A comment on: 'Some problems with calculating the quantum corrections to the classical 't Hooft-Polyakov monopole'

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(June 22, 2003)

In a recent publication we noticed that the Hamiltonian density for fluctuations around the 't Hooft-Polyakov monopole appeared to be non-Hermitian. Here we show that when this Hamiltonian density is integrated into the Hamiltonian all non-Hermitian terms give a vanishing total contribution.

In a recent letter [1] we were looking at the quantum fluctuations around the 't Hooft-Polyakov monopole and noticed that the Hamiltonian density appeared to be non-Hermitian. We then speculated that this could indicate a subtlety in the behaviour of monopoles, although we also said the problem may be merely technical.

The aim of this comment is to show that there are no problems with the Hermiticity of the Hamiltonian. When the Hamiltonian density is integrated to the Hamiltonian the non-Hermitian terms give a vanishing contribution.

For illustration we consider the Hamiltonian density for the gauge fluctuations around the monopole

$$\mathcal{H}_{\rm qu} = \tfrac{1}{2}\pi^i_a\pi^i_a + \tfrac{1}{2}a^i_a\left(-\mathcal{K}^{ij}_{ab} + \mathcal{V}^{ij}_{ab} + \mathcal{D}^{ij}_{ab}\right)a^j_b, \qquad (1)$$

where $\mathcal{V}^{i}_{ab} = \mathcal{V}^{ji}_{ba}$, $\mathcal{K}^{ij}_{ab} = \delta_{ab}\left(\nabla^2\delta^{ij} - \partial^i\partial^j\right)$ and $\mathcal{D}^{ij}_{ab} = D^{ij}_{abl}\partial_l$. The conjugate momenta are $\pi^i_a = \dot{a}^i_a$ in a temporal gauge. Then the first few terms in this Hamiltonian are Hermitian because

$$\int d^3r\, a^i_a\mathcal{K}^{ij}_{ab}b^j_b = \int d^3r\, b^i_a\mathcal{K}^{ij}_{ab}a^j_b, \quad a^i_a\mathcal{V}^{ij}_{ab}b^j_b = b^i_a\mathcal{V}^{ij}_{ab}a^j_b; \quad (2)$$

the first by partial integration, neglecting surface terms, and the second by $\mathcal{V}^{ij}_{ab} = \mathcal{V}^{ji}_{ba}$. However, a partial integration of $\mathcal{D}^{ij}_{ab} = D^{ij}_{abl}\partial_l$ implies

$$(\mathcal{D}^\dagger)^{ij}_{ab} = -\mathcal{D}^{ji}_{ba} - \partial_l D^{ji}_{bal} \neq \mathcal{D}^{ij}_{ab}, \qquad (3)$$

whereby $\mathcal{D}^{ij}_{ab}$ is not Hermitian, as pointed out in ref. [1].

This non-Hermiticity is misleading though. Although the Hamiltonian density $\mathcal{H}_{\rm qu}$ appears to contain non-Hermitian terms, both the Hamiltonian $H_{\rm qu} = \int d^3r\,\mathcal{H}_{\rm qu}$ and the field equations contain only Hermitian operators.

Let us consider the Hamiltonian and the field equations separately:

(i) Notice that $\mathcal{D}^{ij}_{ab}$ in (1) can be split into Hermitian and anti-Hermitian components:

$$\mathcal{D} = \mathcal{D}_H + \mathcal{D}_A, \quad \mathcal{D}^\dagger_H = \mathcal{D}_H,\ \mathcal{D}^\dagger_A = -\mathcal{D}_A, \qquad (4)$$

defined by $\mathcal{D}_H = \frac{1}{2}(\mathcal{D} + \mathcal{D}^\dagger)$, $\mathcal{D}_A = \frac{1}{2}(\mathcal{D} - \mathcal{D}^\dagger)$. Now, the Hamiltonian can be written as (the dots denote other terms in the Hamiltonian)

$$\begin{aligned} H_{\rm qu}[a^i_a, \pi^i_a] &= \int d^3r\,\mathcal{H}_{\rm qu}[a^i_a, \pi^i_a] \\ &= \cdots + \int d^3r\, a^i_a(\mathcal{D}_H)^{ij}_{ab}a^j_b + \int d^3r\,(\mathcal{D}_A)^{ij}_{ab}a^j_b \\ &= \cdots + \int d^3r\, a^i_a(\mathcal{D}_H)^{ij}_{ab}a^j_b \end{aligned} \qquad (5)$$

because of $\int d^3r\, a^i_a(\mathcal{D}_A)^{ij}_{ab}a^j_b = -\int d^3r\, a^i_a(\mathcal{D}_A)^{ij}_{ab}a^j_b = 0$. Thus the non-Hermitian part $\mathcal{D}_A$ does not contribute to the Hamiltonian, which then contains only Hermitian terms.

(ii) The second argument uses the field equations [2]. Consider, for instance, the Lagrangian density for gauge fluctuations in a temporal $a^0_a = 0$ gauge,

$$\begin{aligned} \mathcal{L}_{\rm qu}[a^i_a] = \tfrac{1}{2}\dot{a}^i_a\dot{a}^i_a + \tfrac{1}{2}\partial^i a^j_a\partial^i a^j_a - \tfrac{1}{2}\partial^i a^j_a\partial^j a^i_a \\ + \tfrac{1}{2}a^i_a\mathcal{V}^{ij}_{ab}a^j_b + \tfrac{1}{2}a^i_a D^{ij}_{abl}\partial_l a^j_b. \end{aligned} \qquad (6)$$

One must also impose an additional Gauss's law constraint from this choice of gauge. From (6) the field equations are

$$\begin{aligned} -\ddot{a}^i_a + \mathcal{K}^{ij}_{ab}a^j_b + \tfrac{1}{2}(\mathcal{V}^{ij}_{ab} + \mathcal{V}^{ji}_{ba})a^j_b \\ + \tfrac{1}{2}\mathcal{D}^{ij}_{ab}a^j_b + \tfrac{1}{2}(-\mathcal{D}^{ji}_{ba} - \partial_l D^{ji}_{bal})a^j_b = 0. \end{aligned} \qquad (7)$$

By (3) the last operator is recognized as $(\mathcal{D}^\dagger)^{ij}_{ab}$ and the field equations are

$$-\ddot{a}^i_a + \left(\mathcal{K}^{ij}_{ab} + \mathcal{V}^{ij}_{ab} + (\mathcal{D}_H)^{ij}_{ab}\right)a^j_b = 0. \qquad (8)$$

Again, only the Hermitian component contributes to the equations of motion.

Finally, we note it is not surprising that the operators in the Hamiltonian $H_{\rm qu}$ are Hermitian. The quadratic-order Hamiltonian also describes the scattering of scalar and gauge bosons off the classical monopole background. For the S-matrix to be unitary one should expect a Hermitian Hamiltonian.

[1] N. F. Lepora, Phys. Lett. B **536** (2002) 338 [arXiv:hep-ph/0207286].

[2] H. Weigel, private communication.