbf{q}'',\mathbf{q}')\big)\,\mathrm{e}^{-\mathrm{i}\,\hbar Tl(l+1)/2m} \qquad (4.35)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^{m\,*}(\theta',\phi')Y_l^m(\theta'',\phi'')\,\mathrm{e}^{-\mathrm{i}\,\hbar Tl(l+1)/2m} \quad . \qquad (4.36)$$

The $Y_l^m(\theta,phi)$ are the usual spherical harmonics on the $S^{(2)}$-sphere.

4.2.3. *Sphero-Conical Coordinates.* We consider the general elliptic coordinate system

$$s_0^2 + s_1^2 + s_2^2 = 1 \ ,$$
$$\frac{s_0^2}{\rho_i - a} + \frac{s_1^2}{\rho_i - b} + \frac{s_2^2}{\rho_i - c} = 0 \ , \qquad (4.37)$$

where $c \leq \rho_2 \leq b \leq \rho_1 \leq a$. The connection to the variables $\mathbf{s}$ on the sphere $S^{(2)}$ is given by

$$s_0^2 = \frac{(\rho_1-a)(\rho_2-a)}{(a-c)(a-b)}, \ s_1^2 = \frac{(\rho_1-b)(\rho_2-b)}{(b-a)(b-c)}, \ s_2^2 = \frac{(\rho_1-c)(\rho_2-c)}{(c-a)(c-b)} \ . \qquad (4.38)$$

One now can make the identification $\rho_1 = k^2\,\mathrm{cn}^2(\mu,k)$, $\rho_2 = -k'^2\,\mathrm{cn}^2(\nu,k')$, $b = -k'^2$, $a = k^2$ and $c = 0$. This yields

$$\left.\begin{array}{l} s_0 = \mathrm{sn}(\mu,k)\,\mathrm{dn}(\nu,k') \ , \quad k^2 + k'^2 = 1 \ , \\[4pt] s_1 = \mathrm{dn}(\mu,k)\,\mathrm{sn}(\nu,k') \ , \\[4pt] s_2 = \mathrm{cn}(\mu,k)\,\mathrm{cn}(\nu,k') \ . \end{array}\right\} \qquad (4.39)$$

The functions $\mathrm{sn}(\mu,k)$, $\mathrm{cn}(\mu,k)$ and $\mathrm{dn}(\mu,k)$ denote Jacobi elliptic functions, where e.g. $\mathrm{cn}(\mu,k)$ has the periods $4\mathrm{i}\,K$, $2K + 2\mathrm{i}\,K'$, $4K$, respectively ($K, K'$ the elliptic integrals) [1]. Note the relations $\mathrm{cn}^2\mu + \mathrm{sn}^2\mu = 1$ and $\mathrm{dn}^2\mu = 1 - k^2\,\mathrm{sn}^2\mu$. The metric tensor

$g_{ab}$ in these coordinates is therefore given by $(g_{ab}) = (k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu)\, 1\!\!1_2$. The momentum operators are

$$
p_\mu = \frac{\hbar}{\mathrm{i}} \left( \frac{\partial}{\partial \mu} - \frac{k^2 \operatorname{sn} \mu \operatorname{cn} \mu \operatorname{dn} \mu}{k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu} \right) \;, \quad
p_\nu = \frac{\hbar}{\mathrm{i}} \left( \frac{\partial}{\partial \nu} - \frac{k'^2 \operatorname{sn} \nu \operatorname{cn} \nu \operatorname{dn} \nu}{k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu} \right) \;, \quad (4.40)
$$

and the Hamiltonian has the form

$$
-\frac{\hbar^2}{2m}\Delta_{S^{(2)}} = -\frac{\hbar^2}{2m}\frac{1}{k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu}\left( \frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2} \right)
$$
$$
= \frac{1}{2m}\frac{1}{\sqrt{k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu}}(p_\mu^2 + p_\nu^2)\frac{1}{\sqrt{k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu}}\;. \quad (4.41)
$$

The path integral can therefore be written down yielding [50]

$$
\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)(k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu)
$$
$$
\times \exp\left[ \frac{\mathrm{i}\, m}{2\hbar} \int_{t'}^{t''} (k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu)(\dot\mu^2 + \dot\nu^2)dt \right]
$$
$$
= \sum_\kappa \sum_{l=0}^\infty A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')\, \mathrm{e}^{-\mathrm{i}\,\hbar T l(l+1)/2m} \quad (4.42)
$$
$$
= \frac{1}{4\pi}\sum_{l=0}^\infty (2l+1)P_l\big( \cos\psi_{S^{(2)}}(\mathbf{q}'',\mathbf{q}')\big)\, \mathrm{e}^{-\mathrm{i}\,\hbar T l(l+1)/2m} \quad. \quad (4.43)
$$

The eigen-functions $\Psi_\lambda(\mu,\nu) = A_{l,\kappa}(\mu)B_{l,\kappa}(\nu)$, are Lamé polynomials so that the functions $A_{l,\kappa}$ and $B_{l,\kappa}$ are solutions of the Lamé differential equations [28]

$$
\left.
\begin{aligned}
\frac{\mathrm{d}^2 A_{l,\kappa}(\mu)}{\mathrm{d}\mu^2} + \left[ -\frac{\kappa}{4} + l(l+1) - l(l+1)k^2 \operatorname{sn}^2 \mu \right]A_{l,\kappa}(\mu) = 0 \;, \\
\frac{\mathrm{d}^2 B_{l,\kappa}(\nu)}{\mathrm{d}\nu^2} + \left[ \frac{\kappa}{4} - l(l+1)k'^2 \operatorname{sn}^2 \nu \right]B_{l,\kappa}(\nu) = 0 \;.
\end{aligned}
\right\}
\quad (4.44)
$$

These functions are highly transcendental ones and cannot be expressed in terms of standard power series expansions like (confluent) hypergeometric functions.

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## 4.3. The Pseudosphere $\Lambda^{(2)}$

*4.3.1. General Form of the Propagator and the Green Function.* The evaluation of the Green function on the Pseudosphere $\Lambda^{(2)}$ was subject of a couple of papers. It was first evaluated by means of path integration in [54], followed by [45, 56] (compare also [60, 76]). The propagator and the Green function have the form

$$K^{\Lambda^{(2)}}\left(d_{\Lambda^{(2)}}(\mathbf{q}'', \mathbf{q}'); T\right) = \frac{1}{2\pi} \int_0^\infty p \, dp \, \tanh \pi p$$

$$\times \mathcal{P}_{\mathrm{i}\, p - \frac{1}{2}}\left(\cosh d_{\Lambda^{(2)}}(\mathbf{q}'', \mathbf{q}')\right) \exp\left[-\frac{\mathrm{i}\,\hbar T}{2m}\left(p^2 + \frac{1}{4}\right)\right] , \qquad (4.45)$$

$$G^{\Lambda^{(2)}}\left(d_{\Lambda^{(2)}}(\mathbf{q}'', \mathbf{q}'); E\right) = \frac{m}{\pi\hbar^2} \mathcal{Q}_{-1/2 - \mathrm{i}\sqrt{2mE/\hbar^2 - \frac{1}{4}}}\left(\cosh d_{\Lambda^{(2)}}(\mathbf{q}'', \mathbf{q}')\right) . \qquad (4.46)$$

The invariant hyperbolic distance in $\Lambda^{(2)}$ is given by (for the definition of the coordinates see below)

Pseudo-Polar Coordinates:
$$\cosh d_{\Lambda^{(2)}}(\mathbf{q}'', \mathbf{q}') = \cosh \tau'' \cosh \tau' - \sinh \tau'' \sinh \tau' \cos(\phi'' - \phi') , \qquad (4.47\mathrm{a})$$
Horicyclic Coordinates:
$$= \frac{(x'' - x')^2 + y''^2 + y'^2}{2y'y''} , \qquad (4.47\mathrm{b})$$
Equidistant Coordinates:
$$= \cosh(\tau_2'' - \tau_2') \cosh \tau_1'' \cosh \tau_1' - \sinh \tau_1'' \sinh \tau_1' , \qquad (4.47\mathrm{c})$$
Pseudo-Ellipsoidal Coordinates:
$$= \mathrm{nc}\,\mu''\,\mathrm{nc}\,\mu'\,\mathrm{nc}\,\nu''\,\mathrm{nc}\,\nu' - \mathrm{dc}\,\mu''\,\mathrm{dc}\,\mu'\,\mathrm{sc}\,\nu''\,\mathrm{sc}\,\nu' - \mathrm{sc}\,\mu''\,\mathrm{sc}\,\mu'\,\mathrm{dc}\,\nu''\,\mathrm{dc}\,\nu' . \qquad (4.47\mathrm{d})$$

Because the discussion of the path integral solutions on the pseudosphere $\Lambda^{(2)}$ has been done in some extend in Refs. [45, 54], we just cite the results, with the exception of the case of the ellipsoidal coordinates. In the nomenclature we follow [69].

*4.3.2. Horicyclic Coordinates (Poincaré Upper Half-Plane).* We consider the coordinate system

$$\left. \begin{array}{ll} u_0 = \dfrac{1}{2y}(y^2 + 1 - x) , & u_1 = \dfrac{x}{y} , \qquad y > 0 , \\[2mm] u_2 = \dfrac{1}{2y}(y^2 - 1 - x) , & x \in \mathbb{R} , \end{array} \right\} \qquad (4.48)$$

The metric has the form $(g_{ab}) = \mathbb{1}_2/y^2$, and the momentum operators are

$$p_x = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x} , \qquad p_y = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial y} - \frac{1}{y}\right) . \qquad (4.49)$$

For the Hamiltonian we have

$$-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(2)}} = \frac{\hbar^2}{2m}y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{2m}(y^2 p_x^2 + y p_y^2 y) \ . \tag{4.50}$$

We therefore obtain the path integral identity [45, 54, 60]

$$
\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2} \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\frac{\dot{x}^2+\dot{y}^2}{y^2}dt\right)
$$

$$
= \frac{\sqrt{y'y''}}{\pi^3}\int_{\mathbb{R}} dk \int_0^\infty p\,dp\,\sinh\pi p\,\mathrm{e}^{-\mathrm{i}\,\hbar T(p^2+\frac14)/2m}\,\mathrm{e}^{\mathrm{i}\,k(x''-x')}\,K_{\mathrm{i}\,p}(|k|y')\,K_{\mathrm{i}\,p}(|k|y'') \ . \tag{4.51}
$$

*4.3.3. Equidistant Coordinates (Hyperbolic Strip).* We consider the coordinate system

$$
\begin{aligned}
u_0 &= \cosh\tau_1\cosh\tau_2 \ , & u_1 &= \sinh\tau_1 \ , & \tau_1,\tau_2 \in \mathbb{R} \ , \\
u_2 &= \cosh\tau_1\sinh\tau_2 \ .
\end{aligned}
\tag{4.52}
$$

The metric has the form $(g_{ab}) = \mathrm{diag}(1,\cosh^2\tau_1)$, and the momentum operators are

$$
p_{\tau_1} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau_1} + \frac12\tanh\tau_1\right) \ , \qquad p_{\tau_2} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\tau_2} \ . \tag{4.53}
$$

For the Hamiltonian we have

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(2)}} &= \frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\tau_1^2} + \tanh\tau_1\frac{\partial}{\partial\tau_1} + \frac{1}{\cosh^2\tau_1}\frac{\partial^2}{\partial\tau_2^2}\right) \\
&= \frac{1}{2m}\left(p_{\tau_1}^2 + \frac{1}{\cosh^2\tau_1}p_{\tau_2}^2\right) + \frac{\hbar^2}{8m}\left(1 + \frac{1}{\cosh^2\tau_1}\right) \ . \tag{4.54}
\end{aligned}
$$

Therefore we obtain the path integral identity [45]

$$
\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \cosh\tau_1\,\mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t)
$$

$$
\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{\tau}_1^2 + \cosh^2\tau_1\dot{\tau}_2^2\right) - \frac{\hbar^2}{8m}\left(1 + \frac{1}{\cosh^2\tau_1}\right)\right]dt\right\}
$$

$$
= \left(\cosh\tau_1'\cosh\tau_1''\right)^{-1/2}\mathrm{e}^{-\mathrm{i}\,\hbar T/8m}\int_{\mathbb{R}}\frac{dk}{2\pi}\,\mathrm{e}^{\mathrm{i}\,k(\tau_2''-\tau_2')}
$$

$$
\times \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\tau}_1^2 - \frac{\hbar^2}{2m}\frac{k^2+\frac14}{\cosh^2\tau_1}\right)dt\right]
$$

$$
= \left(\cosh\tau_1'\cosh\tau_1''\right)^{-1/2}\int_{\mathbb{R}}\frac{dk}{2\pi}\,\mathrm{e}^{\mathrm{i}\,k(\tau_2''-\tau_2')}
$$

$$
\times \frac12\int_{\mathbb{R}}\frac{p\sinh\pi p\,dp}{\cosh^2\pi k + \sinh^2\pi p}\,\mathrm{e}^{-\mathrm{i}\,\hbar T(p^2+\frac14)/2m}\,P_{\mathrm{i}\,k-\frac12}^{-\mathrm{i}\,p}(\tanh\tau_1')P_{\mathrm{i}\,k-\frac12}^{\mathrm{i}\,p}(\tanh\tau_1'') \ . \tag{4.55}
$$

In the path integral evaluation one successively uses the path integral solution of the special case of the modified Pöschl-Teller potential (the simple Rosen-Morse potential [46]).

*4.3.4. Pseudospherical Coordinates (Upper Sheet of the Two-Sheeted Hyperboloid, Poincaré Disc).* We consider the coordinate system

$$
\left.
\begin{aligned}
u_0 &= \cosh \tau \ , & \tau &> 0 \ , \\
u_1 &= \sinh \tau \sin \phi \ , & 0 &\leq \phi < 2\pi \ , \\
u_2 &= \sinh \tau \cos \phi \ .
\end{aligned}
\right\}
\tag{4.56}
$$

These are the well-known two-dimensional (pseudo-) spherical coordinates on $\Lambda^{(2)}$. The metric has the form $(g_{ab}) = \mathrm{diag}(1, \sinh^2 \tau_1)$, and the momentum operators are

$$
p_\tau = \frac{\hbar}{i} \left( \frac{\partial}{\partial \tau} + \frac{1}{2} \coth \tau \right) \ , \qquad p_\phi = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \ .
\tag{4.57}
$$

For the Hamiltonian we have

$$
\begin{aligned}
-\frac{\hbar^2}{2m} \Delta_{\Lambda^{(2)}} &= \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial \tau} + \coth \tau \frac{\partial}{\partial \tau} + \frac{1}{\cosh^2 \tau} \frac{\partial^2}{\partial \phi} \right) \\
&= \frac{1}{2m} \left( p_\tau^2 + \frac{1}{\sinh^2 \tau} p_\phi^2 \right) + \frac{\hbar^2}{8m} \left( 1 - \frac{1}{\sinh^2 \tau_1} \right) \ .
\end{aligned}
\tag{4.58}
$$

Therefore we obtain the path integral identity [45, 56]

$$
\begin{aligned}
&\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \sinh \tau \, \mathcal{D}\tau(t) \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t) \\
&\qquad \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} (\dot{\tau}^2 + \sinh^2 \tau \, \dot{\phi}^2) - \frac{\hbar^2}{8m} \left( 1 - \frac{1}{\sinh^2 \tau} \right) \right] dt \right\} \\
&= \frac{1}{2\pi^2} \int_0^\infty dp \sum_{l=-\infty}^{\infty} p \sinh \pi p \exp \left[ -\frac{i\hbar T}{2m} \left( p^2 + \frac{1}{4} \right) \right] \left| \Gamma \left( \frac{1}{2} + i p + l \right) \right|^2 \\
&\qquad \times e^{i\, l(\phi'' - \phi')} \mathcal{P}_{i p - \frac{1}{2}}^{-l} (\cosh \tau') \mathcal{P}_{i p - \frac{1}{2}}^{-l} (\cosh \tau'')
\end{aligned}
\tag{4.59}
$$

(Pseudospherical polar coordinate system on the disc, $0 \leq r < 1$, $\psi \in [0, 2\pi)$:)

$$
\begin{aligned}
&= \frac{i}{\hbar} \int_0^\infty dT \, e^{i\, TE/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \frac{4r \, \mathcal{D}r(t)}{(1-r^2)^2} \int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''} \mathcal{D}\psi(t) \\
&\qquad \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ 2m \frac{\dot{r}^2 + r^2 \dot{\psi}^2}{(1-r^2)^2} + \hbar^2 \frac{(1-r^2)^2}{32 m r^2} \right] dt \right\} \\
&= \frac{1}{2\pi^2} \int_0^\infty dp \sum_{l=-\infty}^{\infty} p \sinh \pi p \exp \left[ -\frac{i\hbar T}{2m} \left( p^2 + \frac{1}{4} \right) \right] \left| \Gamma \left( \frac{1}{2} + i p + l \right) \right|^2 \\
&\qquad \times e^{i\, l(\psi'' - \psi')} \mathcal{P}_{i p - \frac{1}{2}}^{-l} \left( \frac{1 + r'^2}{1 - r'^2} \right) \mathcal{P}_{i p - \frac{1}{2}}^{-l} \left( \frac{1 + r''^2}{1 - r''^2} \right)
\end{aligned}
\tag{4.60}
$$

Here we have also stated the (equivalent) representation in terms of the Poincaré disc via $z = x_1 + i x_2 = r \, e^{i\, \psi} = \tanh \frac{\tau}{2} (\sin \phi + i \cos \phi)$ which is important is various discussions in the theory of quantum chaos [5–8, 61].

*4.3.5.  General Pseudo-Ellipsoidal Coordinates.* We consider the general elliptic coordinate system

$$u_0^2 - u_1^2 - u_2^2 = 1 \ ,$$

$$\frac{u_0^2}{\rho_i - a} - \frac{u_1^2}{\rho_i - b} - \frac{u_2^2}{\rho_i - c} = 0 \ , \qquad (i = 1, 2) \ . \tag{4.61}$$

Explicitly in terms of the variables **u** on the pseudosphere $\Lambda^{(2)}$ they are given by

$$u_0^2 = \frac{(\rho_1 - c)(\rho_2 - c)}{(a - c)(b - c)} \ , \quad u_1^2 = -\frac{(\rho_1 - b)(\rho_2 - b)}{(a - b)(c - b)} \ , \quad u_2^2 = -\frac{(\rho_1 - a)(\rho_2 - a)}{(b - a)(c - a)} \ . \tag{4.62}$$

One now can make the identification $\rho_1 = -k'^2 \operatorname{nc}^2(\mu, k)$, $\rho_2 = k^2 \operatorname{nc}^2(\nu, k')$, $a = -k'^2$, $b = k^2$ and $c = 0$. This yields

$$\left. \begin{aligned} u_0 &= \operatorname{nc}(\mu, k) \operatorname{nc}(\nu, k') \ , \quad k^2 + k'^2 = 1 \ , \\ u_1 &= \operatorname{dc}(\mu, k) \operatorname{sc}(\nu, k') \ , \\ u_2 &= \operatorname{sc}(\mu, k) \operatorname{dc}(\nu, k') \ , \end{aligned} \right\} \tag{4.63}$$

together with $0 < k^2 < k^2 \operatorname{nc}^2 \alpha < -k'^2 < -k'^2 \operatorname{nc}^2 \alpha$, say. The functions $\operatorname{sc}(\mu, k)$, $\operatorname{dc}(\mu, k)$ and $\operatorname{nc}(\mu, k)$ denote Jacobi elliptic functions which correspond to the functions sn, dn and cn by taking the argument imaginary, where e.g. $\operatorname{nc}(\mu, k)$ has the periods $4 \operatorname{i} K$, $2K + 2 \operatorname{i} K'$, $4K$ ($K, K'$ the elliptic integrals) [1]. Note the relations $\operatorname{nc}^2 \mu + \operatorname{sc}^2 \mu = 1$ and $\operatorname{dc}^2 \mu = 1 + k^2 \operatorname{sc}^2 \mu$. The metric tensor $g_{ab}$ in these coordinates is given by

$$(g_{ab}) = (k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu) \, \mathbb{1}_2 \ , \tag{4.64}$$

and the momentum operators are

$$p_\mu = \frac{\hbar}{\operatorname{i}} \left( \frac{\partial}{\partial \mu} - k'^2 \frac{\operatorname{sc} \mu \operatorname{nc} \mu \operatorname{dc} \mu}{k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu} \right) \ , \quad p_\nu = \frac{\hbar}{\operatorname{i}} \left( \frac{\partial}{\partial \nu} - k^2 \frac{\operatorname{sc} \nu \operatorname{nc} \nu \operatorname{dc} \nu}{k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu} \right) \ . \tag{4.65}$$

The Hamiltonian has the form

$$\begin{aligned} -\frac{\hbar^2}{2m} \Delta_{\Lambda^{(2)}} &= -\frac{\hbar^2}{2m} \frac{1}{k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu} \left( \frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2} \right) \\ &= \frac{1}{2m} \frac{1}{\sqrt{k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu}} (p_\mu^2 + p_\nu^2) \frac{1}{\sqrt{k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu}} \ . \end{aligned} \tag{4.66}$$

The path integral can therefore be formulated yielding

$$\frac{\operatorname{i}}{\hbar} \int_0^\infty dE \, \operatorname{e}^{\operatorname{i} ET/\hbar} \int_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) (k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu)$$

$$\times \exp\left[ \frac{\operatorname{i} m}{2\hbar} \int_{t'}^{t''} (k'^2 \operatorname{nc}^2 \mu + k^2 \operatorname{nc}^2 \nu)(\dot{\mu}^2 + \dot{\nu}^2) dt \right]$$

$$= \int_0^\infty ds'' \int_{\mu(0)=\mu'}^{\mu(s'')=\mu''} \mathcal{D}\mu(s) \int_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot\mu^2+\dot\nu^2)+E(k'^2\,\mathrm{nc}^2\,\mu+k^2\,\mathrm{nc}^2\,\nu)\right]ds\right\}$$

$$= \sum_\kappa \int_0^\infty \frac{dp}{\hbar^2(p^2+\frac14)/2m-E}\mathcal{A}^*_{p,\kappa}(\mu')\mathcal{B}^*_{p,\kappa}(\nu')\mathcal{A}_{p,\kappa}(\mu'')\mathcal{B}_{p,\kappa}(\nu'') \tag{4.67}$$

$$= \frac{m}{\pi\hbar^2}\mathcal{Q}_{-1/2-\mathrm{i}\sqrt{2mE/\hbar^2-\frac14}}\big(\cosh d_{\Lambda^{(2)}}(\mathbf{q}'',\mathbf{q}')\big)\ , \tag{4.68}$$

where $d_{\Lambda^{(2)}}(\mathbf{q}'',\mathbf{q}')$ must be expressed in ellipsoidal coordinates. The eigen-functions $\Psi_{p,\kappa}(\mu,\nu):=\mathcal{A}_{p,\kappa}(\mu)\mathcal{B}_{p,\kappa}(\nu)$, are generalized hyperbolic Lamé polynomials in $\mu$ and $\nu$ and are the normalized solution of the Hamiltonian $-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(2)}}$:

$$\left.\begin{aligned}
\frac{\mathrm{d}^2\mathcal{A}_{p,\kappa}(\mu)}{\mathrm{d}\mu^2}+\left[-\frac{\kappa}{4}+(p^2+\tfrac14)k'^2\,\mathrm{nc}^2\,\mu\right]\mathcal{A}_{l,\kappa}(\mu)=0\ ,\\[2mm]
\frac{\mathrm{d}^2\mathcal{B}_{p,\kappa}(\nu)}{\mathrm{d}\nu^2}+\left[\frac{\kappa}{4}+(p^2+\tfrac14)k^2\,\mathrm{nc}^2\,\nu\right]\mathcal{B}_{l,\kappa}(\nu)=0\ .
\end{aligned}\right\} \tag{4.69}$$

Depending on the domain, the values $\mu$ and $\nu$ can take on, one can discriminate six different coordinate system which are encoded in (4.62, 4.63). For this purpose one writes the kinetic energy-term in terms of $\rho_{1,2}$ [69, 90]

$$\dot u_0^2-\dot u_1^2-\dot u_2^2=\frac14(\rho_1-\rho_2)\left[\frac{\dot\rho_1^2}{P_3(\rho_1)}-\frac{\dot\rho_2^2}{P_3(\rho_2)}\right]\ , \tag{4.70}$$

where the polynomial $P_3(\rho)$ is defined by $P_3(\rho)=(\rho-a)(\rho-b)(\rho-c)$. Therefore we obtain the alternative path integral formulation

