

News from the Virasoro Algebra

Karl-Henning Rehren

II. Inst. Theor. Physik, Univ. Hamburg, D-22761 Hamburg (FR Germany)
e-mail: i02reh@dsyibm.desy.de

Abstract: It is shown that the local quantum field theory of the chiral energy-momentum tensor with central charge $c = 1$ coincides with the gauge invariant subtheory of the chiral $SU(2)$ current algebra at level 1, where the gauge group is the global $SU(2)$ symmetry. At higher level, the same scheme gives rise to W -algebra extensions of the Virasoro algebra.

1. Introduction

Let $j^a(x)$ be the local currents of the chiral current algebra of a compact simple Lie group G at level k , defined by the local commutation relations

$$-i[j^a(x), j^b(y)] = \sum_c f_c^{ab} j^c(x) \delta(x-y) + \frac{k}{2\pi} g^{ab} \delta'(x-y). \quad (1)$$

The metric g^{ab} is normalized in terms of the structure constants by $2hg^{ab} = -f_d^{ac} f_c^{bd}$ with h the dual Coxeter number of G ($h = N$ for $G = SU(N)$). Let $T(x)$ be the energy-momentum tensor of the model given by the Sugawara formula

$$T(x) = \frac{\pi}{k+h} \sum_{ab} g_{ab} : j^a(x) j^b(x) : \quad (2)$$

and satisfying the commutation relations

$$-i[T(x), T(y)] = (T(x) + T(y)) \delta'(x-y) - \frac{c}{24\pi} \delta'''(x-y) \quad (3)$$

with central charge $c = \dim(G) \cdot k/(k+h)$. The moments of $T(x)$ generate the conformal symmetry of the model, while the charges $Q^a = \int j^a(x) dx$ generate a global G symmetry. Under this symmetry, the energy-momentum tensor is clearly invariant.

We emphasize the local form of the commutation relations rather than their Fourier transforms, the Kac-Moody and Virasoro algebras. For $I \subset S^1$ an interval of the circle we denote by

$$\mathcal{A}(I) \supset \mathcal{A}_{\text{inv}}(I) \supset \mathcal{A}_{\text{vir}}(I) \quad (4)$$

the local von Neumann algebras given by: the algebra $\{j^a(f) \mid \text{supp } f \subset I\}''$ associated with the currents (1) smeared with test functions with support in the interval I , the subalgebra invariant under G , and the algebra $\{T(f) \mid \text{supp } f \subset I\}''$ associated with the energy-momentum tensor (2),(3) smeared with test functions with support in I , respectively. (The double primes denote the von Neumann closure.) By locality, the algebras of disjoint intervals commute [1]. While the inclusions (4) are obvious, we shall confirm a conjecture by B.Schroer and show that for $G = SU(2)$ at level $k = 1$

$$\mathcal{A}_{\text{vir}}(I) = \mathcal{A}_{\text{inv}}(I). \quad (5)$$

This result is in so far surprising as besides the energy-momentum tensor (2) also bilocal expressions like

$$\sum_{ab} g_{ab} j^a(x) j^b(y) \quad (6)$$

are invariant. Indeed, it can be only true due to the weak closure of the algebras (4), and does not hold for other groups or higher level.

The situation is reminiscent of an early similar result by Langerholc and Schroer [2] who showed that the subalgebra of local “currents” $:\varphi(x)\varphi(x):$ in a four-dimensional theory of free neutral bosons is not smaller than the even subalgebra containing also bilocal operators $\varphi(x)\varphi(y)$. However, the argument used by these authors crucially depends on the non-trivial decay of the commutator function in light-like directions, and can certainly not apply to the present situation with only δ -function commutators (1). Indeed, as mentioned before, our finding is not generic in chiral quantum field theory but very specific to a particular model.

2. The main result

The spectrum of the conformal Hamiltonian $L_0 = \frac{1}{2} \int (1+x^2) T(x) dx$ and the isospin component Q^3 is well known in the positive-energy representations π of the chiral $SU(2)$ current algebra. It is described by the “chiral partition function”

$$\chi_\pi(q, t) := \text{Tr}_\pi(q^{Q^3} t^{L_0}) = \sum_{m, h} N_\pi(m, h) q^m t^h \quad (7)$$

where one may interpret $t = e^{-1/T}$ and $q = e^{-H}$ as a “temperature” and a “magnetic field” respectively. $N_\pi(m, h)$ are the dimensions of the eigenspaces $Q^3 \doteq m$, $L_0 \doteq h$ in the representation π . It is convenient to introduce the functions

$$[n]_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad (8)$$

such that every contribution $[2j+1]_q = q^{-j} + q^{1-j} + \dots + q^{j-1} + q^j$ to the partition function corresponds to a full isospin j multiplet of states.

