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ASYMPTOTIC DISTRIBUTIONS ON SUPER RIEMANN SURFACES

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ABSTRACT

The Selberg super-trace formula for super Riemann surfaces is used to derive asymptotic distributions for the asymptotic distribution of the number $\bar{N}(\lambda)$ ($|\lambda|\to\infty$) of eigenvalues λ of the Dirac-Laplace operator $\square$ on super Riemann surfaces, and for the asymptotic distribution of the number $\bar{N}(N)$ ($N\to\infty$) for the number of the norms N of the hyperbolic conjugacy classes, respectively. In the case that the underlying Riemann surface is compact, we find that the Witten index is determined by the area of the Riemann surface.

1 Introduction.

There are two important problems in the study asymptotic distributions. First, there is the question of prime geodesic theorems according to Riemann [?] in order to find the asymptotic proliferation of prime numbers P, as $P \to \infty$. Second, there is the question of the number of eigenmodes in a cavity, nowadays known as Weyl's law, as the energy tends to infinity. This very important law (one of Hilbert's famous problems) was discovered by Weyl [?]. The latter problem is actually closely related to the question "Can one hear the shape of a drum?" [?], whereas the former could be put into "Can one see the sound of a drum?". Obviously, both asymptotic laws have been the subject of many investigations since, and they both can be put into the context of *index theorems* which makes them mathematically even more attractive and interesting.

In quantum chaos [?, ?] and hyperbolic geometry [?] the prime geodesic theorem translates into an asymptotic distribution of the norms of hyperbolic conjugacy classes, as the norms tend to infinity, respectively the number of periodic orbits, as the length of the periodic orbits tend to infinity, and is generally known as Huber's law [?]. Whereas it is possible by means of the Selberg trace formula [?, ?]–[?] and by the semiclassical Gutzwiller trace formula [?] to establish an intimate connection between periodic orbits and energy-levels, the general outcome is that the energy-levels give information about orbits, but not generally about the shape of the object in question [?], and the statistics of the distribution of the fluctuations of the staircase functions about the asymptotic distribution give evidence of the character of the investigated system [?], may it classically integrable, chaotic, respectively arithmetically chaotic [?]. The corresponding asymptotic laws also provide numerical checks, whether the found (length and energy) spectrum is actually complete up to a chosen value.

In the present paper I apply some of the known techniques for deriving asymptotic distributions, in the following for short denoted by Huber's and Weyl's law, respectively, in the case of super Riemann surfaces. The concept of superspace, e.g. [?] and super Riemann surfaces emerge, for instance, in string theory [?] and quantum super gravity theories [?] and provide a convenient and systematic way to incorporate fermionic degrees of freedom into a field theory. Besides the pure "stringy" aspects of super Riemann surfaces one can also think of a free interacting field theory where an effective potential analysis is made, e.g. Ref.[?, ?].

In the next section I review briefly some classical results concerning Weyl's and Huber's law. In the third section I summarize some results concerning super Riemann surfaces. I have developed in [?]–[?], based on earlier results by Baranov et al. [?], a theory of automorphic forms on super Riemann surfaces which resulted in the formulation of Selberg super-trace formulæ for Dirac-Laplace operators on super Riemann surfaces. It is possible to state Selberg super-trace formulæ for compact and non-compact super Riemann surfaces (according if the underlying usual Riemann surface, the body of the super Riemann surface, is compact or non-compact), and closed and bordered super Riemann surfaces [?, ?], respectively. In order to make the paper self contained I will outline some basic facts about super Riemann surfaces in the third section, including the statement of the relevant Selberg super-trace formulæ. The formulations will be as general as possible to include any super Riemann surfaces.

The fourth section actually then attacks the derivation of the asymptotic law for the proliferation of the periodic orbits, i.e., the norms of the hyperbolic conjugacy classes. The principal result is that we find a similar behaviour as in the classical case.

The fifth section will be concerned with the corresponding Weyl's law. The very construction of the Selberg supertrace formula implies that in the case of a continuous spectrum for a non-compact super Riemann surface. The relevant expression which must be studied has the form $\mathrm{str}[(\mathrm{e}^{-tH}-\mathrm{e}^{-tH_0})]$, where H_0 is the free supersymmetric Hamiltonian. It is found that the positive (continuous) spectrum is canceled out, and we get a relative index theorem, e.g. [?, ?]. Actually, this trace yields a number counting the difference of the bosonic (respectively fermionic) zero-energy states of H minus the difference of the bosonic (respectively fermionic) zero-energy states of H_0 plus an additional number determined by the continuous spectrum. Here, we find quite a different behaviour in comparison the the classical case. Weyl's law is found to be a constant given by the area of the domain. This is in accordance with the Witten index [?] (super-trace of the heat-kernel), formally written as $\mathrm{str}(-1)^F$, where $(-1)^F=1-2a^\dagger a$ is the "Fermion"-number operator, is independent of the parameters of the theory. In the present case, these parameters describe the deformations of the metric, i.e., they are the Teichmüller parameters; therefore we are left with the invariants of the surfaces which are classified according to their genus g, i.e., $\mathrm{str}(-1)^F\propto g\propto\mathcal{A}$, the area of the surface. In the presence of parabolic conjugacy classes, i.e., non-compact domains with cusps, scattering states become dominant and change this picture.

The sixth section is devoted to a discussion and summary of the results. In the three appendices some results as derived from the usual Selberg trace formula are summarized, including some additional notation concerning the general set-up of the Selberg trace formula. Particularly in appendix B, I briefly outline the derivation of Huber's law for $\mathrm{SL}(2,\mathbb{Z})$ which is due to unpublished results of Romani [?].

2 Some classical results

Before I develop the asymptotic distributions for super Riemann surfaces in the sequel, I first want to give a closer look at the classical cases and introduce some notation. In the considerations of the classical asymptotic distribution $\bar N_\Gamma$ of the norms N of the hyperbolic conjugacy classes in a Fuchsian group Γ one knows that (Huber's law [?]) the asymptotic distribution has the form

$$\bar N_\Gamma\propto\mathrm{Ei}(L)\propto\frac{\mathrm{e}^L}{L}\ ,\qquad L\to\infty\ , \tag{2.1}$$

where $N=\mathrm{e}^L$ and $\mathrm{Ei}(z)$ is the exponential integral [?, p.925]. It is the asymptotic distribution (leading term) for any (compact and non-compact) Riemann surface. However, this is a rather rough estimate. As it turns out (??) describes very well the asymptotic distribution in the case of the regular octagon [?]. But in the case of the fundamental domain of the modular group $\mathrm{SL}(2,\mathbb{Z})$ (??) is corrected considerably by the presence of parabolic elements in the Fuchsian group Γ, i.e.,

$$\bar{N}_{\mathrm{SL}(2,\mathbb{Z})} \propto \mathrm{Ei}(L) - \frac{3}{2}\mathrm{Ei}(L/2) \ , \qquad L \to \infty \ . \tag{2.2}$$

Corrections of this order are also present from contributions coming from the eigenvalues of the Laplacian $-\Delta = -y^2(\partial_x^2 + \partial_y^2)$ on the Riemann surface, as already noted by Huber. The appearance of the additional summand $\mathrm{Ei}(L/2)$ is due to the fact that in the derivation of the asymptotic distribution of the mean number of hyperbolic conjugacy classes, $\bar{N}_\Gamma(L)$ is determined by the equation

$$\sum_{k=1}^{\kappa(L)} \frac{1}{k} N_\Gamma\left(\frac{L}{k}\right) = F_\Gamma(L) \ , \tag{2.3}$$

with $\kappa(L) = [L/l_m]$ with some minimum length l_m, i.e., the length of the shortest periodic orbit on the surface, $[x]$ the integer part of the number x, and the function $F_\Gamma(L)$ contains all contributions up to a specific order, say $\mathrm{Ei}(L/2)$, as derived form the Selberg trace formula. Equation (**??**) then is inverted by the Möbius inversion formula [?, p.237], i.e.,

$$N_\Gamma(L) = \sum_{k=1}^{\kappa(L)} \frac{\mu(k)}{k} F_\Gamma\left(\frac{L}{k}\right) \ , \tag{2.4}$$

where $\mu(k)$ is the Möbius function ($\mu(1) = 1$, $\mu(2) = -1$, $\mu(3) = -1, \mu(4) = 0, \mu(5) = -1, \mu(6) = 1, \ldots$). On then applies (**??**) to (**??**), c.f. appendix B for more details. Similar asymptotic laws as (**??**) can be made as long as only compact domains are concerned. If non-compact domains are taken into account, the unknown structure of the scattering matrix generally does not allow any further evaluation. Only in the case of congruence groups $\Gamma(N)$ it is possible to make some estimate based on the fact that the scattering matrix can be expressed in terms of the Γ-function and the Riemann ζ-function, e.g. [?, ?]. These kinds of asymptotic distributions are very similar to the prime number formulæ involving $\pi(x)$, where $\pi(x)$ is the number of prime numbers not exceeding x, compare e.g. [?, ?]). It is also observed in other chaotic systems, e.g., the hyperbola billiard [?]. The leading term is of the form of (**??**), and can be improved by subsequent analysis. The whole matter can be systematically discussed by considering Gutzwiller's periodic orbit formula [?] for billiard systems [?].

