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Particle-ray tracing through misaligned lattice elements in accelerators

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Particle-ray tracing through misaligned lattice elements in accelerators

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Abstract

Particle-ray tracing through misaligned lattice elements can be modeled by sandwiching the transfer-map for the aligned element between two transformations, one applied at the entrance and the other at the exit of the element. We derive the explicit form of these two transformations for an arbitrary misalignment using elementary geometry.

1. Introduction

An accelerator-lattice layout can be thought of as a sequence of geometric objects (the "LEGO blocks" in Forest's pictorial description [1]), placed back-to-back along the design reference orbit. The blocks can be straight or curved, each block enclosing a lattice element and carrying its own local coordinate system. A lattice element may represent a magnet or a drift (i.e. a field-free region); we assume that magnet blocks are interleaved between drift blocks.

By construction, in a lattice with no misalignments the interface between the blocks is a plane normal to the reference orbit. In the presence of misalignments the magnet block may separate from, or encroach into, the adjacent drift block creating positive or negative gaps. Assuming that the fringe-fields of the misaligned element do not overlap with those of the neighbouring magnets, particle propagation through the gap amounts to propagation in field-free space and can be done exactly. Propagation is forward/backward depending on whether the gap is positive or negative.

The particle dynamic through the misaligned element can then be obtained by combining the transfer maps through the gaps at each element end with that for the (aligned) element, leveraging our knowledge of the latter. This is in contrast to the approach where the fields of the misaligned element would be represented relative to, and then tracking would be done in, a non-local reference-frame (e.g. the reference frame of an adjacent element or the machine global reference frame). The latter approach may represent the only viable option when the fields of adjacent magnets overlap but it is generally less convenient.

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While the maps associated with an arbitrary misalignment can be written elegantly in Lie-algebra form [1], those who are not familiar with Lie methods (or perhaps find aspects of the Ref. [1] approach, like the reduction of a thick lattice element to a 'compressed' thin element, not intuitive) may be interested in the alternative but equivalent elementary derivation presented in this Note.

We restrict the analysis to the case of misaligned elements with the design orbit through the element consisting of an arc of circumference or a line (but the description can be easily adjusted to include the more general case where the design orbit through the element is an arbitrary curve). In Sec. 2 misalignments are defined relative to the coordinate system of the drift upstream of the misaligned element. The maps into and out of the misaligned element are worked out in Sec. 3 and 4. An alternate definition of misalignment, which may be more suitable in practical applications, whereby a misalignment is defined relative to the coordinate system with origin at the midpoint of the aligned element, is introduced in Sec. 5.

The appendices report *Mathematica* [2] functions[1] implementing the exact, fully nonlinear maps (Appendix A) and their linear approximation (Appendix B). The codes assume the ultra-relativistic approximation, no deliberate element tilt, and the misalignment specification as described in Sec. 2.

2. Description of element misalignment

Let the Cartesian coordinate system xyz with origin O be defined on the exit face of the drift immediately upstream of the misaligned element; $\hat{z}$ is the unit vector along the longitudinal axis in the direction of the beam motion, $\hat{x}$ points outward in the horizontal plane, and $\hat{y}$ points upwards in the vertical plane. See Fig. 1. In our description the fiducial of the misaligned element is a point P on the element block entry face and is chosen as the origin of the Cartesian coordinate system XYZ. In the absence of misalignments the two frames xyz and XYZ overlap.

A misalignment relative to xyz is defined by the error-displacement vector $\overrightarrow{OP}$ and the error-rotation $\boldsymbol{\mathcal{R}}$ such that

$$\hat{X} = \boldsymbol{\mathcal{R}}\hat{x}, \quad \hat{Y} = \boldsymbol{\mathcal{R}}\hat{y}, \quad \hat{Z} = \boldsymbol{\mathcal{R}}\hat{z}, \tag{1}$$

where $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ are the unit vectors along the XYZ coordinate axes. We specify $\overrightarrow{OP}$ in terms of the coordinate vector $\vec{d} = (d_x, d_y, d_z)$ representing its coordinates in the xyz coordinate system

$$(\overrightarrow{OP})_{xyz} = \vec{d}, \tag{2}$$

[1] The codes are written for clarity rather than efficiency.

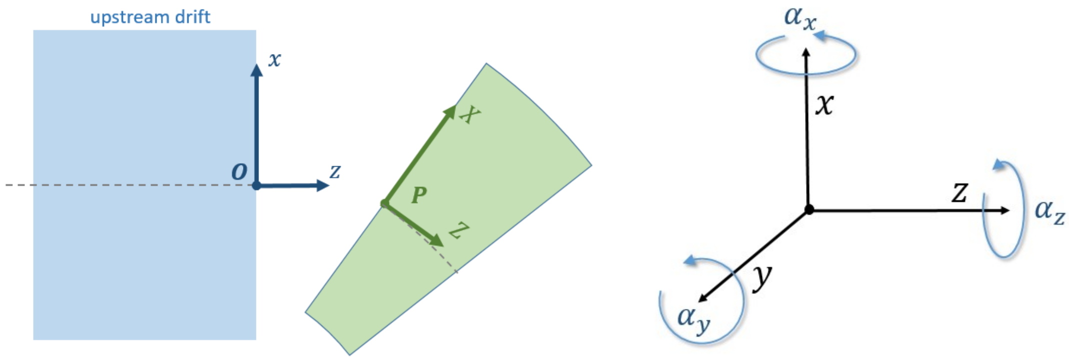

Figure 1: Left: the misalignment of a lattice element (here a bending element, green block) is defined relative to the xyz coordinate system attached to the exit face the upstream drift (the beam travels left to right). In this example the Y axis remains parallel to the y axis, both pointing points towards the reader. Right: the misalignment rotation angles about the coordinate axes are positive in the direction indicated (right-hand rule).