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int_{\rho_1(t')=\rho_1'}^{\rho_1(t'')=\rho_1''}\mathcal{D}\rho_1(t) \int_{\rho_2(t')=\rho_2'}^{\rho_2(t'')=\rho_2''}\mathcal{D}\rho_2(t)\frac{(-\mathrm{i})(\rho_1-\rho_2)}{4\sqrt{P_3(\rho_1)P_3(\rho_2)}}$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{8}(\rho_1-\rho_2)\left(\frac{\dot\rho_1^2}{P_3(\rho_1)}-\frac{\dot\rho_2^2}{P_3(\rho_2)}\right)-\Delta V_{PF}\right]dt\right\}$$

$$= \int_0^\infty ds'' \int_{\rho_1(0)=\rho_1'}^{\rho_1(s'')=\rho_1''}\sqrt{-P_3(\rho_1)}\,\mathcal{D}\rho_1(s) \int_{\rho_2(0)=\rho_2'}^{\rho_2(s'')=\rho_2''}\sqrt{P_3(\rho_2)}\,\mathcal{D}\rho_2(s)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\Big(P_3(\rho_2)\dot\rho_1^2-P_3(\rho_1)\dot\rho_2^2\Big)+ES-\frac{\hbar^2}{8m}\sum_{i=1}^D M_i\Big(\Gamma_i^2+2\Gamma_i'\Big)\right]ds\right\}$$

$$= \sum_\kappa \int_0^\infty \frac{dp}{\hbar^2(p^2+\frac14)/2m-E}\mathcal{A}^*_{p,\kappa}(\rho_1')\mathcal{B}^*_{p,\kappa}(\rho_2')\mathcal{A}_{p,\kappa}(\rho_1'')\mathcal{B}_{p,\kappa}(\rho_2'') \tag{4.71}$$

$$= \frac{m}{\pi\hbar^2}\mathcal{Q}_{-1/2-\mathrm{i}\sqrt{2mE/\hbar^2-\frac14}}\big(\cosh d_{\Lambda^{(2)}}(\mathbf{q}'',\mathbf{q}')\big)\ , \tag{4.72}$$

with the functions $\mathcal{A}_{p,\kappa}$ and $\mathcal{B}_{p,\kappa}$ rewritten in $\rho_{1,2}$ and $\cosh d_{\Lambda^{(2)}}$ expressed in $\rho_{1,2}$. Furthermore denote $M_1 = 1/P_3(\rho_2), M_2 = P_3(\rho_1),\ S = (\rho_1 - \rho_2)/4P_3(\rho_1)P_3(\rho_2)$ and $\Gamma_i = P_3'(\rho_i)/2P_3(\rho_i)$ $(i = 1, 2)$. The six different coordinate systems are now defined by by the roots of the polynomial $P_3(\rho)$ (simple real, complex, double or triple real) with the requirement of the positive definiteness of the two quantities

$$\frac{\rho_1 - \rho_2}{P_3(\rho_1)} > 0 \ , \qquad -\frac{\rho_1 - \rho_2}{P_3(\rho_2)} > 0 \ . \tag{4.73}$$

They are given by [69, 90]

4.3.5.1. *Elliptic Coordinates.*

$$c < b < \rho_2 < a < \rho_1 \ ,$$
$$\frac{u_1^2}{\rho_i - b} + \frac{u_2^2}{\rho_i - a} - \frac{u_0^2}{\rho_i - c} = 0 \ , \qquad (i = 1, 2) \ . \tag{4.74}$$

4.3.5.2. *Hyperbolic Coordinates.*

$$\rho_2 < c < b < a < \rho_1 \ ,$$
$$\frac{u_1^2}{\rho_i - c} + \frac{u_2^2}{\rho_i - a} - \frac{u_0^2}{\rho_i - b} = 0 \ , \qquad (i = 1, 2) \ . \tag{4.75}$$

4.3.5.3. *Semi-Hyperbolic Coordinates.*

$$\rho_2 < a < \rho_1 \ , \qquad b = \gamma + \mathrm{i}\,\delta \ , \qquad c = \gamma - \mathrm{i}\,\delta$$
$$\frac{u_2^2}{\rho_i - a} - \frac{2\delta u_1 u_0 + (\rho_i - \gamma)(u_0^2 - u_1^2)}{[(\rho_i - \gamma)^2 + \delta^2]} = 0 \ , \qquad (i = 1, 2) \ . \tag{4.76}$$

4.3.5.4. *Elliptic-Parabolic Coordinates.*

$$c = b < \rho_2 < a < \rho_1 \ ,$$
$$-\frac{u_2^2}{\rho_i - a} + \frac{u_0^2 - u_1^2}{\rho_i - b} - \frac{(u_0 - u_1)^2}{(\rho_i - b)^2} = 0 \ , \qquad (i = 1, 2) \ . \tag{4.77}$$

4.3.5.5. *Hyperbolic-Parabolic Coordinates.*

$$\rho_2 < c = b < a < \rho_1 \ ,$$
$$-\frac{u_2^2}{\rho_i - a} + \frac{u_0^2 - u_1^2}{\rho_i - b} - \frac{(u_0 - u_1)^2}{(\rho_i - b)^2} = 0 \ , \qquad (i = 1, 2) \ . \tag{4.78}$$

4.3.5.6. *Semi-Circular-Parabolic Coordinates.*

$$\rho_2 < c = b = a < \rho_1 \ ,$$
$$\left(\frac{u_0 - u_1}{\rho_i - a} + u_2\right)^2 = u_0^2 - u_1^2 \ , \qquad (i = 1, 2) \ . \tag{4.79}$$

This concludes the discussion of the separable coordinate systems on two-dimensional spaces of constant curvature.

## 5. Separation of Variables in Three Dimensions

In this Section we give the path integral representations in the three-dimensional spaces of constant curvature $E^{(3)}$, $S^{(3)}$ and $\Lambda^{(3)}$. The notation and presentation will be as in te previous Section, with the only difference that the stating of the invariant distances will be given separately in each case.

Some new path integral evaluations will be presented, however, our emphasize again is more concerned with the connection of the various path integral identities with each other. The wave-function expansions then represent again the various possible group matrix element representations yielding identities connecting the coordinate realizations. We also note the general form a potential must have to be separable in a coordinate system. The two-dimensional analogue then follows from omitting the $z$-coordinate.

### 5.1. The Flat Space $E^{(3)}$

#### 5.1.1. General Form of the Propagator and the Green Function.
Similarly as in the two-dimensional case, the propagator and the Green function on $E^{(3)}$ are well-known, and are given by

$$K^{E^{(3)}}\left(d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}');\right) = \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{3/2}\exp\left[-\frac{m}{2\,\mathrm{i}\,\hbar T}d^2_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')\right]\ , \qquad (5.1)$$

$$G^{E^{(3)}}\left(d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}');E\right) = \frac{m}{4\pi\hbar^2\,d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')}\exp\left(-\frac{d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')}{\hbar}\sqrt{-2mE}\right)\ . \qquad (5.2)$$

$\mathbf{q}$ denotes any of the eleven coordinate systems which allows a separation of variables of the Laplacian in three dimensions. Due to the rather many coordinate systems, the explicit expression for $d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')$ will be given together with the definition of the coordinates. In the nomenclature we follow [69, 87].

#### 5.1.2. Cartesian Coordinates.
Again, we start with the simplest case, cartesian coordinates $(x,y,z) = \mathbf{x} \in \mathbb{R}^3$. Then $(g_{ab}) = \mathbb{1}_3$,

$$d^2_{E^{(3)}}(\mathbf{q}'',\mathbf{q}') = |\mathbf{x}'' - \mathbf{x}'|^2\ , \qquad (5.3)$$

and $\mathbf{p} = -\mathrm{i}\,\hbar\nabla$. This gives for the Hamiltonian

$$-\frac{\hbar^2}{2m}\Delta_{E^{(3)}} = -\frac{\hbar^2}{2m}\nabla^2 = \frac{1}{2m}\mathbf{p}^2\ , \qquad (5.4)$$

and for the path integral we have [31, 32]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2 dt\right) = \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{3/2}\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}|\mathbf{x}''-\mathbf{x}'|^2\right) \qquad (5.5)$$

$$= \int_{\mathbb{R}^3}\frac{d\mathbf{p}}{(2\pi)^3}\exp\left[-\frac{\mathrm{i}\,\hbar T}{2m}\mathbf{p}^2 + \mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}''-\mathbf{x}')\right]\ . \qquad (5.6)$$

Note the formulation via a path integration over the Euclidean group [11]. The most general potential separable in cartesian coordinates has the form

$$V = u(x) + v(y) + w(z)\ . \qquad (5.7)$$

*5.1.3. Circular Cylinder Coordinates.* Next we consider circular cylinder coordinates which are very similar to the two-dimensional polar coordinates

$$
\left.\begin{aligned}
x &= r\cos\phi \ , & r &> 0 \ , \\
y &= r\sin\phi \ , & 0 &\leq \phi < 2\pi \ , \\
z &= z \ , & z &\in \mathbb{R} \ .
\end{aligned}\right\} \tag{5.8}
$$

Here we have for $d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')$

$$
d_{E^{(3)}}^2(\mathbf{q}'',\mathbf{q}') = |z'' - z'|^2 + r'^2 + r''^2 - 2r'r''\cos(\phi'' - \phi') \ . \tag{5.9}
$$

The metric reads $(g_{ab}) = \mathrm{diag}(1, r^2, 1)$, and the momentum operators are given by (4.10) and $p_z = -\mathrm{i}\hbar\partial_z$. This gives for the Hamiltonian

$$
-\frac{\hbar^2}{2m}\Delta_{E^{(3)}} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\phi^2 + p_z^2\right) - \frac{\hbar^2}{8mr^2} \ . \tag{5.10}
$$

We therefore obtain the path integral identity [4, 10, 55, 92, 104]

$$
\int_{r(t')=r'}^{r(t'')=r''} r\mathcal{D}r(t) \int_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t) \int_{z(t')=z'}^{z(t'')=z''} \mathcal{D}z(t)
$$

$$
\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + \frac{\hbar^2}{8mr^2}\right]dt\right\}
$$

$$
= \left(\frac{m}{2\pi\mathrm{i}\hbar T}\right)^{3/2}\exp\left[\frac{\mathrm{i}m}{2\hbar T}\left(|z'' - z'|^2 + r'^2 + r''^2\right)\right]\sum_{l\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}I_l\left(\frac{mr'r''}{\mathrm{i}\hbar T}\right) \tag{5.11}
$$

$$
= \int_{\mathbb{R}}\frac{dp_z}{2\pi}\,\mathrm{e}^{\mathrm{i}\,p_z(z''-z')-\mathrm{i}\hbar p_z^2 T/2m}\sum_{l\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi}\int_0^\infty p\,dp\,J_l(pr')J_l(pr'')\,\mathrm{e}^{-\mathrm{i}\hbar p^2 T/2m} \ . \tag{5.12}
$$

A separable potential has te form

$$
V = u(r) + \frac{1}{r^2}v(\phi) + w(z) \ . \tag{5.13}
$$

*5.1.4. Elliptic Cylinder Coordinates.* We consider the coordinate system

$$
\left.\begin{aligned}
x &= d\cosh\mu\cos\nu \ , & \mu &> 0 \ , \\
y &= d\sinh\mu\sin\nu \ , & -\pi &< \nu \leq \pi \\
z &= z \ , & z &\in \mathbb{R} \ .
\end{aligned}\right\} \tag{5.14}
$$

(alternatively $\mu \in \mathbb{R}$, $0 < \nu < \pi$ [84]). Here the invariant distance is

$$
d_{E^{(3)}}^2(\mathbf{q}'',\mathbf{q}') = |z'' - z'|^2 + d^2\left(\cosh\mu'\cosh\mu''\cos\nu'\cos\nu'' + \sinh\mu'\sinh\mu''\sin\nu'\sin\nu''\right) \ . \tag{5.15}
$$

The metric is $(g_{ab}) = \mathrm{diag}[d^2(\sinh^2\mu + \sin^2\nu), d^2(\sinh^2\mu + \sin^2\nu), 1]$, and we obtain for the momentum operators (4.16), and $p_z = -\mathrm{i}\hbar\partial_z$. Consequently for the Hamiltonian

$$-\frac{\hbar^2}{2m}\Delta_{E^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{1}{d^2(\sinh^2\mu + \sin^2\nu)}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right) + \frac{\partial^2}{\partial z^2}\right]$$

$$= \frac{1}{2m}\left[\frac{1}{d^2}(\sinh^2\mu + \sin^2\nu)^{-1/2}(p_\mu^2 + p_\nu^2)(\sinh^2\mu + \sin^2\nu)^{-1/2} + p_z^2\right] .$$
$$(5.16)$$

The path integral *construction* is straightforward, however, again no explicit path *integration* is possible for the coordinates $\mu$ and $\nu$. The corresponding wave-functions yield again as eigen-functions Mathieu functions. Because we know the kernel in $E^{(3)}$ in terms of the invariant distance $d_{E^{(3)}}$ we can state the following path integral identity

$$\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)d^2(\sinh^2\mu + \sin^2\nu)\int_{z(t')=z'}^{z(t'')=z''}\mathcal{D}z(t)$$

$$\times \exp\left\{\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\left[d^2(\sinh^2\mu + \sin^2\nu)(\dot\mu^2 + \dot\nu^2) + \dot z^2\right]dt\right\}$$

$$= \int_{\mathbb{R}}\frac{dp_z}{2\pi}\,\mathrm{e}^{\mathrm{i}\,p_z(z''-z')-\mathrm{i}\,\hbar p_z^2 T/2m}\frac{1}{2\pi}\sum_{\nu\in\Lambda}\int_0^\infty p\,dp\,\mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m}$$

$$\times \mathrm{me}_\nu^*(\eta', \tfrac{d^2 p^2}{4})\,\mathrm{me}_\nu(\eta'', \tfrac{d^2 p^2}{4})\,\mathrm{Me}_\nu^{(1)*}(\xi', \tfrac{d^2 p^2}{4})\,\mathrm{Me}_\nu^{(1)}(\xi'', \tfrac{d^2 p^2}{4})$$
$$(5.17)$$

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{3/2}\exp\left[\frac{\mathrm{i}\,m}{2\hbar T}d_{E^{(3)}}^2(\mathbf{q}'', \mathbf{q}')\right] ,$$
$$(5.18)$$

and $d_{E^{(3)}}$ must be taken in elliptic cylinder coordinates, and we have used the same notation as for the two-dimensional elliptic coordinates. A potential separable in these coordinates reads

$$V = \frac{u(\cosh\mu) + v(\cos\nu)}{\sinh^2\mu + \sin^2\nu} + v(z) .$$
$$(5.19)$$

*5.1.5. Parabolic Cylinder Coordinates.* The last example for cylinder coordinates in three dimensions are the parabolic cylinder coordinates

$$x = \xi\eta , \qquad y = \tfrac{1}{2}(\eta^2 - \xi^2) , \qquad \xi \in \mathbb{R}, \eta > 0 ,$$
$$z = z , \qquad\qquad\qquad\qquad z \in \mathbb{R} ,$$
$$(5.20)$$

which is the obvious generalization of the two-dimensional case. Therefore

$$d_{E^{(3)}}^2(\mathbf{q}'', \mathbf{q}') = |z'' - z'|^2$$
$$+ \frac{1}{4}\left[(\eta''^2 + \xi''^2)^2 + (\eta'^2 + \xi'^2)^2 - 2(\eta'^2 - \xi'^2)(\eta''^2 - \xi''^2) - 8\eta'\eta''\xi'\xi''\right] . \quad (5.21)$$

We have $(g_{ab}) = \mathrm{diag}(\xi^2 + \eta^2, \xi^2 + \eta^2, 1)$, hence for the momentum operators (4.23) and $p_z = -\mathrm{i}\hbar\partial_z$. This gives for the Hamiltonian

$$-\frac{\hbar^2}{2m}\Delta_{E^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{1}{\xi^2 + \eta^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + \frac{\partial^2}{\partial z^2}\right]$$

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$$= \frac{1}{2m} \left[ (\xi^2 + \eta^2)^{-1/2} (p_\xi^2 + p_\eta^2)(\xi^2 + \eta^2)^{-1/2} + p_z^2 \right] \ . \qquad (5.22)$$

The arising path integral in these coordinates must be treated similarly as in the two-dimensional case, the only difference being the $z$-path integration which is trivial. We obtain

$$\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)(\xi^2 + \eta^2) \int_{z(t')=z'}^{z(t'')=z''} \mathcal{D}z(t)$$

$$\times \exp \left\{ \frac{\mathrm{i}\,m}{2\hbar} \int_{t'}^{t''} \left[ (\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) + \dot{z}^2 \right] dt \right\}$$

$$= \int_{\mathbb{R}} \frac{dp_z}{2\pi} \, \mathrm{e}^{\mathrm{i}\,p_z(z''-z')-\mathrm{i}\,\hbar p_z^2 T/2m}$$

$$\times \sum_{e,o} \int_{\mathbb{R}} d\zeta \int_{\mathbb{R}} dp \, \mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m} \, \Psi_{p,\zeta}^{(e,o)\,*}(\xi',\eta') \Psi_{p,\zeta}^{(e,o)}(\xi'',\eta'') \ , \tag{5.23}$$

in the notation of 4.1.5. A separable potential has the form

$$V = \frac{u(\xi) + v(\eta)}{\xi^2 + \eta^2} + v(z) \ . \tag{5.24}$$

*5.1.6. Sphero-Conical Coordinates.* We consider the coordinate system

$$\left.\begin{aligned} x &= r \operatorname{sn}(\mu, k) \operatorname{dn}(\nu, k') \ , \quad r > 0 \ , \\ y &= r \operatorname{cn}(\mu, k) \operatorname{cn}(\nu, k') \ , \quad k^2 + k'^2 = 1 \ , \\ z &= r \operatorname{dn}(\mu, k) \operatorname{sn}(\nu, k') \ , \end{aligned}\right\} \tag{5.25}$$

in the notation of 4.2.3. The invariant distance is given by

$$d_{E^{(3)}}^2(\mathbf{q}'', \mathbf{q}') = r'^2 + r''^2 - 2r'r'' \Big( \operatorname{sn}\mu'' \operatorname{sn}\mu' \operatorname{dn}\nu'' \operatorname{dn}\nu'$$

$$+ \operatorname{cn}\mu'' \operatorname{cn}\mu' \operatorname{cn}\nu'' \operatorname{cn}\nu' + \operatorname{dn}\mu'' \operatorname{dn}\mu' \operatorname{sn}\nu'' \operatorname{sn}\nu' \Big) \ . \tag{5.26}$$

The metric tensor $g_{ab}$ in these coordinates has the form

$$(g_{ab}) = \operatorname{diag}\left[ 1, r^2 \left( k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu \right), r^2 \left( k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu \right) \right] \ , \tag{5.27}$$

and the momentum operators are $p_r = -\mathrm{i}\,\hbar(\partial/\partial r + 1/r)$, and $p_\mu$, $p_\nu$ as in (4.40). The path integral can now be formulated yielding

$$\int_{r(t')=r'}^{r(t'')=r''} r^2 \mathcal{D}r(t) \int_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)(k^2 \operatorname{cn}^2 \mu + k'^2 \operatorname{cn}^2 \nu)$$

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$$\times \exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\left(\dot{r}^2 + r^2(k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu)(\dot{\mu}^2 + \dot{\nu}^2)\right)dt\right]$$

$$= \sum_{l=0}^{\infty}\sum_{\kappa} A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')$$

$$\times \frac{1}{r'r''}\int_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\frac{(l+\frac{1}{2})^2-\frac{1}{4}}{r^2}\right)dt\right]$$

$$= \frac{m}{\mathrm{i}\,\hbar T\sqrt{r'r''}}\exp\left[-\frac{m}{2\,\mathrm{i}\,\hbar T}(r'^2+r''^2)\right]$$

$$\times \sum_{\kappa}\sum_{l=0}^{\infty} A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')I_{l+1/2}\left(\frac{mr'r''}{\mathrm{i}\,\hbar T}\right) \tag{5.28}$$

$$= \sum_{\kappa}\sum_{l=0}^{\infty} A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')\int_0^{\infty}\frac{p\,dp}{\sqrt{r'r''}}J_{l+\frac{1}{2}}(pr')J_{l+\frac{1}{2}}(pr'')\,\mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m} \tag{5.29}$$

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{3/2}\exp\left[-\frac{m}{2\,\mathrm{i}\,\hbar T}d_{E^{(3)}}^2(\mathbf{q}'',\mathbf{q}')\right]\,, \tag{5.30}$$

with $d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')$ in sphero-conical coordinates. Here a separable potential must have the form

$$V = u(r) + \frac{1}{r^2}\frac{v(\mathrm{cn}\,\alpha)+w(\mathrm{cn}\,\beta)}{k^2\,\mathrm{cn}^2\,\alpha + k'^2\,\mathrm{cn}^2\,\beta}\,. \tag{5.31}$$

### 5.1.7. *Spherical Coordinates.* We consider the spherical coordinates

$$\left.\begin{array}{ll} x = r\sin\theta\cos\phi\,, & r>0\,,\\ y = r\sin\theta\sin\phi\,, & 0<\theta<\pi\,,\\ z = r\cos\theta\,, & 0\le\phi<2\pi\,. \end{array}\right\} \tag{5.32}$$

These are the usual three-dimensional polar coordinates. Here

$$d_{E^{(3)}}^2(\mathbf{q}'',\mathbf{q}') = r'^2 + r''^2 - 2r'r''\Big(\cos\theta'\cos\theta'' - \sin\theta'\sin\theta''\cos(\phi''-\phi')\Big)\,. \tag{5.33}$$

The metric tensor is $(g_{ab}) = \mathrm{diag}(1,r^2,r^2\sin^2\theta)$, and the momentum operators have the form $p_r = -\mathrm{i}\,\hbar(\partial/\partial r + 1/r)$ together with (4.33). For the Hamiltonian we obtain

$$-\frac{\hbar^2}{2m}\Delta_{E^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2\right) - \frac{\hbar^2}{8mr^2}\left(1 + \frac{1}{\sin^2\theta}\right)\,. \tag{5.34}$$

The corresponding path integral is well-known [22, 37, 92] and we have the identity

$$\int_{r(t')=r'}^{r(t'')=r''} r^2\mathcal{D}r(t)\int_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\sin\theta\mathcal{D}\theta(t)\int_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathcal{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + \frac{\hbar^2}{8mr^2}\left(1 + \frac{1}{\sin^2\theta}\right)\right]dt\right\}$$

$$= \frac{m}{4\pi\,i\,\hbar T\sqrt{r'r''}}\exp\left[-\frac{m}{2\,i\,\hbar T}(r'^2 + r''^2)\right]$$

$$\times \sum_{l=0}^{\infty}(2l+1)P_l\left(\cos\psi_{S^{(2)}}(\mathbf{q''},\mathbf{q'})\right)I_{l+1/2}\left(\frac{mr'r''}{i\,\hbar T}\right) \tag{5.35}$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^{m\,*}(\theta',\phi')Y_l^m(\theta'',\phi'')\frac{1}{\sqrt{r'r''}}\int_0^{\infty}p\,dp\,J_{l+\frac{1}{2}}(pr')J_{l+\frac{1}{2}}(pr'')\,e^{-i\,\hbar p^2 T/2m} . \tag{5.36}$$

A potential which is separable in spherical coordinates reads

$$V = u(r) + \frac{1}{r^2}v(\theta) + \frac{1}{r^2\sin^2\theta}w(\phi) . \tag{5.37}$$

*5.1.8. Parabolic Coordinates.* We consider the coordinate system

$$\left.\begin{array}{ll}x = \xi\eta\cos\phi , & y = \xi\eta\sin\phi , \quad \xi,\eta > 0 , \\[4pt] z = \dfrac{1}{2}(\xi^2 - \eta^2) , & 0 \le \phi < 2\pi .\end{array}\right\} \tag{5.38}$$

This gives for the metric tensor $(g_{ab}) = \mathrm{diag}(\xi^2 + \eta^2, \xi^2 + \eta^2, \xi^2\eta^2)$, and

$$d_{E^{(3)}}^2(\mathbf{q''},\mathbf{q'}) = \frac{1}{4}\Big[(\eta''^2 + \xi''^2)^2 + (\eta'^2 + \xi'^2)^2$$
$$- 2(\eta'^2 - \xi'^2)(\eta''^2 - \xi''^2) - 8\eta'\eta''\xi'\xi''\cos(\phi'' - \phi')\Big] . \tag{5.39}$$

For the momentum operators we get

$$p_\xi = \frac{\hbar}{i}\left(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2 + \eta^2} + \frac{1}{2\xi}\right) , \qquad p_\eta = \frac{\hbar}{i}\left(\frac{\partial}{\partial\eta} + \frac{\eta}{\xi^2 + \eta^2} + \frac{1}{2\eta}\right) , \tag{5.40}$$

together with $p_\phi = -i\,\hbar\partial_\phi$. For the Hamiltonian we obtain

$$-\frac{\hbar^2}{2m}\Delta_{E^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{1}{\xi^2 + \eta^2}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\eta^2} + \frac{\partial}{\partial\eta}\right) + \frac{1}{\xi^2\eta^2}\frac{\partial^2}{\partial\phi^2}\right]$$

$$= \frac{1}{2m}\left[\frac{1}{\sqrt{\xi^2 + \eta^2}}(p_\xi^2 + p_\eta^2)\frac{1}{\sqrt{\xi^2 + \eta^2}} + \frac{1}{\xi^2\eta^2}p_\phi^2\right] - \frac{\hbar^2}{8m\xi^2\eta^2} . \tag{5.41}$$