Analogous asymptotic distributions as (**??**) can be derived for higher dimensional groups. One has for instance

$$\bar{N}_\Gamma \propto \frac{\mathrm{e}^{2\rho L}}{2\rho L} \ , \qquad L \to \infty \ , \tag{2.5}$$

where $|2\rho| = p + 2q$. Here p and q specify the number of roots in the algebra. For $\mathrm{SO}_0(2n+1,1) \cong \mathcal{H}^{2n}$ one has $p = 2n$ and $q = 0$, thus $2|\rho| = \sqrt{n}$, say. Compare for $\mathrm{SO}(3,1)$ Elstrod et al. [?], and for the general case Gangoli [?].

The other important asymptotic distribution, i.e. Weyl's law, describes the asymptotic proliferation of the numbers of energy levels $\bar{N}(E)$ not exceeding E, respectively it counts the asymptotic proliferation of the number of modes in a cavity. It provides a useful check whether all modes of a system (as a theoretical or a "hard-ware" model) actually have been found. The leading term (two-dimensional case) has the form

$$\bar{N}(E) \propto \frac{\mathcal{A}}{4\pi}E, \qquad E \to \infty \ , \tag{2.6}$$

with $\mathcal{A}$ the area of the corresponding domain. For general flat compact two-dimensional systems it is possible to give expressions down to powers E^0 [?]:

$$\begin{aligned} \bar{N}(E) &= \frac{\mathcal{A}}{4\pi}E - \frac{\partial\mathcal{A}}{4\pi}\sqrt{E} + \frac{1}{24}\sum_{\text{corners}}\left(\frac{\pi}{\alpha_r} - \frac{\alpha_r}{\pi}\right) \\ &\quad + \frac{1}{12\pi}\int_{\mathcal{A}} K(\sigma)d^2\sigma - \frac{1}{24\pi}\oint_{\partial\mathcal{A}} k(s)ds + O(E^{-1/2}) \ , \qquad E \to \infty \ , \end{aligned} \tag{2.7}$$

Here ∂A denotes the length of the boundary, α_r the angle of the corner r, $K(\sigma)$ is the Gaussian curvature, and $\kappa(s)$ the mean curvature on the boundary. Generally, a Weyl's law can always been derived by studying the short-time behaviour of the heat-kernel $\Theta(t)$ of a system, e.g., [?, ?] for reference. In non-compact domains the leading term is proportional to $E\ln E/4\pi$, say, e.g. [?] and references therein. In the case of a D-dimensional manifolds one obtains [?]

$$\bar{N}(E) \propto \frac{Volume}{\Omega(D)}E^{D/2} \ , \qquad E \to \infty \ , \tag{2.8}$$

where $Volume$ is the volume of the considered domain (cavity) and $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$ is the volume of the D-dimensional unit sphere. Investigations to determine Weyl's law on Riemann surface are due to Venkov [?], Aurich et al. [?], or Müller [?].

3 Super Riemann surfaces and the Selberg super-trace formula

We sketch some facts about $N = 1$ super Riemann surfaces. For more details I refer to e.g. [?]–[?], Baranov et al. [?], Batchelor et al. [?], DeWitt [?], Ninnemann [?], Rabin and Crane [?], and Rogers [?]. Let us start with a (1|1) (complex)-dimensional (not necessarily) flat superspace, parameterized by even coordinates $Z \in \mathbb{C}_c$ and odd (Grassmann) coordinates $\theta \in \mathbb{C}_a$, respectively. Let Λ_∞ be the infinite dimensional vector space generated by elements ζ_a $(a = 1, 2, \ldots)$ with basis $1, \zeta_a, \zeta_a\zeta_b, \ldots$ $(a < b)$ and the anticommuting relation $\zeta_a\zeta_b = -\zeta_b\zeta_a$, $\forall_{a,b}$. Every $Z \in \Lambda_\infty$ can be decomposed as $Z = Z_B + Z_S$ with $Z_B \in \mathbb{C}_c \equiv \mathbb{C}$, $Z_S = \sum_n \frac{1}{n!}c_{a_1,\ldots,a_n}\zeta^{a_n}\ldots\zeta^{a_1}$, with the $c_{a_1,\ldots,a_n} \in \mathbb{C}_a$ totally antisymmetric. Z_B and Z_S, respectively, are called the *body* (sometimes denoted by $Z_B = Z_{red}$) and *soul* of the super-number Z, respectively. In the fermionic string theory one is interested in super-conformal symmetry. The notion of superspace and super-manifolds enables one to represent super-symmetry transformations as pure geometric transformations in the coordinates $Z = (z, \theta) \in \mathbb{C}_c \times \mathbb{C}_a$. As is well-known, a usual complex manifold of complex dimension equal to one is already a Riemann surface. The definition of a super Riemann surface, however, requires the introduction of a super-conformal structure. Let us consider the operator $D = \theta\partial_z + \partial_\theta$ (note $D^2 = \partial_z$). Further we consider a general super-analytic coordinate transformation $\tilde{z} = \tilde{z}(z, \theta)$, $\tilde{\theta} = \tilde{\theta}(z, \theta)$. A super-analytic coordinate transformation is called super-conformal, iff the $(0|1)$-dimensional subspace of the tangential space generated by the action of D is invariant under such a coordinate transformation, i.e. $D = (D\tilde{\theta})\widetilde{D}$. This means that a coordinate transformation is super-conformal iff $Dz' = \theta' D\theta'$.

To study super-symmetric field theories one needs even and odd super-fields. Here the definition of DeWitt of super Riemann manifolds conveniently comes into play. The infinite dimensional algebra Λ_∞ supplies all the required quantities. Domains in $\mathbb{C}^{(1|1)}$ with coordinates (z, θ) are constructed in such a way that the entire Grassmann algebra is attached to the usual complex coordinates. If one considers the universal family of DeWitt super Riemann manifolds with genus g, then only $2g-2$ parameters of Λ_∞ are required, the remaining ones are redundant.

An important property in the theory of supermanifold is when a supermanifold is *split*. This means that for a coordinate transformation $Z \to Z'$ $(Z, Z' \in L_\infty)$ the coefficient functions do not mix which each other. An individual super Riemann can always be considered as being split and corresponds to a usual Riemann surface with a spin structure: at each point of the superspace, S the reduced subgroup $\Gamma_{red} \subset \mathrm{SL}(2, \mathbb{R})$ consists of hyperbolic elements, acts discretely upon $\mathcal{H} = \mathcal{H}^{(1|1)}_{red}$ and has a compact quotient space $\Gamma_{red} \backslash \mathcal{H}$. Then $\Gamma \backslash \mathcal{H} \times S$ is a family of $(1|1)$-dimensional complex compact supermanifolds, parameterized by S [?]. The moduli space of closed super Riemann surfaces of genus g is a family of surfaces labeled by $6g-6$ even and $4g-4$ odd read parameters, and is called the super moduli space.

To generalize the uniformization of usual Riemann surfaces to super Riemann surfaces $\mathcal{M}$, one shows that unique generalizations $\widehat{\mathbb{C}}^{(1|1)}$, $\mathbb{C}^{(1|1)}$ and $\mathcal{H}^{(1|1)} := \{(z, \theta) \in \mathbb{C}^{(1|1)} | \Im(z) > 0\}$ of simple connected Riemann surfaces exist, and endows $U = \widehat{\mathbb{C}}^{(1|1)}$, $\mathbb{C}^{(1|1)}$, $\mathcal{H}^{(1|1)}$ with a super-conformal structure, such that the local coordinate transformations are super-conformal mappings.

In order to construct explicitly a metric on $\mathcal{H}^{(1|1)}$ one starts with the super-vierbeins in flat superspace and performs a super-Weyl transformation to obtain the metric ds^2 in $\mathcal{H}^{(1|1)}$, i.e.,

$$\begin{aligned} ds^2 &= dq^a{}_a g_b dq^b \\ &= \frac{1}{Y^2}[d\bar{z}dz - i\bar{\theta}d\bar{z}d\theta - i\theta d\bar{\theta}dz - (2Y + \theta\bar{\theta})d\theta d\bar{\theta}] \ . \end{aligned} \tag{3.1}$$

The scalar product has the form

$$(\Phi_1, \Phi_2) = \int \frac{dz d\bar{z} d\theta d\bar{\theta}}{2Y} \Phi_1(Z) \bar{\Phi}_2(Z) \ , \tag{3.2}$$

for super-functions $\Phi_1, \Phi_2 \in L^2(\mathcal{H}^{(1|1)}), Y = y + \theta\bar{\theta}/2$.