and similarly we specify $\boldsymbol{\mathcal{R}}$ in terms of the matrix[2] $\boldsymbol{R}_u$ in the same coordinate system with parametrization

$$\boldsymbol{R}_u = \begin{pmatrix} \cos\alpha_y \cos\alpha_z & -\cos\alpha_y \sin\alpha_z & \sin\alpha_y \\ \sin\alpha_x \sin\alpha_y \cos\alpha_z + \cos\alpha_x \sin\alpha_z & \cos\alpha_x \cos\alpha_z - \sin\alpha_x \sin\alpha_y \sin\alpha_z & -\sin\alpha_x \cos\alpha_y \\ \sin\alpha_x \sin\alpha_z - \cos\alpha_x \sin\alpha_y \cos\alpha_z & \cos\alpha_x \sin\alpha_y \sin\alpha_z + \sin\alpha_x \cos\alpha_z & \cos\alpha_x \cos\alpha_y \end{pmatrix}, \quad (3)$$

the result of the ordered sequence $\boldsymbol{R}_u = \boldsymbol{R}_x(\alpha_x)\boldsymbol{R}_y(\alpha_y)\boldsymbol{R}_z(\alpha_z)$ of roll (α_z), yaw (α_y) and pitch (α_x) rotations around the z, y, and x axes respectively:

$$\boldsymbol{R}_y = \begin{pmatrix} \cos\alpha_y & 0 & \sin\alpha_y \\ 0 & 1 & 0 \\ -\sin\alpha_y & 0 & \cos\alpha_y \end{pmatrix}, \quad \boldsymbol{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_x & -\sin\alpha_x \\ 0 & \sin\alpha_x & \cos\alpha_x \end{pmatrix}, \quad \boldsymbol{R}_z = \begin{pmatrix} \cos\alpha_z & -\sin\alpha_z & 0 \\ \sin\alpha_z & \cos\alpha_z & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Mnemonic for the signs: if an observer stands upright with his head pointing toward the direction of positive y and looking forward in the direction of positive z, then the x axis cuts through the observer's ears and:

- positive pitch ($\alpha_x > 0$) corresponds to moving the nose down;
- positive yaw ($\alpha_y > 0$) corresponds to turning the head to the left;
- positive roll ($\alpha_z > 0$) corresponds to tilting the head to the right.

[2]The subscript "u" emphasizes that the coordinate system of reference is that of the upstream drift.

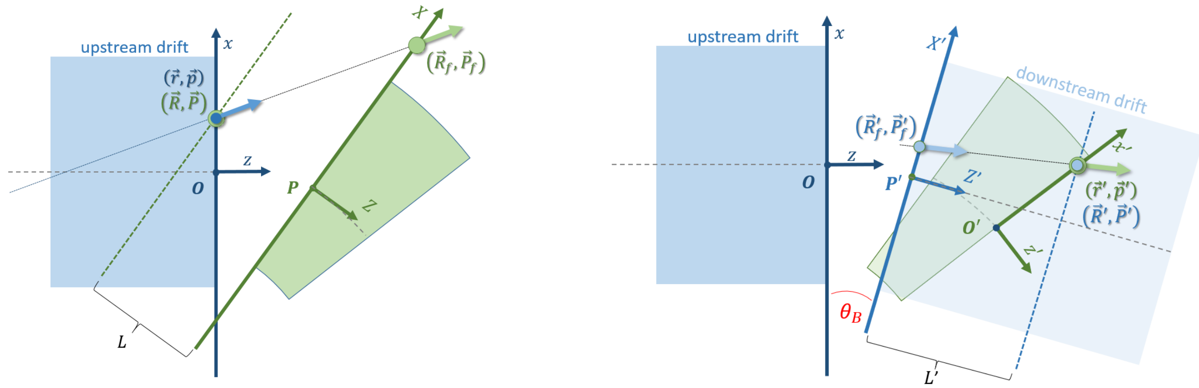

Figure 2: Left: in the coordinate system xyz at the exit of the upstream drift a particle has position and momentum $\vec{r}$ and $\vec{p}$. Propagation to the entrance of the misaligned element entails finding the particle position and momentum components $\vec{R}$ and $\vec{P}$ expressed in the coordinate system XYZ and applying a drift map to determine $\vec{R}_f$ and $\vec{P}_f$. Right: propagation out of the misaligned element to the entrance of the downstream drift. In this example the drift map involves backward propagation.

An alternate and possibly more practical description is to define the misalignment relative to a coordinate system attached not to the exit face of the upstream drift but to the nominal position of the aligned element. This alternate description is discussed in Sec. 5.

3. Propagation into the misaligned element.

3.1. Exact map

Let $\vec{t} = (x, p_x, y, p_y, \ell, \delta)$ be the phase-space coordinates' vector of a particle exiting the drift upstream of the misaligned element, where p_x and p_y are the transverse components of the canonical momentum divided by p_0, the reference-particle design physical momentum; $\ell = ct$ is the scaled time of flight; $\delta = (E - E_0)/(p_0 c)$ the scaled relative energy deviation, with E being the particle (total) energy and $E_0 = \sqrt{p_0^2 c^2 + m^2 c^4}$ the reference-particle design energy. Incidentally, notice that the proper canonical pair is $(\ell, -\delta)$ if ℓ and $-\delta$ are to be interpreted as the configuration-space and canonical-momentum dynamical variables respectively; ℓ acquires the meaning of the orbit path length in the ultrarelativistic limit, where $\delta \simeq (E - E_0)/E_0$.