We obtain the path integral identity [15, 50]

$$\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)(\xi^2 + \eta^2)\xi\eta\int_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathcal{D}\phi(t)$$

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$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big((\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)+\xi^2\eta^2\dot\phi^2\Big)+\frac{\hbar^2}{8m\xi^2\eta^2}\right]dt\right\}$$

$$=(\xi'\xi''\eta'\eta'')^{-1/2}\sum_{l\in\mathbb{Z}}\frac{\mathrm{e}^{-\mathrm{i}\,l(\phi''-\phi')}}{2\pi}\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)(\xi^2+\eta^2)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)-\frac{\hbar^2}{2m}\frac{l^2-\frac14}{\xi^2\eta^2}\right]dt\right\}$$

$$=\sum_{l\in\mathbb{Z}}\frac{\mathrm{e}^{-\mathrm{i}\,l(\phi''-\phi')}}{2\pi}\int_{\mathrm{I\!R}}d\zeta\int_0^\infty\frac{dp}{p}\frac{\left|\Gamma\!\left(\frac{1+|l|}{2}+\frac{\mathrm{i}\zeta}{2p}\right)\right|^4\mathrm{e}^{\pi/ap}}{4\pi^2\,\xi'\xi''\eta'\eta''\Gamma^4(1+|l|)}\,\mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m}$$

$$\times M_{-\mathrm{i}\,\zeta/2p,|l|/2}(-\mathrm{i}\,p\xi''^2)M_{\mathrm{i}\,\zeta/2p,|l|/2}(\mathrm{i}\,p\xi''^2)M_{-\mathrm{i}\,\zeta/2p,|l|/2}(-\mathrm{i}\,p\eta''^2)M_{\mathrm{i}\,\zeta/2p,|l|/2}(\mathrm{i}\,p\eta''^2)\ ,$$

$$(5.42)$$

where the last line was obtained in a similar way as the corresponding result in 4.1.5, and use has been made of (A.2.2). In parabolic coordinates a potential is separable if it has the form

$$V=\frac{u(\xi)+v(\eta)}{\sqrt{x^2+y^2+z^2}}+\frac{w(\phi)}{x^2+y^2}\ .\tag{5.43}$$

*5.1.9. Prolate Spheroidal Coordinates.* We consider the coordinate system

$$\left.\begin{aligned}x&=d\sinh\mu\sin\nu\cos\phi\ ,&\mu&>0\ ,\\y&=d\sinh\mu\sin\nu\sin\phi\ ,&0&<\nu<\pi\ ,\\z&=d\cosh\mu\cos\nu\ ,&0&\le\phi<2\pi\ .\end{aligned}\right\}\tag{5.44}$$

This yields $(g_{ab})=d^2\,\mathrm{diag}(\sinh^2\mu+\sin^2\nu,\sinh^2\mu+\sin^2\nu,\sinh^2\mu\sin^2\nu)$, and for the momentum operators we obtain

$$p_\mu=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\mu}+\frac{\sinh\mu\cosh\mu}{\sinh^2\mu+\sin^2\nu}+\frac12\coth\mu\right),\ p_\nu=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\nu}+\frac{\sin\nu\cos\nu}{\sinh^2\mu+\sin^2\nu}+\frac12\cot\nu\right),$$

$$(5.45)$$

and $p_\phi=-\mathrm{i}\,\hbar\partial_\phi$. The invariant distance is given by

$$d^2_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')=d^2\left[\cosh\mu'\cosh\mu''\cos\nu'\cos\nu''+\sinh\mu'\sinh\mu''\sin\nu'\sin\nu''\cos(\phi''-\phi')\right]\ .$$

$$(5.46)$$

The Hamiltonian has the form

$$-\frac{\hbar^2}{2m}\Delta_{E^{(3)}}$$

$$=-\frac{\hbar^2}{2md^2}\left[\frac{1}{\sinh^2\mu+\sin^2\nu}\left(\frac{\partial^2}{\partial\mu^2}+\coth\mu\frac{\partial}{\partial\mu}+\frac{\partial^2}{\partial\nu^2}+\cot\nu\frac{\partial}{\partial\nu}\right)+\frac{1}{\sinh^2\mu\sin^2\nu}\frac{\partial^2}{\partial^2\phi}\right]$$

$$=\frac{1}{2md^2}\left[\frac{1}{\sqrt{\sinh^2\mu+\sin^2\nu}}(p_\mu^2+p_\nu^2)\frac{1}{\sqrt{\sinh^2\mu+\sin^2\nu}}+\frac{1}{\sinh^2\mu+\sin^2\nu}p_\phi^2\right]$$

$$-\frac{\hbar^2}{8md^2\sinh^2\mu\sin^2\nu}\ .\tag{5.47}$$

The path integral *construction* is straightforward, however no explicit path *integration* is possible. Actually, an expansion into the corresponding wave-functions in the coordinates $\mu$ and $\nu$ yields the spheroidal functions $\mathrm{ps}_n^m(\cos\nu,\gamma)$ and $S_n^{m\,(1)}(\cosh\mu,\gamma)$ ($\gamma^2 = 2mEd^2/\hbar^2 = p^2 d^2$), respectively, as the eigen-function of the Hamiltonian, a specific class of higher transcendental functions [84], similar to the Mathieu functions. However, because we know on the one side the eigen-functions of the Hamiltonian in terms of these functions [84], and on the other the kernel in $E^{(3)}$ in terms of the invariant distance $d_{E^{(2)}}$ we can state the following path integral identity (note the implemented time-transformation)

$$
\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)\, d^3 (\sinh^2\mu + \sin^2\nu)\sinh\mu\sin\nu \int_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)
$$

$$
\times \exp\left\{ \frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[ \frac{m}{2}d^2\left( (\sinh^2\mu+\sin^2\nu)(\dot\mu^2+\dot\nu^2) + \sinh^2\mu\sin^2\nu\,\dot\phi^2 \right) \right.\right.
$$

$$
\left.\left. + \frac{\hbar^2}{8md^2\sinh^2\mu\sin^2\nu} \right]dt \right\}
$$

$$
= (\sinh\mu'\sinh\mu''\sin\nu'\sin\nu'')^{-1/2}\sum_{l\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi}
$$

$$
\times \int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}\,ET/\hbar}\int_0^\infty ds'' \int_{\mu(0)=\mu'}^{\mu(s'')=\mu''}\mathcal{D}\mu(s)\int_{\nu(0)=\nu'}^{\nu(s'')=\nu''}\mathcal{D}\nu(s)
$$

$$
\times \exp\left\{ \frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[ \frac{m}{2}(\dot\mu^2+\dot\nu^2) + Ed^2(\sinh^2\mu+\sin^2\nu) - \frac{\hbar^2}{2m}\left( \frac{l^2-\frac14}{\sinh^2\mu} + \frac{l^2-\frac14}{\sin^2\nu} \right) \right]ds \right\}
$$

$$
\tag{5.48}
$$

$$
= \sum_{l\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi}\sum_{n\in\Lambda}\frac{2}{\pi}\int_0^\infty p^2\,dp\,\frac{2n+1}{2}\frac{(n-l)!}{(n+l)!}\,\mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m}
$$

$$
\times \mathrm{ps}_n^l(\cos\nu',pd)\,\mathrm{ps}_n^l(\cos\nu'',pd)S_n^{l\,(1)\,*}(\cosh\mu',pd)S_n^{l\,(1)\,*}(\cosh\mu'',pd) \tag{5.49}
$$

$$
= \left( \frac{m}{2\pi\,\mathrm{i}\,\hbar T} \right)^{3/2}\exp\left[ \frac{\mathrm{i}\,m}{2\hbar T}d^2_{E^{(3)}}(\mathbf{q}'',\mathbf{q}') \right]\ , \tag{5.50}
$$

and $d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')$ must be taken in prolate spheroidal coordinates. The functions $\mathrm{ps}_n^l(\cos\nu,\gamma)$ and $S_n^{l\,(1)}(\cosh\mu,\gamma)$ are mutually determined by the separation parameter $\lambda = \lambda_n^l(\gamma)$ giving an infinite countable set $\{\lambda_n\}$, $(n\in\Lambda)$. The functions $\mathrm{ps}_n^l(\cos\nu,\gamma)$ yield in the limit $\gamma\to 0$ the associated Legendre-polynomials $P_n^l(\cos\nu)$, and the functions $S_n^{l\,(1)}(\cosh\mu,\gamma)$ the spherical Bessel functions, i.e. $S_n^{l\,(1)}(z/\gamma,\gamma)\propto\sqrt{\pi/2z}\,J_{l+\frac12}(z)$ ($\gamma\to 0$), therefore obeying the correct boundary-conditions. A separable potential must have the form

$$
V = \frac{u(\cosh\mu)+v(\cos\nu)}{\sinh^2\mu+\sin^2\nu} + \frac{w(\phi)}{\sinh^2\mu\sin^2\nu}\ . \tag{5.51}
$$

Note that by the replacement $z\to d(\cosh\mu\cos\nu+1)$ we obtain the spheroidal

coordinate system in which the Coulomb-problem in $\mathbb{R}^3$ is separable [78], e.g. c.f. [50] and references therein.

*5.1.10. Oblate Spheroidal Coordinates.* We consider the coordinate system

$$
\left.
\begin{aligned}
x &= d \cosh \xi \sin \nu \cos \phi \ , & \xi &> 0 \ , \\
y &= d \cosh \xi \sin \nu \sin \phi \ , & 0 &< \nu < \pi \ , \\
z &= d \sinh \xi \cos \nu \ , & 0 &\le \phi < 2\pi
\end{aligned}
\right\}
\tag{5.52}
$$

(alternatively $\mu \in \mathbb{R}$, $0 < \nu < \pi/2$ [84]). This yields $(g_{ab}) = d^2 \operatorname{diag}(\cosh^2 \xi - \sin^2 \nu, \cosh^2 \xi - \sin^2 \nu, \cosh^2 \xi \sin^2 \nu)$, and for the momentum operators we obtain

$$
p_\xi = \frac{\hbar}{\mathrm{i}} \left( \frac{\partial}{\partial \mu} + \frac{\sinh \xi \cosh \xi}{\cosh^2 \xi - \sin^2 \nu} + \frac{1}{2} \tanh \xi \right), \ p_\nu = \frac{\hbar}{\mathrm{i}} \left( \frac{\partial}{\partial \nu} + \frac{\sin \nu \cos \nu}{\cosh^2 \xi - \sin^2 \nu} + \frac{1}{2} \cot \nu \right),
\tag{5.53}
$$

and $p_\phi = -\mathrm{i}\,\hbar \partial_\phi$. The invariant distance is given by

$$
d^2_{E^{(3)}}(\mathbf{q}'', \mathbf{q}') = d^2 \left[ \sinh \xi' \sinh \xi'' \cos \nu' \cos \nu'' + \cosh \xi' \cosh \xi'' \sin \nu' \sin \nu'' \cos(\phi'' - \phi') \right] \ .
\tag{5.54}
$$

The Hamiltonian has the form

$$
-\frac{\hbar^2}{2m} \Delta_{E^{(3)}}
$$

$$
= -\frac{\hbar^2}{2md^2} \left[ \frac{1}{\cosh^2 \xi - \sin^2 \nu} \left( \frac{\partial^2}{\partial \xi^2} + \tanh \xi \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \nu^2} + \cot \nu \frac{\partial}{\partial \nu} \right) + \frac{1}{\cosh^2 \xi \sin^2 \nu} \frac{\partial^2}{\partial^2 \phi} \right]
$$

$$
= \frac{1}{2md^2} \left[ \frac{1}{\sqrt{\cosh^2 \xi - \sin^2 \nu}} (p_\xi^2 + p_\nu^2) \frac{1}{\sqrt{\cosh^2 \xi - \sin^2 \nu}} + \frac{1}{\cosh^2 \xi - \sin^2 \nu} p_\phi^2 \right]
$$

$$
- \frac{\hbar^2}{8md^2 \cosh^2 \xi \sin^2 \nu} \ .
\tag{5.55}
$$

The path integral construction is *straightforward*, however again no explicit *path integration* is possible. An expansion into the corresponding wave-functions in the coordinates $\xi$ and $\nu$ yields eigen-function of the Hamiltonian in terms of spheroidal functions $\operatorname{ps}_n^m(\cos \nu, \mathrm{i}\,\gamma) = \operatorname{psi}_n^m(\cos \nu, \gamma)$ and $S_n^{m\,(1)}(-\mathrm{i}\cosh \xi, \mathrm{i}\,\gamma) = \operatorname{Si}_n^{m\,(1)}(\cosh \xi, \gamma)$ $(\gamma^2 = 2mEd^2/\hbar^2 = pd)$, respectively. Because we know on the one side the eigen-functions of the Hamiltonian in terms of these functions [84], and on the other the kernel in $E^{(3)}$ in terms of the invariant distance $d_{E^{(3)}}$ we can state the following path integral identity

$$
\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) d^3 (\cosh^2 \xi - \sin^2 \nu) \cosh \xi \sin \nu \int_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)
$$

$$
\times \exp \left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} d^2 \left( (\cosh^2 \xi - \sin^2 \nu)(\dot\xi^2 + \dot\nu^2) + \cosh^2 \xi \sin^2 \nu \dot\phi^2 \right) \right. \right.
$$

$$
\left. \left. + \frac{\hbar^2}{8md^2 \cosh^2 \xi \sin^2 \nu} \right] dt \right\}
$$

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$$
= (\cosh \xi' \cosh \xi'' \sin \nu' \sin \nu'')^{-1/2} \sum_{l \in \mathbb{Z}} \frac{e^{\mathrm{i}\, l(\phi'' - \phi')}}{2\pi}
$$

$$
\times \int_{\mathbb{R}} \frac{dE}{2\pi\hbar}\, e^{-\mathrm{i}\, ET/\hbar} \int_0^\infty ds'' \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s) \int\limits_{\nu(0)=\nu'}^{\nu(s'')=\nu''} \mathcal{D}\nu(s)
$$

$$
\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[ \frac{m}{2}(\dot{\xi}^2 + \dot{\nu}^2) + E d^2(\cosh^2 \xi - \sin^2 \nu) - \frac{\hbar^2}{2m}\left( \frac{l^2 - \frac14}{\cosh^2 \xi} - \frac{l^2 - \frac14}{\sin^2 \nu} \right) \right] ds \right\}
\tag{5.56}
$$

$$
= \sum_{l \in \mathbb{Z}} \frac{e^{\mathrm{i}\, l(\phi'' - \phi')}}{2\pi} \sum_{n \in \Lambda} \frac{2}{\pi} \int_0^\infty p^2\, dp\, \frac{2n+1}{2} \frac{(n-l)!}{(n+l)!}\, e^{-\mathrm{i}\,\hbar p^2 T / 2m}
$$

$$
\times \mathrm{psi}_n^l(\cos \nu', pd)\, \mathrm{psi}_n^l(\cos \nu'', pd)\, \mathrm{Si}_n^{l\,(1)\,*}(\cosh \xi', pd)\, \mathrm{Si}_n^{l\,(1)\,*}(\cosh \xi'', pd)
\tag{5.57}
$$

$$
= \left( \frac{m}{2\pi \mathrm{i}\,\hbar T} \right)^{3/2} \exp\left[ \frac{\mathrm{i}\, m}{2\hbar T} d^2_{E^{(3)}}(\mathbf{q}'', \mathbf{q}') \right] ,
\tag{5.58}
$$

and $d_{E^{(3)}}(\mathbf{q}'', \mathbf{q}')$ must be taken in oblate spheroidal coordinates. Due to the construction of the spectral expansion the functions $\mathrm{psi}_n^l(\cos \nu, \gamma)$ and $\mathrm{Si}_n^{l\,(1)}(\cosh \mu, \gamma)$ are obeying the correct boundary-conditions. Here a separable potential must have the form

$$
V = \frac{u(\sinh \mu) + v(\cos \nu)}{\cosh^2 \mu - \sin^2 \nu} + \frac{w(\phi)}{\cosh^2 \mu \sin^2 \nu} .
\tag{5.59}
$$

5.1.11. *Ellipsoidal Coordinates.* The last two coordinate system are the most complicated ones and are similar in some of their features, c.f. Section 3.1. First we consider the coordinate system

$$
\left.
\begin{aligned}
& x = \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{a^2(a^2 - b^2)}} , \quad y = \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{b^2(b^2 - a^2)}} , \\
& z = \frac{\xi_1 \xi_2 \xi_3}{ab} , \quad \xi_1^2 \geq a^2 \geq \xi_2^2 \geq b^2 \geq \xi_3^2 \geq c^2 = 0 .
\end{aligned}
\right\}
\tag{5.60}
$$

The metric tensor is given by

$$
(g_{ab}) = \mathrm{diag}\left( \frac{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}{P_2(\xi_1^2)}, \frac{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}{P_2(\xi_2^2)}, \frac{(\xi_3^2 - \xi_1^2)(\xi_3^2 - \xi_2^2)}{P_2(\xi_3^2)} \right) ,
\tag{5.61}
$$

with $P_2(\xi^2) = (\xi^2 - a^2)(\xi^2 - b^2)$. With the identification $\xi_1 = a\, \mathrm{dn}(\lambda, k) / \mathrm{cn}(\lambda, k)$, $\xi_2 = a\, \mathrm{dn}(\mu, k')$ and $\xi_3 = b\, \mathrm{sn}(\nu, k)$ $(b = ka, \sqrt{a^2 - b^2} = k'a = d)$ this can be rewritten into

$$
\left.
\begin{aligned}
& x = d \frac{\mathrm{sn}(\lambda, k)\, \mathrm{sn}(\mu, k')\, \mathrm{dn}(\nu, k)}{\mathrm{cn}(\lambda, k)} , \quad y = d \frac{\mathrm{cn}(\mu, k')\, \mathrm{cn}(\nu, k)}{\mathrm{cn}(\lambda, k)} , \\
& z = a \frac{\mathrm{dn}(\lambda, k)\, \mathrm{dn}(\mu, k')\, \mathrm{sn}(\nu, k)}{\mathrm{cn}(\lambda, k)} .
\end{aligned}
\right\}
\tag{5.62}
$$

Due to the very complicated structure it is of now use to write down all subsequent necessary quantities for the path integral. Instead we exploit the results of Section 3.2, in particular the separation formula (3.21). We identify $(g_{ab}) \equiv \mathrm{diag}(h_1^2, h_2^2, h_3^2)$, furthermore

$$\Gamma_i = \frac{\xi_i P_2'(\xi_i^2)}{P_2(\xi^2)} \ , \qquad (i = 1, 2, 3) \tag{5.63}$$

$$S = \begin{vmatrix} 1 & 1/(\xi_1^2 - a^2) & 1/(\xi_1^2 - b^2)(a^2 - b^2) \\ 1 & 1/(\xi_2^2 - a^2) & 1/(\xi_2^2 - b^2)(a^2 - b^2) \\ 1 & 1/(\xi_3^2 - a^2) & 1/(\xi_3^2 - b^2)(a^2 - b^2) \end{vmatrix} \ , \tag{5.64}$$

$$\left.\begin{aligned}
M_1 &= \frac{1}{a^2 - b^2} \left( \frac{1}{(\xi_2^2 - a^2)(\xi_3^2 - b^2)} - \frac{1}{(\xi_2^2 - b^2)(\xi_3^2 - a^2)} \right) \ , \\
M_2 &= \frac{1}{a^2 - b^2} \left( \frac{1}{(\xi_1^2 - b^2)(\xi_3^2 - a^2)} - \frac{1}{(\xi_1^2 - a^2)(\xi_3^2 - b^2)} \right) \ , \\
M_3 &= \frac{1}{a^2 - b^2} \left( \frac{1}{(\xi_1^2 - a^2)(\xi_2^2 - b^2)} - \frac{1}{(\xi_1^2 - b^2)(\xi_2^2 - a^2)} \right) \ .
\end{aligned}\right\} \tag{5.65}$$

and obtain the following path integral identity

$$\prod_{i=1}^{3} \int_{\xi_i(t')=\xi_i'}^{\xi_i(t'')=\xi_i''} h_i \mathcal{D}\xi_i(t) \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \sum_{i=1}^{3} h_i^2 \dot\xi_i^2 - \Delta V_{PF}(\{\xi\}) \right] dt \right\}$$

$$= \prod_{i=1}^{3} \int_{\xi_i(t')=\xi_i'}^{\xi_i(t'')=\xi_i''} \sqrt{\frac{S}{M_i}} \mathcal{D}\xi_i(t) \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} S \sum_{i=1}^{3} \frac{\dot\xi_i^2}{M_i} - \Delta V_{PF}(\{\xi\}) \right] dt \right\}$$

$$= (S'S'')^{-1/4} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_0^\infty ds'' \prod_{i=1}^{3} \int_{\xi_i(0)=\xi_i'}^{\xi_i(s'')=\xi_i''} M_i^{-1/2} \mathcal{D}\xi_i(s)$$

$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[ \frac{m}{2} \sum_{i=1}^{3} \frac{\dot\xi_i^2}{M_i} + ES - \frac{\hbar^2}{8m} \sum_{i=1}^{3} M_i \left( \Gamma_i^2 + 2\Gamma_i' \right) \right] ds \right\} \tag{5.66}$$

$$= \int_0^\infty dp \sum_{\kappa, \lambda} \Psi_{\kappa,\lambda,p}^*(\xi_1', \xi_2', \xi_3') \Psi_{\kappa,\lambda,p}(\xi_1'', \xi_2'', \xi_3'') \, \mathrm{e}^{-\mathrm{i}\hbar p^2 T/2m} \tag{5.67}$$

$$= \left( \frac{m}{2\pi \mathrm{i}\hbar T} \right)^{3/2} \exp\left[ -\frac{m}{2\mathrm{i}\hbar T} d_{E^{(3)}}^2(\mathbf{q}'', \mathbf{q}') \right] \ , \tag{5.68}$$

and $d_{E^{(3)}}(\mathbf{q}'', \mathbf{q}')$ must be expressed in ellipsoidal coordinates via $d_{E^{(3)}} = |\mathbf{x}(\{\xi''\} - \mathbf{x}(\{\xi'\})|$. The functions $\Psi_{\kappa,\lambda,p}(\xi_1, \xi_2, \xi_3) = A_{\kappa,\lambda,p}(\xi_1) B_{\kappa,\lambda,p}(\xi_2) C_{\kappa,\lambda,p}(\xi_3)$ are solutions of a three-fold Lamé equation with separation parameters $\lambda, \kappa, p$, respectively [87, p.1305]. A potential separable in ellipsoidal coordinates must have the form

$$V = \frac{(\xi_2^2 - \xi_3^2)u(\xi_1) + (\xi_1^2 - \xi_3^2)u(\xi_2) + (\xi_1^2 - \xi_2^2)u(\xi_3)}{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)(\xi_2^2 - \xi_3^2)} \ . \tag{5.69}$$

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*5.1.12. Paraboloidal Coordinates.* As the last coordinate system in $E^{(3)}$ we now consider

$$
\left.
\begin{aligned}
x &= \sqrt{\frac{(\xi_1^2 - a^2)(\xi_2^2 - a^2)(\xi_3^2 - a^2)}{(a^2 - b^2)}} \quad, \qquad y = \sqrt{\frac{(\xi_1^2 - b^2)(\xi_2^2 - b^2)(\xi_3^2 - b^2)}{(b^2 - a^2)}} \quad, \\
z &= \frac{1}{2}\left(\xi_1^2 + \xi_2^2 + \xi_3^2 - a^2 - b^2\right) \quad, \quad \xi_1^2 \geq a^2 \geq \xi_2^2 \geq b^2 \geq \xi_3^2 \geq c^2 = 0 \quad.
\end{aligned}
\right\}
$$
(5.70)

The metric tensor is given by

$$
(g_{ab}) = \mathrm{diag}\left(\xi_1^2 \frac{(\xi_1^2 - \xi_2^2)(\xi_1^2 - \xi_3^2)}{P_2(\xi_1^2)}, \xi_2^2 \frac{(\xi_2^2 - \xi_1^2)(\xi_2^2 - \xi_3^2)}{P_2(\xi_2^2)}, \xi_3^2 \frac{(\xi_3^2 - \xi_1^2)(\xi_3^2 - \xi_2^2)}{P_2(\xi_3^2)}\right) \quad.
$$
(5.71)