In the case of non-Euclidean harmonic analysis in the context of super Riemann surfaces we consider the group $\mathrm{OSp}(2, \mathbb{C})$ of super conformal automorphisms on super Riemann surfaces as a natural generalization of Möbius transformations which explicitly have the form

$$z' = \frac{az+b}{cz+d} + \theta \frac{\alpha z + \beta}{(cz+d)^2} \ , \qquad \theta' = \frac{\alpha + \beta z}{cz+d} + \frac{\chi_\gamma \theta}{cz+d} \ . \tag{3.3}$$

The χ_γ with $\chi_\gamma = \pm 1$ lead to the description of spin structures on a super Riemann surface. This general super-Möbius transformation does mix the coefficient functions of super-functions $F \in \Lambda_\infty$. Since we work with a super Riemann surfaces which is split, the odd quantities α, β are not necessary and can be omitted. It is sufficient to consider transformations $\gamma \in \mathrm{OSp}(2,1)$ with $\alpha = \beta = 0$ and characters χ_γ which describe spin structures. Furthermore γ and $-\gamma$ describe the same transformation. We thus have that the automorphisms on $\mathcal{H}^{(1|1)}$ are given by $\mathrm{Aut}\mathcal{H}^{(1|1)} = \mathrm{OSp}(2|1, \mathrm{I\!R})/\{\pm \mathbb{1}\}$, and a super-Fuchsian group Γ denotes a discrete subgroup of $\mathrm{Aut}\mathcal{H}^{(1|1)}$. Therefore we obtain for the transformations $z \to z'$ and $\theta \to \theta'$

$$z' = \frac{az+b}{cz+d} \ , \qquad \theta' = \frac{\chi_\gamma \theta}{cz+d} \ . \tag{3.4}$$

$M_{\zeta=0}$ corresponds to the usual Riemann surface M_{red} with some spin-structure, since an element $\gamma \in \mathrm{Aut}\mathcal{H}^{(1|1)}$ is fixed by a $\mathrm{PSL}(2, \mathrm{I\!R})$ transformation and a character $\chi_\gamma = \pm 1$. The properties of the odd coordinates are determined by the properties of M_{red}: θ is the cut of a spinor-bundle.

We introduce the Dirac-Laplace operators $\Box_m$ and $\widehat{\Box}_m$, respectively [?, ?] ($m \in \mathbb{Z}$)

$$\Box_m = 2YD\bar{D} + \mathrm{i}m(\bar\theta - \theta)\bar{D}, \qquad \widehat{\Box}_m = 2YD\bar{D} + \frac{\mathrm{i}m}{2}(\bar\theta - \theta)(D + \bar{D}) \ , \tag{3.5}$$

and $\Box_m$ and $\widehat{\Box}_m$ are related by a linear isomorphism $\Box_m = Y^{-m/2}(\widehat{\Box}_m + \mathrm{i}m/2)Y^{m/2}$. Particularly we have for $m = 0$ [?, ?, ?, ?, ?]

$$\widehat{\Box}_0 = \Box_0 \equiv \Box = 2Y(\partial_\theta \partial_{\bar\theta} + \theta\bar\theta \partial_z \partial_{\bar z} + \theta \partial_{\bar\theta} \partial_z - \bar\theta \partial_\theta \partial_{\bar z}) \ . \tag{3.6}$$

For an even super function $\Psi(Z, \bar{Z}) = A(z, \bar{z}) + \theta\bar\theta B(z, \bar z)/y$ we have the equivalence

$$\widehat{\Box}\Psi(Z, \bar Z) = s\Psi(Z, \bar Z) \Longleftrightarrow -\Delta A(z, \bar z) = s(1-s)A(z, \bar z) \ , \quad B(z, \bar z) = \frac{s}{2}A(z, \bar z) \ , \tag{3.7}$$

and $\Delta = y^2(\partial_x^2 + \partial_y^2)$ is the non-Euclidean Laplacian on the hyperbolic plane $\mathcal{H}$.

Let us introduce the quantities N_γ and l_γ by $2\cosh\frac{l_\gamma}{2} = N_\gamma^{1/2} + N_\gamma^{-1/2} = a + d + \chi_\gamma \alpha\beta$. N_γ is called *norm* of a hyperbolic $\gamma \in \Gamma$ in a super Fuchsian group, and N_{γ_0} will denote the norm of a primitive hyperbolic $\gamma_0 \in \Gamma$, and $l_\gamma = \ln N_\gamma$ denotes the *length* corresponding to a $\gamma \in \Gamma$ and all notions from the bosonic case are interpreted in a straightforward way into their super generalization. Each element $\gamma \in \Gamma/\{\pm\mathbb{1}\}$ is thus uniquely described as $\gamma = k^{-1}\gamma_0 k$ for some primitive γ_0, $n \in \mathrm{I\!N}$ and $k \in \Gamma/\Gamma_{\gamma_0}$. For $\mathrm{OSp}(2, \mathrm{I\!R})/\{\pm\mathbb{1}\}$ in homogeneous coordinates a hyperbolic transformation is always conjugate to the transformation $z' = N_\gamma z$, $\theta' = \chi_\gamma \sqrt{N_\gamma}\,\theta$.

I have proposed in [?] similarly as for the hyperbolic $T \in \Gamma$, elliptic and parabolic $T \in \Gamma$, and appropriate super-fundamental domains $\mathcal{F}^{(1|1)}$, a decomposition of an appropriate $T \in \Gamma$ as follows

$$(T \in \Gamma \text{ conjugate to}) \qquad \gamma \times R \times S \tag{3.8}$$

with $n \in \mathrm{I\!N}$ and $0 < \phi < \pi$, and γ, R and S, respectively, denote hyperbolic, elliptic and parabolic transformations, acting by matrix multiplication, i.e.,

$$(T \in \Gamma \text{ conjugate to}) \qquad \gamma \times R \times S = \begin{pmatrix} N_\gamma^{1/2} & 0 & 0 \\ 0 & N_\gamma^{-1/2} & 0 \\ 0 & 0 & \chi_\gamma \end{pmatrix}$$

$$\times \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & \chi_R \end{pmatrix} \cdot \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \chi_S \end{pmatrix} , \tag{3.9}$$

with $n \in \mathbb{N}$ and $0 < \phi < \pi$, and γ, R and S, respectively, denote hyperbolic, elliptic and parabolic transformations, and $\phi = \pi/\nu_j$, $j = 1, \ldots, s$, where ν_j is the order of the elliptic element R_j.

Concerning non-compact super Riemann surfaces, let V be an h-dimensional complex vector space, $V = \mathbb{C}^h$. Let U be an representation of Γ which acts in the space V and is unitary with respect to the inner product in V. For each $\alpha = 1, \ldots, \kappa$ we have a subspace $V_\alpha \subset V$ of the operator $U(S_\alpha)$, i.e., $V_\alpha = \{v \in V | U(s_j)v = v\}$. Let $k_\alpha = \dim V_\alpha$ and $\kappa_0 = \sum_{\alpha=1}^{\kappa} k_\alpha$. k_α denotes the degree of singularity of the representation U relative to the generator S_α of $\Gamma_\alpha \subset \Gamma$ and $\tilde{\kappa}_0$ denotes the degree of singularity of Γ relative to U. For more details, c.f. [?, ?], and it is sufficient without loss of generality of assume the trivial representation in the following.

One considers similar as in the case of the usual Selberg trace formula a super-automorphic kernel, the super-trace of operator-valued functions $h(\Box)$ (we consider only the case $m = 0$), and can formulate the Selberg super-trace formula in the following way [?, ?, ?]:

Theorem 3.1 *The Selberg super-trace formula on closed super Riemann surfaces for Dirac-Laplace operators $\Box$ for hyperbolic, elliptic and parabolic conjugacy classes is given by*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Big[h(\tfrac{1}{2} + \mathrm{i}p_n^{(B)}) - h(\tfrac{1}{2} + \mathrm{i}p_n^{(F)})\Big] = \mathrm{i} \dim V \frac{\mathcal{A}}{4\pi} \int_{\mathbb{R}} h(\tfrac{1}{2} + \mathrm{i}p) \tanh \pi p \, dp \\
&+ \sum_{\{\gamma\}} \sum_{k=1}^{\infty} \frac{l_\gamma}{2\sinh(k l_\gamma/2)} \Big[g(k l_\gamma) + g(-k l_\gamma) - \chi_\gamma^k \Big(g(k l_\gamma)\mathrm{e}^{-k l_\gamma/2} + g(-k l_\gamma)\mathrm{e}^{k l_\gamma/2}\Big)\Big] \\
&+ \sum_{\{R\}} \sum_{k=1}^{\nu-1} \frac{1}{\nu} \Bigg[\Big(1 - \chi_R^k \cos\frac{k\pi}{\nu}\Big) \\
&\qquad \times \int_0^\infty \frac{g(u)\mathrm{e}^{-u/2} + g(-u)\mathrm{e}^{u/2}}{\cosh u - \cos(2k\pi/\nu)} du + \int_0^\infty \frac{g(u) - g(-u)}{\cosh u - \cos(2k\pi/\nu)} \sinh\frac{u}{2} \, du\Bigg] \\
&-2\Big[\tilde{\kappa}_0 \ln 2 + \kappa_- \ln |\mathrm{sdet}(1 - U(S))|\Big] g(0) \\
&-\frac{\kappa_-}{2} h(\tfrac{1}{2}) \mathrm{tr}[\mathfrak{S}(\tfrac{1}{2})] + \kappa_- \int_0^\infty g(-u) du + \frac{\kappa_-}{2\pi} \int_{\mathbb{R}} h(\tfrac{1}{2} + \mathrm{i}p) \frac{\Delta'(\frac{1}{2} + \mathrm{i}p)}{\Delta(\frac{1}{2} + \mathrm{i}p)} dp \\
&+\frac{\kappa_0}{2} \int_0^\infty [g(u) - g(-u)] du + \frac{\tilde{\kappa}_0}{2\pi} \int_{\mathbb{R}} h(\tfrac{1}{2} + \mathrm{i}p) [\Psi(1 + \mathrm{i}p) + \Psi(1 - \mathrm{i}p)] dp \ ,
\end{aligned} \tag{3.10}$$

where $\tilde{\kappa}_0 = \sum_{\{S_j\}} \kappa_j (1 - \chi_{S_j})$, $\kappa_\pm = \sum_{\{S_j\}} (1 \pm \chi_{S_j})$, *and the other terms similarly interpreted.* $\lambda_n^{(B,F)} = \mathrm{i}p_n^{(B,F)} + \frac{1}{2}$ *on the left runs through the set of all eigenvalues of the Dirac-Laplace operator* $\Box$, *and the summation on the right is taken over all primitive conjugacy classes* $\{\gamma\}$ *with* $\mathrm{str}(\gamma) + \chi_\gamma > 2$, $\{R\}$ *with* $\mathrm{str}(R) + \chi_R < 2$, *and* $\{S\}$ *with* $\mathrm{str}(S) + \chi_S = 2$. *The function* $h(\frac{1}{2} + \mathrm{i}p)$ *must have the following properties*