The first task is to find $\vec{T} = (X, P_x, Y, P_y, \ell, \delta)$, the particle phase-space coordinates at the exit of the upstream drift expressed in the XYZ coordinates system attached to the entry face of the misaligned element, see Fig. 2, left image. For this purpose we introduce the 3D vectors $\vec{r} = (x, y, 0)$ and $\vec{p} = (p_x, p_y, p_z)$ made of the particle position and momentum components in the xyz frame. The transverse coordinates of $\vec{T}$ are derived from the components the two vectors $\vec{R} = (X, Y, Z)$ and $\vec{P} = (P_x, P_y, P_z)$ resulting from the transformation

$$\vec{R} = \boldsymbol{R}^T(\vec{r} - \vec{d}), \tag{4}$$

$$\vec{P} = \boldsymbol{R}^T\vec{p}, \tag{5}$$

with $\boldsymbol{R} = \boldsymbol{R}_u$, as in (3).

Propagation through the gap from $\vec{T}$ to $\vec{T}_f = (X_f, P_{x,,f}Y_f, P_{yf}, \ell_f, \delta_f)$ is accomplished by the drift map (see e.g. [1])

$$\begin{aligned} X_f &= X + \frac{P_x}{P_z}L, \qquad P_{x,f} = P_x, \\ Y_f &= Y + \frac{P_y}{P_z}L, \qquad P_{y,f} = P_y, \\ \ell_f &= \ell + \frac{1/\beta_0 + \delta}{P_z}L, \qquad \delta_f = \delta, \end{aligned} \tag{6}$$

where

$$P_z = \sqrt{1 + 2\delta/\beta_0 + \delta^2 - P_x^2 - P_y^2}, \tag{7}$$

and the drift length (a function of the x, y coordinates) is defined by $L = -Z$, with Z being the third component of the $\vec{R}$ vector in (4).

3.2. Linearized map

Linearization of (6) yields

$$\vec{T}_f = \boldsymbol{L}_{in}\vec{t} + \vec{t}_0, \tag{8}$$

with

$$\boldsymbol{L}_{in} = \begin{pmatrix} \frac{R_{22}}{R_{33}} & \frac{R_{22}L_D}{R_{33}^2} & -\frac{R_{12}}{R_{33}} & -\frac{R_{12}L_D}{R_{33}^2} & 0 & 0 \\ 0 & R_{11} & 0 & R_{21} & 0 & \frac{R_{31}}{\beta_0} \\ -\frac{R_{21}}{R_{33}} & -\frac{R_{21}L_D}{R_{33}^2} & \frac{R_{11}}{R_{33}} & \frac{R_{11}L_D}{R_{33}^2} & 0 & 0 \\ 0 & R_{12} & 0 & R_{22} & 0 & \frac{R_{32}}{\beta_0} \\ -\frac{R_{13}}{\beta_0 R_{33}} & -\frac{R_{13}L_D}{\beta_0 R_{33}^2} & -\frac{R_{23}}{\beta_0 R_{33}} & -\frac{R_{23}L_D}{\beta_0 R_{33}^2} & 1 & -\frac{L_D}{\gamma_0^2\beta_0^2 R_{33}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

where γ_0 is the relativistic factor;

$$L_D = R_{13}d_x + R_{23}d_y + R_{33}d_z, \tag{10}$$

and

$$\vec{t_0} = \left(\frac{d_y R_{12} - d_x R_{22}}{R_{33}}, R_{31}, \frac{d_x R_{21} - d_y R_{11}}{R_{33}}, R_{32}, \frac{L_D}{\beta_0 R_{33}}, 0\right). \tag{11}$$

4. Propagation out of the misaligned element

4.1. Exact map

Consider a bending element with bending angle θ_B, possibly tilted by an angle θ_T. Let the Cartesian coordinate system $X'Y'Z'$ with origin P' be defined at the entry face of the downstream drift element and let the Cartesian coordinate system $x'y'z'$ with origin O' be defined at the exit face of the misaligned element, see the right image of Fig. 2. The task of this section is to determine the map $\vec{t'} \to \vec{T}'_f$ between the 6D coordinate vectors $\vec{t'} = (x', p'_x, y', p'_y, \ell', \delta')$ and $\vec{T}'_f = (X'_f, P'_{x,f}, Y'_f, P'_{y,f}, \ell'_f, \delta'_f)$. We do so by recognizing that we can reduce the calculation to that of the previous section by applying suitable transformations.

Introduce the matrix

$$\boldsymbol{R}_B = \begin{pmatrix} \cos\theta_T & -\sin\theta_T & 0 \\ \sin\theta_T & \cos\theta_T & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_B & 0 & -\sin\theta_B \\ 0 & 1 & 0 \\ \sin\theta_B & 0 & \cos\theta_B \end{pmatrix}, \tag{12}$$

which in the coordinate system xyz represents the orthogonal transformation $\boldsymbol{\mathcal{R}}_B$ that maps the unit vector $\hat{x}$, $\hat{y}$, and $\hat{z}$ to

$$\hat{X}' = \boldsymbol{\mathcal{R}}_B \hat{x}, \quad \hat{Y}' = \boldsymbol{\mathcal{R}}_B \hat{y}, \quad \hat{Z}' = \boldsymbol{\mathcal{R}}_B \hat{z}. \tag{13}$$