Similarly as before we can make the identification $\xi_1 = a\,\mathrm{dn}(\lambda, k)/\mathrm{cn}(\lambda, k)$, $\xi_2 = a\,\mathrm{dn}(\nu, k')$ and $\xi_3 = a\sqrt{1 - k'^2}/\mathrm{cn}^2(\mu, k)$ $(b = ka, \sqrt{a^2 - b^2} = k'a = \sqrt{d})$ and (5.70) can be rewritten into

$$
\left.
\begin{aligned}
x &= d\frac{\mathrm{sn}(\lambda, k)\,\mathrm{sn}(\nu, k')}{\mathrm{cn}(\lambda, k)\,\mathrm{cn}(\mu, k)} \quad, \qquad y = d\frac{\mathrm{sn}(\mu, k)\,\mathrm{cn}(\nu, k')}{\mathrm{cn}(\lambda, k)\,\mathrm{cn}(\mu, k)} \quad, \\
z &= \frac{d}{2}\left[\frac{\mathrm{sn}^2(\lambda, k)}{\mathrm{cn}^2(\lambda, k)} - \frac{\mathrm{sn}^2(\mu, k)}{\mathrm{cn}^2(\mu, k)} + \frac{\mathrm{dn}^2(\nu, k')}{k'^2}\right] \quad.
\end{aligned}
\right\}
$$
(5.72)

We proceed similarly as before and use (3.21) for the path integral formulation. We identify $(g_{ab}) \equiv \mathrm{diag}(h_1^2, h_2^2, h_3^2)$, furthermore

$$
\Gamma_i = \frac{\xi_i P_2'(\xi_i^2)}{P_2(\xi_i^2)} - \frac{1}{\xi_i} \quad, \qquad (i = 1, 2, 3)
$$
(5.73)

$$
S = \begin{vmatrix}
\xi_1^2 & \xi_1^2/(\xi_1^2 - a^2) & \xi_1^2/(\xi_1^2 - b^2)(a^2 - b^2) \\
\xi_2^2 & \xi_2^2/(\xi_2^2 - a^2) & \xi_2^2/(\xi_2^2 - b^2)(a^2 - b^2) \\
\xi_3^2 & \xi_3^2/(\xi_3^2 - a^2) & \xi_3^2/(\xi_3^2 - b^2)(a^2 - b^2)
\end{vmatrix} \quad,
$$
(5.74)

$$
\left.
\begin{aligned}
M_1 &= \frac{\xi_2^2 \xi_3^2}{a^2 - b^2}\left(\frac{1}{(\xi_2^2 - a^2)(\xi_3^2 - b^2)} - \frac{1}{(\xi_2^2 - b^2)(\xi_3^2 - a^2)}\right) \quad, \\
M_2 &= \frac{\xi_1^2 \xi_3^2}{a^2 - b^2}\left(\frac{1}{(\xi_1^2 - b^2)(\xi_3^2 - a^2)} - \frac{1}{(\xi_1^2 - a^2)(\xi_3^2 - b^2)}\right) \quad, \\
M_3 &= \frac{\xi_1^2 \xi_2^2}{a^2 - b^2}\left(\frac{1}{(\xi_1^2 - a^2)(\xi_2^2 - b^2)} - \frac{1}{(\xi_1^2 - b^2)(\xi_2^2 - a^2)}\right) \quad.
\end{aligned}
\right\}
$$
(5.75)

and obtain the following path integral identity

$$
\prod_{i=1}^{3}\int_{\xi_i(t')=\xi_i'}^{\xi_i(t'')=\xi_i''} h_i \mathcal{D}\xi_i(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\sum_{i=1}^{3}h_i^2\dot{\xi}_i^2 - \Delta V_{PF}(\{\xi\})\right]dt\right\}
$$

$$
= \prod_{i=1}^{3}\int_{\xi_i(t')=\xi_i'}^{\xi_i(t'')=\xi_i''} M_i^{-1/2}\mathcal{D}\xi_i(s)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}S\sum_{i=1}^{3}\frac{\dot{\xi}_i^2}{M_i} - \Delta V_{PF}(\{\xi\})\right]dt\right\}
$$

$$= (S'S'')^{-1/4} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}\,ET/\hbar} \int_0^\infty ds'' \prod_{i=1}^3 \int_{\xi_i(0)=\xi_i'}^{\xi_i(s'')=\xi_i''} M_i^{-1/2}\, \mathcal{D}\xi_i(s)$$

$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[ \frac{m}{2}\sum_{i=1}^3 \frac{\dot{\xi}_i^2}{M_i} + ES - \frac{\hbar^2}{8m}\sum_{i=1}^3 M_i\left(\Gamma_i^2 + 2\Gamma_i'\right) \right] ds \right\} \tag{5.76}$$

$$= \int_0^\infty dp \sum_{\kappa,\lambda} \Psi_{\kappa,\lambda,p}^*(\xi_1',\xi_2',\xi_3')\Psi_{\kappa,\lambda,p}(\xi_1'',\xi_2'',\xi_3'')\,\mathrm{e}^{-\mathrm{i}\,\hbar p^2 T/2m} \tag{5.77}$$

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{3/2} \exp\left[ -\frac{m}{2\,\mathrm{i}\,\hbar T}d_{E^{(3)}}^2(\mathbf{q}'',\mathbf{q}') \right] \;, \tag{5.78}$$

and $d_{E^{(3)}}(\mathbf{q}'',\mathbf{q}')$ must be expressed in paraboloidal coordinates via $d_{E^{(3)}} = |\mathbf{x}(\{\xi''\} - \mathbf{x}(\{\xi'\})|$. The functions $\Psi_{\kappa,\lambda,p}(\xi_1,\xi_2,\xi_3) = A_{\kappa,\lambda,p}(\xi_1)B_{\kappa,\lambda,p}(\xi_2)C_{\kappa,\lambda,p}(\xi_3)$ are solutions of a three-fold Lamé equation with separation parameters $\lambda,\kappa,p$ [87, p.1305]. A potential separable in ellipsoidal coordinates has the form of (5.69).

## 5.2. The Sphere $S^{(3)}$

### 5.2.1. General Form of the Propagator and the Green Function.
We consider the sphere $S^{(3)}$ as the other space of constant positive curvature. There exist six coordinate systems which admit separation of variables on $S^{(3)}$ which will be discussed in the following. The propagator and the Green function on $S^{(2)}$ are best calculated in terms of polar coordinates. One obtains [12, 59, 99]

$$K^{S^{(3)}}\left(\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}'); T\right) = \frac{\mathrm{e}^{\mathrm{i}\,\hbar T/2m}}{4\pi^2} \frac{\mathrm{d}}{\mathrm{d}\cos\psi_{S^{(3)}}} \theta_3\left( \frac{\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}')}{2} \middle| -\frac{\hbar T}{2\pi m} \right) \;, \tag{5.79}$$

$$G^{S^{(3)}}\left(\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}'); E\right) = \frac{m}{2\pi\hbar^2} \frac{\sin\left[\left(\pi - \psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}')\right)(a+\frac{1}{2})\right]}{\sin[\pi(a+\frac{1}{2})]\sin\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}')} \;. \tag{5.80}$$

where $a = -\frac{1}{2} + \sqrt{2mE/\hbar^2 + 1}$, and $\theta_3(z|\tau)$ is a Jacobi theta-function [41, p.931]

$$\theta_3(u,q) = \theta_3(u|\tau) = 1 + 2\sum_{n=1}^\infty q^{n^2}\cos(2\pi n u) \;, \qquad (q = \mathrm{e}^{\mathrm{i}\,\pi\tau}) \;. \tag{5.81}$$

### 5.2.2. Spherical Cylinder Coordinates.
We first consider the coordinate system

$$\left.\begin{aligned}
s_0 &= \cos\theta\cos\phi_1 \;, & 0 < \theta < \pi/2 \;, \\
s_1 &= \cos\theta\sin\phi_1 \;, & \\
s_2 &= \sin\theta\cos\phi_2 \;, & 0 \le \phi_{1,2} < 2\pi \;, \\
s_3 &= \sin\theta\sin\phi_2 \;.
\end{aligned}\right\} \tag{5.82}$$

The metric reads $(g_{ab}) = \mathrm{diag}(1, \cos^2\theta, \sin^2\theta)$, and the invariant distance is given by

$$\cos\psi_{S^{(3)}}(\mathbf{q}'', \mathbf{q}') = \cos\theta'\cos\theta''\cos(\phi_1'' - \phi_1') + \sin\theta'\sin\theta''\cos(\phi_2'' - \phi_2') \ . \qquad (5.83)$$

The momentum operators are

$$p_\theta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\theta} + \cot\theta - \tan\theta\right) \ , \qquad (5.84)$$

and $p_{\phi_{1,2}} = -\mathrm{i}\,\hbar\partial_{\phi_{1,2}}$. Therefore for the Hamiltonian

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\Delta_{S^{(3)}} &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\theta^2} + 2(\cot\theta - \tan\theta)\frac{\partial}{\partial\theta} + \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\phi_1^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi_2^2}\right] \\
&= \frac{1}{2m}\left(p_\theta^2 + \frac{p_{\phi_1}^2}{\cos^2\theta} + \frac{p_{\phi_2}^2}{\sin^2\theta}\right) - \frac{\hbar^2}{8m}\left(4 + \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) \ .
\end{aligned}
\qquad (5.85)
$$

We obtain for the path integral

$$
\int_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \sin\theta\cos\theta\,\mathcal{D}\theta(t) \int_{\phi_1(t')=\phi_1'}^{\phi_1(t'')=\phi_1''} \mathcal{D}\phi_1(t) \int_{\phi_2(t')=\phi_2'}^{\phi_2(t'')=\phi_2''} \mathcal{D}\phi_2(t)
$$

$$
\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot\theta^2 + \cos^2\theta\,\dot\phi_1^2 + \sin^2\theta\,\dot\phi_2^2\right) + \frac{\hbar^2}{8m}\left(4 + \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right)\right]dt\right\}
$$

$$
= (\sin\theta'\sin\theta''\cos\theta'\cos\theta'')^{-1/2}\sum_{l,k\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}[l(\phi_1''-\phi_1')+k(\phi_2''-\phi_2')]}}{4\pi^2}
$$

$$
\times \int_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathcal{D}\theta(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot\theta^2 - \frac{\hbar^2}{2m}\left(\frac{l^2-\frac14}{\cos^2\theta} + \frac{k^2-\frac14}{\sin^2\theta}\right)\right]dt + \frac{\mathrm{i}\,\hbar T}{2m}\right\}
$$

$$
= \sum_{l,k\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}[l(\phi_1''-\phi_1')+k(\phi_2''-\phi_2')]}}{4\pi^2}\sum_{n=0}^{\infty}2(|l|!+|k|!+2n+1)\frac{n!\,\Gamma(|l|+|k|+n+1)}{\Gamma(|l|+n+1)\Gamma(|k|+n+1)}
$$

$$
\times (\sin\theta'\sin\theta'')^{|k|}(\cos\theta'\cos\theta'')^{|l|}P_n^{(|k|,|l|)}(1 - 2\sin^2\theta')P_n^{(|k|,|l|)}(1 - 2\sin^2\theta'')
$$

$$
\times \exp\left\{-\frac{\mathrm{i}\,\hbar T}{2m}\left[(2n+|l|+|k|+1)^2 - 1\right]\right\} \ . \qquad (5.86)
$$

*5.2.3. Sphero-Conical Coordinates.* We consider the coordinate system

$$
\left.
\begin{aligned}
s_0 &= \cos\theta \ , & -\pi/2 < \theta < \pi/2 \ , \\
s_1 &= \sin\theta\,\mathrm{sn}(\mu,k)\,\mathrm{dn}(\nu,k') \ , & k^2 + k'^2 = 1 \ , \\
s_2 &= \sin\theta\,\mathrm{cn}(\mu,k)\,\mathrm{cn}(\nu,k') \ , & \\
s_3 &= \sin\theta\,\mathrm{dn}(\mu,k)\,\mathrm{sn}(\nu,k') \ , &
\end{aligned}
\right\}
\qquad (5.87)
$$

in the notation of 4.2.3. The metric has the form $(g_{ab}) = \mathrm{diag}[1, \sin^2\theta(k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu), \sin^2\theta(k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu)]$ and

$$\cos\psi_{S^{(3)}}(\mathbf{q}'', \mathbf{q}') = \cos\theta'\cos\theta'' - \sin\theta'\sin\theta''$$
$$\times \left[\mathrm{sn}\,\mu''\,\mathrm{sn}\,\mu'\,\mathrm{dn}\,\nu''\,\mathrm{dn}\,\nu' + \mathrm{cn}\,\mu''\,\mathrm{cn}\,\mu'\,\mathrm{cn}\,\nu''\,\mathrm{cn}\,\nu' + \mathrm{dn}\,\mu''\,\mathrm{dn}\,\mu'\,\mathrm{sn}\,\nu''\,\mathrm{sn}\,\nu'\right] \ . \tag{5.88}$$

For the momentum operators we have

$$p_\theta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\theta} + \cot\theta\right) \ , \tag{5.89}$$

together with $p_\mu$ and $p_\nu$ as in (4.40). We have for the Hamiltonian

$$-\frac{\hbar^2}{2m}\Delta_{S^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\theta^2} - 2\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta(k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu)}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right)\right]$$
$$= \frac{1}{2m}\left(p_\theta^2 + \frac{1}{\sqrt{k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu}}\frac{p_\mu^2 + p_\nu^2}{\sin^2\theta}\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu}}\right) - \frac{\hbar^2}{2m} \ . \tag{5.90}$$

The path integral can therefore be written down yielding

$$\int_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\sin^2\theta\mathcal{D}\theta(t)\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)(k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot\theta^2 + \sin^2\theta(k^2\,\mathrm{cn}^2\,\mu + k'^2\,\mathrm{cn}^2\,\nu)(\dot\mu^2 + \dot\nu^2)\right) + \frac{\hbar^2}{2m}\right]dt\right\}$$

$$= (\sin\theta'\sin\theta'')^{-1/2}\,\mathrm{e}^{-\mathrm{i}\hbar T/2m}\sum_\kappa\sum_{l=0}^\infty A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')$$

$$\times\int_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\mathcal{D}\theta(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot\theta^2 - \frac{\hbar^2}{2m}\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{\cos^2\theta}\right)dt\right]$$

$$= \sum_\kappa\sum_{l=0}^\infty A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')$$

$$\times\sum_{N=0}^\infty(N+1)\frac{\Gamma(N+l+2)}{(N-l)!}P_{N+1/2}^{-l-1/2}(\sin\theta'')P_{N+1/2}^{-l-1/2}(\sin\theta')\exp\left[-\frac{\mathrm{i}\hbar T}{2m}N(N+2)\right] \ , \tag{5.91}$$

in the notation of 4.2.3, and we have set $N = n + l$, $n \in \mathbb{N}_0$.

5.2.4. *Spherical Coordinates.* We consider the usual three-dimensional polar coordinates

$$\left.\begin{aligned}
s_0 &= \cos\theta_1 \ , & 0 &< \theta_{1,2} < \pi \ , \\
s_1 &= \sin\theta_1\sin\theta_2\sin\phi \ , & 0 &\le \phi < 2\pi \ , \\
s_2 &= \sin\theta_1\sin\theta_2\cos\phi \ , & & \\
s_3 &= \sin\theta_1\cos\theta_2 \ . & &
\end{aligned}\right\} \tag{5.92}$$

Here we have $(g_{ab}) = \mathrm{diag}(1, \sin^2\theta_1, \sin^2\theta_1\sin^2\theta_2)$ and

$$\cos\psi_{S^{(3)}}(\mathbf{q}'', \mathbf{q}') = \cos\theta_1'\cos\theta_1'' + \sin\theta_1'\sin\theta_1''\Big(\cos\theta_2'\cos\theta_2'' + \sin\theta_2'\sin\theta_2''\cos(\phi'' - \phi')\Big)\ .$$
(5.93)

For the momentum operators we have (5.89) for $p_{\theta_1}$, and for $p_{\theta_2}$ and $p_\phi$ as in (4.33). For the Hamiltonian we obtain

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\Delta_{S^{(3)}} &= \frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\theta_1^2} + 2\cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\left(\frac{\partial^2}{\partial\theta_2^2} + \cot\theta_2\frac{\partial}{\partial\theta_2} + \frac{1}{\sin^2\theta_2}\frac{\partial^2}{\partial\phi^2}\right)\right]\\
&= \frac{1}{2m}\left(p_{\theta_1}^2 + \frac{p_\theta^2}{\sin^2\theta_1} + \frac{p_\phi^2}{\sin^2\theta_1\sin^2\theta_2}\right) - \frac{\hbar^2}{8m}\left(4 + \frac{1}{\sin^2\theta_1} + \frac{1}{\sin^2\theta_1\sin^2\theta_2}\right)\ .
\end{aligned}
$$
(5.94)

This gives the path integral formulation

$$
\int_{\theta_1(t')=\theta_1'}^{\theta_1(t'')=\theta_1''}\sin^2\theta_1\,\mathcal{D}\theta_1(t)\int_{\theta_2(t')=\theta_2'}^{\theta_2(t'')=\theta_2''}\sin\theta_2\,\mathcal{D}\theta_2(t)\int_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathcal{D}\phi(t)
$$

$$
\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot\theta_1^2 + \sin^2\theta_1\dot\theta_2^2 + \sin^2\theta_1\sin^2\theta_2\dot\phi^2\right)\right.\right.
$$

$$
\left.\left. + \frac{\hbar^2}{8m}\left(4 + \frac{1}{\sin^2\theta} + \frac{1}{\sin^2\theta_1\sin^2\theta_2}\right)\right]dt\right\}
$$

$$
= \frac{1}{2\pi^2}\sum_{l=0}^\infty (l+1)C_l^1\big(\cos\psi_{S^{(3)}}(\mathbf{q}'', \mathbf{q}')\big)\exp\left[-\frac{\mathrm{i}\hbar T}{2m}l(l+2)\right]
$$
(5.95)

$$
= \sum_{m_1, m_2}\sum_{l=0}^\infty \Psi_{l,m_1,m_2}^{S^{(3)}\,*}(\theta_1', \theta_2', \phi')\Psi_{l,m_1,m_2}^{S^{(3)}}(\theta_1'', \theta_2'', \phi'')\exp\left[-\frac{\mathrm{i}\hbar T}{2m}l(l+2)\right]\ ,
$$
(5.96)

where the wave-functions are given by [28, p.240]

$$
\Psi_{l,m_1,m_2}^{S^{(3)}}(\theta_1, \theta_2, \phi) = N_{S^{(3)}}^{-1/2}\,\mathrm{e}^{\mathrm{i}\,m_2\phi}(\sin\theta_1)^{m_1}(\sin\theta_2)^{m_2}C_{l-m_1}^{m_1+2}(\cos\theta_1)C_{m_1-m_2}^{m_2+\frac{3}{2}}(\cos\theta_2),
$$
(5.97a)

$$
N_{S^{(3)}} = \frac{2\pi^3 2^{-1-2m_1-2m_2}}{(l+1)(m_1+\frac{3}{2}(l-m_1)!(m_1-m_2)!}\frac{\Gamma(l+m_1+2)\Gamma(m_1+m_2+1)}{\Gamma^2(m_1+1)\Gamma^2(m_2+\frac{3}{2})}\ .
$$
(5.97b)

5.2.5. *Twofold Confocal Ellipsoidal Coordinates 1).* We consider the two-fold confocal ellipsoidal coordinates defined by

$$
\frac{s_1^2 + s_3^2}{\rho_i - b} + \frac{s_2^2}{\rho_i - a} + \frac{s_0^3}{\rho_i - c} = 0\ , \qquad (i = 1, 2)\ ,
$$
(5.98)

where $c < \rho_2 < b < \rho_1 < a$. The corresponding metric reads

$$\frac{ds^2}{dt^2} = -\frac{1}{4}(\rho_1 - \rho_2)\left(\frac{\dot{\rho}_1^2}{P_3(\rho_1)} - \frac{\dot{\rho}_2^2}{P_3(\rho_2)}\right) - (\rho_1 - b)(\rho_2 - b)\dot{\rho}_3^2 \ , \qquad (5.99)$$

where $P_3(\rho) = (\rho - a)(\rho - b)(\rho - c)$. Using the notations of the sphero-conical coordinates we can identify

$$\left.\begin{aligned}
s_0 &= \operatorname{sn}(\mu, k)\operatorname{dn}(\nu, k') \ , && k^2 + k'^2 = 1 \ , \\
s_1 &= \operatorname{dn}(\mu, k)\operatorname{sn}(\nu, k')\cos\phi \ , && 0 \le \phi < 2\pi \ , \\
s_2 &= \operatorname{dn}(\mu, k)\operatorname{sn}(\nu, k')\sin\phi \ , && \\
s_3 &= \operatorname{cn}(\mu, k)\operatorname{cn}(\nu, k') \ .
\end{aligned}\right\} \qquad (5.100)$$

We omit details in the following. Obviously, these coordinates correspond to sphero-conical coordinates with an additional circular coordinate. This gives the path integral formulation

$$\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)(k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu)\sqrt{-k'^2}\,\operatorname{dn}\mu\operatorname{sn}\mu \int_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left((k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu)(\dot{\mu}^2 + \dot{\nu}^2) - k'^2\operatorname{dn}^2\mu\operatorname{sn}^2\nu\dot{\phi}^2\right) - \Delta V_{PF}\right]dt\right\}$$

$$= \frac{\mathrm{e}^{i\hbar T/2m}}{4\pi^2}\frac{d}{d\cos\psi_{S^{(3)}}}\theta_3\left(\left.\frac{\psi_{S^{(3)}}(\mathbf{q}'', \mathbf{q}')}{2}\right| - \frac{\hbar T}{2\pi m}\right) \ , \qquad (5.101)$$

and $\psi_{S^{(3)}}(\mathbf{q}'', \mathbf{q}')$ has to be taken in these coordinates. Note that the $\phi$-path integration can be explicitly done, leading to a path integral in $\mu, \nu$ with similar to the sphero-conical one with additional $1/\operatorname{dn}^2\mu$ and $1/\operatorname{sn}^2\nu$-terms, respectively.

*5.2.6. Twofold Confocal Ellipsoidal Coordinates 2).* We consider the two-fold confocal ellipsoidal coordinates defined by

$$\frac{s_1^2}{\rho_i - b} + \frac{s_2^2 + s_3^2}{\rho_i - a} + \frac{s_0^2}{\rho_i - c} = 0 \ , \qquad (i = 1, 2) \ , \qquad (5.102)$$

where $c < \rho_2 < b < \rho_1 < a$. The corresponding metric reads

$$\frac{ds^2}{dt^2} = -\frac{1}{4}(\rho_1 - \rho_2)\left(\frac{\dot{\rho}_1^2}{P_3(\rho_1)} - \frac{\dot{\rho}_2^2}{P_3(\rho_2)}\right) + (\rho_1 - a)(\rho_2 - a)\dot{\rho}_3^2 \ . \qquad (5.103)$$

There is only little difference to the previous case. Proceeding similarly as before we have

$$\left.\begin{aligned}
s_0 &= \operatorname{sn}(\mu, k)\operatorname{dn}(\nu, k')\cos\phi \ , && k^2 + k'^2 = 1 \ , \\
s_1 &= \operatorname{sn}(\mu, k)\operatorname{dn}(\nu, k')\sin\phi \ , && 0 \le \phi < 2\pi \ , \\
s_2 &= \operatorname{dn}(\mu, k)\operatorname{sn}(\nu, k') \ , && \\
s_3 &= \operatorname{cn}(\mu, k)\operatorname{cn}(\nu, k') \ .
\end{aligned}\right\} \qquad (5.104)$$

We omit details in the following. This gives the path integral formulation

$$\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)(k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu)\sqrt{k^2}\operatorname{sn}\mu\operatorname{dn}\mu \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)$$

$$\times \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big((k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu)(\dot{\mu}^2 + \dot{\nu}^2) + k^2\operatorname{sn}^2\mu\operatorname{dn}^2\nu\dot{\phi}^2\Big) - \Delta V_{PF}\right]dt\right\}$$

$$= \frac{e^{i\,\hbar T/2m}}{4\pi^2}\frac{d}{d\cos\psi_{S^{(3)}}}\theta_3\left(\frac{\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}')}{2}\,\Bigg|-\frac{\hbar T}{2\pi m}\right) \;, \tag{5.105}$$

and $\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}')$ has to be taken in these coordinates. The difference of the last two coordinate system lies in the orientation with respect to an axis.

*5.2.7. Confocal Ellipsoidal Coordinates.* Let us consider the coordinate system defined by

$$\frac{s_0^2}{\rho_i - d} + \frac{s_1^2}{\rho_i - c} + \frac{s_2^2}{\rho_i - a} + \frac{s_3^2}{\rho_i - b} = 0 \;, \qquad (i = 1,2,3) \;, \tag{5.106}$$

where $d < \rho_3 < c < \rho_2 < b < \rho_1 < a$ (c.f. Section 3.1). Explicitly

$$\left.\begin{aligned}
s_0^2 &= \frac{(\rho_1 - d)(\rho_2 - d)(\rho_3 - d)}{(a - d)(b - d)(c - d)} \;, & s_1^2 &= \frac{(\rho_1 - c)(\rho_2 - c)(\rho_3 - c)}{(a - c)(b - c)(d - c)} \;, \\
s_2^2 &= \frac{(\rho_1 - a)(\rho_2 - a)(\rho_3 - a)}{(d - a)(c - a)(b - a)} \;, & s_3^2 &= \frac{(\rho_1 - b)(\rho_2 - b)(\rho_3 - b)}{(d - b)(c - b)(a - b)} \;.
\end{aligned}\right\} \tag{5.107}$$

The corresponding line element is given by

$$\frac{ds^2}{dt^2} = -\frac{1}{4}\left[\frac{(\rho_1 - \rho_2)(\rho_1 - \rho_3)}{P_4(\rho_1)}\dot{\rho}_1^2 + \frac{(\rho_2 - \rho_3)(\rho_2 - \rho_1)}{P_4(\rho_2)}\dot{\rho}_2^2 + \frac{(\rho_3 - \rho_1)(\rho_3 - \rho_2)}{P_4(\rho_3)}\dot{\rho}_3^2\right] \;, \tag{5.108}$$

where $P_4(\rho) = (\rho - a)(\rho - b)(\rho - c)(\rho - d)$. We apply the separation formula (3.21). We identify $(g_{ab}) \equiv \operatorname{diag}(h_1^2, h_2^2, h_3^2)$, furthermore

$$\Gamma_i = \frac{\rho_i P_4'(\rho_i)}{P_4(\rho)} \;, \qquad (i = 1,2,3) \tag{5.109}$$

$$S = \frac{(\rho_1 - \rho_2)(\rho_1 - \rho_3)(\rho_2 - \rho_3)}{4P_4(\rho_1)P_4(\rho_2)P_4(\rho_2)} \;, \tag{5.110}$$

$$M_1 = \frac{\rho_2 - \rho_3}{P_4(\rho_2)P_4(\rho_3)} \;, \quad M_2 = \frac{\rho_3 - \rho_1}{P_4(\rho_1)P_4(\rho_3)} \;, \quad M_3 = \frac{\rho_1 - \rho_2}{P_4(\rho_1)P_4(\rho_2)} \;, \tag{5.111}$$

and obtain the following path integral identity

$$\prod_{i=1}^{3}\int\limits_{\rho_i(t')=\rho_i'}^{\rho_i(t'')=\rho_i''} h_i\mathcal{D}\rho_i(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\sum_{i=1}^{3}h_i^2\dot{\rho}_i^2 - \Delta V_{PF}(\{\rho\})\right]dt\right\}$$

$$
= (S'S'')^{-1/4} \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-\mathrm{i}\,ET/\hbar} \int_0^{\infty} ds'' \prod_{i=1}^3 \int_{\rho_i(0)=\rho_i'}^{\rho_i(s'')=\rho_i''} M_i^{-1/2} \mathcal{D}\rho_i(s)
$$

$$
\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[ \frac{m}{2} \sum_{i=1}^3 \frac{\dot{\rho}_i^2}{M_i} + ES - \frac{\hbar^2}{8m} \sum_{i=1}^3 M_i \left( \Gamma_i^2 + 2\Gamma_i' \right) \right] ds \right\}
\tag{5.112}
$$

$$
= \frac{\mathrm{e}^{\mathrm{i}\,\hbar T/2m}}{4\pi^2} \frac{\mathrm{d}}{\mathrm{d}\cos\psi_{S^{(3)}}} \theta_3 \left( \left. \frac{\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}')}{2} \right| - \frac{\hbar T}{2\pi m} \right) \ ,
\tag{5.113}
$$

and $\psi_{S^{(3)}}(\mathbf{q}'',\mathbf{q}')$ must be expressed in confocal ellipsoidal coordinates. Actually, "The linear element of 5. 2.7.5 and 5.2.7.6 can be considered as degenerations of 5.2.7.7 when $b = c$ and $a = b$, respectively ... " [90].

## 5.3. The Pseudosphere $\Lambda^{(3)}$

5.3.1. *General Form of the Propagator and the Green Function.* The propagator, respectively the Green function, on three-dimensional hyperbolic space, i.e. the three-dimensional pseudosphere $\Lambda^{(3)}$ are given by [51, 56, 60]

$$
K^{\Lambda^{(3)}}\left( d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}'); T \right) = \left( \frac{m}{2\pi\,\mathrm{i}\,\hbar T} \right)^{\frac{3}{2}} \frac{d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}')}{\sinh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}')} \exp\left[ \frac{\mathrm{i}\,m}{2\hbar T} d_{\Lambda^{(3)}}^2(\mathbf{q}'',\mathbf{q}') - \frac{\mathrm{i}\,\hbar T}{2m} \right]
\tag{5.114}
$$

$$
G^{\Lambda^{(3)}}\left( d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}'); E \right) = \frac{-m}{\pi^2\hbar^2 \sinh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}')}
$$

$$
\times \mathcal{Q}_{-\mathrm{i}\sqrt{2mE/\hbar^2-1}-1/2}^{1/2}\left( \cosh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}') \right) \ .
\tag{5.115}
$$

where $\mathbf{q}$ can denote any of the 34 coordinate systems which separate on $\Lambda^{(3)}$.

5.3.2. *Spherical Coordinates.* We consider the coordinate system

$$
\left.\begin{aligned}
u_0 &= \cosh\tau_1 \cosh\tau_2 \ , \quad \tau_1, \tau_2 \in \mathbb{R} \ , \\
u_1 &= \cosh\tau_1 \sinh\tau_2 \ , \\
u_2 &= \sinh\tau_1 \sin\phi \ , \qquad 0 \le \phi < 2\pi \ , \\
u_3 &= \sinh\tau_1 \cos\phi \ .
\end{aligned}\right\}
\tag{5.116}
$$

The metric tensor is $(g_{ab}) = \mathrm{diag}(1, \cosh^2\tau_1, \sinh^2\tau_1)$. For the momentum operators we have

$$
p_{\tau_1} = \frac{\hbar}{\mathrm{i}}\left( \frac{\partial}{\partial\tau_1} + \frac{1}{2}\coth\tau_1 + \frac{1}{2}\tanh\tau_1 \right) \ , \qquad p_{\tau_2} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\tau_2} \ , \qquad p_\phi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\phi} \ .
\tag{5.117}
$$

The hyperbolic distance is given by

$$
\cosh d_{\Lambda^{(3)}}\mathbf{q}'',\mathbf{q}') = \cosh\tau_1' \cosh\tau_1'' \cosh(\tau_2'' - \tau_2') - \sinh\tau_1' \sinh\tau_1'' \cos(\phi'' - \phi') \ .
\tag{5.118}
$$

For the Hamiltonian we have

$$
-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\tau_1^2} + \left(\tanh\tau_1 + \coth\tau_1\right)\frac{\partial}{\partial\tau_1} + \frac{1}{\cosh^2\tau_1}\frac{\partial^2}{\partial\tau_2^2} + \frac{1}{\sinh^2\tau_1}\frac{\partial^2}{\partial\phi^2}\right]
$$
$$
= \frac{1}{2m}\left(p_{\tau_1}^2 + \frac{1}{\cosh^2\tau_1}p_{\tau_2}^2 + \frac{1}{\sinh^2\tau_1}p_\phi^2\right) + \frac{\hbar^2}{8m}\left(4 + \frac{1}{\cosh^2\tau_1} - \frac{1}{\sinh^2\tau_1}\right) .
$$
$$(5.119)$$

We obtain the path integral formulation

$$
\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\cosh\tau_1\sinh\tau_1\,\mathcal{D}\tau_1(t)\int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\int_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathcal{D}\phi(t)
$$
$$
\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot\tau_1^2 + \cosh^2\tau_1\dot\tau_2^2 + \sinh^2\tau_1\dot\phi^2\right)\right.\right.
$$
$$
\left.\left.- \frac{\hbar^2}{8m}\left(4 + \frac{1}{\cosh^2\tau_1} - \frac{1}{\sinh^2\tau_1}\right)\right]dt\right\}
$$
$$
= \left(\sinh\tau_1'\sinh\tau_1''\cosh\tau_1'\cosh\tau_1''\right)^{-1/2}\mathrm{e}^{-\mathrm{i}\,\hbar T/2m}\sum_{l\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi}\int_{\mathbb{R}}\frac{dp_1}{2\pi}\mathrm{e}^{\mathrm{i}\,p_1(\tau_2''-\tau_2')}
$$
$$
\times \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot\tau_1^2 - \frac{\hbar^2}{2m}\left(\frac{p_1^2 + \frac{1}{4}}{\cosh^2\tau_1} + \frac{l^2 - \frac{1}{4}}{\sinh^2\tau_1}\right)\right]dt\right\}
$$
$$
= \left(\sinh\tau_1'\sinh\tau_1''\cosh\tau_1'\cosh\tau_1''\right)^{-1/2}\sum_{l\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}}{2\pi}\int_{\mathbb{R}}\frac{dp_1}{2\pi}\mathrm{e}^{\mathrm{i}\,p_1(\tau_2''-\tau_2')}
$$
$$
\times \int_0^\infty dp\,\Psi_p^{(p_1,l)\,*}(\tau_1')\Psi_p^{(p_1,l)}(\tau_1'')\exp\left[-\frac{\mathrm{i}\,\hbar T}{2m}(p^2 + 1)\right] .
$$
$$(5.120)$$

In the path integration, the cylindrical $\phi$- and free particle $\tau_2$-path integration have been separated straightforwardly. In the last path integral the path integral solutions for modified Pöschl-Teller potential with $\nu = p_1$, $\eta = |l|$ have been used. The wavefunctions $\Psi_p^{(p_1,l)}(\tau_1)$ are given by

$$
\Psi_p^{(p_1,l)}(\tau_1) = N_p^{(p_1,l)}(\cosh\tau_1)^{\mathrm{i}\,p_1-1/2}(\sinh\tau_1)^{|l|-1/2}
$$
$$
\times {}_2F_1\left[\frac{1}{2}(|l| + \mathrm{i}\,p_1 - \mathrm{i}\,p + 1), \frac{1}{2}(|l| + \mathrm{i}\,p_1 + \mathrm{i}\,p + 1); |l| + 1; -\sinh^2\tau_1\right] ,
$$
$$(5.121a)$$
$$
N_p^{(p_1,l)} = \frac{1}{|l|!}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(\frac{1 + |l| + \mathrm{i}\,p_1 + \mathrm{i}\,p}{2}\right)\Gamma\left(\frac{1 + |l| + \mathrm{i}\,p_1 - \mathrm{i}\,p}{2}\right) .
$$
$$(5.121b)$$

*5.3.3. Horicyclic Coordinates.* We consider the coordinate system

$$\left.\begin{aligned}
u_0 &= \frac{1}{2}\left(\frac{1}{y} + y + \frac{x_1^2 + x_2^2}{y}\right) \ , \quad y > 0 \ , \\
u_1 &= \frac{x_1}{y} \ , \quad u_2 = \frac{x_2}{y} \ , \qquad (x_1, x_2) = \mathbf{x} \in \mathbb{R}^2 \ , \\
u_3 &= \frac{1}{2}\left(\frac{1}{y} - y - \frac{x_1^2 + x_2^2}{y}\right) \ ,
\end{aligned}\right\} \tag{5.122}$$

The metric tensor is $(g_{ab}) = \mathbb{1}/y^2$. For the momentum operators we have $p_{x_{1,2}} = -i\hbar\partial_{x_{1,2}}$ and

$$p_y = \frac{\hbar}{i}\left(\frac{\partial}{\partial y} - \frac{3}{2y}\right) \ . \tag{5.123}$$

The hyperbolic distance is given by

$$\cosh d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}') = \frac{y'^2 + y''^2 + (x_1'' - x_1')^2 + (x_2'' - x_2')^2}{2y'y''} \ . \tag{5.124}$$

For the Hamiltonian we have

$$\begin{aligned}
-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(3)}} &= -\frac{\hbar^2}{2m}y^2\left(\frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \\
&= \frac{1}{2m}\left(yp_y^2 y + y^2 p_{x_1}^2 + y^2 p_{x_2}^2\right) + \frac{3\hbar^2}{8m} \ . \tag{5.125}
\end{aligned}$$

The path integral discussion for this coordinate system and its $D$-dimensional generalization has been discussed in [51]. The path integral is solved by exploiting the path integral solution of the radial harmonic oscillator; we cite the result. We obtain the identity [51]

$$\begin{aligned}
\int_{y(t')=y'}^{y(t'')=y''} &\frac{\mathcal{D}y(t)}{y^3} \int_{x_1(t')=x_1'}^{x_1(t'')=x_1''} \mathcal{D}x_1(t) \int_{x_2(t')=x_2'}^{x_2(t'')=x_2''} \mathcal{D}x_2(t) \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\frac{\dot{\mathbf{x}}^2 + \dot{y}^2}{y^2} - \frac{3\hbar^2}{8m}\right)dt\right] \\
&= y'y'' \int_{\mathbb{R}^2}\frac{d\mathbf{k}}{(2\pi)^2}\,e^{i\,\mathbf{k}\cdot(\mathbf{x}''-\mathbf{x}')} \\
&\quad \times \frac{2}{\pi^2}\int_0^\infty dp\,p\sinh\pi p\exp\left[-\frac{i\hbar T}{2m}(p^2 + 1)\right]K_{i\,p}(|\mathbf{k}|y')K_{i\,p}(|\mathbf{k}|y'') \ . \tag{5.126}
\end{aligned}$$

*5.3.4. Pseudospherical-Conical Coordinates.* We consider the coordinate system

$$\left.\begin{aligned}
u_0 &= \cosh\tau \ , &\tau > 0 \ , \\
u_1 &= \sinh\tau\,\mathrm{sn}(\mu, k)\,\mathrm{dn}(\nu, k') \ , &k^2 + k'^2 = 1 \ , \\
u_2 &= \sinh\tau\,\mathrm{cn}(\mu, k)\,\mathrm{cn}(\nu, k') \ , \\
u_3 &= \sinh\tau\,\mathrm{dn}(\mu, k)\,\mathrm{sn}(\nu, k') \ ,
\end{aligned}\right\} \tag{5.127}$$

in the notation of 4.2.3. The hyperbolic distance reads

$$
\cosh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}') = \cosh\tau''\cosh\tau' - \sinh\tau''\sinh\tau'\Big(\operatorname{sn}\mu''\operatorname{sn}\mu'\operatorname{dn}\nu''\operatorname{dn}\nu'
$$
$$
+\operatorname{cn}\mu''\operatorname{cn}\mu'\operatorname{cn}\nu''\operatorname{cn}\nu' + \operatorname{dn}\mu''\operatorname{dn}\mu'\operatorname{sn}\nu''\operatorname{sn}\nu'\Big)\ .
\tag{5.128}
$$

The metric tensor $g_{ab}$ is given by

$$
(g_{ab}) = \operatorname{diag}\big[1,\sinh^2\tau(k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu),\sinh^2\tau(k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu)\big]\ ,
\tag{5.129}
$$

and the momentum operators are

$$
p_\tau = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau} + \coth\tau\right)\ ,
\tag{5.130}
$$

and with $p_\mu$, $p_\nu$ as in (4.40). The Hamiltonian has the form

$$
-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\tau^2} + 2\coth\tau\frac{\partial}{\partial\tau} + \frac{1}{\sinh^2\tau}\frac{1}{k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\partial^2}{\partial\nu^2}\right)\right]
$$
$$
= \frac{1}{2m}\left[p_\tau^2 + \frac{1}{\sinh^2\tau}\frac{1}{\sqrt{k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu}}(p_\mu^2 + p_\nu^2)\frac{1}{\sqrt{k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu}}\right] + \frac{\hbar^2}{2m}\ .
\tag{5.131}
$$

The path integral can be written down yielding

$$
\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\sinh^2\tau\,\mathcal{D}\tau(t)\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)(k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu)
$$
$$
\times\exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\left[\dot\tau^2 + \sinh^2\tau(k^2\operatorname{cn}^2\mu + k'^2\operatorname{cn}^2\nu)(\dot\mu^2 + \dot\nu^2)\right]dt - \frac{\mathrm{i}\hbar T}{2m}\right\}
$$
$$
= (\sinh\tau'\sinh\tau'')^{-1}\,\mathrm{e}^{-\mathrm{i}\hbar T/2m}\sum_{l=0}^{\infty}\sum_{\kappa}A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')
$$
$$
\times\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{\sinh^2\tau}\right)dt\right]
$$
$$
= (\sinh\tau'\sinh\tau'')^{-1/2}\sum_{l=0}^{\infty}\sum_{\kappa}A_{l,\kappa}^*(\mu')B_{l,\kappa}^*(\nu')A_{l,\kappa}(\mu'')B_{l,\kappa}(\nu'')
$$
$$
\times\frac{1}{\pi}\int_0^\infty dp\,p\sinh\pi p\,|\Gamma(\mathrm{i}p + l + 1)|^2\exp\left[-\frac{\mathrm{i}\hbar T}{2m}(p^2 + 1)\right]
$$
$$
\times\mathcal{P}_{\mathrm{i}\,p-1/2}^{-\frac{1}{2}-l}(\cosh\tau')\mathcal{P}_{\mathrm{i}\,p-1/2}^{-\frac{1}{2}-l}(\cosh\tau'')\ .
\tag{5.132}
$$

### 5.3.5. Hyperbolic-Pseudo-Conical Coordinates.

$$
\left.\begin{array}{ll}
u_0 = \cosh\tau\,\mathrm{nc}(\mu,k)\,\mathrm{nc}(\nu,k') \;, & k^2 + k'^2 = 1 \;, \\[4pt]
u_1 = \sinh\tau \;, & \tau \in \mathbb{R} \;, \\[4pt]
u_2 = \cosh\tau\,\mathrm{dc}(\mu,k)\,\mathrm{sc}(\nu,k') \;, & \\[4pt]
u_3 = \cosh\tau\,\mathrm{sc}(\mu,k)\,\mathrm{dc}(\nu,k') \;. &
\end{array}\right\}
\tag{5.133}
$$

Actually, these coordinates encode six different coordinate systems in a similar way as discussed in 4.3.5. We omit the details and write down the corresponding path integral formulation which reads as follows

$$
\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \cosh^2\tau\,\mathcal{D}\tau(t) \int_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t)(k'^2\,\mathrm{nc}^2\,\mu + k^2\,\mathrm{nc}^2\,\nu)
$$

$$
\times \exp\left\{ \frac{\mathrm{i}\,m}{2\hbar} \int_{t'}^{t''} \left[ \dot\tau^2 + \cosh^2\tau(k'^2\,\mathrm{nc}^2\,\mu + k^2\,\mathrm{nc}^2\,\nu)(\dot\mu^2 + \dot\nu^2) \right] dt - \frac{\mathrm{i}\,\hbar T}{2m} \right\}
$$

$$
= (\cosh\tau'\cosh\tau'')^{-1}\,\mathrm{e}^{-\mathrm{i}\,\hbar T/2m} \sum_\kappa \int_0^\infty dk\, \mathcal{A}_{k,\kappa}^*(\mu')\mathcal{B}_{k,\kappa}^*(\nu')\mathcal{A}_{k,\kappa}(\mu'')\mathcal{B}_{k,\kappa}(\nu'')
$$

$$
\times \int_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t) \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\frac{k^2 + \tfrac{1}{4}}{\cosh^2\tau} \right) dt \right]
$$

$$
= (\cosh\tau'\cosh\tau'')^{-1} \sum_\kappa \int_0^\infty dk\, \mathcal{A}_{k,\kappa}^*(\mu')\mathcal{B}_{k,\kappa}^*(\nu')\mathcal{A}_{k,\kappa}(\mu'')\mathcal{B}_{k,\kappa}(\nu'')
$$

$$
\times \frac{1}{2} \int_{\mathbb{R}} \frac{p\sinh\pi p\,dp}{\cosh^2\pi k + \sinh^2\pi p}\, \mathrm{e}^{-\mathrm{i}\,\hbar T(p^2+1)/2m}\, P_{\mathrm{i}\,k-\frac{1}{2}}^{-\mathrm{i}\,p}(\tanh\tau')P_{\mathrm{i}\,k-\frac{1}{2}}^{\mathrm{i}\,p}(\tanh\tau'')
$$

$$
\tag{5.134}
$$

$$
= \left( \frac{m}{2\pi\,\mathrm{i}\,\hbar T} \right)^{\frac{3}{2}} \frac{d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}')}{\sinh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}')} \exp\left[ \frac{\mathrm{i}\,m}{2\hbar T}d_{\Lambda^{(3)}}^2(\mathbf{q}'',\mathbf{q}') - \frac{\mathrm{i}\,\hbar T}{2m} \right] \;,
\tag{5.135}
$$

where $d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}')$ must be expressed in hyperbolic pseudo-conical coordinates.

### 5.3.6. Pseudospherical Coordinates. We consider the coordinate system

$$
\left.\begin{array}{ll}
u_0 = \cosh\tau \;, & \tau > 0 \\[4pt]
u_1 = \sinh\tau\,\sin\theta\,\sin\phi \;, & 0 < \theta < \pi \;, \\[4pt]
u_2 = \sinh\tau\,\sin\theta\,\cos\phi \;, & 0 \le \phi < 2\pi \;, \\[4pt]
u_3 = \sinh\tau\,\cos\theta \;. &
\end{array}\right\}
\tag{5.136}
$$

This is the usual three-dimensional (pseudo-) spherical polar coordinates system on $\Lambda^{(3)}$. We have $(g_{ab}) = \mathrm{diag}(1, \sinh^2\tau, \sinh^2\tau\sin^2\theta)$ and the invariant distance is given by

$$
\cosh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}')
$$
$$
= \cosh\tau''\cosh\tau' - \sinh\tau''\sinh\tau'\left[ \sin\theta''\sin\theta'\cos(\phi'' - \phi') + \cos\theta''\cos\theta' \right] \;.
\tag{5.137}
$$

For the momentum operators we have the operators (4.33,5.130). This gives for the Hamiltonian

$$-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\tau^2} + 2\coth\tau\frac{\partial}{\partial\tau} + \frac{1}{\sinh^2\tau}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right]$$

$$= \frac{1}{2m}\left(p_\tau^2 + \frac{p_\theta^2}{\sinh^2\tau} + \frac{p_\phi^2}{\sinh^2\tau\sin^2\theta}\right) + \frac{\hbar^2}{8m}\left(4 - \frac{1}{\sinh^2\tau} - \frac{1}{\sinh^2\tau\sin^2\theta}\right) .$$

$$(5.138)$$

In the path integral evaluation one successively uses the path integral solution related special case of the modified Pöschl-Teller potential. Therefore we obtain the path integral formulation (and c.f. [10, 56] for its $D$-dimensional generalization)

$$\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\sinh^2\tau\mathcal{D}\tau(t)\int_{\theta(t')=\theta'}^{\theta(t'')=\theta''}\sin\theta\mathcal{D}\theta(t)\int_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathcal{D}\phi(t)$$

$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{\tau}^2 + \sinh^2\tau\dot{\theta}^2 + \sinh^2\tau\sin^2\theta\dot{\phi}^2\right)\right.\right.$$

$$\left.\left. - \frac{\hbar^2}{8m}\left(4 - \frac{1}{\sinh^2\tau} - \frac{1}{\sinh^2\tau\sin^2\theta}\right)\right]dt\right\}$$

$$= (\sinh\tau'\sinh\tau'')^{-1}\,e^{-i\hbar T/2m}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^{m\,*}(\theta',\phi')Y_l^m(\theta'',\phi'')$$

$$\times\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{2m}\frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{\sinh^2\tau}\right)dt\right]$$

$$= (\sinh\tau'\sinh\tau'')^{-1/2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^{m\,*}(\theta',\phi')Y_l^m(\theta'',\phi'')$$

$$\times\frac{1}{\pi}\int_0^\infty dp\,p\sinh\pi p\left|\Gamma(i\,p+l+1)\right|^2\exp\left[-\frac{i\hbar T}{2m}(p^2+1)\right]$$

$$\times\mathcal{P}_{i\,p-1/2}^{-\frac{1}{2}-l}(\cosh\tau')\mathcal{P}_{i\,p-1/2}^{-\frac{1}{2}-l}(\cosh\tau'') .$$

$$(5.139)$$

*5.3.7. Hyperbolic-Spherical Coordinates.* We consider the coordinate system

$$\left.\begin{array}{ll}u_0 = \cosh\tau_1\cosh\tau_2 , & \tau_1 \in \mathbb{R} , \\ u_1 = \sinh\tau_1 , & \tau_2 > 0 , \\ u_2 = \cosh\tau_1\sinh\tau_2\sin\phi , & 0 \le \phi < 2\pi , \\ u_3 = \cosh\tau_1\sinh\tau_2\cos\phi .\end{array}\right\}$$

$$(5.140)$$

The hyperbolic distance is given by

$$\cosh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}') = \cosh\tau_1''\cosh\tau_1'\cosh\tau_2''\cosh\tau_2'$$

$$- \sinh\tau_1''\sinh\tau_1 - \cosh\tau_1''\cosh\tau_1'\sinh\tau_2''\sinh\tau_2'\cos(\phi''-\phi') . \quad (5.141)$$

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Here $(g_{ab}) = \mathrm{diag}(1, \cosh^2 \tau_1, \cosh^2 \tau_1 \sinh^2 \tau_2)$, and for the momentum operators we have

$$p_{\tau_1} = \frac{\hbar}{i} \left( \frac{\partial}{\partial \tau_1} + \tanh \tau_1 \right) \ , \tag{5.142}$$

and $p_{\tau_2}, p_\phi$ as in (4.57). For the Hamiltonian we have

$$-\frac{\hbar^2}{2m} \Delta_{\Lambda^{(3)}} = \frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial \tau_1} + 2\tanh \tau_1 \frac{\partial}{\partial \tau_1} + \frac{1}{\cosh^2 \tau_1} \left( \frac{\partial^2}{\partial \tau_2^2} + \coth \tau_2 \frac{\partial}{\partial \tau_2} + \frac{1}{\sinh^2 \tau_2} \frac{\partial^2}{\partial \phi} \right) \right]$$

$$= \frac{1}{2m} \left( p_{\tau_1}^2 + \frac{p_\tau^2}{\cosh^2 \tau_1} + \frac{p_\phi^2}{\cosh^2 \tau_1 \sinh^2 \tau_2} \right) + \frac{\hbar^2}{8m} \left( 4 + \frac{1}{\cosh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1 \sinh^2 \tau_2} \right) \ . \tag{5.143}$$

Therefore we obtain the path integral formulation

$$\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \cosh^2 \tau_1 \mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \sinh \tau_2 \mathcal{D}\tau_2(t) \int_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)$$

$$\times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \left( \dot{\tau}_1^2 + \cosh^2 \tau_1 \dot{\tau}_2^2 + \cosh^2 \tau_1 \sinh^2 \tau_2 \dot{\phi}^2 \right) \right. \right.$$

$$\left. \left. - \frac{\hbar^2}{8m} \left( 4 + \frac{1}{\cosh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1 \sinh^2 \tau_2} \right) \right] dt \right\}$$

$$= \left( \cosh \tau_1' \cosh \tau_1'' \right)^{-1} \sum_{l \in \mathbb{Z}} \frac{e^{i\, l(\phi'' - \phi')}}{2\pi}$$

$$\times \frac{1}{\pi} \int_{\mathbb{R}} dp_1\, p_1 \sinh \pi p_1 \left| \Gamma(\tfrac{1}{2} + i\, p_1 - l) \right|^2 \mathcal{P}_{i\,p_1 - \frac{1}{2}}^{-l}(\cosh \tau_2') \mathcal{P}_{i\,p_1 - \frac{1}{2}}^{-l}(\cosh \tau_2'')$$

$$\times \frac{1}{2} \int_{\mathbb{R}} \frac{p \sinh \pi p\, dp}{\cosh^2 \pi p_1 + \sinh^2 \pi p} \exp \left[ -\frac{i\,\hbar T}{2m}(p^2 + 1) \right] P_{i\,p_1 - \frac{1}{2}}^{-i\,p}(\tanh \tau_3') P_{i\,p_1 - \frac{1}{2}}^{i\,p}(\tanh \tau_3'') \ . \tag{5.144}$$

In the path integral evaluation first the path integral solution of the simple Manning-Rosen potential, and second of the simple Rosen-Morse potential was used.

5.3.8. *Equidistant Coordinates.* We consider the coordinate system

$$\left. \begin{aligned} u_0 &= \cosh \tau_1 \cosh \tau_2 \cosh \tau_3 \ , & \tau_1, \tau_2, \tau_3 \in \mathbb{R} \ , \\ u_1 &= \sinh \tau_1 \ , \quad u_2 = \cosh \tau_1 \sinh \tau_2 \ , \\ u_3 &= \cosh \tau_1 \cosh \tau_2 \sinh \tau_3 \ . \end{aligned} \right\} \tag{5.145}$$

The hyperbolic distance is given by

$$\cosh d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}') = \cosh \tau_1' \cosh \tau_1'' \left[ \cosh \tau_2' \cosh \tau_2'' \right.$$

$$+ \sinh \tau_2'' \sinh \tau_2' \cos(\tau_3'' - \tau_3') \Big] - \sinh \tau_1' \sinh \tau_1'' \ .$$

$$(5.146)$$

This coordinate system was called in [111] the hyperbolic, respectively the Lobachevskian one. Here $(g_{ab}) = \mathrm{diag}(1, \cosh^2 \tau_1, \cosh^2 \tau_1 \cosh^2 \tau_2)$, and for the momentum operators we have for $p_{\tau_1}$ as in (5.142), and for $p_{\tau_2}, p_{\tau_3}$ as in (4.53). For the Hamiltonian we get

$$-\frac{\hbar^2}{2m} \Delta_{\Lambda^{(3)}} = \frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial \tau_1} + 2 \tanh \tau_1 \frac{\partial}{\partial \tau_1} + \frac{1}{\cosh^2 \tau_1} \left( \frac{\partial^2}{\partial \tau_2} + \tanh \tau_2 \frac{\partial}{\partial \tau_2} + \frac{1}{\cosh^2 \tau_2} \frac{\partial^2}{\partial \tau_3} \right) \right]$$

$$= \frac{1}{2m} \left( p_{\tau_1}^2 + \frac{p_{\tau_2}^2}{\cosh^2 \tau_1} + \frac{p_{\tau_3}^2}{\cosh^2 \tau_1 \cosh^2 \tau_2} \right) + \frac{\hbar^2}{8m} \left( 4 + \frac{1}{\cosh^2 \tau_1} + \frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \right) \ .$$

$$(5.147)$$

Therefore we obtain the path integral formulation

$$\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \cosh^2 \tau_1 \mathcal{D}\tau_1(t) \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \cosh \tau_2 \mathcal{D}\tau_2(t) \int\limits_{\tau_3(t')=\tau_3'}^{\tau_3(t'')=\tau_3''} \mathcal{D}\tau_3(t)$$

$$\times \exp \left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \left( \dot{\tau}_1^2 + \cosh^2 \tau_1 \dot{\tau}_2^2 + \cosh^2 \tau_1 \cosh^2 \tau_2 \dot{\tau}_3^2 \right) \right. \right.$$

$$\left. \left. - \frac{\hbar^2}{8m} \left( 4 + \frac{1}{\cosh^2 \tau_1} + \frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \right) \right] dt \right\}$$

$$= \left( \cosh^2 \tau_1' \cosh^2 \tau_1'' \cosh \tau_2' \cosh \tau_2'' \right)^{-1/2} \int_{\mathbb{R}} \frac{dk}{2\pi} e^{\mathrm{i} \, k(\tau_3'' - \tau_3')}$$

$$\times \frac{1}{2} \int_{\mathbb{R}} \frac{p_1 \sinh \pi p_1 \, dp_1}{\cosh^2 \pi k + \sinh^2 \pi p_1} P_{\mathrm{i} \, k - \frac{1}{2}}^{-\mathrm{i} \, p_1} (\tanh \tau_2') P_{\mathrm{i} \, k - \frac{1}{2}}^{\mathrm{i} \, p_1} (\tanh \tau_2'')$$

$$\times \frac{1}{2} \int_{\mathbb{R}} \frac{p \sinh \pi p \, dp}{\cosh^2 \pi p_1 + \sinh^2 \pi p} e^{-\mathrm{i} \, \hbar T(p^2+1)/2m} \, P_{\mathrm{i} \, p_1 - \frac{1}{2}}^{-\mathrm{i} \, p} (\tanh \tau_3') P_{\mathrm{i} \, p_1 - \frac{1}{2}}^{\mathrm{i} \, p} (\tanh \tau_3'') \ .$$

$$(5.148)$$

In the path integral evaluation one successively uses the path integral solution related special case of the modified Pöschl-Teller potential. In Appendix 3 we give the path integral solution of the D-dimensional generalization of this coordinate system.

*5.3.9. Hyperbolic-Horicyclic Coordinates.* We consider the coordinate system

$$\left. \begin{aligned} u_0 &= \frac{\cosh \tau}{2} \left( \frac{1}{y} + y + \frac{x^2}{y} \right) \ , \qquad \tau \in \mathbb{R} \ , \\ u_1 &= \sinh \tau \ , \qquad u_2 = \frac{x}{y} \cosh \tau \ , \qquad x \in \mathbb{R} \ , \\ u_3 &= \frac{\cosh \tau}{2} \left( \frac{1}{y} - y - \frac{x^2}{y} \right) \ , \qquad y > 0 \ . \end{aligned} \right\}$$

$$(5.149)$$

The hyperbolic distance is given by

$$\cosh d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}') = \frac{y'^2 + y''^2 + (x'' - x')^2}{2 y' y''} \cosh \tau' \cosh \tau'' - \sinh \tau' \sinh \tau'' \ . \quad (5.150)$$

Here $(g_{ab}) = \text{diag}(1, \cosh^2 \tau/y^2, \cosh^2 \tau/y^2)$ and for the momentum operators we have the operators (4.49,5.142), therefore for the Hamiltonian

$$-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(3)}} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\tau^2} + 2\tanh\tau\frac{\partial}{\partial\tau} + \frac{y^2}{\cosh^2\tau}\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)\right]$$

$$= \frac{1}{2m}\left[p_\tau^2 + \frac{1}{\cosh^2\tau}\left(yp_y^2 y + y^2 p_x^2\right)\right] + \frac{\hbar^2}{2m} \quad . \tag{5.151}$$

We therefore obtain the path integral formulation

$$\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \cosh^2\tau\mathcal{D}\tau(t) \int\limits_{y(t')=y'}^{y(t')=y''} \frac{\mathcal{D}y(t)}{y^2} \int\limits_{x(t')=x'}^{x(t')=x''} \mathcal{D}x(t)$$

$$\times \exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\left(\dot\tau^2 + \cosh^2\tau\frac{\dot x^2 + \dot y^2}{y^2}\right)dt - \frac{\mathrm{i}\,\hbar T}{2m}\right]$$

$$= (\cosh\tau'\cosh\tau'')^{-1}\frac{\sqrt{y'y''}}{\pi^3}\int_0^\infty dp_0\,\mathrm{e}^{\mathrm{i}\,p_0(x''-x')}$$

$$\times \int_0^\infty dp_1\,p_1\sinh\pi p_1 K_{\mathrm{i}\,p_1}(|p_0|y')K_{\mathrm{i}\,p_1}(|p_0|y'')$$

$$\times \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot\tau^2 - \frac{\hbar^2}{2m}\frac{p_1^2 + \frac{1}{4}}{\cosh^2\tau}\right)dt - \frac{\mathrm{i}\,\hbar T}{2m}\right]$$

$$= (\cosh\tau'\cosh\tau'')^{-1}\frac{\sqrt{y'y''}}{\pi^3}\int_0^\infty dp_0\,\mathrm{e}^{\mathrm{i}\,p_0(x''-x')}$$

$$\times \int_0^\infty dp_1\,p_1\sinh\pi p_1 K_{\mathrm{i}\,p_1}(|p_0|y')K_{\mathrm{i}\,p_1}(|p_0|y'')$$

$$\times \frac{1}{2}\int_{\mathbb{R}}\frac{p\sinh\pi p\,dp}{\cosh^2\pi p_2 + \sinh^2\pi p}P_{\mathrm{i}\,p_1 - \frac{1}{2}}^{-\mathrm{i}\,p}(\tanh\tau_3')P_{\mathrm{i}\,p_1 - \frac{1}{2}}^{\mathrm{i}\,p}(\tanh\tau_3'')\,\mathrm{e}^{-\mathrm{i}\,\hbar T(p^2+1)/2m} \quad . \tag{5.152}$$

In this path integral solution the path integral identities from the two-dimensional Poincaré upper half-plane, and the simple Rosen-Morse potential have been used.

*5.3.10. Horicyclic-Spherical Coordinates.* We consider the coordinate system

$$\left.\begin{aligned} u_0 &= \frac{1}{2}\left(\frac{1}{y} + y + \frac{r^2}{y}\right) \ , & y &> 0 \ , \\ u_1 &= \frac{r\sin\phi}{y} \ , \quad u_2 = \frac{r\cos\phi}{y} \ , & r &> 0 \ , \\ u_3 &= \frac{1}{2}\left(\frac{1}{y} - y - \frac{r^2}{y}\right) \ , & 0 &\le \phi < 2\pi \ . \end{aligned}\right\} \tag{5.153}$$

The hyperbolic distance is given by

$$\cosh d_{\Lambda^{(3)}}(\mathbf{q}'',\mathbf{q}') = \frac{y'^2 + y''^2 + r'^2 + r''^2 - 2r'r''\cos(\phi''-\phi')}{2y'y''} \quad . \tag{5.154}$$

Here $(g_{ab}) = \mathrm{diag}(1, 1, r^2)/y^2$ and the momentum operators are given by (4.10,5.123). For the Hamiltonian we obtain

$$-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(3)}} = -\frac{\hbar^2}{2m}y^2\left[\frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + y^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\right]$$

$$= \frac{1}{2m}\left(yp_y^2 y + y^2 p_r^2 + \frac{y^2}{r^2}p_\phi^2\right) - \frac{\hbar^2 y^2}{8mr^2} + \frac{3\hbar^2}{8m} \ . \tag{5.155}$$

This coordinate system is, of course very similar to the horicyclic system and has been considered in its $D$-dimensional generalization in [51] as well. We just cite the result: We obtain the path integral identity

$$\int\limits_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^3} \int\limits_{r(t')=r'}^{r(t'')=r''} r\mathcal{D}r(t) \int\limits_{\phi(t')=\phi'}^{\phi(t'')=\phi''} \mathcal{D}\phi(t)$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\frac{\dot{y}^2 + \dot{r}^2 + r^2\dot{\phi}^2}{y^2} + \frac{\hbar^2 y^2}{8mr^2}\right)dt - \frac{3\,\mathrm{i}\,\hbar T}{8m}\right]$$

$$= \frac{y'y''}{\pi^3}\sum_{l\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\,l(\phi''-\phi')}\int_0^\infty dk\,kJ_l(kr')J_l(kr'')$$

$$\times \int_0^\infty dp\,p\sinh\pi pK_{\mathrm{i}\,p}(ky')K_{\mathrm{i}\,p}(ky'')\exp\left[-\frac{\mathrm{i}\,\hbar T}{2m}(p^2+1)\right] \ . \tag{5.156}$$

In the path integral evaluation after the trivial $\phi$-path integration the path integral solution of the free radial two-dimensional motion has been used.

5.3.11. *Horicyclic-Spheroidal Coordinates.* We consider the coordinate system

$$\left.\begin{aligned} u_0 &= \frac{1}{2}\left(\frac{1}{y} + y + \frac{\cosh^2\mu - \sin^2\nu}{y}\right) \ , && y > 0 \ , \\ u_1 &= \frac{\cosh\mu\cos\nu}{y} \ , \quad u_2 = \frac{\sinh\mu\sin\nu}{y} \ , && \mu > 0 \ , \\ u_3 &= \frac{1}{2}\left(\frac{1}{y} - y - \frac{\cosh^2\mu - \sin^2\nu}{y}\right) \ , && -\pi \le \nu < \pi \ . \end{aligned}\right\} \tag{5.157}$$

The hyperbolic distance is given by

$$\cosh d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}') = \frac{1}{2y'y''}\left[y'^2 + y''^2 + (\cosh^2\mu'' - \sin^2\nu'') + (\cosh^2\mu' - \sin^2\nu')\right.$$

$$\left. - 2\cosh\mu''\cosh\mu'\cos\nu''\cos\nu' - 2\sinh\mu''\sinh\mu'\sin\nu''\sin\nu'\right] \ . \tag{5.158}$$

Here we have $(g_{ab}) = \mathrm{diag}(1, \sinh^2\mu + \sin^2\nu, \sinh^2\mu + \sin^2\nu)$ together with the momentum operator $p_y$ (5.123), and for $p_\mu$ and $p_\nu$ as in (5.45). The Hamiltonian has the form

$$-\frac{\hbar^2}{2m}\Delta_{\Lambda^{(3)}} - \frac{\hbar^2}{2m}y^2\left[\frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \frac{1}{\sinh^2\mu + \sin^2\nu}\left(\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial \nu^2}\right)\right]$$

$$= \frac{1}{2m} \left[ y p_y^2 y + y^2 (\sinh^2 \mu + \sin^2 \nu)^{-1/2} (p_\mu^2 + p_\nu^2)(\sinh^2 \mu + \sin^2 \nu)^{-1/2} \right] - \frac{3\hbar^2}{8m} \ . \tag{5.159}$$

Using the result form the two-dimensional elliptic coordinate system in $E^{(2)}$ we obtain the path integral identity

$$\int\limits_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^3} \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) \int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) (\sinh^2 \mu + \sin^2 \nu)$$

$$\times \exp\left[ \frac{\mathrm{i}\, m}{2\hbar} \int_{t'}^{t''} \frac{\dot{y}^2 + (\sinh^2 \mu + \sin^2 \nu)(\dot{\mu}^2 + \dot{\nu}^2)}{y^2} dt - \frac{3\,\mathrm{i}\,\hbar T}{8m} \right]$$

$$= \frac{y'y''}{\pi^3} \sum_{\nu \in \Lambda} \int_0^\infty k\,dk \ \mathrm{me}_\nu^*(\eta', \tfrac{k^2}{4}) \,\mathrm{me}_\nu(\eta'', \tfrac{k^2}{4}) \,\mathrm{Me}_\nu^{(1)\,*}(\xi', \tfrac{k^2}{4}) \,\mathrm{Me}_\nu^{(1)}(\xi'', \tfrac{k^2}{4})$$

$$\times \int_0^\infty dp\,p \sinh \pi p K_{\mathrm{i}\,p}(ky') K_{\mathrm{i}\,p}(ky'') \,\mathrm{e}^{-\mathrm{i}\,\hbar T(p^2+1)/2\,m} \tag{5.160}$$

$$= \left( \frac{m}{2\pi\,\mathrm{i}\,\hbar T} \right)^{3/2} \frac{d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}')}{\sinh d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}')} \exp\left[ \frac{\mathrm{i}\,m}{2\hbar T} d_{\Lambda^{(3)}}^2(\mathbf{q}'', \mathbf{q}') - \frac{\mathrm{i}\,\hbar T}{2m} \right] \ , \tag{5.161}$$

where $d_{\Lambda^{(3)}}(\mathbf{q}'' \, \mathbf{q}')$ must be expressed in hyperbolic-spheroidal coordinates.

*5.3.12. Horicyclic-Parabolic Coordinates.* We consider the coordinate system

$$\left. \begin{aligned} u_0 &= \frac{1}{2}\left( \frac{1}{y} + y + \frac{(\xi^2 + \eta^2)^2}{4y} \right) \ , \quad y > 0 \ , \\ u_1 &= \frac{\xi\eta}{y} \ , \quad u_2 = \frac{\eta^2 - \xi^2}{2y} \ , \qquad \xi \in \mathrm{I\!R}, \eta > 0 \ , \\ u_3 &= \frac{1}{2}\left( \frac{1}{y} - y - \frac{(\xi^2 + \eta^2)^2}{4y} \right) \ . \end{aligned} \right\} \tag{5.162}$$

The hyperbolic distance is given by

$$\cosh d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}') = \frac{1}{2y'y''} \left[ y'^2 + y''^2 + \frac{1}{4}(\xi''^2 + \eta''^2)^2 \right.$$

$$\left. + \frac{1}{4}(\xi'^2 + \eta'^2)^2 - \frac{1}{2}(\xi''^2 - \eta''^2)(\xi'^2 - \eta'^2) - 2\xi''\xi'\eta''\eta' \right] \ . \tag{5.163}$$

Here we have $(g_{ab}) = \mathrm{diag}(1, \xi^2 + \eta^2, \xi^2 + \eta^2)$ together with the momentum operators (4.22,5.123). The Hamiltonian has the form

$$-\frac{\hbar^2}{2m} \Delta_{\Lambda^{(3)}} - \frac{\hbar^2}{2m} y^2 \left[ \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} + \frac{1}{\xi^2 + \eta^2}\left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \right]$$

$$= \frac{1}{2m} \left[ y p_y^2 y + y^2 (\xi^2 + \eta^2)^{-1/2} (p_\xi^2 + p_\eta^2)(\xi^2 + \eta^2)^{-1/2} \right] - \frac{3\hbar^2}{8m} \ . \tag{5.164}$$

Using the path integral solution corresponding to the two-dimensional parabolic coordinates in $E^{(2)}$ we obtain the identity

$$
\int_{y(t')=y''}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^3} \int_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \int_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t)(\xi^2 + \eta^2)
$$

$$
\times \exp\left[ \frac{\mathrm{i}\,m}{2\hbar} \int_{t'}^{t''} \frac{\dot{y}^2 + (\xi^2+\eta^2)(\dot{\xi}^2+\dot{\eta}^2)}{y^2}dt - \frac{3\hbar^2}{8m} \right]
$$

$$
= \frac{2y'y''}{\pi^2} \sum_{e,o} \int_{\mathbb{R}} d\zeta \int_{\mathbb{R}} dk\, \Psi^{(e,o)\,*}_{k,\zeta}(\xi',\eta')\Psi^{(e,o)}_{k,\zeta}(\xi'',\eta'')
$$

$$
\times \int_0^\infty dp\, p \sinh \pi p K_{\mathrm{i}\,p}(|k|y')K_{\mathrm{i}\,p}(|k|y'') \exp\left[ -\frac{\mathrm{i}\,\hbar T}{2m}(p^2 + 1) \right] , \tag{5.165}
$$

in the notation of 4.1.5.

*5.3.13. Confocal Ellipsoidal Coordinates.* Let us consider the coordinate system defined by

$$
\frac{u_0^2}{\rho_i - d} - \frac{u_1^2}{\rho_i - c} - \frac{u_2^2}{\rho_i - a} - \frac{u_3^2}{\rho_i - b} = 0 , \qquad (i = 1,2,3) , \tag{5.166}
$$

where $d < c < \rho_3 < b < \rho_2 < a < \rho_1$ (c.f. Section 3.1). Explicitly

$$
\left. \begin{aligned}
u_0^2 &= \frac{(\rho_1 - d)(\rho_2 - d)(\rho_3 - d)}{(a-d)(b-d)(c-d)} , & u_1^2 &= -\frac{(\rho_1 - c)(\rho_2 - c)(\rho_3 - c)}{(a-c)(b-c)(d-c)} , \\
u_2^2 &= -\frac{(\rho_1 - a)(\rho_2 - a)(\rho_3 - a)}{(d-a)(c-a)(b-a)} , & u_3^2 &= -\frac{(\rho_1 - b)(\rho_2 - b)(\rho_3 - b)}{(d-b)(c-b)(a-b)} .
\end{aligned} \right\} \tag{5.167}
$$

The corresponding line element is given by

$$
\frac{ds^2}{dt^2} = \frac{1}{4}\left[ \frac{(\rho_1 - \rho_2)(\rho_1 - \rho_3)}{P_4(\rho_1)}\dot{\rho}_1^2 + \frac{(\rho_2 - \rho_3)(\rho_2 - \rho_1)}{P_4(\rho_2)}\dot{\rho}_2^2 + \frac{(\rho_3 - \rho_1)(\rho_3 - \rho_2)}{P_4(\rho_3)}\dot{\rho}_3^2 \right] , \tag{5.168}
$$

where $P_4(\rho) = (\rho - a)(\rho - b)(\rho - c)(\rho - d)$. Note the similarities as well the differences in comparison to the confocal ellipsoidal coordinates on the sphere $S^{(3)}$. We apply the separation formula (3.21). We identify $(g_{ab}) \equiv \mathrm{diag}(h_1^2, h_2^2, h_3^2)$, furthermore

$$
\Gamma_i = \frac{\rho_i P_4'(\rho_i)}{P_4(\rho)} , \qquad (i = 1,2,3) \tag{5.169}
$$

$$
S = \frac{(\rho_1 - \rho_2)(\rho_1 - \rho_3)(\rho_2 - \rho_3)}{4P_4(\rho_1)P_4(\rho_2)P_4(\rho_2)} , \tag{5.170}
$$

$$
M_1 = \frac{\rho_2 - \rho_3}{P_4(\rho_2)P_4(\rho_3)} , \qquad M_2 = \frac{\rho_3 - \rho_1}{P_4(\rho_1)P_4(\rho_3)} , \qquad M_3 = \frac{\rho_1 - \rho_2}{P_4(\rho_1)P_4(\rho_2)} , \tag{5.171}
$$

and obtain the following path integral identity

$$
\prod_{i=1}^{3} \int_{\rho_i(t')=\rho_i'}^{\rho_i(t'')=\rho_i''} h_i \mathcal{D}\rho_i(t) \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2}\sum_{i=1}^{3} h_i^2 \dot{\rho}_i^2 - \Delta V_{PF}(\{\rho\}) \right] dt \right\}
$$

$$= (S' S'')^{-1/4} \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \, e^{-iET/\hbar} \int_0^{\infty} ds'' \prod_{i=1}^{3} \int_{\rho_i(0)=\rho_i'}^{\rho_i(s'')=\rho_i''} M_i^{-1/2} \mathcal{D}\rho_i(s)$$

$$\times \exp\left\{ \frac{i}{\hbar} \int_0^{s''} \left[ \frac{m}{2} \sum_{i=1}^{3} \frac{\dot{\rho}_i^2}{M_i} + ES - \frac{\hbar^2}{8m} \sum_{i=1}^{3} M_i \left( \Gamma_i^2 + 2\Gamma_i' \right) \right] ds \right\} \qquad (5.172)$$

$$= \left( \frac{m}{2\pi\, i\, \hbar T} \right)^{\frac{3}{2}} \frac{d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}')}{\sinh d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}')} \exp\left[ \frac{i\,m}{2\hbar T} d_{\Lambda^{(3)}}^2(\mathbf{q}'', \mathbf{q}') - \frac{i\,\hbar T}{2m} \right] , \qquad (5.173)$$

and $d_{\Lambda^{(3)}}(\mathbf{q}'', \mathbf{q}')$ must be expressed in confocal ellipsoidal coordinates. Actually, the relation $d < c < \dots$ is only one possibility to obtain a coordinate system on $\Lambda^{(3)}$. We can distinguish eighteen further cases, which fall into two classes with eleven and seven coordinate systems, respectively. In the first class emerges when ever two of the $\{e_i\} = \{a, b, c, d\}$ are equal. They represent the negative curvature cases in comparison to the positive curvature ones as noted in 5.2.6. The second class is obtained by a similar consideration as for the pseudo-conical coordinate systems in the two-dimensional case. C.f. Section 3.1 for the classification into the four general classes A,B,C,D and [90] for the explicit construction. Therefore we obtain a total of 34 different coordinate systems. This concludes the discussion of the separable coordinate systems on three-dimensional spaces of constant curvature.