1. $h(\frac{1}{2} + \mathrm{i}p) \in C^\infty(\mathbb{R})$,

2. *$h(p)$ vanishes faster than $1/|p|$ for $p \to \pm\infty$.*
3. *$h(\frac{1}{2} + \mathrm{i}p)$ is holomorphic in the strip $\Im(p) \leq \frac{1}{2} + \epsilon$, $\epsilon > 0$, to guarantee absolute convergence in the summation over $\{\gamma\}$.*
4. *$g(u) = (2\pi)^{-1} \int_{\mathbb{R}} h(\frac{1}{2} + \mathrm{i}p)\mathrm{e}^{-\mathrm{i}up} dp$,*
5. *$\Psi(z) = \Gamma(z)/\Gamma'(z)$ with $\Gamma(z)$ the Eulerian Γ-function [?, p.943],*
6. *$\Delta(z)$ denotes the determinant of the scattering matrix $\mathfrak{S}$ as constructed form the Eisenstein series,*
7. *and the remaining quantities describe in detail the cusp-space with κ_0 its dimension, and κ the number of cusps.*

As far as the hyperbolic conjugacy classes are concerned, the result of Theorem 2.1 can be generalized to m-weighted Dirac-Laplace operators $\hat{\square}_m$ [?].

4 Derivation of Huber's law

In order to analyze the Selberg super-trace formula we have to start by considering a properly chosen test-function in the $\sum_{\{\gamma\}}$ terms to eliminate the χ_γ contribution. We set

$$g(u,t) = \frac{1}{4\sqrt{\pi t}} \mathrm{e}^{u/2} \left[\mathrm{e}^{-(u-L)^2/4t} - \mathrm{e}^{-(u+L)^2/4t} \right] \quad , \tag{4.1}$$

and together with the abbreviation $s = \frac{1}{2} + \mathrm{i}p$ we obtain

$$h(p,t) = \int_{\mathbb{R}} g(u) \mathrm{e}^{\mathrm{i}up} du = \sinh(sL)\mathrm{e}^{s^2 t} \quad . \tag{4.2}$$

which is a sin-hyperbolic modulated heat kernel, and fulfills the condition of a proper test-function. In the following we use for convenience both the notations $h(\frac{1}{2} + \mathrm{i}p) \equiv h(p)$ for the test-function h. Note the property

$$\lim_{t \to 0^+} \int_{\mathbb{R}} g(u) f(u) = \frac{1}{2} \left[f(L)\mathrm{e}^{L/2} - f(-L)\mathrm{e}^{-L/2} \right] \tag{4.3}$$

for a test function $f(u)$. We furthermore find

$$\frac{g(u) + g(-u)}{\sinh(u/2)} = \frac{\mathrm{e}^{-(u-L)^2/4t} - \mathrm{e}^{-(u+L)^2/4t}}{2\sqrt{\pi t}} \equiv G_1(u,t) \quad , \tag{4.4}$$

which gives the obvious identity

$$\lim_{t \to 0^+} G_1(u,t) = \delta(u - L) - \delta(u + L) \quad . \tag{4.5}$$

Now we have $h(p, t = 0) = \sinh(sL) = \sinh[(\frac{1}{2} + \mathrm{i}p)L]$, $h(0,0) = \sinh(L/2)$, and $g(u = 0, t) = 0$.

We find for the super-trace formula of the test-function (??)

$$
\begin{aligned}
\sum_{n=0}^{\infty}\Big[h(\tfrac{1}{2}+\mathrm{i}p_n^{(B)},t)-h(\tfrac{1}{2}+\mathrm{i}p_n^{(F)},t)\Big] &= \mathrm{i}\frac{\mathcal{A}}{4\pi}\int_{\mathbb{R}} h(\tfrac{1}{2}+\mathrm{i}p,t)\tanh\pi p\,dp \\
&+\sum_{\{\gamma\}}\sum_{k=1}^{\infty} l_\gamma G_1(kl_\gamma,t)+\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{1}{\nu}\int_0^\infty \frac{g(u,t)-g(-u,t)}{\cosh u-\cos(2k\pi/\nu)}\sinh\frac{u}{2}\,du \\
&-\frac{\kappa_-}{2}\mathrm{tr}[\mathfrak{S}(\tfrac{1}{2})]\sinh\frac{L}{2}\mathrm{e}^{t/4}+\kappa_-\int_0^\infty g(-u,t)du+\frac{\kappa_-}{2\pi}\int_{\mathbb{R}} h(\tfrac{1}{2}+\mathrm{i}p,t)\frac{\Delta'(\frac{1}{2}+\mathrm{i}p)}{\Delta(\frac{1}{2}+\mathrm{i}p,t)}dp \\
&+\frac{\kappa_0}{2}\int_0^\infty [g(u,t)-g(-u,t)]du+\frac{\tilde{\kappa}_0}{2\pi}\int_{\mathbb{R}} h(\tfrac{1}{2}+\mathrm{i}p,t)[\Psi(1+\mathrm{i}p)+\Psi(1-\mathrm{i}p)]dp\ . \qquad (4.6)
\end{aligned}
$$

We know that if s is an eigenvalue of $\square$, then $s(1-s)$ is an eigenvalue of $-\Delta$ [?, ?, ?]. Now let us consider the zero-mode of $-\Delta$, the usual hyperbolic Laplacian, i.e., $s(1-s)=0$. We have two possibilities in choosing a proper s, $s=0$ and $s=1$, respectively, and both values correspond to zero-modes of $\square$ and we can replace $s\to 1-s$ in all equations involving $\square$. Let us denote the multiplicities of these zero-modes by $\Delta n_0^{(s=0)}\equiv\Delta n_0^{(0)}$ and $\Delta n_0^{(s=1)}\equiv\Delta n_0^{(1)}$, respectively. Therefore

$$
\sum_{\text{zero-modes}}[h(\tfrac{1}{2}+\mathrm{i}p_n^{(B)},t)-h(\tfrac{1}{2}+\mathrm{i}p_n^{(F)},t)]=\Delta n_0^{(1)}\sinh L\ . \qquad (4.7)
$$

Let us first consider compact super Riemann surfaces. Investigating the various terms we find for the zero-length term

$$
\lim_{t\to 0^+}\mathrm{i}\int_{\mathbb{R}} h(p,t)\tanh\pi p dp=-\lim_{t\to 0^+}\int_0^\infty\frac{g(u)-g(-u)}{\sinh(u/2)}du=-\coth\frac{L}{2}\ . \qquad (4.8)
$$

In the term involving the hyperbolic conjugacy classes we obtain in the limit $t\to 0^+$:

$$
\begin{aligned}
&\Delta n_0^{(1)}\sinh L+\sum_{n=1}^{\infty}[h(p_n^{(B)})-h(p_n^{(F)})] \\
&\qquad=-\frac{\mathcal{A}}{4\pi}\coth\frac{L}{2}+\frac{1}{2}\sum_{\{\gamma\}}\sum_{k=1}^{\infty} l_n\delta(L-l_nk)\ . \qquad (4.9)
\end{aligned}
$$

Operating by $2\int_{l_0}^L dl/l$ on both sides, where $0<l_0<l_1<L$ we get $(\kappa(L)=[L/l_1]$

$$
\begin{aligned}
&\Delta n_0^{(1)}\int_{l_0}^L\frac{dl}{l}(\mathrm{e}^l-\mathrm{e}^{-l})+\sum_{n=1}^{\infty}\int_{l_0}^L\frac{dl}{l}\Big[\Big(\mathrm{e}^{s_n^{(B)}l}-\mathrm{e}^{-s_n^{(B)}l}\Big)-\Big(\mathrm{e}^{s_n^{(F)}l}-\mathrm{e}^{-s_n^{(F)}l}\Big)\Big] \\
&\qquad=-\frac{\mathcal{A}}{2\pi}\int_{l_0}^L\frac{dl}{l}\coth\frac{l}{2}+\sum_{k=1}^{\kappa(L)}\frac{1}{k}\sum_{n=1}^{\infty}kl_n\int_{l_0}^L\frac{dl}{l}\delta(l-l_nk)\ . \qquad (4.10)
\end{aligned}
$$