The sign convention for θ_B is the same as in MAD [3] (positive-angle bending is towards negative x for a particle travelling in the direction of positive z). It should be obvious that $\hat{x}' = \boldsymbol{\mathcal{R}}\hat{X}'$, $\hat{y}' = \boldsymbol{\mathcal{R}}\hat{Y}'$, and $\hat{z}' = \boldsymbol{\mathcal{R}}\hat{Z}'$, where $\boldsymbol{\mathcal{R}}$ is the error rotation, or equivalently

$$\hat{X}' = \boldsymbol{\mathcal{R}}^T \hat{x}', \quad \hat{Y}' = \boldsymbol{\mathcal{R}}^T \hat{y}', \quad \hat{Z}' = \boldsymbol{\mathcal{R}}^T \hat{z}'. \tag{14}$$

Inspection of Eq. (14) and comparison with Eq. (1) show that for the map *out* of the misaligned element the operator $\boldsymbol{\mathcal{R}}^T$ plays the same role played by $\boldsymbol{\mathcal{R}}$ for propagation *into* the misaligned element: the latter represents the rotation that brings the exit face xy of the upstream drift to the entry face XY of the misaligned element while $\boldsymbol{\mathcal{R}}^T$ represents the rotation of the exit face $x'y'$ of the misaligned element to the entry face $X'Y'$ of the downstream drift. To proceed we need to express $\boldsymbol{\mathcal{R}}^T$ as a matrix in the $x'y'z'$ coordinate system (below we will denote this matrix as $\boldsymbol{R}'$) from knowledge of the orthogonal transformation

$$\boldsymbol{\mathcal{T}} : (\hat{x}, \hat{y}, \hat{z}) \to (\hat{x}', \hat{y}', \hat{z}') = (\boldsymbol{\mathcal{T}}\hat{x}, \boldsymbol{\mathcal{T}}\hat{y}, \boldsymbol{\mathcal{T}}\hat{z}) \tag{15}$$

that maps the coordinate-axis unit vectors of the xyz frame to those of $x'y'z'$. The form of $\boldsymbol{\mathcal{T}}$ is deduced by combining (14) and (13):

$$\hat{x}' = \boldsymbol{\mathcal{R}}\hat{X}' = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}_B \hat{x}, \quad \hat{y}' = \boldsymbol{\mathcal{R}}\hat{Y}' = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}_B \hat{y}, \quad \hat{z}' = \boldsymbol{\mathcal{R}}\hat{Z}' = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}_B \hat{z}, \tag{16}$$

yielding $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}_B$. In general, if the basis vectors transform according to the orthogonal transformation $\boldsymbol{\mathcal{T}}$, the matrix $\boldsymbol{A}$ representing a linear operator $\boldsymbol{\mathcal{A}}$ transforms according to

$$\boldsymbol{A}' = \boldsymbol{T}^T \boldsymbol{A}\boldsymbol{T}, \tag{17}$$

where $\boldsymbol{T}$ is the matrix representation of $\boldsymbol{\mathcal{T}}$. Therefore, the matrix representation of $\boldsymbol{\mathcal{R}}^T$ in the $x'y'z'$ coordinates is

$$\boldsymbol{R}' = \boldsymbol{R}_B^T \boldsymbol{R}_u^T \boldsymbol{R}_B, \tag{18}$$

having identified $\boldsymbol{A} = \boldsymbol{R}_u^T$ and $\boldsymbol{T} = \boldsymbol{R}_u \boldsymbol{R}_B$

A moment's thought should now convince us that we can simply apply the expressions in Sec. 3 upon the substitutions $(x, y, z) \to (x', y', z')$, $(X, Y, Z) \to (X', Y', Z')$, $\boldsymbol{R} \to \boldsymbol{R}'$, $L \to L'$, and $\vec{d} \to \vec{d'}$ (where $\vec{d'} = (\overrightarrow{O'P'})_{x'y'z'}$ is the vector of the three components of $\overrightarrow{O'P'}$ in the $x'y'z'$ coordinate system) to obtain the desired transfer map out of the misaligned element $\vec{t'} \to \vec{T}'_f$.

To write this map in an explicit form define the 3D vectors $\vec{r'} = (x', y', z')$ and $\vec{p'} = (p'_x, p'_y, p'_z)$ containing the particle position and momentum components at the exit of the misaligned element in the $x'y'z'$ coordinate system. The transformed vectors

$$\begin{aligned}
\vec{R'} &\equiv (X', Y', Z') = \boldsymbol{R}'^T(\vec{r'} - \vec{d'}), \qquad (19)\\
\vec{P'} &\equiv (P'_x, P'_y, P'_z) = \boldsymbol{R}'^T \vec{p'}. \qquad (20)
\end{aligned}$$

yield the components of the particle 6D phase-space coordinates' vector $\vec{T'} = (X', P'_x, Y', P'_y, \ell', \delta')$ in the $X'Y'Z'$ coordinate system. Propagation through the gap from $\vec{T'}$ to $\vec{T}'_f$ results from applying the drift map

$$\begin{aligned}
X'_f &= X' + \frac{P'_x}{P'_z}L', \qquad P'_{x,f} = P'_x,\\
Y'_f &= Y' + \frac{P'_y}{P'_z}L', \qquad P'_{y,f} = P'_y,\\
\ell'_f &= \ell' + \frac{1/\beta_0 + \delta}{P'_z}L', \qquad \delta'_f = \delta'.
\end{aligned} \tag{21}$$

where

$$P'_z = \sqrt{1 + 2\delta'/\beta_0 + \delta'^2 - {P'_x}^2 - {P'_y}^2}. \tag{22}$$

The drift length is defined by $L' = -Z'$, with Z' being the third component of the $\vec{R'}$ vector in (19).