## 6. Discussion and Summary

In this paper I have discussed path integrals in two- and three-dimensional spaces of constant curvature, i.e. the flat Euclidean spaces $E^{(2)}$ and $E^{(3)}$, the spheres $S^{(2)}$ and $S^{(3)}$, and the pseudospheres $\Lambda^{(2)}$ and $\Lambda^{(3)}$. In many cases I could perform the path integration explicitly, thanks to some basic path integral solutions. In the majority of the cases, in particular on the pseudospheres $\Lambda^{(2)}$ and $\Lambda^{(3)}$, only an indirect reasoning was possible. However, in any case, the *explicit* expressions of the propagators and the Green functions are known in terms of the invariant distance in this space. This is possible for any dimension, c.f. [31, 32] for the Euclidean space, c.f. [10, 55] for the spheres, and c.f. [10, 56, 107] for the pseudospheres, thus providing coordinate-independent expressions. Knowing them gives rise to numerous identities connecting the path integral formulations, explicit solutions in terms of the spectral expansions, and the coordinate-independent general formulæ.

There are several generalizations possible one can think of. First, one can study complex Riemannian manifolds [69, 71] and try to study corresponding path integral formulations. At least, the complex Riemannian manifold of constant negative curvature $\Lambda^{(\mathbb{C}_n)}$ has already attracted some attention in the theory of automorphic forms, in order to construct a Selberg trace formula [108] similar to the usual one [62, 102, 109]. Here also a path integral evaluation is possible [51]. However, it is obvious that complexity increases in these more elaborated cases.

Second, one can introduce potential problems and ask for the separable ones. There exist some studies of this kind in the literature, e.g. [49], where the most important one is the Kepler-Coulomb problem in spaces of constant (positive and negative) curvature

[9, 47]. However, no systematic study has been done until now. In the case of the Coulomb-problem, one is on the one hand interested in the symmetry properties and transformations between the bases of the system [17, 79, 95, 97], where the O(4)-symmetry of the hydrogen atom is the best known one and lies at the origin of the separability of this problem in four coordinate systems. On the other one is interested in the relation [63, 95] of the Kepler-problem and the harmonic oscillator in spaces of constant curvature. Whereas in flat space the transformation which relates the Coulomb problem in $\mathbb{R}^3$ and the isotropic harmonic oscillator in $\mathbb{R}^4$ is the Kustaanheimo-Stiefel transformation [23, 24], such a transformation is not known for constant (positive and negative) curvature. On $S^{(3)}$ and $\Lambda^{(3)}$ the Coulomb problem and the harmonic oscillator are only separable in two coordinate system, the spherical and the sphero-conical, respectively. the explict form on $E^{(3)}$, $S^{(3)}$ and $\Lambda^{(3)}$ is given in the following table:

| | Kepler problem | Harmonic Oscillator |
|---|---|---|
| $E^{(3)}$ | $-\dfrac{\alpha}{r}$ | $\dfrac{m}{2}\omega^2 r^2$ |
| $S^{(3)}$ | $-\dfrac{\alpha}{R}\cot\theta_1$ | $\dfrac{m}{2}\omega^2 R^2 \tan^2\theta_1$ |
| $\Lambda^{(3)}$ | $-\dfrac{\alpha}{R}\coth\tau$ | $-\dfrac{m}{2}\omega^2 R^2 \tanh^2\tau$ |

(where we have reintroduced $R$, and the coordinate notation is as in Section 5). And as a third, one can introduce constant magnetic fields on spheres and pseudospheres [44, 48]. The corresponding cases in flat space are well-known, e.g. [31, 32, 58, 74].

A systematic study for the search of separable potentials in $\mathbb{R}^2$ and $\mathbb{R}^3$ does exist [26, 30], a project which has been started by Smorodinsky, Winternitz et al. [81]. These potentials can be put into two classes, which are called maximally and minimally superintegrable, respectively. A Hamiltonian system in three degrees of freedom is called maximally superintegrable if it admits five globally defined and single-valued integrals of motion (the Coulomb-problem and the harmonic oscillator belong to this class). A Hamiltonian system in three degrees of freedom is called minimally superintegrable if it admits four globally defined and single-valued integrals of motion (with the ring-potential as an example [51]). The maximally superintegrable potentials can be stated explicitly, whereas the (eight) minimally superintegrable potentials allow in addition to an explicit expression an arbitrary function of the coordinates according to either $\propto F(r)$, $\propto F(z)$ or $\propto F(y/x)$, respectively. The corresponding super-integrable potentials in $\mathbb{R}^2$ can be obtained in a similar way, and there are a total number of four independent potentials [34]. The five maximally superintegrable potentials in $\mathbb{R}^3$ are:

$$V_1(x,y,z) = \frac{m}{2}\omega^2(x^2+y^2+z^2) + \frac{\hbar^2}{2m}\left(\frac{k_1}{x^2}+\frac{k_2}{y^2}+\frac{k_3}{z^2}\right) ; \qquad (6.1)$$

this potential is separable in (with the coordinates systems which admit explicit path integral evaluation in *italic*) *cartesian, spherical, circular cylindrical*, elliptic cylindrical, oblate spheroidal, prolate spheroidal, conical, and ellipsoidal coordinates.

$$V_2(x,y,z) = -\frac{\alpha}{\sqrt{x^2+y^2+z^2}} + \frac{\hbar^2}{2m}\left(\frac{k_1}{x^2}+\frac{k_2}{y^2}\right) ; \qquad (6.2)$$

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this potential is separable in *spherical, parabolic*, and conical coordinates.

$$V_3(x,y,z) = \frac{\hbar^2}{2m}\left(\frac{k_1 x}{\sqrt{x^2 + y^2}} + \frac{k_2}{y^2} + \frac{k_3}{z^2}\right) \ ; \qquad (6.3)$$

this potential is separable in *spherical and parabolic cylindrical coordinates*.

$$V_4(x,y,z) = \frac{\hbar^2}{2m}\left(\frac{k_1 x}{\sqrt{x^2 + y^2}} + \frac{k_2}{y^2}\right) + k_3\,z \ ; \qquad (6.4)$$

this potential is separable in *parabolic cylindrical* and parabolic coordinates.

$$V_5(x,y,z) = \frac{m}{2}\omega^2(x^2 + y^2 + 4z^2) + \frac{\hbar^2}{2m}\left(\frac{k_1}{x^2} + \frac{k_2}{y^2}\right); \qquad (6.5)$$

this potential is separable in *cartesian*, parabolic (one obtains an intractable quadratic plus sextic oscillator), and elliptic cylindrical coordinates.

In each case it is easy to construct the various path integral formulations, based on the coordinate systems as listed in Section 5.1, and obtain therefore the various identities connecting them. The corresponding path integral evaluations then can be performed by using the basic path integrals as given in Appendix 1, which is left as an exercise to the interested reader. Note that in comparison to the free motion in spaces of constant curvature the statement of the propagator in a coordinate independent way is not possible. This concludes the discussion.

**Acknowledgments**

**Appendix 1. Some Important Path Integral Solutions**

In this Appendix we cite some important path integral solutions, in particular for the radial harmonic oscillator and for the (modified) Pöschl-Teller potential, including some special cases. These three path integral solutions are the most important building blocks for almost all other path integrals.

*A.1.1. The path integral for the radial harmonic oscillator.*
The calculation of the path integral for the radial harmonic oscillator has first been performed by Peak and Inomata [92]. A more general case is due to Goovaerts [37] (c.f. also [22]). Path integrals related to the radial harmonic oscillator may be called of

Besselian type [65]. Here we are not going into the subtleties of the Besselian functional measure due to the Bessel functions which appear in the lattice approach [33, 55, 104] which is actually necessary for the explicit evaluation of the radial harmonic oscillator path integral [21, 37, 92]. One obtains (modulo the mentioned subtleties) ($r > 0$)

$$
\int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2}\left( \dot{r}^2 - \omega^2 r^2 \right) - \hbar^2 \frac{\lambda^2 - \frac{1}{4}}{2mr^2} \right] dt \right\}
$$

$$
:= \int_{r(t')=r'}^{r(t'')=r''} \mu_\lambda[r^2] \mathcal{D}r(t) \exp\left[ \frac{i\,m}{2\hbar} \int_{t'}^{t''} \left( \dot{r}^2 - \omega^2 r^2 \right) dt \right]
$$

$$
= \frac{m\omega\sqrt{r'r''}}{i\,\hbar\sin\omega T} \exp\left[ -\frac{m\omega}{2\,i\,\hbar}(r'^2 + r''^2)\cot\omega T \right] I_\lambda\left( \frac{m\omega r' r''}{i\,\hbar\sin\omega T} \right) \qquad (A1.1)
$$

with the nontrivial functional weight $\mu_\lambda[r^2]$:

$$
\mu_\lambda[r_j r_{j-1}] = \sqrt{\frac{2\pi m r_j r_{j-1}}{i\,\epsilon\hbar}} \; e^{-m r_j r_{j-1}/i\,\epsilon\hbar} \, I_\lambda\left( \frac{m r_j r_{j-1}}{i\,\epsilon\hbar} \right) \;. \qquad (A1.2)
$$

In the special case that $\omega = 0$, i.e. for the free radial motion, we obtain

$$
\int_{r(t')=r'}^{r(t'')=r''} \mu_\lambda[r^2] \mathcal{D}r(t) \exp\left( \frac{i\,m}{2\hbar} \int_{t'}^{t''} \dot{r}^2 \, dt \right)
$$

$$
= \sqrt{r'r''}\, \frac{m}{i\,\hbar T} \exp\left[ -\frac{m}{2\,i\,\hbar T}(r'^2 + r''^2) \right] I_\lambda\left( \frac{m r' r''}{i\,\hbar T} \right) \qquad (A1.3)
$$

$$
= \sqrt{r'r''} \int_0^\infty dp\, p\, e^{-i\,\hbar T p^2/2m} \, J_\lambda(pr') J_\lambda(pr'') \;. \qquad (A1.4)
$$

A.1.2. *The modified Pöschl-Teller Potential.*
The path integral solution for the Pöschl-Teller potential can be achieved by means of the SU(2)-path integral. We have [10, 21, 66]

$$
\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left( \frac{\alpha^2 - \frac{1}{4}}{\sin^2 x} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 x} \right) \right] dt \right\}
$$

$$
= \sum_{l=0}^{\infty} \exp\left[ -\frac{i\,\hbar T}{2m}(\alpha + \beta + 2l + 1)^2 \right] \Psi_l^{(\alpha,\beta)\,*}(x') \Psi_l^{(\alpha,\beta)}(x'') \;, \qquad (A1.5)
$$

with the wavefunctions given by

$$
\Psi_n^{(\alpha,\beta)}(x) = \left[ 2(\alpha + \beta + 2l + 1)\frac{l!\,\Gamma(\alpha + \beta + l + 1)}{\Gamma(\alpha + l + 1)\Gamma(\beta + l + 1)} \right]^{1/2}
$$

$$
\times (\sin x)^{\alpha + 1/2}(\cos x)^{\beta + 1/2} P_n^{(\alpha,\beta)}(\cos 2x) \;. \qquad (A1.6)
$$

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*A.1.3. The modified Pöschl-Teller Potential.*

The path integral solution for the modified Pöschl-Teller potential can be achieved by means of the $SU(1,1)$-path integral. We have [10, 66, 68]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r}-\frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right)\right]dt\right\}$$

$$=\sum_{n=0}^{N_M}\Phi_n^{(\eta,\nu)*}(r')\Phi_n^{(\eta,\nu)}(r'')\exp\left\{-\frac{i\hbar T}{2m}\left[2(k_1-k_2-n)-1\right]^2\right\}$$

$$+\int_0^\infty dp\,\Phi_p^{(\eta,\nu)*}(r')\Phi_p^{(\eta,\nu)}(r'')\exp\left(-\frac{i\hbar T}{2m}p^2\right)\ . \tag{A1.7}$$

Now introduce the numbers $k_1,k_2$ defined by: $k_1 = \frac{1}{2}(1\pm\nu)$, $k_2=\frac{1}{2}(1\pm\eta)$, where the correct sign depends on the boundary-conditions for $r\to 0$ and $r\to\infty$, respectively. In particular for $\eta^2=\frac{1}{4}$, i.e. $k_2=\frac{1}{4},\frac{3}{4}$, we obtain wavefunctions with even and odd parity, respectively. The number $N_M$ denotes the maximal number of states with $0,1,\dots,N_M < k_1-k_2-\frac{1}{2}$. The bound state wavefunctions read as ($\kappa = k_1-k_2-n$)

$$\left.\begin{aligned}
\Phi_n^{(k_1,k_2)}(r) &= N_n^{(k_1,k_2)}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{-2k_1+\frac{3}{2}}\\
&\quad\times {}_2F_1(-k_1+k_2+\kappa,-k_1+k_2-\kappa+1;2k_2;-\sinh^2 r)\\
N_n^{(k_1,k_2)} &= \frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa-1)\Gamma(k_1+k_2-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2}
\end{aligned}\right\} \tag{A1.8}$$

The scattering states are given by:

$$\left.\begin{aligned}
\Phi_p^{(k_1,k_2)}(r) &= N_p^{(k_1,k_2)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}}\\
&\quad\times {}_2F_1(k_1+k_2-\kappa,k_1+k_2+\kappa-1;2k_2;-\sinh^2 r)\\
N_p^{(k_1,k_2)} &= \frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\\
&\quad\times\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2},
\end{aligned}\right\} \tag{A1.9}$$

$[\kappa=\frac{1}{2}(1+ip)]$. Of course, in the path integral formulation of the modified Pöschl-Teller potential a similar functional weight interpretation must be used as for the Pöschl-Teller potential in order to have a proper short-time behaviour, respectively a lattice regularization [46].

I also cite two special cases, where first only the $1/\sinh^2 r$, and second where only the $1/\cosh r$ term is present. The special case $V^{(sh)}(r)=V_0/\sinh^2 r$ ($V_0=\frac{\hbar^2}{2m}(\lambda^2-\frac{1}{4})$, $r>0$) is given by [46, 56, 75] (simple Manning-Rosen potential)

$$\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2-\frac{\hbar^2}{2m}\frac{\lambda^2-\frac{1}{4}}{\sinh^2 r}\right)dt\right]$$

$$= \frac{1}{\pi} \sqrt{\sinh r' \sinh r''} \int_0^\infty dp\, p \sinh \pi p \left| \Gamma(\tfrac{1}{2} + \mathrm{i}\, p - \lambda) \right|^2$$

$$\times \left| \Gamma(\tfrac{1}{2} + \mathrm{i}\, p - \lambda) \right|^2 \mathcal{P}^{-\lambda}_{\mathrm{i}\, p - 1/2}(\cosh r') \mathcal{P}^{-\lambda}_{\mathrm{i}\, p - 1/2}(\cosh r'') \, \mathrm{e}^{-\mathrm{i}\, \hbar p^2 T / 2m} \quad .$$

(A 1.10)

The special case $V^{(ch)}(x) = V_0 / \cosh^2 x$ $(V_0 = \frac{\hbar^2}{2m}(k^2 + \frac{1}{4}), \ x \in \mathrm{I\!R})$ on the other is given by [45, 46, 75] (simple Rosen-Morse potential)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \dot{x}^2 - \frac{\hbar^2}{2m} \frac{k^2 + \frac{1}{4}}{\cosh^2 x} \right) dt \right]$$

$$= \frac{1}{2} \int_{\mathrm{I\!R}} dp\, p \sinh \pi p \, \frac{P^{-\mathrm{i}\, p}_{\mathrm{i}\, k - 1/2}(\tanh x') P^{\mathrm{i}\, p}_{\mathrm{i}\, k - 1/2}(\tanh x'')}{\cosh^2 \pi k + \sinh^2 \pi p} \, \mathrm{e}^{-\mathrm{i}\, \hbar p^2 T / 2m} \quad .$$

(A 1.11)

Note that for $V_0 < 0$ also bound state solutions are allowed. These, however, we do not need.

## Appendix 2. Discussion of a Dispersion Relation

We consider the integral representation as given by Buchholz [13, p.158]:

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{-\sigma - \mathrm{i}\, \infty}^{-\sigma + \mathrm{i}\, \infty} \Gamma\left( \frac{1 + \mu}{2} - s \right) \Gamma\left( \frac{1 + \mu}{2} + s \right) \left( \tan \frac{\phi}{2} \right)^{2s}$$

$$\times \mathcal{M}_{\chi_1 + s - \frac{1 + \mu}{2}, \frac{\mu}{2}}(-\mathrm{i}\, x) \mathcal{M}_{\chi_2 + s - \frac{1 + \mu}{2}, \frac{\mu}{2}}(+\mathrm{i}\, y) ds$$

$$= \frac{\sqrt{xy}}{2} \sin \phi \exp\left[ \frac{\mathrm{i}}{2}(x - y) \cos \phi \right] J_\mu(\sqrt{xy} \sin \phi) \quad ,$$

where $\quad \left| \arctan \frac{\phi}{2} \right| < \frac{\pi}{2} \quad , \qquad |\sigma| = \frac{1 + \Re(\mu)}{2} \quad , \qquad \Re(\mu) > -1 \quad .$

(A2.1)

In order to apply (A2.1) for the determination of continuous spectra we perform some manipulations. We replace $x \to 2x$, $y \to -2y$, $s \to +\mathrm{i}\, p$, $\chi_{1,2} \to (1 + \mu)/2$, $\sin \phi \to 1/\sin \alpha$, and $\mu \to 2\mu$. This gives

$$\frac{1}{\sin \alpha} \exp\left[ -(x + y) \cot \alpha \right] I_{2\mu}\left( \frac{2\sqrt{xy}}{\sin \alpha} \right)$$

$$= \frac{1}{2\pi \sqrt{xy}} \int_{\mathrm{I\!R}} \frac{\Gamma(\tfrac{1}{2} + \mu + \mathrm{i}\, p) \Gamma(\tfrac{1}{2} + \mu - \mathrm{i}\, p)}{\Gamma^2(1 + 2\mu)} \, \mathrm{e}^{-2\alpha p + \pi p} M_{+\mathrm{i}\, p, \mu}(-2\mathrm{i}\, x) M_{-\mathrm{i}\, p, \mu}(+2\mathrm{i}\, y) dp \quad ,$$

(A2.2)

where use has been made of some properties of the Whittaker functions $M_{\mu, \nu}(z) = \Gamma(1 + 2\mu) \mathcal{M}_{\mu, \nu}$, and [13, p.11] $M_{\chi, \mu/2}(z \, \mathrm{e}^{\pm \mathrm{i}\, \pi}) = \mathrm{e}^{\pm \mathrm{i}\, \pi(1+\mu)/2} M_{-\chi, \mu/2}(z)$, respectively. Equation (A2.2) is the desired expansion formula for the determination of continuous

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spectra. Note the additional $e^{\pi p}$-factor in (A2.2). This representation can also be deduced from the integral formula [29, p.414; 41, p.884]

$$\int_{\mathbb{R}} dx \; e^{-2\,i\,x\rho}\, \Gamma\left(\tfrac{1}{2}+\nu+i\,x\right)\Gamma\left(\tfrac{1}{2}+\nu-i\,x\right)\mathcal{M}_{i\,x,\nu}(\mu)\mathcal{M}_{i\,x,\nu}(\nu)$$
$$= \frac{2\pi\sqrt{\mu\nu}}{\cosh\rho}\exp\left[-(\mu+\nu)\tanh\rho\right]J_{2\nu}\left(\frac{2\sqrt{\mu\nu}}{\cosh\rho}\right) \; . \quad \text{(A2.3)}$$

We consider the special case of (A2.2), where the Bessel function is replaced by an exponential, i.e.:

$$I_\mu\left(\frac{2\sqrt{xy}}{\sin\alpha}\right) \to \exp\left(\frac{2\sqrt{xy}}{\sin\alpha}\right) \; .$$

Together with the properties of the Whittaker functions that for $\mu=\pm\tfrac{1}{2}$ they can be expressed by parabolic cylinder functions $E_\nu^{(0,1)}$ we obtain

$$\frac{1}{\sqrt{2\pi\sin\alpha}}\exp\left[-(x+y)\cot\alpha\right]\exp\left(\frac{2\sqrt{xy}}{\sin\alpha}\right)$$
$$= \frac{\sqrt{\tfrac{1}{2}\sqrt{xy}}}{\sin\alpha}\exp\left[-(x+y)\cot\alpha\right]\left[I_{\frac{1}{2}}\left(\frac{2\sqrt{xy}}{\sin\alpha}\right)+I_{-\frac{1}{2}}\left(\frac{2\sqrt{xy}}{\sin\alpha}\right)\right]$$
$$= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} dp \; e^{-2\,\alpha p+\pi p}$$
$$\times\left[\left|\Gamma\left(\frac{1}{4}-i\,p\right)\right|^2 E^{(0)}_{-\frac{1}{2}+2\,i\,p}\left(e^{-i\,\pi/4}\,2\sqrt{x}\right)E^{(0)}_{-\frac{1}{2}-2\,i\,p}\left(e^{i\,\pi/4}\,2\sqrt{y}\right)\right.$$
$$\left.+\left|\Gamma\left(\frac{3}{4}-i\,p\right)\right|^2 E^{(1)}_{-\frac{1}{2}+2\,i\,p}\left(e^{-i\,\pi/4}\,2\sqrt{x}\right)E^{(1)}_{-\frac{1}{2}-2\,i\,p}\left(e^{i\,\pi/4}\,2\sqrt{y}\right)\right] \; , $$
$$\text{(A2.4)}$$

The functions $E_\nu^{(0)}(z)$ and $E_\nu^{(1)}(z)$ are even and odd in the variable $z$, respectively. We can establish the connection to the parabolic cylinder functions $D_\nu$ and (A2.4) is equivalent with the expansion [41, p.896]:

$$\int_{c-i\infty}^{c+i\infty} \left[D_\nu(x)D_{-\nu-1}(i\,y)+D_\nu(-x)D_{-\nu-1}(-i\,y)\right]\frac{t^{-\nu-1}d\nu}{\sin(-\nu\pi)}$$
$$= 2\,i\,\sqrt{\frac{2\pi}{1+t^2}}\exp\left[\frac{1-t^2}{1+t^2}\cdot\frac{x^2+y^2}{4}+\frac{i\,txy}{1+t^2}\right] \; . \quad \text{(A2.5)}$$