The integration over the δ-function gives the step-function $\Theta(x)$, which in turn by the summation over n gives the number of geodesics with length L/k not exceeding L/k. Skipping all terms and powers proportional to e^{-L} we get, as $L\to\infty$

$$
\Delta n_0^{(1)}\mathrm{Ei}(L)+\sum_{n=1}^{\infty}\Big[\mathrm{Ei}(s_n^{(B)}L)-\mathrm{Ei}(s_n^{(F)}L)\Big]=\frac{\mathcal{A}}{2\pi}\ln L+\sum_{k=1}^{\kappa(L)}\frac{1}{k}N\Big(\frac{L}{k}\Big)+O(\mathrm{e}^{-L}/L)\ . \qquad (4.11)
$$

Denoting now

$$F(L) := \Delta n_0^{(1)}\mathrm{Ei}(L) + \sum_{n=1}^{\infty}\Big[\mathrm{Ei}(s_n^{(B)}) - \mathrm{Ei}(s_n^{(F)})\Big] - \frac{\mathcal{A}}{2\pi}\ln L \tag{4.12}$$

we obtain the asymptotic distribution for the length of the periodic orbits on a closed compact super Riemann surface according to

$$\bar{N}(L) \propto \sum_{k=1}^{\kappa(L)} \frac{\mu(k)}{k} F\Big(\frac{L}{k}\Big) \simeq F(L) - \frac{1}{2}F\Big(\frac{L}{2}\Big) \quad , \qquad L \to \infty \quad , \tag{4.13}$$

which gives in second order due to the properties of the Möbius function

$$\bar{N}(L) \propto \Delta n_0^{(1)}\Big[\mathrm{Ei}(L) - \frac{1}{2}\mathrm{Ei}\Big(\frac{L}{2}\Big)\Big] + \sum_{n=1}^{\infty}\Big[\mathrm{Ei}(s_n^{(B)}L) - \mathrm{Ei}(s_n^{(F)}L)\Big] \quad , \qquad L \to \infty \quad . \tag{4.14}$$

I have included the $\mathrm{Ei}(L/2)$-term because $\Re(s_n^{(B,F)}) = \frac{1}{2}$ gives a contribution of the same order.

Taking into account the elliptic terms, we find that the contributions corresponding to the first term vanishes, and for the second we find

$$\lim_{t\to 0^+}\int_0^\infty \frac{g(u)-g(-u)}{\cosh u - \cos(2k\pi/\nu)}\sinh\frac{u}{2}du = \frac{1}{2}\frac{\sinh L}{\cosh L - \cos(2k\pi/\nu)} \to \frac{1}{2} \quad , \qquad L \to \infty \quad , \tag{4.15}$$

and thus this expression gives for $L \to \infty$ only a constant, thus leading only to terms proportional to $\ln L$ in $F(L)$ similarly as in (??), which can be neglected.

Let us extend our investigation to non-compact super Riemann surfaces. First, we observe that with the test function (??) we have $g(0,t) = 0$, and corresponding terms in the Selberg super-trace formula do not give a contribution. Furthermore we have $h(\frac{1}{2}, 0^+) = \sinh\frac{L}{2}$. This gives a $\mathrm{Ei}(L/2)$ contribution in $\bar{N}(L)$. The remaining terms are of similar order except for the term coming form the scattering matrix $\Delta(s)$. This term is completely undetermined, except for the property that it does not exceed the contributions of $O(\mathrm{e}^l/L)$, i.e., it must be lower than of order $O(\mathrm{e}^L/L)$. This general statement is well-known in the theory of the Selberg zeta function, e.g. [?]). Therefore, the only statement which can be made is that on non-compact super Riemann surfaces the asymptotic distribution of the length of geodesics, i.e. the asymptotic proliferation of hyperbolic conjugacy classes, is according to

$$\bar{N}(L) \propto \mathrm{Ei}(\Delta n_0^{(1)}L) \propto \frac{\mathrm{e}^{\Delta n_0^{(1)}L}}{\Delta n_0^{(1)}L} \quad , \qquad L \to \infty \quad , \tag{4.16}$$

and no further statement can be made.

5 Derivation of Weyl's law

In order to investigate Weyl's law on super Riemann surfaces, we consider the heat-kernel function

$$h_{HK}(p,t) = \mathrm{e}^{s^2 t} = \exp[(-p^2 + \mathrm{i}p + 1/4)t] \quad , \tag{5.1}$$

and we have to investigate the Selberg super-trace formula in the limit $t \to 0^+$. First of all, we easily find

$$g_{HK}(u,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{u^2}{4t} + \frac{u}{2} \right) \ . \tag{5.2}$$

Let us start with compact super Riemann surfaces. The relevant expressions concerning the zero-length term and the contribution from the hyperbolic conjugacy classes have been calculated in [?, ?], and in particular we get for the zero-length term (in the following we drop the index in h_{HK})

$$\mathrm{i} \dim V \frac{\mathcal{A}}{4\pi} \int_{\mathbb{R}} h(p,t) \tanh \pi p \, dp = -\dim V \frac{\mathcal{A}}{4\pi} \ . \tag{5.3}$$

We obtain the following Selberg super-trace formula for the heat kernel

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left[\mathrm{e}^{t(\lambda_n^{(B)})^2} - \mathrm{e}^{t(\lambda_n^{(F)})^2} \right] = -\dim V \frac{\mathcal{A}}{4\pi} + \frac{1}{\sqrt{4\pi t}} \sum_{\{\gamma\}} \sum_{k=1}^{\infty} \frac{l_\gamma \mathrm{e}^{-k^2 l_\gamma^2/4t}}{\sinh(k l_\gamma/2)} \left(\cosh \frac{k l_\gamma}{2} - \chi_\gamma^k \right) \\
&+ \sum_{\{R\}} \sum_{k=1}^{\nu-1} \frac{1}{\nu} \Bigg[\left(1 - \chi_R^k \cos \frac{k\pi}{\nu} \right) \\
&\qquad \times \int_0^\infty du \frac{g(u,t)\mathrm{e}^{-u/2} + g(-u,t)\mathrm{e}^{u/2}}{\cosh u - \cos(2k\pi/\nu)} + \int_0^\infty \frac{g(u,t) - g(-u,t)}{\cosh u - \cos(2k\pi/\nu)} \sinh \frac{u}{2} \, du \Bigg] \\
&- \frac{1}{\sqrt{\pi t}} \Big[\tilde{\kappa}_0 \ln 2 + \kappa_- \ln |\mathrm{sdet}(1 - U(S))| \Big] \\
&- \frac{\kappa_-}{2} \mathrm{tr}[\mathfrak{S}(\tfrac{1}{2})] \mathrm{e}^{t/4} + \kappa_- \int_0^\infty g(-u,t) du + \frac{\kappa_-}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{(\frac{1}{2}+\mathrm{i}p)^2 t} \frac{\Delta'(\frac{1}{2}+\mathrm{i}p)}{\Delta(\frac{1}{2}+\mathrm{i}p)} dp \\
&+ \frac{\kappa_0}{2} \int_0^\infty [g(u,t) - g(-u,t)] du + \frac{\tilde{\kappa}_0}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{(\frac{1}{2}+\mathrm{i}p)^2 t} [\Psi(1+\mathrm{i}p) + \Psi(1-\mathrm{i}p)] dp \ .
\end{aligned} \tag{5.4}$$

For small t the heat kernel $\Theta(t) = \mathrm{str}(\mathrm{e}^{-tH})$ has the expansion given by the integral

$$\Theta(t) \simeq \int_0^\infty \mathrm{e}^{-tE} dN(E) = \int_0^\infty \mathrm{e}^{-z} dN(z/t) \ , \qquad t \to 0^+ \ , \tag{5.5}$$

hence for small t only the asymptotic behaviour for large E will be of importance. For the usual bosonic case $E \propto p^2$, and in case of super Riemann surfaces $E \propto p$. The term corresponding to the hyperbolic conjugacy classes can be exploited to yield the fine-structure of the spectral staircase [?], which however is of no interest here, and the term is omitted.

Let us try to derive some convenient tools for our analysis. Following Aurich and Steiner [?], we consider $\bar{N}(s)$, as $s \to \infty$ (d_n denotes the multiplicity of the level s_n)

$$\begin{aligned}
\bar{N}(s) &= \int_s^{s_0} ds' \sum_n d_n \delta(s' - s_n) = \int_0^p dp' \sum_n d_n \Big[\delta(p' - p) + \delta(p' + p) \Big] \\
&= \lim_{\epsilon \to 0} \sum_n d_n \int_0^p dp' \frac{1}{\sqrt{\pi}\, \epsilon} \Big[\mathrm{e}^{-(p'-p)^2/\epsilon^2} + \mathrm{e}^{-(p'-p)^2/\epsilon^2} \Big] \ .
\end{aligned} \tag{5.6}$$

This gives for an integral term with respect to p in the trace formula

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dp'' f(p'') \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^p dp' \Big[\mathrm{e}^{-(p'-p)^2/\epsilon^2} + \mathrm{e}^{-(p'-p)^2/\epsilon^2} \Big]$$

$$
\begin{aligned}
&= \int_{\rm I\!R} dp'' f(p'') \int_0^p dp' \Big[\delta(p'-p)+\delta(p'+p)\Big] \\
&= \int_{-p}^{p} dp' f(p') \ . \qquad (5.7)
\end{aligned}
$$

We are therefore lead to the following correspondence

$$
\begin{aligned}
&\text{Term in the asymptotic expansion in the heat kernel:} \\
&\lim_{t\to 0+} \int_{\rm I\!R} h_{HK}(p,t) f(p) dp \\
\Longleftrightarrow\ &\text{Term in the asymptotic expansion of Weyl's law:} \\
&\lim_{p\to\infty} \int_{-p}^{p} h_{HK}(p',0) f(p') dp' \ . \qquad (5.8)
\end{aligned}
$$

We then can consider the expression

$$
\bar N^{(B)}(p) - \bar N^{(F)}(p) - \frac{\kappa_-}{2\pi} \int_{-p}^{p} \frac{\Delta'(\frac12+{\rm i}q)}{\Delta(\frac12+{\rm i}q)} dq \ , \qquad p\to\infty \ , \qquad (5.9)
$$

in order to study the asymptotic proliferation.