The remaining task now is to work out the expression for displacement vector $\vec{d'}$. Recalling that under a coordinate transformation as in (15) a coordinate vector transforms according to the transpose of the transformation matrix ($\boldsymbol{T} = \boldsymbol{R}_u \boldsymbol{R}_B$) and in view of (16), we can write the displacement vector as

$$\begin{aligned}
\vec{d'} &= (\overrightarrow{O'P'})_{x'y'z'}\\
&= (\boldsymbol{R}_u \boldsymbol{R}_B)^T (\overrightarrow{O'P'})_{xyz}\\
&= (\boldsymbol{R}_u \boldsymbol{R}_B)^T (\overrightarrow{OP'} - \overrightarrow{OO'})_{xyz},
\end{aligned} \tag{23}$$

where in the xyz frame the vectors $\overrightarrow{OP'}$ and $\overrightarrow{OO'}$ read (with $R_c = L_B/\theta_B$ being the bending element radius of curvature):

$$
\begin{aligned}
(\overrightarrow{OP'})_{xyz} &= \begin{pmatrix} \cos\theta_T & -\sin\theta_T & 0 \\ \sin\theta_T & \cos\theta_T & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_c\cos\theta_B - R_c \\ 0 \\ R_c\sin\theta_B \end{pmatrix}, && (24) \\
(\overrightarrow{OO'})_{xyz} &= \boldsymbol{R}_u(\overrightarrow{OP'})_{xyz} + \vec{d}, && (25)
\end{aligned}
$$

where $\vec{d}$ is as in (2). The matrix $\boldsymbol{R}_u$ appearing in (23) and (25) is as in (3).

Finally, the case of a misaligned, straight (non-bending) element of length L_s is obtained in the limit $\theta_B \to 0$, in which case $\boldsymbol{R}_B$ is the identity and

$$(\overrightarrow{OP'})_{xyz} = (0, 0, L_s). \tag{26}$$

4.2. Linearized map

The linearized form of (21) is

$$\vec{T}'_f = \boldsymbol{L}_{out}\vec{t'} + \vec{t}'_0, \tag{27}$$

where $\boldsymbol{L}_{out}$ has the same form of $\boldsymbol{L}_{in}$ in Eq. (9) but with the entries of the matrix $\boldsymbol{R}$ replaced by those of the matrix $\boldsymbol{R}'$ defined in (18); L_D is replaced by

$$L'_D = R'_{13}d'_x + R'_{23}d'_y + R'_{33}d'_z, \tag{28}$$

with $\vec{d'}$ as defined in (23), and

$$\vec{t}'_0 = \left(\frac{d'_y R'_{12} - d'_x R'_{22}}{R'_{33}}, R'_{31}, \frac{d'_x R'_{21} - d'_y R'_{11}}{R'_{33}}, R'_{32}, \frac{L'_D}{\beta_0 R'_{33}}, 0 \right). \tag{29}$$

5. Alternate description of the element misalignment

Let us introduce the $x_0y_0z_0$ coordinate system shown in Fig. 3 with origin O_0 at the midpoint of the chord subtended by the design-trajectory arc in the aligned bend magnet. The longitudinal axis $\hat{z}_0$ is in the direction of the chord. (The coordinate system $X_0Y_0Z_0$ with origin P_0 is attached to the misaligned bend.) In this section the misalignment is defined relative to the element nominal (i.e. without misalignment) position and is specified by the displacement-error vector $\vec{d}_a = (d_{x0}, d_{y0}, d_{z0})$ and the rotation errors' matrix $\boldsymbol{R}_a$ in the $x_0y_0z_0$ coordinate system; $\boldsymbol{R}_a$ is parametrized as in Eq. (3) but now in terms of the angles α_{x0}, α_{y0}, and α_{z0}, for rotations about the x_0, y_0, and z_0 axes respectively. Denote with $\boldsymbol{\mathcal{R}}$ the abstract linear operator corresponding to the matrix $\boldsymbol{R}_a$.

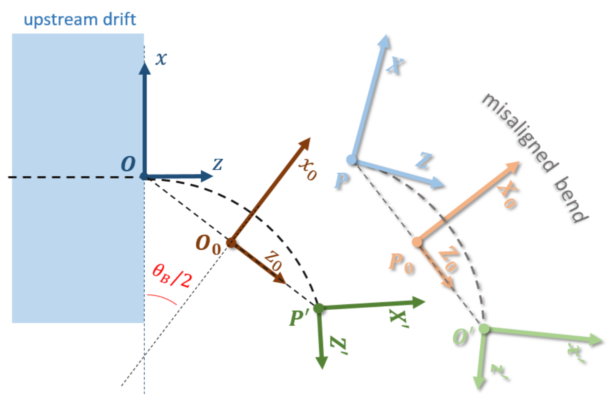

Figure 3: It may be more practical to define a misalignment relative to the $x_0y_0z_0$ coordinate system attached to the aligned magnet rather than the xyz coordinate system attached to the upstream drift (as done in Sec. 2). The origin O_0 of the $x_0y_0z_0$ system is chosen to be the midpoint of the chord subtended by the design-trajectory arc in the aligned magnet (for straight magnets this is simply the the midpoint along the longitudinal axis). The longitudinal axis z_0 is in the direction of the chord. The coordinate system attached to the misaligned bend element is $X_0Y_0Z_0$ with origin in P_0. For completeness, the coordinate systems $X'Y'Z'$ at the exit of the aligned element (and entrance of the downstream drift) and $x'y'z'$ at the exit of the misaligned bend are also shown.