## Appendix 3. The $D$-Dimensional Hyperbolic System

We consider the $D$-dimensional generalization of the hyperbolic coordinate system:

$$\left.\begin{aligned}
u_0 &= \cosh\tau_1\ldots\cosh\tau_{D-1} \ , \\
u_1 &= \sinh\tau_1 \ , \\
u_2 &= \cosh\tau_1\sinh\tau_2 \ , \\
&\ \vdots \\
u_{D-1} &= \cosh\tau_1\ldots\sinh\tau_{D-1} \ .
\end{aligned}\right\} \tag{A3.1}$$

Therefore we obtain the path integral formulation by successively applying (A1.11) $(\tau_1,\ldots,\tau_{D-1}\in\mathbb{R})$

$$\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \cosh^{D-2}\tau_1\,\mathcal{D}\tau_1(t)\cdots \int_{\tau_{D-1}(t')=\tau_{D-1}'}^{\tau_{D-1}(t'')=\tau_{D-1}''} \mathcal{D}\tau_{D-1}(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot\tau_1^2 + \cosh^2\tau_1\dot\tau_2^2 + \ldots(\cosh^2\tau_1\ldots\cosh^2\tau_{D-2})\dot\tau_{D-1}^2\right)\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{8m}\left((D-2)^2 + \frac{1}{\cosh^2\tau_1} + \cdots + \frac{1}{\cosh^2\tau_1\ldots\cosh^2\tau_{D-2}}\right)\right]dt\right\}$$

$$= \left(\cosh^{D-2}\tau_1'\cosh^{D-2}\tau_1''\ldots\cosh\tau_{D-2}'\cosh\tau_{D-2}''\right)^{-1/2}\int_{\mathbb{R}}\frac{dp_0}{2\pi}\,\mathrm{e}^{\mathrm{i}\,p_0(\tau_{D-1}''-\tau_{D-1}')}$$

$$\times\prod_{j=1}^{D-2}\frac{1}{2}\int_{\mathbb{R}}\Psi_{\mathrm{i}\,p_j-1}^{-\mathrm{i}\,p_j}(\tau_{D-1-j}')\Psi_{\mathrm{i}\,p_j-1}^{\mathrm{i}\,p_j}(\tau_{D-1-j}'')\exp\left[-\frac{\mathrm{i}\,\hbar T}{2m}\left(p_{D-2}^2 + \frac{(D-2)^2}{4}\right)\right] \ . \tag{A3.2}$$

with the wave-functions $\Psi_{\mathrm{i}\,k}^{\mathrm{i}\,p}(\tau)$ given by

$$\Psi_{\mathrm{i}\,k}^{\mathrm{i}\,p}(\tau) = \sqrt{\frac{p\sinh\pi p}{2(\cosh^2\pi k + \sinh^2\pi p)}}\,P_{\mathrm{i}\,k-\frac{1}{2}}^{\mathrm{i}\,p}(\tanh\tau) \ . \tag{A3.3}$$

This solution represents an alternative path integral solutions as already outlined for two other coordinate systems in [51]. Note that irrespective which coordinate system for the path integral formulation on $\Lambda^{(D-1)}$ is chosen (and there are many indeed) the Green function *always* has the form [51, 56]

$$G^{\Lambda^{(D-1)}}\left(d_{\Lambda^{(D-1)}}(\mathbf{q}'',\mathbf{q}');E\right) = \frac{m}{\pi\hbar^2}\left(\frac{-1}{2\pi\sinh d_{\Lambda^{(D-1)}}(\mathbf{q}'',\mathbf{q}')}\right)^{(D-3)/2}$$

$$\times\mathcal{Q}_{-\mathrm{i}\sqrt{2mE/\hbar^2-(D-2)^2/4}-1/2}^{(D-3)/2}\left(\cosh d_{\Lambda^{(D-1)}}(\mathbf{q}'',\mathbf{q}')\right) \ . \tag{A3.4}$$

# References

[1] M.Abramowitz and I.A.Stegun (Eds.): Pocketbook of Mathematical Functions (*Verlag Harry Deutsch*, Frankfurt/Main, 1984)

[2] S.Albeverio, P.Blanchard and R.Høegh-Krohn: Some Applications of Functional Integration; *Lecture Notes in Physics* **153** (*Springer-Verlag*, Berlin, 1982)

[3] S.Albeverio and R.J.Høegh-Krohn: Mathematical Theory of Feynman Path Integrals; *Lecture Notes in Mathematics* **523** (*Springer-Verlag*, Berlin, 1976)

[4] A.M.Arthurs: Path Integrals in Polar Coordinates; *Proc.Roy.Soc.(London)* **A 313** (1969) 445

[5] R.Aurich, C.Matthies, M.Sieber and F.Steiner: Novel Rule for Quantizing Chaos; *Phys.Rev.Lett.* **68** (1992) 1629

[6] R.Aurich, M.Sieber and F.Steiner: Quantum Chaos of the Hadamard-Gutzwiller Model; *Phys.Rev. Lett.* **61** (1988) 483

[7] R.Aurich and F.Steiner: From Classical Periodic Orbits to the Quantization of Chaos; *Proc.Roy.Soc. (London)* **A 437** (1992) 693; Statistical Properties of Highly Excited Quantum Eigenstates of a Strongly Chaotic System; *Physica* **D 64** (1993) 185

[8] N.L.Balazs and A.Voros: Chaos on the Pseudosphere; *Phys.Rep.* **143** (1986) 109

[9] A.O.Barut, A.Inomata and G.Junker: Path Integral Treatment of the Hydrogen Atom in a Curved Space of Constant Curvature; *J.Phys.A: Math.Gen.* **20** (1987) 6271; Path Integral Treatment of the Hydrogen Atom in a Curved Space of Constant Curvature II; *J.Phys.A: Math.Gen.* **23** (1990) 1179

[10] M.Böhm and G.Junker: Path Integration Over Compact and Noncompact Rotation Groups; *J.Math.Phys.* **28** (1987) 1978

[11] M.Böhm and G.Junker: Path Integration Over the n-Dimensional Euclidean Group; *J.Math.Phys.* **30** (1989) 1195

[12] M.Böhm and G.Junker: Group Theoretical Approach to Path Integrations on Spheres; in "Path Summation: Achievements and Goals", Trieste, 1987, p.469, eds.: S.Lundquist et al. (*World Scientific*, Singapore, 1988)

[13] H.Buchholz: The Confluent Hypergeometric Function, Springer Tracts in Natural Philosophy, Volume 15 (*Springer-Verlag*, Berlin, 1969)

[14] D.P.L.Castrigiano and F.Stärk: New Aspects of the Path Integrational Treatment of the Coulomb Potential; *J.Math.Phys.* **30** (1989) 2785.

[15] L.Chetouani et T.F.Hammann: Traitement Exact des Système Coulombiens, dans le Formalisme des Intégrales de Feynman, en Coordonnées Parabolique; *Nuovo Cimento* **B 98** (1987) 1

[16] G.Darboux: Leçons sur les Systéme Orthogonaux et les Coordonées Curvilignes (Paris, 1910)

[17] L.S.Davtyan, G.S.Pogosyan, A.N.Sisakyan, and V.M.Ter-Antonyan: Transformations Between Parabolic Bases of the Two-Dimensional Hydrogen Atom in the Continuous Spectrum; *Theor.Math. Phys.* **74** (1988) 157;
L.G.Mardoyan, G.S.Pogosyan, A.N.Sisakyan, and V.M.Ter-Antonyan: Hidden Symmetry, Separation of Variables and Interbasis Expansion in the Two-Dimensional Hydrogen Atom; *J.Phys.A: Math,Gen.* **18** (1985) 455

[18] B.S.DeWitt: Point Transformations in Quantum Mechanics; *Rev.Mod.Phys.* **29** (1957) 377.

[19] W.Dittrich and M.Reuter: Classical and Quantum Dynamics. From Classical Paths to Path Integrals (*Springer-Verlag*, Berlin, 1992)

[20] J.C.D'Olivio and M.Torres: The Canonical Formalism and Path Integrals in Curved Spaces; *J.Phys.A: Math.Gen.* **21** (1988) 3355;

[21] I.H.Duru: Path Integrals Over SU(2) Manifold and Related Potentials; *Phys.Rev.* **D 30** (1984) 2121

[22] I.H.Duru: On the Path Integral for the Potential $V = ar^{-2} + br^2$; *Phys.Lett.* **A 112** (1985) 421

[23] I.H.Duru and H.Kleinert: Solution of the Path Integral for the H-Atom; *Phys.Lett.* **B 84** (1979) 185

[24] Quantum Mechanics of H-Atoms From Path Integrals; *Fortschr.Phys.* **30** (1982) 401

[25] S.F.Edwards and Y.V.Gulyaev: Path Integrals in Polar Co-ordinates; *Proc.Roy.Soc.(London)* **A 279** (1964) 229

[26] L.P.Eisenhart: Enumeration of Potentials for Which One-Particle Schroedinger Equations Are Separable; *Phys.Rev.* **74** (1948) 87

[26] J.Elstrod, F.Grunewald and J.Mennicke: The Selberg Zeta-Function for Cocompact Discrete Subgroups of PSL(2, ℂ); *Elementary Theory of Numbers, Banach Center Publications* **17** (1985) 83

[28] A.Erdélyi, W.Magnus, F.Oberhettinger and F.G.Tricomi (Eds.): Higher Transcendental Functions, Vol.II (*McGraw Hill*, New York, 1985)

[29] A.Erdélyi, W.Magnus, F.Oberhettinger and F.G.Tricomi (Eds.): Tables of Integral Transforms, Vol.II (*McGraw Hill*, New York, 1954)

[30] N.W.Ewans: Superintegrability in Classical Mechanics; *Phys.Rev.* **A 41** (1990) 5666; Group Theory of the Smorodinsky-Winternitz System; *J.Math.Phys.* **32** (1991) 3369

[31] R.P.Feynman: Space-Time Approach to Non-Relativistic Quantum Mechanics; *Rev.Mod.Phys.* **20** (1948) 367.

[32] R.P.Feynman and A.Hibbs: Quantum Mechanics and Path Integrals (*McGraw Hill, New York*, 1965)

[33] W.Fischer, H.Leschke and P.Müller: Changing Dimension and Time: Two Well-Founded and Practical Techniques for Path Integration in Quantum Physics; *J.Phys.A: Math.Gen.* **25** (1992) 3835; Path Integration in Quantum Physics by Changing the Drift of the Underlying Diffusion Process; *Universität Erlangen-Nürnberg preprint*, February 1992

[34] J.Friš, V.Mandrosov, Ya.A.Smorodinsky, M.Uhlir and P.Winternitz: On Higher Symmetries in Quantum Mechanics; *Phys.Lett.* **16** (1965) 354;
J.Friš, Ya.A.Smorodinskiĭ, M.Uhlíř and P.Winternitz: Symmetry Groups in Classical and Quantum Mechanics; *Sov.J.Nucl.Phys.* **4** (1967) 444

[35] I.M.Gelfand and A.M.Jaglom: Die Integration in Funktionenräumen und ihre Anwendung in der Quantentheorie; *Fortschr.Phys.* **5** (1957) 517;
I.M.Gel'fand and A.M.Yaglom: Integration in Functional Spaces and its Applications in Quantum Physics; *J.Math.Phys.* **1** (1960) 48

[36] J.-L.Gervais and A.Jevicki: Point Canonical Transformations in the Path Integral; *Nucl.Phys.* **B 110** (1976) 93.

[37] M.J.Goovaerts: Path-Integral Evaluation of a Nonstationary Calogero Model; *J.Math.Phys.* **16** (1975) 720

[38] I.M.Gel'fand and M.I.Graev: Geometry of Homogeneous Spaces, Representations of Groups in Homogeneous Spaces and Related Questions. I; *Amer.Math.Soc.Transl., Ser.2* **37** (1964) 351

[39] I.M.Gel'fand, M.I.Graev, and N.Ya.Vilenkin: Generalized Functions, Vol.5 (*Academic Press*, New York, 1966)

[40] J.Glimm and A.Jaffe: Quantum Physics: A Functional Point of View (*Springer-Verlag*, Berlin, 1981)

[41] I.S.Gradshteyn and I.M.Ryzhik: Table of Integrals, Series, and Products, Academic Press, New York, 1980

[42] M.B.Green, J.H.Schwarz and E.Witten: Superstring Theory I&II (*Cambridge University Press*, Cambridge, 1988)

[43] C.Grosche: The Product Form for Path Integrals on Curved Manifolds; *Phys.Lett.* **A 128** (1988) 113.

[44] C.Grosche: The Path Integral on the Poincaré Upper Half-Plane With a Magnetic Field and for the Morse Potential; *Ann.Phys.(N.Y.)* **187** (1988) 110

[45] C.Grosche: The Path Integral on the Poincaré Disc, the Poincaré Upper Half-Plane and on the Hyperbolic Strip; *Fortschr.Phys.* **38** (1990) 531

[46] C.Grosche: Path Integral Solution of a Class of Potentials Related to the Pöschl-Teller Potential; *J.Phys.A: Math.Gen.* **22** (1989) 5073

[47] C.Grosche: The Path Integral for the Kepler Problem on the Pseudosphere; *Ann.Phys.(N.Y.)* **204** (1990) 208

[48] C.Grosche: Path Integration on the Hyperbolic Plane With a Magnetic Field; *Ann.Phys.(N.Y.)* **201** (1990) 258

[49] C.Grosche: Separation of Variables in Path Integrals and Path Integral Solution of Two Potentials on the Poincaré Upper Half-Plane; *J.Phys.A: Math.Gen.* **23** (1990) 4885

[50] C.Grosche: Coulomb Potentials by Path-Integration; *Fortschr.Phys.* **40** (1992) 695

[51] C.Grosche: Path Integration on Hyperbolic Spaces; *J.Phys.A: Math.Gen.* **25** (1992) 4211

[52] C.Grosche: Path Integral Solution of Two Potentials Related to the SO(2,1) Dynamical Algebra; *J.Phys.A: Math.Gen.* **26** (1993) L279

[53] C.Grosche: Path Integral Solution of a Class of Explicitly Time-Dependent Potentials; *Trieste preprint*, SISSA/2/93/FM, *Phys.Lett.* **A**, in press

[54] C.Grosche and F.Steiner: The Path Integral on the Poincaré Upper Half Plane and for Liouville Quantum Mechanics; *Phys.Lett.* **A 123** (1987) 319

[55] C.Grosche and F.Steiner: Path Integrals on Curved Manifolds; *Zeitschr.Phys.* **C 36** (1987) 699

[56] C.Grosche and F.Steiner: The Path Integral on the Pseudosphere; *Ann.Phys.(N.Y.)* **182** (1988) 120

[57] C.Grosche and F.Steiner: Classification of Solvable Feynman Path Integrals; *DESY preprint* DESY 92-189, to appear in the *Proceedings of the "Fourth International Conference on Path Integrals from meV to MeV", May 1992, Tutzing, Germany*, World Scientific, Singapore

[58] C.Grosche and F.Steiner: Table of Feynman Path Integrals; to appear in: *Springer Tracts in Modern Physics*

[59] C.Grosche and F.Steiner: Feynman Path Integrals; to appear in: *Springer Lecture Notes in Physics*

[60] M.C.Gutzwiller: The Geometry of Quantum Chaos; *Physica Scripta* **T 9** (1985) 184

[61] M.C.Gutzwiller: Chaos in Classical and Quantum Mechanics (*Springer-Verlag*, New York, 1990)

[63] P.W.Higgs: Dynamical Symmetries in a Spherical Geometry; *J.Phys.A: Math.Gen.* **12** (1979) 309

[62] D.A.Hejhal: The Selberg Trace Formula for PSL(2, ℝ), I&II; *Lecture Notes in Mathematics* **548, 1001** (*Springer-Verlag*, Berlin, 1976)

[64] A.Inomata: Exact Path-Integration for the Two Dimensional Coulomb Problem; *Phys.Lett.* **A 87** (1982) 387

[65] A.Inomata: Roles of Dynamical Groups in Path Integration; to appear in the proceedings of the "International Workshop on 'Symmetry Methods in Physics' in Memory of Prof. Ya. A. Smorodinsky", Dubna, July 1993

[66] A.Inomata and R.Wilson: Path Integral Realization of a Dynamical Group; *Lecture Notes in Physics* **261**, p.42 (*Springer*, Berlin-Heidelberg, 1985); Factorization-Algebraization-Path Integration and Dynamical Groups; in "Symmetries in Science II", eds.: B.Gruber and R.Lenczewski, p.255 (*Plenum Press*, New York, 1986)

[67] G.Junker: Remarks on the Local Time Rescaling in Path Integration; *J.Phys.A: Math.Gen.* **23** (1990) L881

[68] G.Junker and M.Böhm: The SU(1,1) Propagator as a Path Integral Over Noncompact Groups; *Phys.Lett.* **A 117** (1986) 375

[69] E.G.Kalnins: Separation of Variables for Riemannian Spaces of Constant Curvature (*Longman Scientific & Technical*, Essex, 1986)

[70] E.G.Kalnins and W.Miller, Jr.: Separation of Variables on $n$-Dimensional Riemannian Manifolds 1. The $n$-Sphere and Euclidean $n$-Space; *J.Math.Phys.* **27** (1986) 1721

[71] E.G.Kalnins, W.Miller, Jr., and G.J.Reid: Separation of Variables for Complex Riemannian Spaces of Constant Curvature 1. Orthogonal Separable Coordinates for $S_{n\mathbb{C}}$ and $E_{n\mathbb{C}}$; *Proc.Roy.Soc. (London)* **A 394** (1984) 183

[72] H.Kleinert: How to do the Time Sliced Path Integral for the H Atom; *Phys.Lett.* **A 120** (1987) 361

[73] H.Kleinert: Path Integrals in Quantum Mechanics, Statistics and Polymer Physics (*World Scientific*, Singapore, 1990)

[75] H.Kleinert and I.Mustapic: Summing the Spectral Representations of Pöschl-Teller and Rosen-Morse Fixed-Energy Amplitudes; *J.Math.Phys.* **33** (1992) 643

[74] N.Kokiantonis and D.P.L.Castrigiano: Propagator for a Charged Anisotropic Oscillator in a Constant Magnetic Field; *J.Phys.A: Math.Gen.* **18** (1985) 45

[76] R.Kubo: Path Integration on the Upper Half-Plane; *Prog.Theor.Phys.* **78** (1987) 755; Geometry, Heat Equation and Path Integrals on the Poincaré Upper Half-Plane; *Prog.Theor.Phys.* **79** (1988) 217

[77] V.Kuznetsov: private communication

[78] L.D.Landau and E.M.Lifschitz: Lehrbuch der Theoretischen Physik, Vol.I - III (*Akademie Verlag*, Berlin, 1979)

[79] I.V.Lutsenko, G.S.Pogosyan, A.N.Sisakyan, and V.M.Ter-Antonyan: Hydrogen Atom as Indicator of Hidden Symmetry of a Ring-Shaped Potential; *Theor.Math.Phys.* **83** (1990) 633

[80] D.W.McLaughlin and L.S.Schulman: Path Integrals in Curved Spaces; *J.Math.Phys.* **12** (1971) 2520

[81] A.A.Makarov, J.A.Smorodinsky, Kh.Valiev and P.Winternitz: A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries; *Nuovo Cimento* **A 52** (1967) 1061

[82] P.T.Matthews and A.Salam: Covariant Fock Equations; *Proc.Roy.Soc.(London)* **A 221** (1953) 128; The Green's Function of Quantized Fields; *Nuovo Cimento* **12** (1954) 563; Propagators of Quantized Field; *Nuovo Cimento* **2** (1955) 120

[83] I.M.Mayes and J.S.Dowker: Canonical Functional Integrals in General Coordinates; *Proc.Roy.Soc. (London)* **A 327** (1972) 131

[84] J.Meixner and F.W.Schäfke: Mathieusche Funktionen und Sphäroidfunktionen (*Springer-Verlag*, Berlin, 1954)

[85] M.M.Mizrahi: The Weyl Correspondence and Path Integrals; *J.Math.Phys.* **16** (1975) 2201

[86] C.Morette: On the Definition and Approximation of Feynman's Path Integrals; *Phys.Rev.* **81** (1951) 848; Feynman's Path Integral: Definition Without Limiting Procedure; *Commun.Math.Phys.* **28** (1972) 47

[87] P.M.Morse and H.Feshbach: Methods of Theoretical Physics (*McGraw-Hill*, New York, 1953)

[88] E.Nelson: Feynman Integrals and the Schrödinger Equation; *J.Math.Phys.* **5** (1964) 332

[89] M.Omote: Point Canonical Transformations and the Path Integral; *Nucl.Phys.* **B 120** (1977) 325

[90] N.N.Olevskiĭ: Triorthogonal Systems in Spaces of Constant Curvature in which the Equation $\Delta_2 u + \lambda u = 0$ Allows the Complete Separation of Variables; *Math.Sb.* **27** (1950) 379 (in russian)

[91] N.K.Pak and I.Sökmen: General New-Time Formalism in the Path Integral; *Phys.Rev.* **A 30** (1984) 1629

[92] D.Peak and A.Inomata: Summation Over Feynman Histories in Polar Coordinates; *J.Math.Phys.* **10** (1969) 1422

[93] A.Pelster and A.Wunderlin: On the Generalization of the Duru-Kleinert-Propagator Transformations; *Zeitschr.Phys.* **B 89** (1992) 373

[94] B.Podolsky: Quantum-Mechanically Correct Form of Hamiltonian Function for Conservative Systems; *Phys.Rev.* **32** (1928) 812.

[95] G.S.Pogosyan, A.N.Sissakian and S.I.Vinitsky: The Super-Integrable Systems of the Smorodinsky-Winternitz Type on the Three Dimensional Sphere; to appear in the proceedings of the "International Workshop on 'Symmetry Methods in Physics' in Memory of Prof. Ya. A. Smorodinsky", Dubna, July 1993

[96] A.M.Polyakov: Quantum Geometry of Bosonic Strings; *Phys.Lett.* **B 103** (1981) 207

[97] C.Quesne: A New Ring-Shaped Potential and its Dynamical Invariance Algebra; *J.Phys.A: Math.Gen.* **21** (1988) 3093

[98] G.Roepstorff: Pfadintegrale in der Quantenphysik (*Vieweg*, Braunschweig, 1992)

[99] L.Schulman: A Path Integral for Spin; *Phys.Rev.* **176** (1968) 1558

[100] L.S.Schulman: Techniques and Applications of Path Integration (*John Wiley & Sons*, New York, 1981)

[101] J.Schwinger (ed.): Quantum Electrodynamics (ed.J.Schwinger) *Dover*, New York, 1958

[102] A.Selberg: Harmonic Analysis and Discontinuous Groups in Weakly Symmetric Riemannian Spaces with Application to Dirichlet Series; *J.Indian Math.Soc.* **20** (1956) 47

[103] B.Simon: Functional Integration and Quantum Physics (*Academic Press*, New York, 1979)

[104] F.Steiner: Path Integrals in Polar Coordinates From eV to GeV; in "Bielefeld Encounters in Physics and Mathematics VII; Path Integrals From meV to MeV", 1985, p.335; eds.: M.C.Gutzwiller et al. (*World Scientific*, Singapore, 1986)

[105] V.V.Stepanov: The Laplace Equation in Certain Trirectangular Systems; *Math.Sb.* **11** (1942) 204 (in russian, cited after [90])

[106] S.N.Storchak: Rheonomic Homogeneous Point Transformation and Reparametrization in the Path Integral; *Phys.Lett.* **A 135** (1989) 77

[107] A.B.Venkov: Expansion in Automorphic Eigenfunctions of the Laplace Operator and the Selberg Trace Formula in the Space $SO_0(n, 1)/SO(n)$; *Soviet Math.Dokl.* **12** (1971), 1363.

[108] A.B.Venkov: Expansion in Automorphic Eigenfunctions of the Laplace-Beltrami Operator in Classical Symmetric Spaces of Rank One, and the Selberg Trace Formula; *Proc.Math.Inst.Steklov* **125** (1973), 6.

[109] A.B.Venkov: Spectral Theory of Automorphic Functions; *Proc.Math.Inst. Steklov* **153** (1981) 1

[110] N.Ja.Vilenkin: Special Functions Connected with Class 1 Representations of Groups of Motions in Spaces of Constant Curvature; *Trans.Moscow Math.Soc.* (1963) 209

[111] N.Ya.Vilenkin and Ya.A.Smorodinsky: Invariant Expansions of Relativistic Amplitudes; *Sov.Phys. JETP* **19** (1964) 1209

[112] F.W.Wiegel: Introduction to Path-Integral Methods in Physics and Polymer Science (*World Scientific*, Singapore, 1986)

[113] A.Young and C.DeWitt-Morette: Time Substitution in Stochastic Processes as a Tool in Path Integration; *Ann.Phys.* **169** (1986) 140