Compact super Riemann surfaces with only hyperbolic conjugacy classes

The case of compact super Riemann surfaces with only hyperbolic conjugacy classes is the simplest one and we immediately obtain

$$
\bar N^{(B)}(p) - \bar N^{(F)}(p) = -\dim V \frac{\mathcal A}{4\pi} \ , \qquad (5.10)
$$

which is therefore evidence for exact super symmetry of the spectrum, all energy levels but the ground state are degenerate and cancel.

Compact super Riemann surfaces with hyperbolic and elliptic conjugacy classes

Exploiting the δ-function properties of $g(u,t)$ it is not difficult to incorporate also the elliptic terms. The first term in the summation over elliptic conjugacy classes yields

$$
\begin{aligned}
&\int_0^\infty du \frac{g(u,t){\rm e}^{-u/2} + g(-u,t){\rm e}^{u/2}}{\cosh u - \cos(2k\pi/\nu)} = \frac{1}{\sqrt{\pi t}} \int_{\rm I\!R} \frac{{\rm e}^{-u^2/4t}}{\cosh u - \cos(2k\pi/\nu)} du \\
&\to 2 \int_{\rm I\!R} \frac{\delta(u)}{\cosh u - \cos(2k\pi/\nu)} du = \frac{2}{1-\cos(2k\pi/\nu)} \ , \qquad t\to 0^+ \ . \qquad (5.11)
\end{aligned}
$$

For the second term we obtain

$$
\begin{aligned}
&\int_0^\infty \frac{g(u)-g(-u)}{\cosh u - \cos(2k\pi/\nu)} \sinh\frac{u}{2} du = \int_{\rm I\!R} \frac{g(u)\sinh\frac{u}{2} du}{\cosh u - \cos(2k\pi/\nu)} \\
&= \frac{1}{2\sqrt{\pi t}} \int_{\rm I\!R} {\rm e}^{-u^2/4t+u/2} \sinh\frac{u}{2} du \to \int_{\rm I\!R} \frac{\delta(u){\rm e}^{u/2}\sinh\frac{u}{2} du}{\cosh u - \cos(2k\pi/\nu)} = 0 \ , \qquad t\to 0^+ \ . \qquad (5.12)
\end{aligned}
$$

Summarizing we get

$$
\bar N^{(B)}(p) - \bar N^{(F)}(p) = -\dim V \frac{\mathcal A}{4\pi} + \sum_{\{R\}} \sum_{k=1}^{\nu-1} \frac{1}{\nu} \Big(1 - \chi_R^k \cos\frac{k\pi}{\nu}\Big) \frac{2}{1-\cos(2k\pi/\nu)} \ , \qquad (5.13)
$$

and the supersymmetric properties of the system are almost the same as before.

Non-comapct super Riemann surfaces

Turning to the case where also cups are present, we have to evaluate some additional contributions. Integral expressions over $g(u,t)$ are determined by means of the limit $t \to 0^+$, similarly as the two integrals in the elliptic conjugacy classes, and give either a constant or zero. Considering the equivalence relation

$$\begin{aligned} &\lim_{t\to 0^+} \frac{1}{2\pi} \int_{\mathbb{R}} h(p,t)[\Psi(1+\mathrm{i}p)+\Psi(1-\mathrm{i}p)]dp \\ \Longleftrightarrow &\lim_{p\to\infty} \int_{-p}^{p} h(q,0)[\Psi(1+\mathrm{i}q)+\Psi(1-\mathrm{i}q)]dq \ , \end{aligned} \tag{5.14}$$

we obtain for the term involving the Ψ-functions

$$\begin{aligned} &\frac{1}{2\pi}\int_{-p}^{p} [\Psi(1+\mathrm{i}q)+\Psi(1-\mathrm{i}q)]dq \\ &= \frac{1}{\pi}\int_{-p}^{p} \Big(\frac{d}{dt}\ln\Gamma(1+\mathrm{i}q)\Big)dq = \frac{2}{\pi}(p\ln p - p) + \frac{1}{2} + O(p^{-1}) \ , \qquad p\to\infty \ , \end{aligned} \tag{5.15}$$

and we have used $\ln\Gamma(z) \propto (z-\frac{1}{2})\ln z - z + \ln\sqrt{2\pi}$, $(|z|\to\infty$, [?, p.940]. All remaining terms are simple. Therefore we get the corrections to Weyl's law in the presence of cusps, as $p \to \infty$

$$\begin{aligned} &\bar{N}^{(B)}(p) - \bar{N}^{(F)}(p) - \frac{\kappa_-}{2\pi}\int_{-p}^{p} \frac{\Delta'(\frac{1}{2}+\mathrm{i}q)}{\Delta(\frac{1}{2}+\mathrm{i}q)}dq \\ &= -\dim V\frac{\mathcal{A}}{4\pi} + 2\sum_{\{R\}}\sum_{k=1}^{\nu-1} \frac{1-\chi_R^k\cos(k\pi/\nu)}{\nu[1-\cos(2k\pi/\nu)]} - \kappa_-\Big(\tfrac{1}{2}\mathrm{tr}[\mathfrak{S}(\tfrac{1}{2})]+1\Big) + \frac{\kappa_0}{2} \\ &\qquad -\frac{4p}{\pi}\Big[\tilde{\kappa}_0\ln 2 + \kappa_-\ln|\mathrm{sdet}(1-U(S))| + \frac{\tilde{\kappa}_0}{2}\Big] - \frac{2\tilde{\kappa}_0}{\pi}p\ln p + O(p^{-1}) \ . \end{aligned} \tag{5.16}$$

We see clearly that additional states due to cusps increase this expression to $-\infty$, in contrast to the compact case, where it is approaching a constant.

6 Discussion and summary

In this paper I have discussed how to derive asymptotic distributions for the number of eigenvalues of the Laplacian (for short Weyl's law) and for the number of norms of hyperbolic conjugacy classes (for short Huber's law) on super Riemann surfaces by means of the Selberg super trace formula. In the case of Huber's law I have found a similar behaviour as for the corresponding case on usual Riemann surfaces. The case of the Weyl's law, however, was quit different. For compact super Riemann surfaces Weyl's law is essentially equivalent with the Witten index of the quantum mechanically system. For non-compact super Riemann surfaces this result was modified by the additional presence of scattering states which yield energy-dependent (respectively momentum dependent) terms, a similar effect as in the classical case.

However, there are several obstacles for a more thorough investigation which would cover all cases of super Riemann surfaces, i.e. compact, respectively non-compact, and closed, respectively bordered super Riemann surfaces. In the case of non-compact super

Riemann surfaces any further refinement of the leading term in Huber's law faces the completely unknown behaviour of the logarithmic derivative of the scattering matrix. The exact form of the scattering matrix is only known for congruence groups [?]. In the case of bordered super Riemann surfaces there seems no obvious discussion how to treat the asymptotic number of the norms of the inverse hyperbolic conjugacy classes. The corresponding case of the classical result for usual bordered Riemann surfaces exploits the known behaviour of the heat-kernel for $t \to 0$ and $t \to \infty$, respectively. Summarizing, our model has the advantage that it is possible to determine many quantities explicitly nevertheless, in comparison to the generally flat superspace models as discussed, e.g, by Borisov et al. [?] and Fuchs [?].

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A Summary of the classical Selberg trace formula

The most general form of the Selberg trace formula for closed Riemann surfaces was given by Venkov [?] and Hejhal [?, ?], completing the original work of Selberg [?].

Let us consider an *fundamental domain (polygon)* $\mathcal{F}$ of a *Fuchsian group* Γ, realised on the Poincaré upper half-plane $\mathcal{H}$. In the case that $\mathcal{F}$ is compact, the corresponding Fuchsian group Γ is cocompact, otherwise Γ is non-cocompact. An arbitrary Fuchsian group is generated by

1. Hyperbolic elements denoted by γ_i, $i = 0, \dots, 2g-1$, including their inverses, which are characterized by $\mathrm{tr}(\gamma_i) > 2$ $(i = 1, \dots, 2g-1)$; these transformations are also called boosts or dilatations, respectively.
2. Elliptic elements denoted by R_j $(j = 1, \dots s)$ characterized by $\mathrm{tr}(R_j) < 2$ $(j = 1, \dots, s)$ with s the number of inequivalent fixed points of the group action; these transformations are also called rotations. ν_j is called the order of the elliptic element R_j for $R_j^{\nu_j} = 1$ $(j = 1, \dots, s)$.
3. Parabolic elements S_j $(j = 1, \dots, \kappa)$ characterized by $\mathrm{tr}(S_j) = 2$ $(j = 1, \dots, \kappa)$ with κ the number of inequivalent cusps; these transformations are also called translations.

The generators $\{\gamma_1, \dots, \gamma_{2g-1}, R_1, \dots, R_s, S_1, \dots, S_\kappa\}$ obey the constraint

$$(\gamma_0\gamma_1^{-1}\dots\gamma_{2g-2}\gamma_{2g-1}^{-1})(\gamma_0^{-1}\gamma_1\dots\gamma_{2g-2}^{-1}\gamma_{2g-1})R_1\dots R_sS_1\dots S_\kappa = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{A.1}$$

The non-Euclidean area of such a fundamental domain is given by (Gauss-Bonnet theorem, e.g. [?])

$$\mathcal{A} = 2\pi \left[2(g-1) + \kappa + \sum_{j=1}^{s} \left(1 - \frac{1}{\nu_j} \right) \right] \quad . \tag{A.2}$$

To each cusp there is associated an *Eisenstein series*

$$e(z,s,\alpha)=\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma} y^s(\gamma z) \tag{A.3}$$

$z\in\mathcal{H}$, $\Re(s)>1$, $\alpha=1,\dots,\kappa$, with Γ_α being the stabilizer of the cusp α. In the spectral decomposition of the hyperbolic Laplacian these Eisenstein series span the continuous spectrum.

An element $T\in\Gamma$ (hyperbolic, elliptic, parabolic) is realized as a two times two matrix acting on $z\in\mathcal{H}$ as follows (*Möbius transformations*)

$$Tz=\frac{az+b}{cz+d}\qquad\text{with}\qquad T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\ . \tag{A.4}$$

Let us introduce the quantities N_γ and l_γ by $2\cosh(l_\gamma/2)=N_\gamma^{1/2}+N_\gamma^{-1/2}=a+d$. N_γ is called the *norm* of a hyperbolic $\gamma\in\Gamma$ in a Fuchsian group and l_γ the *length* of a periodic orbit corresponding to γ. Each element $\gamma\in\Gamma/\{\pm\mathbb{1}\}$ is uniquely described as $\gamma=k^{-1}\gamma_0 k$ for some primitive γ_0, $n\in\mathbb{N}$ and $k\in\Gamma/\Gamma_{\gamma_0}$.

We introduce the *point pair invariant* $R(z,w)=|z-w|^2/[\Re(z)\Re(w)]$ and and construct the *integral kernel* $k[R(z,w)]\equiv k(z,w)$ for operator valued functions $\Lambda(\Delta_m)$

$$k(z,w)=\left(\frac{w-\bar z}{z-\bar w}\right)^{m/2}\Phi\left(\frac{|z-w|^2}{\Im(z)\Im(w)}\right) \tag{A.5}$$

for some $\Phi\in C_c^2(\mathbb{R})$ $(z,w\in\mathcal{H})$. We introduce furthermore the *automorphic kernel* by

$$K(z,w)=\frac{1}{2}\sum_{\gamma\in\bar\Gamma}\chi_\gamma^m j(\gamma,w)k(z,\gamma w)\ . \tag{A.6}$$

This construction of the automorphic kernel is valid for any Fuchsian group. The trace of the operator $\Lambda(\Delta_m)$ is then constructed by considering $\int_{\mathcal{F}}K(z,z)dxdy/y^2=\sum_n\Lambda(E_n)$, including an appropriate regularization if necessary by subtracting the continuous spectrum contribution as represented by the Eisenstein series. $\chi(\gamma)\equiv\chi_\gamma$ denotes a multiplier system acting according to $\bar\Gamma\to\{\pm1\}$ with $\chi(-\mathbb{1})=-1$ $(\gamma\in\bar\Gamma$, such that $\bar\Gamma\in\mathrm{SL}(2,\mathbb{R})$, $\widehat{\Gamma}=\bar\Gamma/\{\pm1\})$, and $j(\gamma,z)$ denotes the automorphic weight [?]. We formulate [?, ?, ?]:

Theorem A.1 *The Selberg trace formula on arbitrary Riemann surfaces for automorphic forms of weight m, $m\in\mathbb{Z}$, is given by*

$$\begin{aligned}\sum_{n=1}^\infty h(p_n)=&-\frac{\mathcal{A}\dim V}{16\pi^2}\int_0^\infty\frac{\cosh(mu/2)}{\sinh(u/2)}g'(u)du+\frac{1}{2}\sum_{\{\gamma\}}\sum_{k=1}^\infty\frac{\chi_\gamma^{mk}\mathrm{tr}_V[U^k(\gamma)]l_\gamma}{\sinh(kl_\gamma/2)}g(kl_\gamma)\\
&+\frac{\mathrm{i}}{2}\sum_{\{R\}}\sum_{k=1}^{\nu-1}\chi_R^{mk}\mathrm{tr}_V[U^k(R)]\frac{\exp[\mathrm{i}(m-1)k\pi/\nu]}{\nu\sin(k\pi/\nu)}\int_{\mathbb{R}}du\,g(u)\frac{\mathrm{e}^{(m-1)u/2}(\mathrm{e}^u-\mathrm{e}^{2\mathrm{i}k\pi/\nu})}{\cosh u+\cos[\pi-2(k\pi/\nu)]}\\
&-\Big[\kappa_0\ln 2+\ln|\det(1-U(S))|\Big]g(0)+\frac{\kappa_0}{2}\int_0^\infty\frac{g(u)}{\sinh(u/2)}\left(1-\cosh\frac{um}{2}\right)du\\
&+\frac{1}{4}[\kappa_0-\mathrm{tr}(\mathcal{S})]h(0)+\frac{1}{4\pi}\int_{\mathbb{R}}\frac{\Delta'(\frac{1}{2}+\mathrm{i}p)}{\Delta(\frac{1}{2}+\mathrm{i}p)}h(p)dp-\frac{\kappa_0}{2\pi}\int_{\mathbb{R}}h(p)\Psi(1+\mathrm{i}p)dp\ .\end{aligned} \tag{A.7}$$

$E_n = p_n^2 + \frac{1}{4}$ on the left runs through the set of all eigenvalues of the Maass Laplacian, and the summation on the right is taken over all primitive conjugacy classes $\{\gamma\}$ with $\mathrm{tr}(\gamma) > 2$, *$\{R\}$ with* $\mathrm{tr}(R) < 2$, *R elliptic, and $\{S\}$ with* $\mathrm{tr}(S) = 2$, *S parabolic. The test function h must satisfy the following properties*

1. *$h(p)$ is an even function in p,*
2. *$h(p)$ is analytic in the strip $\Im(p) < \frac{1}{2} + \epsilon$ for some $\epsilon > 0$,*
3. *and $h(p)$ vanishes according to $h(p) = O[1/(1+p^2)^{1+\epsilon}]$ for some $\epsilon > 0$ for $p \to \pm\infty$,*
4. *$g(u) = \pi^{-1} \int_0^\infty h(p)\cos(up)dp$.*
5. *$\Psi(z) = \Gamma(z)/\Gamma'(z)$ with $\Gamma(z)$ the Eulerian Γ-function,*
6. *$\Delta(z)$ denotes the determinant of the scattering matrix $\mathfrak{S}$ as constructed form the Eisenstein series,*
7. *$U(\gamma)$, $U(R)$ denote unitary representations of the hyperbolic and elliptic conjugacy classes, respectively, V a (multidimensional) complex vector space,*
8. *and the remaining quantities describe in detail the cusp-space with κ_0 its dimension, and κ the number of cusps.*

B Huber's law from the Selberg trace formula

To examine the classical Selberg trace formula in order to obtain a Huber's law from it one starts by considering the test-function (cosine-modulated heat-kernel function)

$$h(p,t) = \cos(pl)\exp\Big[-\Big(p^2+\frac{1}{4}\Big)t\Big] \ , \tag{B.1}$$

which in turn gives the Fourier transformed

$$g(u,t) = \frac{1}{4\sqrt{\pi t}}\mathrm{e}^{-t/4}\Big[\mathrm{e}^{-(L-u)^2/4t} + \mathrm{e}^{-(L+u)^2/4t}\Big] \ . \tag{B.2}$$

We consider the trace formula (**??**) for $m = 0$ and assume a trivial representation U. Due to the zero-mode of the non-Euclidean Laplacian we get

$$\sum_{n=0}^{\infty} h(p_n,t) = \cosh\frac{L}{2} + \sum_{n=1}^{\infty} h(p_n,t) \ . \tag{B.3}$$

Furthermore it is not difficult to derive the following equivalences (compare e.g. Romani [**?**])

$$\lim_{t\to 0^+} \frac{\mathcal{A}}{4\pi}\int_{\mathbb{R}} p\tanh(\pi p)h(p,t)dp = -\frac{\mathcal{A}}{8\pi}\frac{\coth\frac{L}{2}}{\sinh\frac{L}{2}}, \tag{B.4}$$

$$\lim_{t\to 0^+} \frac{\mathrm{e}^{-t/4}}{8\sqrt{\pi t}}\sum_{\{\gamma\}}\sum_{k=1}^{\infty}\frac{l_\gamma g(kl_\gamma)}{\sinh(kl_\gamma/2)} = \frac{1}{4}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{g_n l_n}{\sinh(kl_n/2)}[\delta(L+kl_n)+\delta(L-kl_n)] \tag{B.5}$$

$$\lim_{t\to 0^+} \frac{1}{4\nu\sin\phi}\int_{\mathbb{R}}\frac{\cosh[\pi(1-2\phi)p]}{\cosh\pi p}h(p,t)dp = \frac{1}{2\nu}\frac{\cosh\frac{L}{2}}{\cosh L - \cos 2\phi}, \tag{B.6}$$

$$\lim_{t\to 0^+} \frac{\mathrm{e}^{-t/4}}{2\pi}\int_{\mathbb{R}} h(p,t)\Psi(1+\mathrm{i}p)dp = -\frac{1}{2}\frac{1}{\mathrm{e}^L - 1} \ . \tag{B.7}$$

We have denoted by l_n the length of the n-th orbit (ordered according to their value) with g_n their multiplicity. Trivially we have $h(p=0,t)=1$ and $\lim_{t\to0^+} g(u=0,t)=0$. The really difficult term is now

$$\lim_{t\to0^+}\frac{\mathrm{e}^{-t/4}}{4\pi}\int_{\mathbb{R}}\cos(pL)\mathrm{e}^{-p^2t}\frac{\Delta'(\frac{1}{2}+\mathrm{i}p)}{\Delta(\frac{1}{2}+\mathrm{i}p)}dp\equiv J(L), \tag{B.8}$$

provided the term exists. In general no definite statement can be made up to the estimate that the asymptotic distribution of norms $\bar{N}_\Gamma(N)$ for any Fuchsian group of the first kind behaves according to [?]

$$\bar{N}_\Gamma(N)=\mathrm{li}N+O(N^{3/4}/\log N)\ ,\qquad N\to\infty\ , \tag{B.9}$$

where li is the logarithm integral [?, p.929]. Let us consider only the terms (??) and the contribution coming from the hyperbolic conjugacy classes. Operating with $4\int_{l_0}^{L}\sinh(L/2)/L$ then yields

$$\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{\infty}g_n\Theta(L-kl_n)=\mathrm{Ei}(L)+O(\ln L)\ . \tag{B.10}$$

According to the reasoning in the introduction, Huber's law (??), follows and is valid for any Riemann surface. If one considers only compact Riemann surfaces, this can be refined, where one can exploit the full impact of the Möbius inversion formula (??).

In the case of congruence groups (subgroups of $\mathrm{SL}(2,\mathbb{Z})$) and related groups it is now possible to give explicit expressions for the scattering matrix and its determinant. The remaining terms in the trace formula are not difficult to evaluate. This has the consequence that we can state the following identity for the case of $\mathrm{SL}(2,\mathbb{Z})$ (compare e.g. [?, ?, ?]; the general case is similar, and the terms are not considerably altered; we consider this case for simplicity)

$$\begin{aligned}&\frac{1}{4\pi}\int_{\mathbb{R}}h(p,t)\frac{\Delta'(\frac{1}{2}+\mathrm{i}p)}{\Delta(\frac{1}{2}+\mathrm{i}p)}dp\\&=g(0)\ln\pi-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\int_{\mathbb{R}}\cos pL\,\mathrm{e}^{-2\mathrm{i}p\ln n}dp-\frac{1}{2\pi}\int_{\mathbb{R}}h(p,t)\Psi(\tfrac{1}{2}+\mathrm{i}p)dp-\frac{h(0)}{4}\ .\end{aligned} \tag{B.11}$$

Here $\Lambda(n)$ is the von Mangoldt function [?] which is equal to $\ln n$ if $n=P^k$ $(k\geq1)$, where P is a prime number, and zero otherwise. Therefore we can explicitly state $J(L)$

$$J_{\mathrm{SL}(2,\mathbb{Z})}(L)=\delta(L)\ln\pi+\frac{1}{2}\frac{\mathrm{e}^{L/2}}{\mathrm{e}^{L}-1}+\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}[\delta(L+2\ln n)+\delta(L-2\ln n)]\ . \tag{B.12}$$

One then finds [?] for the dominating contribution, as $L\to\infty$ $(c>0)$

$$J_{\mathrm{SL}(2,\mathbb{Z})}(L)=-\sum_{n=2}^{\infty}\Lambda(n)\bigg(\frac{1}{\ln n}-\frac{1}{n^2\ln n}\bigg)\Theta(L-2\ln n)\propto\mathrm{Ei}\Big(\frac{L}{2}\Big)+O\bigg(\frac{\mathrm{e}^{\frac{1}{2}(L-c\sqrt{L/2})}}{L}\bigg)\ , \tag{B.13}$$

Putting all relevant terms together, one obtains [?]

$$\bar{N}(L)=\mathrm{Ei}(L)-\frac{3}{2}\mathrm{Ei}\Big(\frac{L}{2}\Big)-\frac{1}{3}\mathrm{Ei}\Big(\frac{L}{3}\Big)+\frac{1}{2}\mathrm{Ei}\Big(\frac{L}{4}\Big)+\frac{1}{3}\mathrm{Ei}\Big(\frac{L}{6}\Big)$$

$$+2\frac{\mathrm{e}^{L/2}}{L}\sum_{k=1}^{\infty}\frac{1}{\sqrt{p_k^2+1/4}}\Big[\cos(p_kL-\alpha_k)-\mathrm{e}^{-L/4}\cos\Big(\frac{p_kL}{2}-\alpha_k\Big)-\mathrm{e}^{-L/3}\cos\Big(\frac{p_kL}{3}-\alpha_k\Big)\Big]$$
$$+O\Big(\frac{\mathrm{e}^{L/2}}{L^2}\Big)\ , \qquad L\to\infty\ . \tag{B.14}$$

Here, $\alpha_k=\arctan(2p_k)$ with $E_k=p_k^2+1/4$ the k^{th} energy eigenvalue of the Laplacian on modular domain. In (??) I have displayed just the first two terms.

C Weyl's law from the Selberg trace formula

In the general case where one has a non-compact Riemann surface one must consider the combination of the discrete and continuous states of the Laplacian on the hyperbolic plane. Generally, Weyl's law can always been derived by studying the short-time behaviour of the heat-kernel $\Theta(t)$ of a system. If one has for instance (see e.g. [?, ?] for reference)

$$\Theta(t)\propto c_\alpha t^\alpha+c_\beta t^\beta+\dots\ , \qquad t\to 0^+\ , \tag{C.1}$$

then it follows, c.f. (??)

$$\bar{N}(E)\propto c_\alpha\Gamma(1-\alpha)E^{-\alpha}+c_\beta\Gamma(1-\beta)E^{-\beta}+\dots\ , \qquad E\to\infty\ . \tag{C.2}$$

For open systems, the scattering states must be subtracted. For an appropriate test-function one can choose the heat kernel function $h(p,t)=\mathrm{e}^{-(p^2+1/4)t}$. Alternatively, the following result can also be derived by means of the functional equation for the Selberg zeta-function [?]. One finds for the number of energy levels $\bar{N}(E)$ of Energy E parameterized by $E_n=p^2+\frac{1}{4}$ not exceeding E minus the number of resonances in the same range [?, ?]

$$\begin{aligned}&\bar{N}(p)-\frac{1}{4\pi}\int_{-p}^{p}\frac{\Delta'(\frac{1}{2}+\mathrm{i}p)}{\Delta(\frac{1}{2}+\mathrm{i}p)}dp\\ &=\frac{\mathcal{A}}{4\pi}\Big(p^2-\frac{1}{3}\Big)-\frac{\kappa}{\pi}p\ln p+\frac{p}{\pi}\Big[\kappa(1-\ln 2)-\ln|\det(1-U(S))|\Big]\\ &\qquad+\frac{\mathrm{tr}(\mathcal{S})}{4}+\sum_{\{R\}}\sum_{k=1}^{\nu-1}\frac{1}{4\nu\sin^2(k\pi/\nu)}+O(\ln p/p)\ , \qquad p\to\infty\ .\end{aligned} \tag{C.3}$$

Here I have included all terms down to the constants. I have used that for the trace of the heat kernel one has $\mathrm{tr}(\mathrm{e}^{-(p^2+1/4)t})\propto(\mathcal{A}/4\pi)[1/t-1/3+O(t)]$ $(t\to 0^+)$ [?, ?], and one uses (??).

As an example, we consider the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$, i.e., the modular domain. Weyl's law has the form [?, ?]

$$\bar{N}(E)=\frac{E}{12}-\frac{1}{\pi}\sqrt{E}\ln E-\frac{2+\ln\frac{\pi}{2}}{\pi}\sqrt{E}+\frac{5}{72}+\frac{1}{8\pi}\frac{\ln E}{E}+O(E^{-1})\ , \qquad E\to\infty\ . \tag{C.4}$$

Equation (??) as well as the cases of Dirichlet or Neumann boundary conditions are nicely confirmed by the numerical data, e.g. [?, ?].

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