To take advantage of the results from the two previous sections, we want to relate the pair $(\vec{d}_a, \boldsymbol{R}_a)$ to the pairs $(\vec{d}, \boldsymbol{R})$ for the into-the-element and $(\vec{d'}, \boldsymbol{R}')$ for the out-of-the-element map. We start by recognizing that

$$\hat{x}_0 = \boldsymbol{\mathcal{R}}_{B/2}\hat{x}, \quad \hat{y}_0 = \boldsymbol{\mathcal{R}}_{B/2}\hat{y}, \quad \hat{z}_0 = \boldsymbol{\mathcal{R}}_{B/2}\hat{z}, \tag{30}$$

with $\boldsymbol{\mathcal{R}}_{B/2}$ being the operator associated with the matrix of the form defined in (12) in the xyz coordinate system but with $\theta_B/2$ replacing θ_B. Consider the inverse of (30), i.e. the transformation $\boldsymbol{\mathcal{T}} : (\hat{x_0}, \hat{y_0}, \hat{z_0}) \to (\hat{x}, \hat{y}, \hat{z}) = (\boldsymbol{\mathcal{T}}\hat{x}_0, \boldsymbol{\mathcal{T}}\hat{y}_0, \boldsymbol{\mathcal{T}}\hat{z}_0)$ from the coordinates system $x_0y_0z_0$ to the coordinate system xyz, with $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{R}}^T_{B/2}$. By applying Eq. (17), we find the matrix $\boldsymbol{R}$ representing the error-rotation $\boldsymbol{\mathcal{R}}$ when expressed in the xyz coordinate:

$$\boldsymbol{R} = \boldsymbol{R}_{B/2}\boldsymbol{R}_a\boldsymbol{R}^T_{B/2}; \tag{31}$$

this is the matrix relevant for propagation into the misaligned element. Similarly, from Eq. (18) we can write the rotation matrix relevant for propagation out of the misaligned element:

$$\begin{aligned} \boldsymbol{R}' &= \boldsymbol{R}^T_B\boldsymbol{R}^T\boldsymbol{R}_B = \boldsymbol{R}^T_B(\boldsymbol{R}_{B/2}\boldsymbol{R}_a\boldsymbol{R}^T_{B/2})^T\boldsymbol{R}_B \\ &= (\boldsymbol{R}^T_B\boldsymbol{R}_{B/2})\boldsymbol{R}^T_a(\boldsymbol{R}^T_{B/2}\boldsymbol{R}_B) = \boldsymbol{R}^T_{B/2}\boldsymbol{R}^T_a\boldsymbol{R}_{B/2}, \end{aligned} \tag{32}$$

where in the second equality we made use of (31). As for the displacement vector relevant for the into-the-element map, we have

$$\vec{d} = (\overrightarrow{OP})_{xyz} = (\overrightarrow{OO_0})_{xyz} + (\overrightarrow{O_0P_0})_{xyz} + (\overrightarrow{P_0P})_{xyz}, \tag{33}$$

where

$$(\overrightarrow{OO_0})_{xyz} = R_c \sin(\theta_B/2)\,(\hat{z}_0)_{xyz} = R_c \sin(\theta_B/2)\,\boldsymbol{R}_{B/2}(\hat{z})_{xyz}, \tag{34}$$

$$(\overrightarrow{O_0P_0})_{xyz} = \boldsymbol{T}^T(\overrightarrow{O_0P_0})_{x_0y_0z_0} = \boldsymbol{R}_{B/2}\vec{d}_a, \tag{35}$$

$$\begin{aligned}(\overrightarrow{P_0P})_{xyz} &= -R_c \sin(\theta_B/2)\,(\hat{Z}_0)_{xyz} = -R_c \sin(\theta_B/2)\,\boldsymbol{R}(\hat{z}_0)_{xyz} \\ &= -R_c \sin(\theta_B/2)\,\boldsymbol{R}\boldsymbol{R}_{B/2}(\hat{z})_{xyz}.\end{aligned} \tag{36}$$

Again, R_c is the bend-magnet radius of curvature. In the above expressions, $(\hat{z})_{xyz} = (0, 0, 1)$. Similarly, for the displacement vector relevant for the out-the-element map, see Eq. (23),

$$\begin{aligned}\vec{d'} &= (\overrightarrow{O'P'})_{x'y'z'} \\ &= (\boldsymbol{R}\boldsymbol{R}_B)^T(\overrightarrow{OP'} - \overrightarrow{OO'})_{xyz} \\ &= (\boldsymbol{R}\boldsymbol{R}_B)^T[(\overrightarrow{OP'})_{xyz} - \boldsymbol{R}(\overrightarrow{OP'})_{xyz} - \vec{d}],\end{aligned} \tag{37}$$

where for $\overrightarrow{OO'}$ we made use of Eq. (25); $\overrightarrow{OP'}$ is defined as in (24) and $\vec{d}$ as in (33). The matrix $\boldsymbol{R}$ appearing in (36) and (37) is as in (31).

6. Conclusions

In this Note we have worked out the exact nonlinear maps bridging the gaps at the two ends of a misaligned element in their exact nonlinear form. Typical usage of these maps entails small rotation angles but the expressions reported here remain valid for $-\pi/2 < \alpha_i < \pi/2$, $i = x, y, z$; however, if the misalignment is extreme and the canonical momenta large one may run into problems with square roots of negative numbers. There may be applications where using the exact nonlinear map may be important but in high-energy machines the linear approximation is generally adequate. The linear approximation does not suffer from the square-root of negative-number problem but, of course, will become inaccurate if the misalignments are too large.

We carried out numerical tests to validate the formulas, including the trivial example of a misaligned drift, and verified that the exact nonlinear maps converge to the linearized maps in the small-amplitude limit.

7. Acknowledgements

Work supported by the Office of Science of the US Department of Energy under Contract No. DEAC02-05CH11231.