The irreducible positive-energy representations are determined by the representation of G in which the subspace of lowest conformal energy transforms. For $SU(2)$ this is an isospin $s = 0, \frac{1}{2}, \dots, \frac{k}{2}$. The corresponding chiral partition functions are [3, 4, 5]

$$\chi_s^{(k)}(q, t) = \frac{\sum_{n \in \mathbb{Z}} [2(k+2)n + 2s + 1]_q \cdot t^{h+n((k+2)n+2s+1)}}{\sum_{n \in \mathbb{Z}} [4n + 1]_q \cdot t^{n(2n+1)}} \quad (9)$$

with $h = s(s+1)/(k+2)$. For the vacuum representation at level $k=1$ this is

$$\chi_0^{(1)}(q, t) = \frac{\sum_{n \in \mathbb{Z}} [6n + 1]_q \cdot t^{n(3n+1)}}{\sum_{n \in \mathbb{Z}} [4n + 1]_q \cdot t^{n(2n+1)}}. \quad (10)$$

Lemma 1: The partition function (10) equals

$$\chi_0^{(1)}(q, t) = \sum_{j \in \mathbb{N}_0} [2j + 1]_q \cdot t^{j^2} (1 - t^{2j+1}) p(t). \quad (11)$$

Here $p(t) = \prod_{n \in \mathbb{N}} (1 - t^n)^{-1}$. Lemma 1 follows directly from [6, Prop. 6.1].

Now, upon restriction to the $SU(2)$ -invariant subalgebra \mathcal{A}_{inv} , the vacuum representation π^0 of the current algebra \mathcal{A} decomposes into subrepresentations π_j^{inv} characterized by the eigenvalue of the Casimir operator $C \doteq j(j+1)$. Clearly, only integer isospins occur, and every subrepresentation π_j^{inv} arises with a full isospin j multiplet as its multiplicity space. By inspection of Lemma 1, this implies

Lemma 2: The partition function $\chi_j^{\text{inv}}(t) = \text{Tr}_{\pi_j} t^{L_0}$ of the representation π_j^{inv} of the gauge invariant subalgebra \mathcal{A}_{inv} (for $G = SU(2)$, $k = 1$) is

$$\chi_j^{\text{inv}}(t) = t^{j^2} (1 - t^{2j+1}) p(t). \quad (12)$$

But (12) equals the partition function $\chi_h^{c=1}(t) = \text{Tr}_{\pi_h} t^{L_0}$ of the representation $\pi_h^{c=1}$ with lowest conformal energy $L_0 \doteq h = j^2$ of the Virasoro algebra \mathcal{A}_{Vir} with central charge $c = 1$, given by the partition function $t^h p(t)$ of the Verma module and a correction factor $(1 - t^\nu)$ for the null space at level $\nu = 2j + 1$ [5]. Since $\mathcal{A}_{\text{Vir}} \subset \mathcal{A}_{\text{inv}}$, we conclude

Lemma 3: Upon restriction to the subalgebra \mathcal{A}_{Vir} , the irreducible representations π_j^{inv} of \mathcal{A}_{inv} remain irreducible and coincide with $\pi_h^{c=1}$, $h = j^2$.

In order to arrive at our main conclusion (5), we need one further information. Namely, the local quantum field theory of the energy-momentum tensor satisfies Haag duality in the vacuum representation [1]:

Proposition 4: In the vacuum representation π_0 of \mathcal{A}_{Vir}

$$\pi_0(\mathcal{A}_{\text{Vir}}(I)) = \pi_0(\mathcal{A}_{\text{Vir}}(I^c))'. \quad (13)$$

Here I^c denotes the complementary interval $S^1 \setminus I$, and the prime denotes the commutant of an algebra of operators. Now we are ready to prove

Proposition 5: For $G = SU(2)$ and $k = 1$, one has

$$\mathcal{A}_{\text{inv}}(I) = \mathcal{A}_{\text{Vir}}(I). \quad (14)$$

Proof: Let π_0 be the vacuum representation of \mathcal{A}_{inv} which by Lemma 3 restricts to the vacuum representation of \mathcal{A}_{Vir} . In the following chain of inclusions, the first inclusion is evident, the second is due to locality of \mathcal{A}_{inv} , the third is the commutant of the first (with I^c instead of I), and the last equality is Prop. 4:

$$\pi_0(\mathcal{A}_{\text{Vir}}(I)) \subset \pi_0(\mathcal{A}_{\text{inv}}(I)) \subset \pi_0(\mathcal{A}_{\text{inv}}(I^c))' \subset \pi_0(\mathcal{A}_{\text{Vir}}(I^c))' = \pi_0(\mathcal{A}_{\text{Vir}}(I)). \quad (15)$$

Since the vacuum representation is locally faithful, this implies Prop. 5.

Corollary 6: The representations $\pi_h^{c=1}$ of the quantum field theory of the energy-momentum tensor with $c = 1$ have statistical dimension $d(\pi_h^{c=1}) = 2j + 1$ provided the lowest conformal energy is $h = j^2$, $j \in \mathbb{N}_0$.

Namely, the statistical dimension of a sector of a gauge invariant subtheory which is contained in the restriction of the vacuum representation (with unbroken symmetry) equals the dimension of the corresponding representation of the symmetry group [7].

This seems to be the first non-trivial case that the statistical dimension of a sector of the Virasoro algebra with $c \geq 1$ has been computed. The value we found coincides with the asymptotic dimension defined by the “high-temperature” limit $\frac{1}{T} \searrow 0, t \nearrow 1$

$$d_{\text{as}}(\pi) := \lim \frac{\chi_{\pi}(t)}{\chi_0(t)}. \quad (16)$$

It is widely expected albeit never proven that $d(\pi) = d_{\text{as}}(\pi)$ holds quite generally for chiral theories. By [5, Prop. 8.2, 8.3], this formula would yield infinite statistics for all sectors of \mathcal{A}_{Vir} with $c > 1$ except the vacuum sector (namely $\chi_h^c(t) = t^h p(t)$ for $h > 0$ and $= (1-t)p(t)$ for $h = 0$), as well as for all sectors π_h of \mathcal{A}_{Vir} with $c = 1$ except when $h = j^2$ for $j \in \frac{1}{2}\mathbb{N}_0$; in the latter case, $d(\pi_h) = 2j + 1$.

3. Extended algebras

One may repeat the analysis at higher level, or for other groups. One has to expand the partition functions (9) (for $SU(2)$) of the vacuum representation into the form

$$\chi_0^{(k)}(q, t) = \sum_{j \in \mathbb{N}_0} [2j + 1]_q \cdot \chi_j^{\text{inv}}(t) \quad (17)$$

and in turn expand $\chi_j^{\text{inv}}(t)$ into the partition functions of the corresponding Virasoro algebra with $c > 1$. One finds that the irreducible representations of \mathcal{A}_{inv} no longer remain irreducible for \mathcal{A}_{Vir} . E.g., at level 2 one finds an infinite vacuum branching

$$\chi_0^{\text{inv}} = \chi_0^{c=3/2} + \chi_4^{c=3/2} + \chi_6^{c=3/2} + \chi_8^{c=3/2} + \dots \quad (18)$$

showing that \mathcal{A}_{Vir} is a true subtheory of \mathcal{A}_{inv} . Thus \mathcal{A}_{inv} is a conformally invariant chiral theory containing apart from the energy-momentum tensor further primary fields of scaling dimensions which can be read off the character expansion. The level 2 example (18) exhibits the lowest lying primary fields with dimensions 4, 6, and 8.

This is a general scheme to construct new local extensions (W -algebras) of the Virasoro algebra.

References

- [1] D. Buchholz, H. Schulz-Mirbach: *Haag duality in conformal quantum field theory*, Rev. Math. Phys. **2**, 105 (1990).
- [2] J. Langerholc, B. Schroer: *Can current operators determine a complete theory?*, Commun. Math. Phys. **4**, 123 (1967).
- [3] V.G. Kac: *Infinite-dimensional Lie algebras and Dedekind’s η -function*, Funct. Anal. Appl. **8**, 68 (1974).
- [4] A. Pressley, G. Segal: *Loop Groups*, Oxford Univ. Press (1986);
J. Fuchs: *Affine Lie Algebras and Quantum Groups*, Cambridge Univ. Press (1992).
- [5] V.G. Kac, A.K. Raina: *Highest Weight Representations of Infinite Dimensional Lie Algebras*, World Scientific (1987).
- [6] G. Segal: *Unitary representations of some infinite dimensional groups*, Commun. Math. Phys. **80**, 301 (1981).
- [7] S. Doplicher, R. Haag, J.E. Roberts: *Fields, observables, and gauge transformations. I*, Commun. Math. Phys. **13**, 1 (1969), and: *Local observables and particle statistics. I*, Commun. Math. Phys. **23**, 199 (1971).