Appendix A. Mathematica code: exact map (ultrarelativistic limit)

```
misalign[x_, px_, y_, py_, ell_, delta_, (* particle phase-space coordinates *)
         ax_, ay_, az_,      (* rotation misalignment angles *)
         dx_, dy_, dz_,      (* translation misalignment *)
         thetaB_, lenB_,     (* misaligned-element bend-angle and length *)
         face_               (* face=1 => element entry; face=2 => element exit *)
        ] := Module[{},
  (* An element with bend-angle smaller than "smallAngle" is treated as straight *)
  smallAngle = 10^-12;
  (* displacement-error vector in the xyz coordinate system *)
  OP = {dx, dy, dz};
  (* rotational-error matrix in the xyz coordinate system *)
  cx = Cos[ax]; sx = Sin[ax];
  cy = Cos[ay]; sy = Sin[ay];
  cz = Cos[az]; sz = Sin[az];
  RX = {{            cy cz,            - cy sz,     sy},
        { cz sx sy + cx sz,  cx cz - sx sy sz, -cy sx},
        {-cx cz sy + sx sz,  cz sx + cx sy sz,  cx cy}};

  If[ (* case face=1: *)
      face == 1, R = RX; d = OP,
      (* case face=2: *)
      If[Abs[thetaB] < smallAngle,  OPp = {0, 0, lenB},
         OPp = {lenB/thetaB*(Cos[thetaB] - 1), 0, lenB*Sin[thetaB]/thetaB}  ];
      RB = {{Cos[thetaB], 0,   -Sin[thetaB]},
            {          0, 1,              0},
            {Sin[thetaB], 0,    Cos[thetaB]}};
      RBT = Transpose[RB];  RXT = Transpose[RX];
      R = RBT.RXT.RB; (* rotational-error matrix in the x'y'z' coord. system *)
      OOp = RX.OPp + OP; (* object in the xyz coord. system *)
      OpPp = (OPp - OOp); (* object in the xyz coord. system *)
      d = RBT.RXT.OpPp   (* object in the x'y'z' coord. system *) ];

  RT = Transpose[R];
```

```
 pz = Sqrt[ (1 + delta)^2 - px^2 - py^2 ];
 r = {x, y, 0};  p = {px, py, pz};
 rr = RT.(r - d); pp = RT.p; L = -rr[[3]];

 Xf = rr[[1]] + L*pp[[1]]/pp[[3]];  Pxf = pp[[1]];
 Yf = rr[[2]] + L*pp[[2]]/pp[[3]];  Pyf = pp[[2]];
 ellf = ell + L*(1 + delta)/pp[[3]];  deltaf = delta;

 {Xf , Pxf , Yf , Pyf, ellf, deltaf} ]
```

Appendix B. Mathematica code: linear approximation (ultrarelativistic limit)

```
misalignLinear[x_, px_, y_, py_, ell_, delta_, (* particle phase-space coordinates *)
               ax_, ay_, az_,   (* rotation misalignment angles *)
               dx_, dy_, dz_,   (* translation misalignment *)
               thetaB_, lenB_,  (* misaligned-element bend-angle and length *)
               face_            (* face=1 => element entry; face=2 => element exit *)
              ] := Module[{},
 (* An element with bend-angle smaller than "smallAngle" is treated as straight *)
 smallAngle = 10^-12;
 (* particle initial coordinates *)
 t = {x, px, y, py, ell, delta};
 (* displacement-error vector in the xyz coordinate system *)
 OP = {dx, dy, dz};
 (* rotational-error matrix in the xyz coordinate system *)
 cx = Cos[ax]; sx = Sin[ax];
 cy = Cos[ay]; sy = Sin[ay];
 cz = Cos[az]; sz = Sin[az];
 RX = {{ cy cz,                        -cy sz,     sy},
       { cz sx sy + cx sz,   cx cz - sx sy sz, -cy sx},
       {-cx cz sy + sx sz,   cz sx + cx sy sz,  cx cy}};
 RXT = Transpose[RX];

 If[ (* face=1 *)
```

```
    face == 1, R = RX; d = {dx, dy, dz},
    (* face=2 *)
    RB =   {{Cos[thetaB], 0, -Sin[thetaB]},
           {          0, 1,            0},
           {Sin[thetaB], 0,  Cos[thetaB]}};
    RBT = Transpose[RB];
    R = RBT.RXT.RB; (* rotational-error matrix in the x'y'z' coordinate system *)
    If[ Abs[thetaB] < smallAngle,  OPp = {0, 0, lenB},
        OPp = {lenB/thetaB*(Cos[thetaB] - 1), 0, lenB*Sin[thetaB]/thetaB} ];
    OOp = RX.OPp + OP;
    OpPp = (OPp - OOp);
    d = RBT.RXT.OpPp;  ];

d1 = d[[1]]; d2 = d[[2]]; d3 = d[[3]];
R11 = R[[1,1]]; R12 = R[[1,2]]; R13 = R[[1,3]];
R21 = R[[2,1]]; R22 = R[[2,2]]; R23 = R[[2,3]];
R31 = R[[3,1]]; R32 = R[[3,2]]; R33 = R[[3,3]];
LD = R13*d1 + R23*d2 + R33*d3;
t0 = {(d2*R12 - d1*R22)/R33, R31, (d1*R21 - d2*R11)/R33, R32, LD/R33, 0};
LinMat = {{ R22/R33, LD*R22/R33^2, -R12/R33,-LD*R12/R33^2, 0,   0},
          {       0,          R11,        0,          R21, 0, R31},
          {-R21/R33,-LD*R21/R33^2,  R11/R33, LD*R11/R33^2, 0,   0},
          {       0,          R12,        0,          R22, 0, R32},
          {-R13/R33,-LD*R13/R33^2, -R23/R33,-LD*R23/R33^2, 1,   0},
          {       0,            0,        0,            0, 0,   1}};
tf = LinMat.t + t0;

{tf[[1]], tf[[2]], tf[[3]], tf[[4]], tf[[5]], tf[[6]]}  ]
```

References

[1] E. Forest, Beam Dynamics, Harwood Academic Publishers, (1998).

[2] Wolfram Research, Inc., Mathematica, Champaign, IL (2021).

[3] H. Grote and F.C. Iselin, The MAD program, CERN/SL/90-13 (1993).

Response to Reviewer #2.

Q: Line 83: "... possibly tilted by an angle θ_T." θ_T of Eq. (12) and α_Z of Eq. (3) seem to be equivalent. If so, the relevant matrix is being multiplied twice at line 107 and related transformations. Please clarify this.

A: The error rotation and the R_B rotation are conceptually different. R_B can be thought of as the map relating the momentum vector of the reference trajectory at the entrance to that at the exit of the element in its aligned position; it represents a geometric transformation used to express the rotation error in the correct coordinate frame at the exit of the element. Although the two matrices that make up R_B (a tilt about the z axis and a rotation about the y axis) are similar in form to the matrices representing the rotational misalignment-errors about the same axes, they have different meaning; there is no "double counting".

Q: Line 93: "It should be obvious that ..."
The rotation matrix of Eq. (3) is first being used to move the particle from the upstream drift into the misaligned element (Sec. 2 and 3), then, from the end point of the misaligned element to the entry face of the downstream drift (Sec. 4.) This assumes that passing through the drift and then the bending section is equivalent to passing through the bending section and then the drift.

1) Under what conditions are these two configurations commutative?

2) Please state the amount of the error with this assumption. Note that it will accumulate at every revolution.

3) Can quadrupoles and sextupoles also be treated like this?

A: The referee's phrasing does not quite capture what is going on here: there is no issue with the commutativity (or lack thereof) between drift and non-drift mappings. Let’s see if a more descriptive explanation helps. Consider the case represented by the right image in Fig. 2 (map out of the misaligned element); having arrived at this point, we know the particle coordinates at the location where the particle exits the misaligned element. Call this point A. Now, forget how the particle arrived here and remove the misaligned bend from this picture. The idea of the method is to find the putative coordinates at the entrance of the downstream drift (call this point B) such that if the misaligned magnet were not present the particle orbit originating from B would reach A. Looking at the figure we see that point B can be found by backward propagation in free space over the drift length L'. Now, having succeeded in finding B, we are in business because the particle coordinates at the end of the downstream drift (call them point C), which ultimately are what we need to proceed further with tracking down the lattice, are simply obtained by applying the full downstream-drift transfer map to B. Note that C is the correct physical result even if point A, the starting point for tracking through the downstream drift, is only a "putative" initial condition and is not where the physical particle actually entered the downstream drift. This trick works regardless of the nature (curved, straight) of the misaligned element.

Q: Eq (34): The linear distance between O and O_0 is expressed by $R_c \sin(\theta_B/2)$. This is half of the arc-length of the bending section, which is slightly larger than what is intended. This difference may be small for one turn. But after many turns (in many cases billions), the error can accumulate. Please state the amount of the numerical error regarding this assumption. Same with Eq. (36.).

A: As a matter of fact, Rc*sin(thetaB/2) is half the chord not half the bend arclength; it is exactly the intended quantity. There is no "cumulative error" involved.

Q: Eq (35): The definition of $\vec d_a$ is not given.

A: \vec d_a is defined on line 137.

Q: Lines 166-168: A short section should be provided to show the level of discrepancy between those analytical solutions (or Lie algebra based solutions as mentioned at line 20) and the two methods described here, as a function of number of passages. This will help the reader understand the term "appropriate limit" w.r.t. the misalignment parameters at line 168.

A: The results presented here are meant to be (and are) numerically equivalent to those obtained by Lie methods. The approximation alluded to in the Conclusion is the approximation between the fully non-linear maps and their linearization. We verified that, as expected, the numerical error we make by using the linearized maps vanishes and scales as a power-law as the amplitude of the particle motion goes to zero.

The revised manuscript fixes a typo (line 62, t-> \ell) and contains a couple of minor stylistic changes (commas added, etc). The conclusions have been slightly modified for clarity.

9/12/2022

Further response to Reviewer#2

Reviewer #2: The comment regarding Eq's. 34 and 36 is not addressed. Those equations are inconsistent with Fig. 3, and the definition of R_c.

The comment by the Referee with regard to Eq. (34) and (36) was addressed (so, the issue is why the response is not satisfactory); see below:

> *Q: Eq (34): The linear distance between O and O_0 is expressed by $R_c \sin(\theta_B/2)$. This is half of the arc-length of the bending section, which is slightly larger than what is intended. This difference may be small for one turn. But after many turns (in many cases billions), the error can accumulate. Please state the amount of the numerical error regarding this assumption. Same with Eq. (36.).*
>
> *A: As a matter of fact, Rc*sin(thetaB/2) is half the chord not half the bend arclength; it is exactly the intended quantity. There is no "cumulative error" involved.*

Since the referee incorrectly referred to the length of the chord as an "arc-length", in our response we alluded to the distinction between "chord" and "arc", guessing that this was perhaps the source of the misunderstanding but, apparently, we did not catch the sense of the Referee’s objection. To make progress from here, we would like to ask the Referee to clarify his/her thinking. The Referee said that the distance between O and O_0 was larger than what was intended. It’d be helpful if the Referee could indicate what he/she thinks the "intended" quantity should be and specifically how Eq. (34) is inconsistent with Fig. 3.

Declaration of interests

☒The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: