THE RESOLUTION OF FIELD IDENTIFICATION FIXED POINTS IN DIAGONAL COSET THEORIES

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Abstract.
The fixed point resolution problem is solved for diagonal coset theories. The primary fields into which the fixed points are resolved are described by submodules of the branching spaces, obtained as eigenspaces of the automorphisms that implement field identification. To compute the characters and the modular S-matrix we use 'orbit Lie algebras' and 'twining characters', which were introduced in a previous paper [1]. The characters of the primary fields are expressed in terms branching functions of twining characters. This allows us to express the modular S-matrix through the S-matrices of the orbit Lie algebras associated to the identification group. Our results can be extended to the larger class of 'generalized diagonal cosets'.

£ Heisenberg fellow

1   Introduction

The coset construction has been proposed already a long time ago as an important tool to construct two-dimensional conformal field theories. Surprisingly, however, several crucial problems concerning the consistency of this construction have not been solved so far. In this paper we present an answer to one of the key questions, the problem of field identification and of the resolution of field identification fixed points, for an important class of coset theories, namely those based on diagonal subalgebras. Our results show in particular that for these coset theories the complete information needed in the coset construction is already encoded in the underlying WZW theories. Our methods and results can in fact be easily extended to the larger class of 'generalized diagonal' embeddings.

The main idea of the coset construction is the following. One starts with a finite-dimensional Lie algebra $\bar{\mathsf{g}}$, which is the direct sum of simple and abelian subalgebras, i.e. a so-called reductive Lie algebra. In addition, one chooses a subalgebra $\bar{\mathsf{g}}'$ of $\bar{\mathsf{g}}$ which is reductive as well. Using the loop space construction and incorporating central extensions, one obtains the corresponding untwisted Kac–Moody algebras g and g', along with a natural embedding of g' into g. Also, performing the Sugawara construction, one obtains Virasoro generators L_n for g and L_n' for g', with central charges c and c', respectively.

The crucial observation [2] underlying the coset construction is that the generators $\overset{\bullet}{L}_n :=
L_n - L_n'$ satisfy again the commutation rules of the Virasoro algebra, this time with central charge $\overset{\bullet}{c} := c - c'$; they are referred to as the generators as the coset Virasoro algebra. In order to be able to associate a conformal field theory to this Virasoro algebra one would like to find the representation spaces on which this algebra acts. The hope is, of course, to express these spaces in terms of representation spaces of g and g', for this allows us to use the representation theory of affine Lie algebras, which is by now a well-developed tool. In this article we will show that this is in fact possible; however, in addition to the well-known concepts of the representation theory of affine Lie algebras, such as characters and branching functions, one also has to use aspects of the representation theory that were developed recently, namely the theory [1] of twining characters and orbit Lie algebras.

A hint on what these representation spaces might be is given by the observation that the generators $\overset{\bullet}{L}_n$ commute with any element of the subalgebra g',

$$[\overset{\bullet}{L}_n, J_n^a] = 0 \qquad \text{for all} \qquad J_n^a \in \mathsf{g}'. \tag{1.1}$$

This suggests the following ansatz for the representation spaces of the coset Virasoro algebra: fix non-negative integral level for g and take any unitary highest weight module $\mathcal{H}_\Lambda$ of g at this level. Equation (1.1) suggests that one should regard all vectors of $\mathcal{H}_\Lambda$ which differ only by the action of some element of g' as the same vector for the putative representation space of the coset conformal field theory. We therefore decompose $\mathcal{H}_\Lambda$ into modules $\mathcal{H}_{\Lambda'}$ of g', $\mathcal{H}_\Lambda =
\bigoplus_{\Lambda'} \mathcal{H}_{\Lambda,\Lambda'} \otimes \mathcal{H}_{\Lambda'}$, and tentatively consider $\mathcal{H}_{\Lambda,\Lambda'}$ as part of the Hilbert space of the coset conformal field theory. On the level of characters this corresponds to decomposing the character $\chi_\Lambda^{\mathsf{g}}$ of $\mathcal{H}_\Lambda$ into characters $\chi_{\Lambda'}^{\mathsf{g}'}$ of g', according to

$$\chi_\Lambda^{\mathsf{g}}(\tau) = \sum_{\Lambda'} b_{\Lambda,\Lambda'}(\tau)\, \chi_{\Lambda'}^{\mathsf{g}'}(\tau) \tag{1.2}$$

and taking the *branching functions* $b_{\Lambda,\Lambda'}$ as the characters of the coset conformal field theory.

This guess, however, turns out to be too naive. Namely, first, several branching functions vanish, and second, certain non-vanishing branching functions are identical, in particular the putative vacuum primary field seems to be present several times [3,4,5]. The main reason for a branching function to vanish are conjugacy class selection rules, arising from the embedding $\bar{g}' \hookrightarrow \bar{g}$ of finite-dimensional Lie algebras.

A convenient formalism to implement these two observations is provided by simple currents (for a review see [6]). It can be shown [7] that there is a subgroup G_{id} of the group of integer spin simple currents of the tensor product theory described by $g \oplus (g')^*$, the so-called *identification group*, such that the group theoretical selection rules can be expressed by the condition that the monodromy charges Q_J of any allowed branching function with respect to all simple currents J in the identification group vanish. Moreover, S-matrix elements only change by phases on simple current orbits,

$$S_{J \star \Lambda, M} = e^{2\pi i Q_J(M)} \cdot S_{\Lambda, M}, \tag{1.3}$$

where '$\star$' denotes the fusion product (note that since J is a simple current, the product $J \star \Lambda$ contains just one primary field so that the notation in (1.3) makes sense). By standard simple current arguments [6] these relations and the corresponding relations for the T-matrix imply that there is a modular invariant in which only allowed branching functions occur,

$$Z(\tau) = \sum_{\substack{[\Lambda; \Lambda'] \\ Q=0}} |G_{\Lambda, \Lambda'}| \cdot \Big| \sum_{J \in G_{\mathrm{id}} / G_{\Lambda, \Lambda'}} b_{J \star (\Lambda; \Lambda')}(\tau) \Big|^2. \tag{1.4}$$

This formula is to be read as follows: the branching functions for which the monodromy charge Q with respect to all elements of G_{id} vanishes are organized into orbits by the action of the group G_{id}. The square bracket indicates that the first summation is over a representative for each such orbit (Λ and Λ' are integrable highest weights of g and g', respectively, at the relevant levels). Now for any orbit we can define a stabilizer group $G_{\Lambda, \Lambda'}$ which is the subgroup of G_{id} consisting of those elements that leave $(\Lambda; \Lambda')$ invariant. In the complete square finally we have a sum over the branching functions in the respective orbit.

Note that the stabilizer of the putative vacuum module, i.e. the one with highest weights ($\Lambda = 0$, $\Lambda' = 0$), is trivial, which implies that the vacuum would appear $|G_{\mathrm{id}}|^2$ times, due to the $|G_{\mathrm{id}}|^2$ terms in the complete square containing the identity. Therefore we would like to divide the expression (1.4) by this factor. However, as was first pointed out in [4], this will inevitably lead to problems such as fractional coefficients in the partition functions as soon as there are orbits for which the stabilizer is non-trivial. Such orbits are termed *fixed points*.

The factor $|G_{\Lambda, \Lambda'}|$ in front of the complete square in (1.4) suggests that any fixed point should in fact correspond to $|G_{\Lambda, \Lambda'}|$ many rather than to a single primary field; this is called the *resolution* of the fixed point. As far as representations of the modular group are concerned, a solution of the problem of how to resolve fixed points was proposed in [7]; it prescribes a modification of the elements of the modular matrix S that involve fixed points in terms of the S-matrix of another (putative) conformal field theory, called the *fixed point theory*. In this description it seems as if some necessary information for the coset conformal field theory is missing which cannot be obtained from data of g and g' alone: it is unclear how the S-matrix of the fixed point theory can be interpreted in terms of the underlying Kac–Moody algebras g and g'. A closer analysis also reveals [7] that the branching functions have to be changed as well in order to obtain the characters of the coset conformal field theory; this is known as

character modification. [1] In [7] it was checked that the prescription in terms of the fixed point theory yields consistent results at the level of representations of the modular group SL(2,**Z**).

In this paper, we will use the action of certain outer automorphisms on the branching spaces to implement field identification and fixed point resolution for diagonal cosets directly on the candidate Hilbert space of the theory. The procedure of fixed point resolution is then explained by the idea [5,8] that the branching spaces which correspond to fixed points carry *reducible* representations of the coset Virasoro algebra. These reducible representations can be split into $|G_{\Lambda,\Lambda'}|$ many irreducible representations.

The most important new insight of this paper is that this splitting is governed by natural structures present in the Kac–Moody algebras **g** and **g**′, namely by the recently introduced [1] twining characters and orbit Lie algebras. This implies in particular that the fixed point theories which describe this splitting in fact do not constitute an independent input, but play the role of summarizing various data that are already obtainable from the algebras **g** and **g**′. Using the results of [1], we can show that at the level of representations of the modular group our implementation of fixed point resolution is consistent with the results of [7]. In particular, we can compute both the modifications of the S-matrix and the characters explicitly.

This paper is organized as follows. In section 2 briefly review the construction of diagonal coset conformal field theories and generalized diagonal cosets and then introduce the automorphisms relevant for field identification, the so-called diagram automorphisms. Section 3 is devoted to the implementation of field identification and a closer characterization of the coset chiral algebra. To this end branching spaces and branching functions are introduced and described in subsection 3.1, and in subsection 3.2 it is also explained how each field identification implies a conjugacy class selection rule as well as an invariance property of the coset Virasoro algebra. Field identification is then described by the use of certain automorphisms of **g** which have the special property that they give also rise to automorphisms of the subalgebra **g**′; we call the latter property the factorization property. In section 4 we implement fixed point resolution directly on the branching spaces. In particular, we derive explicit formulæ for the characters of the primary fields into which the fixed points are resolved, as well as a formula for the S-matrix of the coset conformal field theory. In section 5 we illustrate our formalism by applying it to diagonal cosets describing **Z**₂ orbifolds of the free boson compactified on a circle. Finally, we discuss how our results can be generalized to more general cosets.

In two appendices we present some additional information. In appendix A we derive some properties of automorphisms in general cosets that fulfill the factorization property. Our proof provides in particular additional insight into the uniqueness of modular anomalies of characters and branching functions. In appendix B we show that the S-matrix which is obtained for the coset theory is indeed a unitary, symmetric matrix, whose square is a conjugation of the primary fields and which, together with a diagonal unitary T-matrix, generates a representation of the group SL(2,**Z**).

Our results show that the fixed points of coset conformal field theories supply a surprisingly rich structure. It would be interesting to compare our algebraic approach to the description of

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[1] The terms 'field identification' and 'character modification' are actually misnomers. Not the fields are to be identified, but rather, several distinct combinations of highest weights must be identified because they describe one and the same field of the coset theory; and it is not the characters which get modified, but rather the branching functions have to be modified in order that they coincide with the true characters of the coset conformal field theory.

coset conformal field theories in terms of gauged WZW models so as to identify the analogous structures in the geometric approach. A first step towards the geometric interpretation of character modifications has been undertaken recently in [9].

2 Embeddings and diagram automorphisms

2.1 Diagonal embeddings

In this section we briefly recall the definition of diagonal cosets. We fix a simple Lie algebra $\bar{\mathsf{h}}$; then we choose $\bar{\mathsf{g}}$ as the direct sum of two copies of $\bar{\mathsf{h}}$, $\bar{\mathsf{g}} \cong \bar{\mathsf{h}} \oplus \bar{\mathsf{h}}$, and as a subalgebra $\bar{\mathsf{g}}'$ we take $\bar{\mathsf{g}}' \cong \bar{\mathsf{h}}$ embedded diagonally into $\bar{\mathsf{g}}$. As a general convention, quantities referring to the subalgebra g' will be denoted by the same symbol as the corresponding quantities for g, but with a prime added. However, in order to keep the notation at a reasonable level, we will suppress the prime whenever the context (such as primed indices) already makes it clear that one is dealing with an object referring to $\bar{\mathsf{g}}'$.

The loop space construction provides an embedding of the corresponding untwisted Kac–Moody algebra $\mathsf{g}' \cong \mathsf{h}$ into $\mathsf{g} \cong \mathsf{h} \oplus \mathsf{h}$; it is given by

$$E_m^{\pm\bar{\alpha}} = E_{(1),m}^{\pm\bar{\alpha}} + E_{(2),m}^{\pm\bar{\alpha}}, \qquad H_m^i = H_{(1),m}^i + H_{(2),m}^i. \tag{2.1}$$

Moreover, one has to introduce two central extensions $K_{(1)}$ and $K_{(2)}$ for the two ideals of g and a central extension K' for g', for which the embedding is

$$K' = K_{(1)} + K_{(2)}. \tag{2.2}$$

Diagonal cosets have been studied extensively in the literature. Many interesting conformal field theories can be realized in terms of diagonal cosets: the Virasoro minimal series, the $N = 1$ superconformal minimal series [2,10], and the rational $\mathbf{Z}_2$ orbifolds of $c = 1$ theories. It was also in the framework of diagonal cosets that extensions of the chiral algebra have been discussed, leading to realizations of $\mathcal{W}$ algebras in terms of affine Lie algebras (see [11] and references cited there). Modular invariants for diagonal cosets have been studied in [12,13].

To study the implications of this situation for the representation theory, let us assume that V is a unitary highest weight module of g on which the central elements $K_{(1)}$ and $K_{(2)}$ act as non-negative integer multiples $k_{(1)}, k_{(2)}$ of the identity. To obtain also an action of the Virasoro algebra on V, we define $L_m := L_{(1),m} + L_{(2),m}$, where

$$L_{(p),m} := \sum_{a,b=1}^{\dim\bar{\mathsf{h}}} \frac{\kappa_{ab}}{2\left(k_{(p)} + g^{\vee}\right)} \sum_{n\in\mathbf{Z}} :T_n^a T_{m-n}^b: \tag{2.3}$$

is the Sugawara construction of the Virasoro generators of h at level $k_{(p)}$, i.e. the T_n^a are the generators of h, κ_{ab} is the Killing form of $\bar{\mathsf{h}}$, and the colons ': :' denote a normal ordering.

Due to the embedding $\mathsf{g}' \hookrightarrow \mathsf{g}$ we can view V as a g'-module, on which the central element K' acts as a multiple $k'\mathbb{1}_V$ of the identity where

$$k' = k_{(1)} + k_{(2)}. \tag{2.4}$$

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To obtain a representation of the Virasoro algebra for $\mathbf{g}'$ as well, we also perform the Sugawara construction for $\mathbf{g}'$. [2] By direct computation [2] (compare also [14]), one checks that the difference

$$\overset{\bullet}{L}_m := L_m - L'_m \tag{2.5}$$

of the two Virasoro algebras commutes with all elements of $\mathbf{g}'$ and represents again a Virasoro algebra, with central charge

$$\overset{\bullet}{c} := c - c', \tag{2.6}$$

the *coset Virasoro algebra*.

Diagonal coset theories have the advantage that various properties of the embedding can be checked in a straightforward manner. One property of diagonal cosets that can be seen immediately is that the triangular decomposition

$$\mathbf{g} = \mathbf{g}_+ \oplus \mathbf{g}_\circ \oplus \mathbf{g}_- \tag{2.7}$$

of $\mathbf{g}$ into the Cartan subalgebra $\mathbf{g}_\circ$ and the Borel subalgebras $\mathbf{g}_\pm$ spanned by the step operators of positive and negative roots, and the analogous decomposition

$$\mathbf{g}' = \mathbf{g}'_+ \oplus \mathbf{g}'_\circ \oplus \mathbf{g}'_- \tag{2.8}$$

of $\mathbf{g}'$ are compatible in the sense that

$$\mathbf{g}'_\# \subseteq \mathbf{g}_\# \tag{2.9}$$

for $\# \in \{+, \circ, -\}$. By construction, we then have in fact

$$\mathbf{g}'_\# = \mathbf{g}_\# \cap \mathbf{g}', \tag{2.10}$$

for $\# \in \{+, \circ, -\}$.

The results we present in this paper can be generalized in several directions. First of all, the algebra $\bar{\mathbf{h}}$ we use to construct need not be simple; rather, it can be the direct sum of several simple and abelian $\mathbf{u}(1)$ algebras. Moreover, one can also consider embeddings of more than one copy of $\bar{\mathbf{h}}$ into more than two copies of $\bar{\mathbf{h}}$. It is this class of cosets that we call 'generalized diagonal cosets'. However, to keep the presentation as simple as possible, we will concentrate in this paper on diagonal cosets and assume that $\bar{\mathbf{h}}$ is simple; the extension of our results to the case of generalized diagonal cosets is straightforward.

2.2 Strictly outer automorphisms

Let us now introduce the class of automorphisms of the Kac–Moody algebra $\mathbf{g}$ which we will use to solve the fixed point problem. A *strictly outer* automorphism (or *diagram* automorphism) of an affine Lie algebra $\mathbf{g}$ is the automorphism ω that is induced via

$$\omega(E^i_\pm) = E^{\dot\omega i}_\pm, \qquad \omega(H^i) = H^{\dot\omega i}, \tag{2.11}$$

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[2] These Virasoro operators act in a well-defined way on V, because for any vector $v \in V$ there are only finitely many positive roots of $\mathbf{g}'$ for which Ev, with E the associated step operator, does not vanish. (Modules with this property are known as 'restricted modules'.) Note, however, that even though V is a unitary highest weight module over $\mathbf{g}$, it is typically not a highest weight module over $\mathbf{g}'$ any more, but rather a (in general infinite) direct sum of unitary highest weight modules.

by an automorphism of the Dynkin diagram of $\mathbf{g}$, i.e. by a permutation $\dot{\omega}$ of the labels $i \in \{0, 1, ..., r\}$ of the nodes which leaves the Cartan matrix A invariant, $A^{\dot{\omega}i, \dot{\omega}j} = A^{i,j}$. Since the number of nodes of the Dynkin diagram is finite, any strictly outer automorphism has finite order, which we denote by N. As an example for such an automorphism consider the Dynkin diagram of $A_n^{(1)}$: it is a regular $n+1$-gon; any rotation with an angle a multiple of $2\pi/(n+1)$ corresponds to the action of a simple current and is a symmetry of the Dynkin diagram.

By construction, any diagram automorphism respects the triangular decomposition of $\mathbf{g}$, i.e.

$$\omega(\mathbf{g}_\#) = \mathbf{g}_\# \tag{2.12}$$

for $\# \in \{+, \circ, -\}$. Conversely, if ω is some automorphism of an affine Lie algebra $\mathbf{g}$ which obeys (2.12), then ω is induced by a symmetry $\dot{\omega}$ of the Dynkin diagram. Namely, let E^α be some step operator corresponding to a positive root α, so that $[h, E^\alpha] = \alpha(h)E^\alpha$ for all $h \in \mathbf{g}_\circ$. Applying the map ω to both sides of this equation, we learn that $[\omega(h), \omega(E^\alpha)] = \alpha(h)\,\omega(E^\alpha)$. Since $\omega(\mathbf{g}_\circ) = \mathbf{g}_\circ$, this shows that $\omega(E^\alpha)$ is again a step operator, corresponding to a root $\omega^\star(\alpha)$, which obeys $\omega^\star(\alpha)(\omega(h)) = \alpha(h)$.

The same argument also shows that, in case the root space corresponding to the root α has dimension larger than one, this mapping still provides a mapping of root spaces. Thus the automorphism ω induces a mapping $\omega^\star$ of the root system; in fact, $\omega^\star$ is even an automorphism of the root system. Moreover, (2.12) implies that for α a positive root, $\omega(E^\alpha)$ is in $\mathbf{g}_+$, and hence $\omega^\star$ is an automorphism of the root system which maps positive roots on positive roots. The latter automorphisms are precisely the symmetries of the Dynkin diagram of $\mathbf{g}$ (compare e.g. [15]).

The automorphism $\omega^\star$ of the root (or weight) space of $\mathbf{g}$ acts on the simple $\mathbf{g}$-roots $\alpha^{(i)}$ and the fundamental $\mathbf{g}$-weights $\Lambda_{(i)}$ as

$$\omega^\star(\alpha^{(i)}) = \alpha^{(\dot{\omega}i)}, \qquad \omega^\star(\Lambda_{(i)}) = \Lambda_{(\dot{\omega}i)}. \tag{2.13}$$

Also, there is a unique extension of ω to the semidirect sum of $\mathbf{g}$ with the Virasoro algebra, namely via

$$\omega(L_m) = L_m - (\bar{\Lambda}_{(\dot{\omega}0)}, H_m) + \tfrac{1}{2}(\bar{\Lambda}_{(\dot{\omega}0)}, \bar{\Lambda}_{(\dot{\omega}0)})\,\delta_{m,0}\,K \tag{2.14}$$

and $\omega(C) = C$, where C is the central element of the Virasoro algebra. (For a more detailed discussion of these matters we refer the reader to [1].)

In the case of coset theories, we are interested in a situation in which the strictly outer automorphisms of an untwisted Kac–Moody algebra $\mathbf{g}$ and a subalgebra $\mathbf{g}'$ are linked. Namely, consider a strictly outer automorphism ω of $\mathbf{g}$ and its restriction

$$\omega' := \omega\big|_{\mathbf{g}'} \tag{2.15}$$

to the subalgebra $\mathbf{g}'$. Assume that ω' has the non-trivial property that it maps $\mathbf{g}'$ to itself

$$\omega': \quad \mathbf{g}' \to \mathbf{g}'; \tag{2.16}$$

for reasons that will become apparent soon, we will in this case say that ω fulfills the *factorization property*.

For diagonal cosets the strictly outer automorphisms that fulfill the factorization property can be described explicitly as follows:[3] fix any strictly outer automorphism ω_{h} of h, then $\omega := \omega_{\mathsf{h}} \oplus \omega_{\mathsf{h}}$ is a strictly outer automorphism of $\mathsf{g} = \mathsf{h} \oplus \mathsf{h}$; this automorphism restricts to a strictly outer automorphism ω' of $\mathsf{g}' = \mathsf{h}$, which just coincides with ω_{h}. In order to implement field identification and fixed point resolution, we will use those strictly outer automorphisms that correspond to simple currents; we remark that if ω_{h} corresponds to a simple current of h, then ω and ω' correspond to the action of a simple current of g respectively g'.

Also note that any two diagram automorphisms ω_1 and ω_2 of g corresponding to simple currents commute,

$$[\omega_1, \omega_2] = 0 \, . \tag{2.17}$$

This holds because the action of ω_1 and ω_2 on all of g is already fully determined by their action on the subalgebra g_+ spanned by step operators of positive roots, and hence by their action on the generators corresponding to simple roots [1]; the latter actions, in turn, just correspond to the actions of the automorphisms $\dot\omega_1$ and $\dot\omega_2$ of the Dynkin diagram of g which commute. Analogous statements apply to g' and hence, as we will see later, if ω_1 and ω_2 fulfill the factorization property, to the action on the chiral algebra of the coset theory.

3 Branching spaces and selection rules

3.1 Branching spaces

From the fact (1.1) that the coset Virasoro algebra and the subalgebra g' commute and from the decomposition (1.2) of characters it is apparent that the objects of basic interest in the coset construction are the spaces whose 'characters' are the branching functions. As we will see, these spaces, known as *branching spaces*, are in fact the building blocks of the Hilbert space of the coset theory. However, we emphasize that, in contrast to some claims in the literature, in the presence of fixed points they do not provide the irreducible representation spaces of the theory; rather, the construction of the Hilbert space also requires some of the tools developed in [1]. More concretely, branching spaces can be introduced as follows.

Any irreducible highest weight module $\mathcal{H}_\Lambda$ corresponding to a primary field of g decomposes [16, §12.10] into (in general infinitely many) irreducible highest weight modules of g'. We write this decomposition as

$$\mathcal{H}_\Lambda = \bigoplus_\ell \mathcal{H}_{\ell;\Lambda'} \, . \tag{3.1}$$

Here we label the g'-modules in the decomposition by an integer $\ell = 1, 2, \ldots$ and supply as a (redundant) label of these modules their highest weight $\Lambda' \equiv \Lambda'(\ell)$ with respect to g'. All highest weights Λ' appearing in the decomposition have the same level.

Denote by $v_{\ell;\lambda'}$ the highest weight vector of $\mathcal{H}_{\ell;\lambda'}$, and consider for any g'-weight Λ' the complex vector space

$$\mathcal{H}_{\Lambda,\Lambda'} := \mathrm{span}_{\mathbf{C}} \{ v_{\ell;\lambda'} \mid \lambda'(\ell) = \Lambda' \} \, , \tag{3.2}$$

i.e. the span of those highest weight vectors $v_{\ell;\lambda'}$ which have weight Λ' with respect to g' (or, equivalently, of those irreducible highest weight modules of g' appearing in (3.1) whose highest

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weight is Λ'). The space $\mathcal{H}_{\Lambda,\Lambda'}$ is called the *branching space* that corresponds to the combination $(\Lambda; \Lambda')$ of highest weights. Using the branching spaces we can write the decomposition (3.1) in the form

$$\mathcal{H}_\Lambda = \bigoplus_\ell \mathcal{H}_{\ell;\Lambda'} \cong \bigoplus_{\Lambda'} \mathcal{H}_{\Lambda,\Lambda'} \otimes \mathcal{H}_{\Lambda'}, \tag{3.3}$$

where the summation is over all integrable highest weights of $\mathbf{g}'$ at the relevant level. It must be noted that this definition does *not* guarantee that a given branching space is non-empty. We will see later that typically some branching spaces are empty; the main reason for this are group theoretical selection rules.

To describe field identification, we need to describe maps on branching spaces. Our strategy will be to construct these maps from maps on representation spaces of $\mathbf{g}$ and $\mathbf{g}'$; let us therefore describe in some detail these maps first. As was shown in [1], any strictly outer automorphism ω of $\mathbf{g}$ induces a map τ_ω: $\mathcal{H}_\Lambda \to \mathcal{H}_{\omega^\star\Lambda}$. The action of this map on a highest weight module can be described as follows: denote by v_Λ the highest weight vector of $\mathcal{H}_\Lambda$ and by $v_{\omega^\star\Lambda}$ the highest weight vector of $\mathcal{H}_{\omega^\star\Lambda}$. Any vector of $\mathcal{H}_\Lambda$ is of the form $R_\Lambda(x)v_\Lambda$, where x is an element of $\mathsf{U}(\mathbf{g}_-)$, the universal enveloping algebra of $\mathbf{g}_-$, i.e. x is a linear combination of products of lowering operators. Such a vector is mapped by τ_ω to the vector $R_{\omega^\star\Lambda}(\omega(x))v_{\omega^\star\Lambda}$ in $\mathcal{H}_{\omega^\star\Lambda}$. Note that we have here $\omega(x)$ acting on the highest weight vector of some other module; it can be checked [1] that this prescription ensures that the map τ_ω is compatible with the null vector structure of the corresponding irreducible highest weight module, i.e. τ_ω provides a one-to-one correspondence between the respective null vectors.

From this definition it follows that τ_ω 'ω-twines' the action of $\mathbf{g}$ in the sense that

$$\tau_\omega(R_\Lambda(x) \cdot v) = R_{\omega^\star\Lambda}(\omega(x)) \cdot \tau_\omega(v) \tag{3.4}$$

for all $x \in \mathbf{g}$ and all $v \in \mathcal{H}_\Lambda$, i.e. that the diagram

$$
\begin{array}{ccc}
\mathcal{H}_\Lambda & \xrightarrow{\;R_\Lambda(x)\;} & \mathcal{H}_\Lambda \\
{\scriptstyle \tau_\omega} \downarrow & & \downarrow {\scriptstyle \tau_\omega} \\
\mathcal{H}_{\omega^\star\Lambda} & \xrightarrow{\;R_{\omega^\star\Lambda}(\omega(x))\;} & \mathcal{H}_{\omega^\star\Lambda}
\end{array}
\tag{3.5}
$$

commutes. From now on we will for simplicity drop the symbol 'R.' which indicates the representation in which the various generators act, as usual (anyway, in the notations like xv, the vector space of which v is an element already uniquely determines the representation of $\mathbf{g}$ that acts on it).

Let us now turn to those automorphisms which fulfill the factorization property; then τ_ω maps highest weight vectors with respect to $\mathbf{g}'$ in the module $\mathcal{H}_\Lambda$ to highest weight vectors with respect to $\mathbf{g}'$ in the module $\mathcal{H}_{\omega^\star\Lambda}$. This holds since the factorization property $\omega(\mathbf{g}') \subseteq \mathbf{g}'$ and (2.10) imply that we also have $\omega'(\mathbf{g}'_+) \subseteq \mathbf{g}'_+$, so that the property $\mathbf{g}'_+ \cdot v = 0$ for some $v \in \mathcal{H}_\Lambda$ implies that also $\mathbf{g}'_+ \cdot (\tau_\omega(v)) = \tau_\omega(\omega(\mathbf{g}'_+) \cdot v) = \tau_\omega(\mathbf{g}'_+ \cdot v) = 0$. Moreover, if v_1 and v_2 lie in one and the same irreducible highest weight module of $\mathbf{g}'$, then so do $\tau_\omega(v_1)$ and $\tau_\omega(v_2)$. To see this, we can assume that v_1 is a highest weight vector with respect to $\mathbf{g}'$. Then $v_2 = u \cdot v_1$ for some $u \in \mathsf{U}(\mathbf{g}'_-)$, and hence $\tau_\omega(v_2) = \tau_\omega(u \cdot v_1) = \omega(u) \cdot \tau_\omega(v_1)$; because of $\omega(u) \in \mathsf{U}(\mathbf{g}'_-)$ this shows that $\tau_\omega(v_2)$ is an element of the Verma module with highest weight vector $\tau_\omega(v_1)$. Since the map τ_ω respects the null vector structure of the modules, i.e. $\tau_\omega(v)$ is a null vector if and only if v is a

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null vector, $\tau_\omega(v_2)$ is in fact an element of the irreducible highest weight module with highest weight vector $\tau_\omega(v_1)$.

Comparing the decomposition (3.1) with the analogous decomposition of $\tau_\omega(\mathcal{H}_\Lambda) \equiv \mathcal{H}_{\omega \star \Lambda}$, it then follows that the map τ_ω induces a mapping $\overset{\bullet}{\tau}_\omega$, of the same order N, of the labels of the $\mathbf{g}'$-modules in (3.1),

$$\ell \mapsto \overset{\bullet}{\tau}_\omega \ell \,, \tag{3.6}$$

such that $\tau_\omega(\mathcal{H}_{\ell;\Lambda'}) = \mathcal{H}_{\overset{\bullet}{\tau}_\omega \ell;\omega' \star \Lambda'}$, i.e. the decomposition of $\mathcal{H}_{\omega \star \Lambda}$ reads

$$\mathcal{H}_{\omega \star \Lambda} \equiv \tau_\omega(\mathcal{H}_\Lambda) = \bigoplus_\ell \tau_\omega(\mathcal{H}_{\ell;\Lambda'}) = \bigoplus_\ell \mathcal{H}_{\overset{\bullet}{\tau}_\omega \ell;\omega' \star \Lambda'} \,. \tag{3.7}$$

This implies that the action of the map τ_ω on an irreducible $\mathbf{g}$-module $\mathcal{H}_\Lambda$ can be described as the combination of two maps: first, a map $\overset{\bullet}{\tau}_\omega$ acting on $\mathbf{g}'$-submodules of $\mathcal{H}_\Lambda$, which maps any irreducible $\mathbf{g}'$-submodule of $\mathcal{H}_\Lambda$ to some $\mathbf{g}'$-submodule of $\mathcal{H}_{\omega \star \Lambda}$ according to $\mathcal{H}_{\ell;\Lambda'} \mapsto \overset{\bullet}{\tau}_\omega(\mathcal{H}_{\ell;\Lambda'}) = \mathcal{H}_{\overset{\bullet}{\tau}_\omega \ell;\omega' \star \Lambda'} = \tau_\omega(\mathcal{H}_{\ell;\Lambda'})$ (or in terms of highest weight vectors, $v_{\ell;\Lambda'} \mapsto \overset{\bullet}{\tau}_\omega(v_{\ell;\Lambda'}) = v_{\overset{\bullet}{\tau}_\omega \ell;\omega' \star \Lambda'}$), i.e., a map for which, roughly speaking, only the positions of the $\mathbf{g}'$-modules $\mathcal{H}_{\ell;\Lambda'}$ and $\mathcal{H}_{\overset{\bullet}{\tau}_\omega \ell;\omega' \star \Lambda'}$ in the respective $\mathbf{g}$-modules matter; and second, a map $\tau_{\omega'}$ (defined via the automorphism ω' of $\mathbf{g}'$ in the same way as τ_ω is defined via ω), which describes how a specific vector in $\mathcal{H}_{\ell;\Lambda'}$, considered just as a $\mathbf{g}'$-module, is mapped to a vector in the $\mathbf{g}'$-module $\mathcal{H}_{\overset{\bullet}{\tau}_\omega \ell;\omega' \star \Lambda'}$. It is this observation which is the origin of the name factorization property that we chose to describe the situation (2.16).

3.2 Branching functions

Since the generators $\overset{\bullet}{L}_n$ of the coset Virasoro algebra commute with all elements of the subalgebra $\mathbf{g}'$, they act in a natural way on the branching spaces. We can therefore introduce the *branching functions* $b_{\Lambda,\Lambda'}$ as the traces

$$b_{\Lambda,\Lambda'}(\tau) = \mathrm{tr}_{\mathcal{H}_{\Lambda,\Lambda'}} q^{\overset{\bullet}{L}_0 - \overset{\bullet}{C}/24} \,. \tag{3.8}$$

over the branching spaces; here we have set as usual $q := \exp(2\pi\mathrm{i}\tau)$. From equation (3.3) and the relations $L_0 = \overset{\bullet}{L}_0 + L'_0$ it is also apparent that the branching functions appear in the decomposition

$$\chi_\Lambda(\tau) = \sum_{\Lambda'} b_{\Lambda,\Lambda'}(\tau)\, \chi_{\Lambda'}(\tau) \tag{3.9}$$

of (Virasoro specialized [4] characters $\chi_\Lambda(\tau)$ of $\mathbf{g}$ into (Virasoro specialized) characters $\chi_{\Lambda'}$ of $\mathbf{g}'$. The modular transformation properties of the characters of $\mathbf{g}$ and $\mathbf{g}'$ imply that the branching functions transform as

$$b_{\Lambda,\Lambda'}(-\tfrac{1}{\tau}) = \sum_{M,M'} S_{\Lambda,M} S^*_{\Lambda',M'}\, b_{M,M'}(\tau) \,, \tag{3.10}$$

where the sum on M and M' is over all integrable highest weights of $\mathbf{g}$ and $\mathbf{g}'$, respectively, at the relevant levels.

---

[4] Results similar to the ones derived below hold for the full characters including Cartan angles, provided that for the $\mathbf{g}$-characters one includes only the exponentials of the specific linear combinations $H^i_{(1),n} + H^i_{(2),n}$ of the generators of the Cartan subalgebra $\bar{\mathbf{g}}_\circ$.)

It is easy to check that for diagonal cosets the relation

$$(\bar{\Lambda}_{(\omega 0)}, \bar{H}) = (\bar{\Lambda}'_{(\dot{\omega}'0')_{(1)}}, \bar{H}'_{(1)}) + (\bar{\Lambda}'_{(\dot{\omega}'0')_{(2)}}, \bar{H}'_{(2)}) \tag{3.11}$$

holds. (For generalized diagonal cosets one has to sum over the various ideals that make up $\mathbf{g}$ and $\mathbf{g}_+$; then the analogous relation takes the form $\sum_p (\bar{\Lambda}_{(\omega 0)_{(p)}}, \bar{H}_{(p)}) = \sum_{p'} (\bar{\Lambda}'_{(\dot{\omega}'0')_{(p')}}, \bar{H}'_{(p')})$.) [5]

The relation (3.11) has two important consequences: first, it leads to a conjugacy class selection rule. This arises because $c_\lambda := (\bar{\Lambda}_{(\omega 0)}, \bar{\lambda}) \bmod \mathbf{Z}$ defines a conjugacy class of the $\mathbf{g}$-weight λ; in conformal field theory terms, this class is the monodromy charge [17] of λ with respect to the simple current, and the identity (3.11) asserts that only those branching functions can be non-zero which correspond to combinations of weights which have zero monodromy charge with respect to all identification currents. Conversely, all selection rules in a diagonal coset can be described by identification currents which are all of the form $(J_\mathbf{h}, J_\mathbf{h}; J_\mathbf{h})$, where $J_\mathbf{h}$ can be any simple current of $\mathbf{h}$, because for diagonal cosets the conjugacy class selection rules are precisely those which come from the conjugacy class selection rules for the tensor products of $\bar{\mathbf{h}}$-representations. Therefore all selection rules can be implemented by strictly outer automorphisms of $\mathbf{g}$ and $\mathbf{g}'$ that fulfill the factorization property. Also note that the selection rule is an identity for conformal weights only modulo integers, while the condition (3.11) is valid exactly.

A second important consequence of the relation (3.11) which follows from the transformation (2.14) of the Virasoro algebra under strictly outer automorphisms is that the coset Virasoro generators $\overset{\bullet}{L}_m = L_m - L'_m$ are invariant under ω:

$$
\begin{aligned}
\omega(\overset{\bullet}{L}_m) &= \omega(L_m) - \omega'(L'_m) \\
&= \left[L_m - \sum_p (\bar{\Lambda}_{(\omega 0)_{(p)}}, \bar{H}_{(p),m}) \right] - \left[L'_m - \sum_{p'} (\bar{\Lambda}'_{(\dot{\omega}'0')_{(p')}}, \bar{H}'_{(p'),m}) \right] = \overset{\bullet}{L}_m \, .
\end{aligned}
\tag{3.12}
$$

3.3   Identification of branching spaces and the coset chiral algebra

Together with the ω-twining property (3.4) of $\overset{\bullet}{\tau}_\omega$, the invariance property (3.12) of the coset Virasoro algebra implies that

$$\overset{\bullet}{\tau}_\omega \overset{\bullet}{L}_n = \omega(\overset{\bullet}{L}_n) \overset{\bullet}{\tau}_\omega = \overset{\bullet}{L}_n \overset{\bullet}{\tau}_\omega \, , \tag{3.13}$$

i.e. the map $\overset{\bullet}{\tau}_\omega$ intertwines the action of the coset Virasoro algebra on the spaces $\mathcal{H}_{\Lambda,\Lambda'}$ and $\mathcal{H}_{\omega\star\Lambda,\omega'\star\Lambda'}$, i.e. we have $\overset{\bullet}{\tau}_\omega : \ \mathcal{H}_{\Lambda,\Lambda'} \to \mathcal{H}_{\omega\star\Lambda,\omega'\star\Lambda'}$ with

$$[\overset{\bullet}{\tau}_\omega, \overset{\bullet}{L}_m] = 0 \tag{3.14}$$

for all $m \in \mathbf{Z}$. (At this point it is worth while recalling our general convention that we do not display the representations R in which $\overset{\bullet}{L}_m$ acts explicitly; thus equation (3.14) stands for the

---

[5] A relation of this type can actually be derived in a much more general setting for any automorphism that fulfills the factorization property. The derivation, which is presented in appendix A, yields as a by-product the assertion that the modular anomaly of branching functions is uniquely determined.

commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}_{\Lambda,\Lambda'} & \xrightarrow{\;R_{\Lambda,\Lambda'}(\overset{\bullet}{L}_m)\;} & \mathcal{H}_{\Lambda,\Lambda'} \\
\overset{\bullet}{\tau}_\omega \downarrow & & \downarrow \overset{\bullet}{\tau}_\omega \\
\mathcal{H}_{\omega\star\Lambda,\omega'\star\Lambda'} & \xrightarrow{\;R_{\omega\star\Lambda,\omega'\star\Lambda'}(\overset{\bullet}{L}_m)\;} & \mathcal{H}_{\omega\star\Lambda,\omega'\star\Lambda'}
\end{array}
\tag{3.15}
$$

i.e. for $\overset{\bullet}{\tau}_\omega \circ R_{\Lambda,\Lambda'}(\overset{\bullet}{L}_m) = R_{\omega\star\Lambda,\omega'\star\Lambda'}(\overset{\bullet}{L}_m)\circ \overset{\bullet}{\tau}_\omega$.) This shows in particular that $\mathcal{H}_{\Lambda,\Lambda'}$ and $\mathcal{H}_{\omega\star\Lambda,\omega'\star\Lambda'}$ carry isomorphic representations of the coset Virasoro algebra. Thus our result is a major step towards implementing field identification directly on the representation spaces (up to now it was only known that the Virasoro specialized characters of $\mathcal{H}_{\Lambda,\Lambda'}$ and $\mathcal{H}_{\omega\star\Lambda,\omega'\star\Lambda'}$, i.e. the branching functions, are identical).

In order for the coset theory to be a fully consistent conformal field theory, the intertwining property of $\overset{\bullet}{\tau}_\omega$ must be valid not only for the coset Virasoro algebra, but for all of the (maximally extended) chiral algebra $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$ of the coset theory. The coset chiral algebra is commonly defined as the commutant of g' in some suitable extension $\tilde{\mathsf{U}}(\mathsf{g})$ of the universal enveloping algebra $\mathsf{U}(\mathsf{g})$ of g; this extension must contain certain normal-ordered infinite sums of elements of $\mathsf{U}(\mathsf{g})$, like those occurring in the Sugawara construction. Obviously, the automorphism ω of g, which extends to an automorphism of order N of the enveloping algebra $\mathsf{U}(\mathsf{g})$, should also extend to an automorphism of $\tilde{\mathsf{U}}(\mathsf{g})$ of the same order. Such an extension of ω must, of course, respect the normal ordering prescription.

Let us denote by $\mathcal{W}_{\mathsf{g},\mathsf{h}}$ the subspace of $\tilde{\mathsf{U}}(\mathsf{g})$ which contains all elements that commute with $\mathsf{g}' \subset \tilde{\mathsf{U}}(\mathsf{g})$. This subspace is in fact a subalgebra of $\tilde{\mathsf{U}}(\mathsf{g})$ which contains the coset Virasoro algebra. All elements of $\mathcal{W}_{\mathsf{g},\mathsf{h}}$ have a well-defined action on branching spaces. Moreover, the factorization property of ω implies that $\omega(\mathcal{W}_{\mathsf{g},\mathsf{h}}) \subseteq \mathcal{W}_{\mathsf{g},\mathsf{h}}$ for any automorphism ω that we need for the description of field identification; by an analogous argument as in the case of the endomorphism ω' (2.16), ω therefore induces an automorphism of order N of $\mathcal{W}_{\mathsf{g},\mathsf{h}}$. Thus we can decompose $\mathcal{W}_{\mathsf{g},\mathsf{h}}$ into eigenspaces with respect to all automorphisms ω that fulfill the factorization property.

The eigenspace $\mathcal{W}_{\mathsf{g},\mathsf{h}}^{(0)}$ of $\mathcal{W}_{\mathsf{g},\mathsf{h}}$ that is left invariant under all these automorphisms plays a particularly important role. It is a subalgebra of $\mathcal{W}_{\mathsf{g},\mathsf{h}}$ which according to (2.14) contains the coset Virasoro algebra. It contains all elements of $\mathcal{W}_{\mathsf{g},\mathsf{h}}$ that are intertwined by $\overset{\bullet}{\tau}_\omega$. Hence, the implementation of field identification by the maps $\overset{\bullet}{\tau}_\omega$ implies that $\mathcal{W}_{\mathsf{g},\mathsf{h}}^{(0)}$ rather than $\mathcal{W}_{\mathsf{g},\mathsf{h}}$ is the coset chiral algebra $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$:

$$
\mathcal{W}_{\mathsf{g}/\mathsf{h}} = \mathcal{W}_{\mathsf{g},\mathsf{h}}^{(0)} \ .
\tag{3.16}
$$

Note that this imposes additional constraints on a consistent coset chiral algebra, which have not been realized in the literature so far. In the next section we will see that precisely these constraints enforce a consistent prescription for the resolution of field identification fixed points.

4   Fixed point characters and S-matrices

We are now in a position to discuss the resolution of field identification fixed points in generalized diagonal coset conformal field theories. To this end we first recall that G_{id}, the *identification*

group, is the abelian group generated by all combinations of simple currents of $\mathbf{g}$ and $\mathbf{g}'$ that describe selection rules of the theory. Recall that for generalized diagonal cosets all of them stem from strictly outer automorphisms that fulfill the factorization property (2.16). A non-empty branching space is called a *fixed point* if the relevant highest weights Λ and Λ' satisfy

$$\omega^{\star}(\Lambda) = \Lambda \,, \qquad \omega'^{\star}(\Lambda') = \Lambda' \,. \tag{4.1}$$

For diagonal cosets the fixed points are described by a combination of three weights of $\mathbf{h}$ that are fixed under a simple current automorphism $\omega_{\mathbf{h}}$ of $\mathbf{h}$.

4.1 Twining branching functions

Recall that for any $\omega \in G_{\mathrm{id}}$ there is a linear map

$$\overset{\bullet}{\tau}_{\omega}: \quad \mathcal{H}_{\Lambda,\Lambda'} \to \mathcal{H}_{\omega^{\star}\Lambda,\omega'^{\star}\Lambda'} \,; \tag{4.2}$$

in case Λ, Λ' corresponds to a fixed point of ω, the map $\overset{\bullet}{\tau}_{\omega}$ is even an endomorphism, and we can define the *twining branching function* $b^{[\omega]}_{\Lambda,\Lambda'}$ as the trace

$$b^{[\omega]}_{\Lambda,\Lambda'}(\tau) := \mathrm{tr}_{\mathcal{H}_{\Lambda,\Lambda'}} \overset{\bullet}{\tau}_{\omega} \, q^{\overset{\bullet}{L}_0 - \overset{\bullet}{C}/24} \,. \tag{4.3}$$

In other words, the twining branching function can be seen as the generating functional of the trace of $\overset{\bullet}{\tau}_{\omega}$ on the subspaces having definite grade with respect to the coset Virasoro algebra. They are thus generalized character-valued indices.

These twining branching functions turn out to be key ingredients in the description of fixed point resolution. In this section, we will therefore derive some of their properties using the theory of twining characters of WZW theories [1]. The *twining character* $\chi^{[\omega]}_{\Lambda}$ of the irreducible highest weight module $\mathcal{H}_{\Lambda}$ of $\mathbf{g}$ with respect to the strictly outer automorphism ω is defined as [1]

$$\chi^{[\omega]}_{\Lambda}(h) := \mathrm{tr}_{\mathcal{H}_{\Lambda}} \tau_{\omega} \, \mathrm{e}^{2\pi \mathrm{i} R_{\Lambda}(h)} \,. \tag{4.4}$$

In the present context we are mainly interested in the Virasoro specialized (and anomaly-modified) twining character which for simplicity we denote by the same symbol, i.e.

$$\chi^{[\omega]}_{\Lambda}(\tau) = \mathrm{tr}_{\mathcal{H}_{\Lambda}} \tau_{\omega} \, \mathrm{e}^{2\pi \mathrm{i} \tau (L_0 - c/24))} \,, \tag{4.5}$$

and as usual we simply write L_0 in place of $R_{\Lambda}(L_0)$. (In [1] only the case when $\bar{\mathbf{g}}$ is simple has been described explicitly; in the general case, one has to take a suitable product of twining characters of the various summands of $\mathbf{g}$.) For notational convenience we define formally the twining branching function $b^{[\omega]}_{\Lambda,\Lambda'}$ to be zero if $(\Lambda; \Lambda')$ is not a fixed point of ω.

The main result of [1] was that the twining character coincides with the character of another Kac–Moody algebra $\breve{\mathbf{g}}$, the so-called *orbit Lie algebra*. The orbit Lie algebra corresponding to $\mathbf{g}$ and its diagram automorphism ω is obtained by a simple prescription which corresponds to folding the Dynkin diagram of $\mathbf{g}$ according to the action of $\dot{\omega}$. Namely, the nodes of the Dynkin diagram of the orbit Lie algebra $\breve{\mathbf{g}}$ correspond to the orbits of $\dot{\omega}$ on the Dynkin diagram of

g; the relative length of the corresponding simple roots is given by the relative length of the orbits. More precisely, the Cartan matrix $\breve{A}$ of the orbit Lie algebra $\breve{\mathbf{g}}$ is given by the formula

$$\breve{A}^{[i],[j]} := s_i \frac{N_i}{N} \sum_{l=0}^{N-1} A^{\dot{\omega}^l i, j} \,, \tag{4.6}$$

where

$$s_i := 1 - \sum_{l=1}^{N_i - 1} A^{\dot{\omega}^l i, i} \,, \tag{4.7}$$

N is the order of ω and N_i the length of the $\dot{\omega}$-orbit through i, and where $[i], [j]$ take values in the set of $\dot{\omega}$-orbits (or, alternatively, in a label set which contains one chosen representative for each $\dot{\omega}$-orbit). All diagram automorphisms of affine Lie algebras satisfy $s_i \in \{1, 2\}$ for all i, except for the automorphism of order N of $\mathbf{g} = A_{N-1}^{(1)}$ that rotates the Dynkin diagram, which has $s = 0$ and leads to a trivial twining character [1].

Except for a special class of theories, for which the orbit Lie algebras are twisted Kac–Moody algebras, these orbit Lie algebras correspond precisely to the fixed point theories introduced in [7]. It is a highly non-trivial property of the twining characters that, in case ω is a diagram automorphism that corresponds to a simple current of $\mathbf{g}$, the twining characters form (up to a shift in the modular anomaly) a unitary representation of the twofold covering $\mathrm{SL}(2,\mathbf{Z})$ of the modular group.

To be able to apply these results to the description of twining branching functions, we will show that $b_{\Lambda,\Lambda'}^{[\omega]}(\tau)$ arises in the decomposition of the twining characters of $\mathbf{g}$ with respect to the automorphism τ_ω into twining characters of $\mathbf{g}'$ with respect to $\tau_{\omega'}$, i.e. they obey

$$\chi_\Lambda^{[\omega]}(\tau) = \sum_{\Lambda'} b_{\Lambda,\Lambda'}^{[\omega]}(\tau) \, \chi_{\Lambda'}^{[\omega']}(\tau) \,, \tag{4.8}$$

which also justifies the name 'twining branching function'.

To prove (4.8), we first note that (with the convention that the 'trace' is zero if τ_ω maps $\mathcal{H}_{\ell;\omega'\star\Lambda'}$ to $\mathcal{H}_{m;\omega'\star\Lambda'}$ with $m \neq \ell$)

$$\chi_\Lambda^{[\omega]}(\tau) = \mathrm{tr}_{\mathcal{H}_\Lambda} \tau_\omega \, q^{L_0 - C/24} = \sum_\ell \mathrm{tr}_{\mathcal{H}_{\ell,\Lambda'(\ell)}} \tau_\omega \, q^{L_0' - C'/24} q^{\overset{\bullet}{L}_0 - \overset{\bullet}{C}/24} \,, \tag{4.9}$$

because the coset Virasoro algebra commutes with L_n'. Owing to the fact that $\overset{\bullet}{\tau}_\omega$ and the coset Virasoro algebra commute (3.14), this can be recast in the form

$$\chi_\Lambda^{[\omega]}(\tau) = \sum_\ell \mathrm{tr}_{\mathcal{H}_{\ell,\Lambda'}} \tau_{\omega'} \, q^{L_0' - C'/24} \overset{\bullet}{\tau}_\omega \, q^{\overset{\bullet}{L}_0 - \overset{\bullet}{C}/24} \,, \tag{4.10}$$

or, rewriting the summation indices and using the fact that non-vanishing contributions to the right hand side can only arise for $\overset{\bullet}{\tau}_\omega \ell = \ell$, as

$$\chi_\Lambda^{[\omega]}(\tau) = \sum_{M'} \sum_{\substack{\ell \\ \Lambda'(\ell) = M'}} \mathrm{tr}_{\mathcal{H}_{\ell,\Lambda'}} \tau_{\omega'} \, q^{L_0' - C'/24} \overset{\bullet}{\tau}_\omega \, q^{\overset{\bullet}{L}_0 - \overset{\bullet}{C}/24}$$

$$= \sum_{M'} \mathrm{tr}_{\mathcal{H}_{M'}} \big(\tau_{\omega'} \, q^{L_0' - C'/24}\big) \cdot \mathrm{tr}_{\mathcal{H}_{\Lambda,M'}} \big(\overset{\bullet}{\tau}_\omega \, q^{\overset{\bullet}{L}_0 - \overset{\bullet}{C}/24}\big) \,. \tag{4.11}$$

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By the defining relations of the twining characters (4.5) and the twining branching functions (4.3), this is nothing but (4.8).

The formula (4.8) means in particular that, analogously to the case of ordinary branching functions, the twining branching functions $b^{[\omega]}_{\Lambda,\Lambda'}$ are computable from the twining characters $\chi^{[\omega]}_{\Lambda}$ and $\chi^{[\omega']}_{\Lambda'}$. Since the latter are equivalent to the characters of the orbit Lie algebras, the twining branching functions can be computed as the ordinary branching function

$$b^{[\omega]}_{\Lambda,\Lambda'} = \breve{b}_{\breve{\Lambda},\breve{\Lambda}'} \tag{4.12}$$

of the coset $\breve{\mathsf{h}} \oplus \breve{\mathsf{h}}/\breve{\mathsf{h}}$, where $\breve{\mathsf{h}}$ is the orbit Lie algebra of h with respect to ω_{h}.

Moreover, just as for the ordinary branching functions, the twining branching functions $b^{[\omega]}_{\Lambda,\Lambda'}$ have nice modular transformation properties. Given the modular transformation properties of the twining characters of g,

$$\chi^{[\omega]}_{\lambda}\left(-\tfrac{1}{\tau}\right) = \sum_{\mu} S^{[\omega]}_{\lambda,\mu} \, \chi^{[\omega]}_{\mu}(\tau) \,, \tag{4.13}$$

and of the twining characters of g',

$$\chi^{[\omega']}_{\lambda'}\left(-\tfrac{1}{\tau}\right) = \sum_{\mu'} S^{[\omega']}_{\lambda',\mu'} \, \chi^{[\omega']}_{\mu'}(\tau) \,, \tag{4.14}$$

we find that

$$b^{[\omega]}_{\Lambda,\Lambda'}\left(-\tfrac{1}{\tau}\right) = \sum_{(\mu;\mu')} \mathcal{S}^{[\omega]}_{(\lambda;\lambda'),(\mu;\mu')} \, b^{[\omega]}_{\mu,\mu'}(\tau) \,. \tag{4.15}$$

Here we introduced

$$\mathcal{S}^{[\omega]}_{(\lambda;\lambda'),(\mu;\mu')} := S^{[\omega]}_{\lambda,\mu} \, \big(S^{[\omega']}_{\lambda',\mu'} \big)^{*} \,, \tag{4.16}$$

where $S^{[\omega]}_{\lambda,\mu}$ respectively $S^{[\omega']}_{\lambda',\mu'}$ stands for the product of the corresponding S-matrices of the twining characters for all the ideals of g respectively g'. It would be rather difficult to compute the matrix elements $S^{[\omega]}_{\lambda,\mu}$ and $S^{[\omega']}_{\lambda',\mu'}$ from the definition (4.5) of the twining characters. However, using the results of [1] these matrices can be easily computed as the S-matrices of the respective orbit Lie algebras. Also, the matrix $\mathcal{T}^{[\omega]}$ which describes the behaviour of the twining branching functions under $\tau \mapsto \tau + 1$ is the diagonal unitary matrix obtained as the restriction of the ordinary (untwined) T-matrix $\mathcal{T}$ to the fixed points. This matrix is defined as the restriction to orbits of non-vanishing branching functions of the product $T^{\mathsf{g}} \otimes (T^{\mathsf{g}'})^{*}$ of T-matrices for the characters of g and g'.

We can summarize that the twining branching functions are a set of functions which can be described as the branching functions (4.12) of the diagonal coset of the orbit Lie algebra $\breve{\mathsf{h}}$ of h. In particular they have nice modular transformation properties; the corresponding S- and T-matrices are given by the product of the S- and T-matrices of the orbit Lie algebras.

4.2   Identities for twining branching functions and S-matrices

Before describing the implementation of fixed point resolution, we still need two more relations. Namely, it can happen that a fixed point is left fixed only by part of the identification group, while the rest of the identification group induces field identifications. We would like to identify

the corresponding twining branching functions and their S-matrix elements also in this situation. In the present section we will derive relations for the twining branching functions (cf. (4.21)) and their S-matrix (cf. (4.20)) which show that this is indeed possible.

The first important observation we can make is that when performing the field identification required for some simple current J_1, the simple current of $\mathbf{g}$ with highest weight $J_2 := k_{(p)}\Lambda_{(\dot\omega_2 0)}$ gives rise to a simple current of the orbit Lie algebra $\breve{\mathbf{g}}$. Namely, simple currents correspond to so-called cominimal fundamental weights $\Lambda_{(\dot\omega_2 0)}$ of $\mathbf{g}$ (except for the isolated simple current of E_8 at level two), i.e. the associated Coxeter label has the value $a_{\dot\omega_2 0} = 1$. Because of the identity [1] $\breve{a}_{[i]} = a_i/s_i$, it follows that for the $\dot\omega_1$-orbit $[\dot\omega_2 0]$ we have $\breve{a}_{[\dot\omega_2 0]} = 1$ as well, which proves the observation. Note, however, that the simple current $\breve{J}_2$ of $\breve{\mathbf{g}}$ on which J_2 is projected can be trivial, i.e. the identity primary field, as is definitely the case for the orbit of the current J_1 itself. Furthermore, the converse of the statement does not hold, i.e. it is not true that any simple current of $\breve{\mathbf{g}}$ arises in this way. Namely, in case there is an orbit with $s_i = 2$, it can give rise to an additional simple current of $\breve{\mathbf{g}}$ (as an example, take $\mathbf{g} = C_{2n+1}^{(1)}$ and its unique non-trivial simple current, which leads to the orbit Lie algebra $\breve{\mathbf{g}} = C_n^{(1)}$; the non-trivial simple current of $\breve{\mathbf{g}}$ arises precisely from the unique orbit of $\breve{\mathbf{g}}$ that has $s_i = 2$).

Further, since the group of simple currents is abelian so that any two diagram automorphisms commute (see (2.17)), the action of $\breve{J}_2$ on the primary fields of $\breve{\mathbf{g}}$ reproduces the action of J_2 on J_1-orbits of primary fields of $\mathbf{g}$. To describe coset theories, we have to combine the corresponding statements for $\mathbf{g}$ and $\mathbf{g}'$; in the case of fixed points, the pair $(\Lambda; \Lambda')$ of weights of $\mathbf{g}$ and $\mathbf{g}'$ that characterizes a branching function gets projected down to a pair $(\breve{\Lambda}; \breve{\Lambda}')$ of weights of $\breve{\mathbf{g}}$ and $\breve{\mathbf{g}}'$.

To analyze the situation for the identification current $(J_{\mathsf{h}}, J_{\mathsf{h}}; J_{\mathsf{h}})$, we can assume that the orbit Lie algebra $\breve{\mathsf{h}}$ is again an untwisted affine Lie algebra and that there is again a coset conformal field theory associated to the embedding (4.17). In the cases (compare e.g. section 5) for which the orbit Lie algebra is a twisted affine algebra, the simple current has order 2 and the stabilizer is either trivial or the whole identification group. It is therefore natural to look at the diagonal embedding

$$\breve{\mathsf{h}} \hookrightarrow \breve{\mathsf{h}} \oplus \breve{\mathsf{h}}\,, \tag{4.17}$$

where $\breve{\mathsf{h}}$ is the orbit Lie algebra of h with respect to the simple current J_{h}. We claim that the fixed points of $\mathbf{g}/\mathbf{g}'$ which are allowed by the selection rules are in one-to-one correspondence to the branching functions allowed by the selection rules of the embedding (4.17).

To compute the monodromy charge

$$\breve{Q}_2((\breve{\Lambda}; \breve{\Lambda}')) \equiv \breve{h}_{(\breve{\Lambda}; \breve{\Lambda}')} - \breve{h}_{\breve{J}_2 \star (\breve{\Lambda}; \breve{\Lambda}')} + \breve{h}_{\breve{J}_2} \bmod \mathbf{Z} \tag{4.18}$$

of the field labeled by $(\breve{\Lambda}; \breve{\Lambda}')$ with respect to the simple current $\breve{J}_2$, we note that in the transition from $\mathbf{g}$ to $\breve{\mathbf{g}}$ the conformal weights are shifted by a constant that does not depend on the branching function [1], so that we have

$$\breve{Q}_2((\breve{\Lambda}; \breve{\Lambda}')) = h_{(\Lambda; \Lambda')} - h_{J_2 \star (\Lambda; \Lambda')} + \breve{h}_{\breve{J}_2} \bmod \mathbf{Z} = \breve{h}_{\breve{J}_2} \bmod \mathbf{Z}\,. \tag{4.19}$$

Here in the last equality, we use that J_2 is an identification current, which implies that $h_{(\Lambda; \Lambda')} - h_{J_2 \star (\Lambda; \Lambda')} \in \mathbf{Z}$. According to the relation (4.19), the monodromy charge of all allowed fixed points is a constant that does not depend on the specific fixed point. Since the simple current

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$\breve{J}_2$ describes a selection rule for the embedding (4.17), it has spin 0. [6] It follows that the right hand side of (4.19) vanishes. Thus we have derived that the monodromy charges of the projected fixed points with respect to the projected simple currents are zero; in other words, selection rules get projected down to selection rules.

Two immediate consequences of this statement are the following. First, using (1.3) we have

$$\mathcal{S}^{[\omega_1]}_{\omega_2^\star(\Lambda;\Lambda'),(M;M')} = \breve{\mathcal{S}}_{\breve{J}_{2\star}(\breve{\Lambda};\breve{\Lambda}'),(\breve{M};\breve{M}')} = \breve{\mathcal{S}}_{(\breve{\Lambda};\breve{\Lambda}'),(\breve{M};\breve{M}')} = \mathcal{S}^{[\omega_1]}_{(\Lambda;\Lambda'),(M;M')}, \tag{4.20}$$

i.e. $\mathcal{S}^{[\omega_1]}_{(\Lambda;\Lambda'),(M;M')}$ does not depend on the choice of representative of the ω_2-orbit. Second, from the identity (4.20) we conclude further that

$$
\begin{aligned}
b^{[\omega_1]}_{\omega_2^\star\Lambda,\omega_2'^\star\Lambda'}\left(-\tfrac{1}{\tau}\right) &= \sum_{(M;M')} \mathcal{S}^{[\omega_1]}_{\omega_2^\star(\Lambda;\Lambda'),(M;M')} \, b^{[\omega_1]}_{M,M'}(\tau) \\
&= \sum_{(M;M')} \mathcal{S}^{[\omega_1]}_{(\Lambda;\Lambda'),(M;M')} \, b^{[\omega_1]}_{M,M'}(\tau) = b^{[\omega_1]}_{\Lambda,\Lambda'}\left(-\tfrac{1}{\tau}\right).
\end{aligned}
\tag{4.21}
$$

In other words, just as ordinary branching functions, the twining branching functions are identified as well.

4.3 Resolution of fixed points

As seen in section 3, field identification requires that all elements of the coset chiral algebra $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$ have to commute with the map τ_ω for all automorphisms $\omega \in G_{\mathrm{id}}$. If a branching space $\mathcal{H}_{\Lambda,\Lambda'}$ with branching function $b_{\Lambda,\Lambda'}$ has a non-trivial stabilizer group

$$G_{\Lambda,\Lambda'} = \left\{ \omega \in G_{\mathrm{id}} \mid \omega^\star\Lambda = \Lambda, \ \omega'^\star\Lambda' = \Lambda' \right\}, \tag{4.22}$$

then the corresponding mappings τ_ω constitute endomorphisms of $\mathcal{H}_{\Lambda,\Lambda'}$. The fact that the action of the coset algebra and the mappings τ_ω commute then implies immediately that $\mathcal{H}_{\Lambda,\Lambda'}$ carries a reducible representation of the coset chiral algebra $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$, since already the eigenspaces with respect to τ_ω form modules over $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$. This way the implementation of field identification by the maps τ_ω enforces fixed point resolution: one has to split the branching spaces $\mathcal{H}_{\Lambda,\Lambda'}$ into the eigenspaces with respect to the action of the stabilizer $G_{\Lambda,\Lambda'}$.

More precisely, let us consider any fixed branching space $\mathcal{H}_{\Lambda,\Lambda'}$, with branching function $b_{\Lambda,\Lambda'}$ and stabilizer group $G_{\Lambda,\Lambda'} \subseteq G_{\mathrm{id}}$. Any character

$$\Psi : \quad G_{\Lambda,\Lambda'} \to \mathbf{C} \tag{4.23}$$

of the stabilizer group $G_{\Lambda,\Lambda'}$ gives rise to an eigenspace $\mathcal{H}^{(\Psi)}_{\Lambda,\Lambda'}$ of $\mathcal{H}_{\Lambda,\Lambda'}$ on which τ_ω has eigenvalue $\Psi(\omega)$, i.e. acts like $\Psi(\omega)$ times the identity map. Just as the branching functions for the spaces $\mathcal{H}_{\Lambda,\Lambda'}$ are denoted by $b_{\Lambda,\Lambda'}$, we denote the (Virasoro specialized) character of the eigenspace $\mathcal{H}^{(\Psi)}_{\Lambda,\Lambda'}$ by $b^{(\Psi)}_{\Lambda,\Lambda'}$. By construction, we have

$$b_{\Lambda,\Lambda'}(\tau) = \sum_{\Psi \in G^\star_{\Lambda,\Lambda'}} b^{(\Psi)}_{\Lambda,\Lambda'}(\tau), \tag{4.24}$$

---

[6] Note that in the case when $\breve{\mathsf{h}}$ is a twisted affine Lie algebra the identification current has order two so that fixed points with different stabilizer do not exist.

where $G^\star_{\Lambda,\Lambda'}$ is the group of characters of $G_{\Lambda,\Lambda'}$. Since the coefficients in the expansion of $b^{(\Psi)}_{\Lambda,\Lambda'}$ in powers of $q = \exp(2\pi i \tau)$ count the number of eigenstates at the relevant grade, they are manifestly non-negative integers, and hence they are natural candidates for the characters of the resolved fixed points.

Since $\overset{\bullet}{\tau}_\omega$ acts on $\mathcal{H}^{(\Psi)}_{\Lambda,\Lambda'}$ as the multiplication with $\Psi(\omega)$, it follows that

$$b^{[\omega]}_{\Lambda,\Lambda'} = \sum_{\Psi \in G^\star_{\Lambda,\Lambda'}} \mathrm{tr}_{\mathcal{H}^{(\Psi)}_{\Lambda,\Lambda'}} q^{\overset{\bullet}{L}_0 - \overset{\bullet}{C}/24} \cdot \Psi(\omega) = \sum_{\Psi \in G^\star_{\Lambda,\Lambda'}} \Psi(\omega)\, b^{(\Psi)}_{\Lambda,\Lambda'}\,. \tag{4.25}$$

The orthogonality relation

$$\sum_{\omega \in G_{\Lambda,\Lambda'}} \Psi_1^*(\omega)\, \Psi_2(\omega) = |G_{\Lambda,\Lambda'}|\, \delta_{\Psi_1,\Psi_2} \tag{4.26}$$

of the characters of $G_{\Lambda,\Lambda'}$ then implies that

$$b^{(\Psi)}_{\Lambda,\Lambda'} = \frac{1}{|G_{\Lambda,\Lambda'}|} \sum_{\omega \in G_{\Lambda,\Lambda'}} \Psi^*(\omega)\, b^{[\omega]}_{\Lambda,\Lambda'}\,. \tag{4.27}$$

Now recall that $b^{[\omega]}_{\Lambda,\Lambda'}$ is zero whenever $\omega \notin G_{\Lambda,\Lambda'}$. It is therefore consistent to extend $\Psi \in G^\star_{\Lambda,\Lambda'}$ to a function on the full identification group G_{id} by setting $\Psi(\omega) = 0$ for $\omega \notin G_{\Lambda,\Lambda'}$. (This extended function is *not* a character of G_{id}.) With this convention, which will simplify various formulæ later on, we can rewrite (4.27) as

$$b^{(\Psi)}_{\Lambda,\Lambda'} = \frac{1}{|G_{\Lambda,\Lambda'}|} \sum_{\omega \in G_{\mathrm{id}}} \Psi^*(\omega)\, b^{[\omega]}_{\Lambda,\Lambda'}\,. \tag{4.28}$$

At this point the following comments concerning field identification are in order. As already seen in (2.17), diagram automorphisms corresponding to simple currents commute (the group of simple currents is abelian); this implies that the maps τ_{ω_1} and τ_{ω_2} induced by any two simple current automorphisms ω_1 and ω_2 commute as well. Suppose now that $\omega_2 \notin G_{\Lambda,\Lambda'}$ so that ω_2 induces a field identification. We then have

$$G_{\Lambda,\Lambda'} = G_{\omega_2^\star \Lambda, \omega_2'^\star \Lambda'}\,, \tag{4.29}$$

i.e. the stabilizers of all branching functions that have to be identified are identical. Moreover, τ_{ω_2} respects the decomposition into eigenspaces. Namely, if $v \in \mathcal{H}^{(\Psi)}_{\Lambda,\Lambda'}$, i.e. $\tau_{\omega_1} v = \Psi(\omega_1)\, v$ for all $\omega_1 \in G_{\Lambda,\Lambda'}$, then also

$$\tau_{\omega_1}(\tau_{\omega_2} v) = \tau_{\omega_2} \tau_{\omega_1} v = \Psi(\omega_1)\, \tau_{\omega_2} v\,. \tag{4.30}$$

This shows that τ_{ω_2} implements in fact the field identification between $\mathcal{H}^{(\Psi)}_{\Lambda,\Lambda'}$ and $\mathcal{H}^{(\Psi)}_{\omega_2^\star \Lambda, \omega_2'^\star \Lambda'}$, and the corresponding characters coincide. The characters of resolved fixed points of the coset theory are therefore

$$\mathcal{X}^{(\Psi)}_{\Lambda,\Lambda'} := \frac{1}{|G_{\mathrm{id}}|} \sum_{\omega \in G_{\mathrm{id}}} b^{(\Psi)}_{\omega^\star \Lambda, \omega'^\star \Lambda'} = \frac{1}{|G_{\mathrm{id}}| \cdot |G_{\Lambda,\Lambda'}|} \sum_{\omega_1,\omega_2 \in G_{\mathrm{id}}} \Psi^*(\omega_1)\, b^{[\omega_1]}_{\omega_2^\star \Lambda, \omega_2'^\star \Lambda'}\,, \tag{4.31}$$

or, inserting the identity (4.21),

$$\mathcal{X}^{(\Psi)}_{\Lambda,\Lambda'} = \frac{1}{|G_{\Lambda,\Lambda'}|} \sum_{\omega \in G_{\mathrm{id}}} \Psi^*(\omega)\, b^{[\omega]}_{\Lambda,\Lambda'} = b^{(\Psi)}_{\Lambda,\Lambda'}\,. \tag{4.32}$$

Thus the true characters of diagonal coset conformal field theories can be expressed entirely in terms of branching functions and twining branching functions.

4.4 Computation of the S-matrix

With the above results we are now also in a position to compute the S-matrix $\mathcal{S}^{[\omega]}$ explicitly. To this end we denote by the pair $[\Lambda; \Lambda']$ of highest weights in square brackets the orbit of $(\Lambda; \Lambda')$ with respect to the identification group G_{id}. Then we find

$$
\begin{aligned}
\mathcal{X}_{\Lambda,\Lambda'}^{(\Psi)}\left(-\tfrac{1}{\tau}\right) &= \frac{1}{|G_{\mathrm{id}}| \cdot |G_{\Lambda,\Lambda'}|} \sum_{\omega_1,\omega_2 \in G_{\mathrm{id}}} \Psi^*(\omega_1)\, b^{[\omega_1]}_{\omega_2^\star \Lambda, \omega_2'^\star \Lambda'}\left(-\tfrac{1}{\tau}\right) \\
&= \frac{1}{|G_{\mathrm{id}}| \cdot |G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|} \sum_{\substack{\omega_1,\omega_2,\omega_3 \\ \in G_{\mathrm{id}}}} \sum_{[M;M']} \Psi^*(\omega_1)\, \mathcal{S}^{[\omega_1]}_{(\omega_2^\star \Lambda;\omega_2'^\star \Lambda'),(\omega_3^\star M;\omega_3'^\star M')}\, b^{[\omega_1]}_{\omega_3^\star M, \omega_3'^\star M'}(\tau) \\
&= \frac{1}{|G_{\mathrm{id}}| \cdot |G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|} \sum_{\substack{\omega_1,\omega_2,\omega_3 \\ \in G_{\mathrm{id}}}} \sum_{[M;M']} \Psi^*(\omega_1)\, \mathcal{S}^{[\omega_1]}_{(\omega_2^\star \Lambda;\omega_2'^\star \Lambda'),(\omega_3^\star M;\omega_3'^\star M')} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \sum_{\tilde\Psi \in G^\star_{M,M'}} \tilde\Psi(\omega_1)\, b^{(\tilde\Psi)}_{\omega_3^\star M, \omega_3'^\star M'}(\tau) \\
&= \frac{1}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|} \sum_{[M;M']} \sum_{\tilde\Psi \in G^\star_{M,M'}} \sum_{\omega_1,\omega_3 \in G_{\mathrm{id}}} \Psi^*(\omega_1)\, \mathcal{S}^{[\omega_1]}_{(\Lambda;\Lambda'),(M;M')}\, \tilde\Psi(\omega_1)\, b^{(\tilde\Psi)}_{\omega_3^\star M, \omega_3'^\star M'}(\tau) \\
&= \frac{|G_{\mathrm{id}}|}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|} \sum_{[M;M']} \sum_{\tilde\Psi \in G^\star_{M,M'}} \Big[\sum_{\omega_1 \in G_{\mathrm{id}}} \Psi^*(\omega_1)\, \mathcal{S}^{[\omega_1]}_{(\Lambda;\Lambda'),(M;M')}\, \tilde\Psi(\omega_1) \Big] \mathcal{X}^{(\tilde\Psi)}_{M,M'}(\tau) \, .
\end{aligned}
$$

(4.33)

From this formula we read off that the actual S-matrix for a coset conformal field theory with fixed points is

$$
\mathcal{S}_{([\Lambda;\Lambda'],\Psi),([M;M'],\tilde\Psi)} := \frac{|G_{\mathrm{id}}|}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|} \sum_{\omega \in G_{\mathrm{id}}} \Psi^*(\omega)\, \mathcal{S}^{[\omega]}_{[\Lambda;\Lambda'],[M;M']}\, \tilde\Psi(\omega) \, .
\tag{4.34}
$$

Note that here $\Psi \in G^\star_{\Lambda,\Lambda'}$ and $\tilde\Psi \in G^\star_{M,M'}$, so that according to our conventions, the summation is effectively only over the subgroup $G_{\Lambda,\Lambda'} \cap G_{M,M'}$ of G_{id}.

In appendix B we will check that the matrix (4.34) is symmetric and unitary and that its square gives a permutation of order two on the primary fields of the coset conformal field theory. Moreover, as already mentioned we obtain a diagonal unitary matrix $\mathcal{T}$ by using the conformal weights of the branching functions. Thus the different fields into which a fixed point is resolved have the same T-matrix elements (although due to the character modifications their conformal weights might differ by integers):

$$
\mathcal{T}_{([\Lambda;\Lambda'],\Psi),([M;M'],\tilde\Psi)} = \delta_{[\Lambda;\Lambda'],[M;M']}\, \delta_{\Psi,\tilde\Psi}\, T_{(\Lambda;\Lambda')} \, .
\tag{4.35}
$$

In the appendix we will also check $\mathcal{S}$ and $\mathcal{T}$ generate a unitary representation of SL(2,$\mathbf{Z}$). Furthermore, in all cases we have checked explicitly, the matrix $\mathcal{S}$ gives rise via the Verlinde formula to non-negative integral fusion coefficients.

As an illustration, consider the case where the identification group G_{id} is isomorphic to the cyclic group $\mathbf{Z}_N$ with N prime. Then the identification group G_{id} has N characters $\Psi = \Psi_k$, $k = 0, 1, ..., N-1$, acting on $n \in \mathbf{Z}_N$ as

$$
\Psi_k(n) = \zeta^{kn} \, ,
\tag{4.36}
$$

where

$$\zeta := \exp\left(\frac{2\pi i}{N}\right) \tag{4.37}$$

is a primitive Nth root of unity. Then for any automorphism $\omega \neq id$ in G_{id} the matrix $\mathcal{S}^{[\omega^n]}$ is the same for all $n = 1, 2, \ldots, N-1$,

$$\mathcal{S}^{[\omega^n]}_{(\lambda;\lambda'),(\mu;\mu')} =: \mathring{\mathcal{S}}_{(\lambda;\lambda'),(\mu;\mu')} \tag{4.38}$$

for $n \neq 0$, while of course

$$\mathcal{S}^{[\omega^0]}_{(\lambda;\lambda'),(\mu;\mu')} = \mathcal{S}_{(\lambda;\lambda'),(\mu;\mu')} \equiv S_{\lambda,\mu}\left(S'_{\lambda',\mu'}\right)^* \tag{4.39}$$

is the S-matrix describing the modular behavior of the ordinary branching functions.

If either $(\Lambda; \Lambda')$ or $(M; M')$ is not a fixed point, the sum in (4.34) has only one term, corresponding to the identity element of G_{id}, and the S-matrix is a multiple of the ordinary S-matrix element for the corresponding branching functions. More interesting is the case when both branching functions are fixed points; then we have

$$
\begin{aligned}
\mathcal{S}_{([\lambda;\lambda'],\Psi_k),([\mu;\mu'],\Psi_l)} &= \frac{1}{N} \sum_{n=0}^{N-1} \Psi_k^*(n)\, \mathcal{S}^{[\omega^n]}_{(\lambda;\lambda'),(\mu;\mu')}\, \Psi_l(n) = \frac{1}{N} \sum_{n=0}^{N-1} \zeta^{n(l-k)}\, \mathcal{S}^{[\omega^n]}_{(\lambda;\lambda'),(\mu;\mu')} \\[2mm]
&= \frac{1}{N} \mathcal{S}_{(\lambda;\lambda'),(\mu;\mu')} + \frac{1}{N} \mathring{\mathcal{S}}_{(\lambda;\lambda'),(\mu;\mu')} \cdot \sum_{n=1}^{N-1} \zeta^{n(l-k)} \\[2mm]
&= \begin{cases} \dfrac{1}{N}\, \mathcal{S}_{(\lambda;\lambda'),(\mu;\mu')} + \left(1 - \dfrac{1}{N}\right) \mathring{\mathcal{S}}_{(\lambda;\lambda'),(\mu;\mu')} & \text{for } l = k\,, \\[3mm] \dfrac{1}{N}\, \mathcal{S}_{(\lambda;\lambda'),(\mu;\mu')} - \dfrac{1}{N}\, \mathring{\mathcal{S}}_{(\lambda;\lambda'),(\mu;\mu')} & \text{for } l \neq k\,. \end{cases}
\end{aligned} \tag{4.40}
$$

This is precisely the expression for the S-matrix that has been conjectured in [7].

5   The diagonal B_n embeddings

To illustrate our results presented in the previous sections we study now the diagonal coset theories based on

$$\mathbf{g} = (B_{n+1}^{(1)})_1 \oplus (B_{n+1}^{(1)})_1\,, \qquad \mathbf{g}' = (B_{n+1}^{(1)})_2 \tag{5.1}$$

in some detail. In these cosets, we have the identification current $(J, J; J)$, where J denotes the unique non-trivial simple current of $B_n^{(1)}$.

The coset theories (5.1) have Virasoro central charge $\mathring{c} = 1$, so that we can determine their characters also via a different route, namely by constructing them out of a free boson. The $c = 1$ conformal field theories have been classified, and looking at the spectrum, it turns out that the series $SO(N)_1 \oplus SO(N)_1 / SO(N)_2$ can be identified with $\mathbf{Z}_2$ orbifolds of the $U(1)$ theory with $4N$ primary fields. The comparison of the known character formulæ for the latter theories to the character modifications implied by our prescription provides us with a rather non-trivial consistency check. The coset description of these theories has fixed point problems only for odd N, and in (5.1) we have set $N = 2n + 3$.

An analysis of these cosets has already been presented in [7]. The main results of that work can be summarized as follows. The set of conformal weights of the branching functions was shown to be equal to the set of conformal weights of the non-unitary minimal Virasoro models with $(p, q) = (2, N)$. Hence, according to the conjecture formulated in [7], the characters of those models are natural candidates for the character modifications. This conjecture was verified explicitly by computing the relevant orbifold characters, and showing that the difference between certain pairs of them is indeed equal to the minimal model characters.

Our new results improve the description of these coset theories in two important ways:

1. We can *derive* the character modifications, rather than conjecturing them.

2. We can extend the analysis in principle to the entire series

$B(n + 1, k, l) := (B^{(1)}_{n+1})_k \oplus (B^{(1)}_{n+1})_l / (B^{(1)}_{n+1})_{k+l}$, for arbitrary k and l.

The reason why the procedure of [7] was limited to $k = l = 1$ is that except for $k = l = 1$ the fixed point conformal weights could not be identified with the spectrum of any known conformal field theory. Now according to the results of [1] the orbit Lie algebra of $(B^{(1)}_{n+1})_k$ is given by the twisted affine Lie algebra $(\tilde{B}^{(2)}_n)_k$. The results of the present paper then imply that the character modifications for $B(n + 1, k, l)$ must be equal to branching functions of [7] $(\tilde{B}^{(2)}_n)_k \oplus (\tilde{B}^{(2)}_n)_l / (\tilde{B}^{(2)}_n)_{k+l} =: \tilde{B}(n, k, l)$. While this is true for arbitrary k and l, it is only for $k = l = 1$ that we have an explicit check.

The coset theories $B(n + 1, k, l)$ have an identification current $(J, J; J)$ which satisfies the factorization property. For $k = l = 1$ the fixed points are the combinations $(s, s; r)$, where s and $r \in \{1, 2, \ldots, n + 1\}$ denote the primary fields that carry the fundamental spinor representation of B_{n+1} and the antisymmetric tensor representations, respectively. As was shown in [7], after fixed point resolution the combination $(s, s; r)$ is resolved into two primary fields of the orbifold conformal field theory. Each of these two representations corresponds in its turn to two representations of the $U(1)$ theory. This can be described diagrammatically as follows:

$$(s, s; r) \xrightarrow{\text{FP}} \begin{cases} |q| = N - 2r \xrightarrow{\text{D}} \begin{cases} q = N - 2r, \\ q = -N + 2r, \end{cases} \\ \\ |q| = N + 2r \xrightarrow{\text{D}} \begin{cases} q = N + 2r, \\ q = -N - 2r. \end{cases} \end{cases} \tag{5.2}$$

Here the first step represents fixed point resolution of the coset theory, whereas the second step represents the extension of the orbifold chiral algebra by a spin-1 simple current, which becomes the $U(1)$-current of the theory on the circle. The primary fields of the $U(1)$ theory are uniquely identified by their $U(1)$-charge q. The orbifold procedure identifies opposite charges and projects out the $U(1)$ current, but in the untwisted sector the absolute value of the charge can still be used to label the fields. Since opposite charges are identified in orbifolding, it follows that in the inverse process of extending the orbifold chiral algebra the corresponding fields are fixed points of the spin-1 simple current.

---

[7] Although the notation used here is the same as the one we employ for coset conformal field theories, we are not proposing here a definition of coset theories of twisted affine algebras, but only of branching functions for the embedding of the algebras.

It is extremely important to be aware of the difference between the two steps in the diagram (5.2). The first step represents fixed point resolution in a coset theory, and hence involves character modifications, while the second represents fixed point resolution in a 'D-type' modular invariant, which does not involve any character modification. Hence we have

$$b_{s,s;r} = \chi^{\text{orb.}}_{N-2r} + \chi^{\text{orb.}}_{N+2r} \tag{5.3}$$

and

$$\chi^{\text{orb.}}_{N-2r} = \chi^{\text{circ.}}_{N-2r} = \chi^{\text{circ.}}_{-N+2r}, \qquad \chi^{\text{orb.}}_{N+2r} = \chi^{\text{circ.}}_{N+2r} = \chi^{\text{circ.}}_{-N-2r}, \tag{5.4}$$

where $\chi^{\text{orb.}}_q$ and $\chi^{\text{circ.}}_q$ are the Virasoro specialized characters of the orbifold and of the $U(1)$ theory, respectively; the latter are just ratios of θ-functions for the circle theory and the η-function. The character modification can thus be written as

$$\Delta_{s,s;r} = \chi^{\text{circ.}}_{N-2r} - \chi^{\text{circ.}}_{N+2r}, \tag{5.5}$$

where the overall sign was chosen in such a way that all Δ's are positive. We have now expressed the character modifications in terms of known functions, which are ratios of θ- and η-functions. It was shown in [7] that $\Delta_{s,s;r} = \chi^{2,N}_{r,1}$, where $\chi^{p,q}_{r,s}$ denotes a character of a Virasoro minimal model.

Now we turn to the description of these character modifications in terms of branching functions of twisted Kac-Moody algebras. A fixed point of the simple current of $B^{(1)}_{n+1}$ has Dynkin labels $(a_0, a_1, a_2, \ldots, a_{n+1})$ with $a_0 = a_1$, where the first label refers to the extended simple root. The twining character of the simple current automorphism was shown in [1] to be equal to the $\tilde{B}^{(2)}_n$-character for the highest weight with Dynkin labels $(a_1, a_2, \ldots, a_{n+1})$, where the first label corresponds to the longest simple root and the last one to the shortest. The unique fixed point s of $B^{(1)}_{n+1}$ at level 1 has highest weight $(0,0,0,\ldots,0,1)$; it is thus related to the $\tilde{B}^{(2)}_n$-representation with Dynkin labels $(0,\ldots,0,1)$, which we will denote by $\check{s}$. At level 2 of $B^{(1)}_{n+1}$ there are $n+1$ fixed points, which we denote by $r \in \{1,2,\ldots,n+1\}$, with highest weight $(1,1,0,\ldots,0)$ for $r = 1$, $(0,0,0,\ldots,0,1,0,\ldots,0)$ (with $a_{r+1} = 1$) for $1 \leq r \leq n$, and $(0,0,\ldots,0,2)$ for $r = n+1$. They are mapped respectively to the $\tilde{B}^{(2)}_n$-representations with highest weights $(1,0,\ldots,0)$, $(0,\ldots,0,1,0,\ldots,0)$ and $(0,\ldots,0,2)$, to which we will refer by $\check{r}$. The identity we wish to check is then

$$\left[\chi^{\tilde{B}^{(2)}_{n,1}}_{\check{s}} \right]^2 = \sum_{r=1}^{n+1} \Delta_{s,s;r} \, \chi^{\tilde{B}^{(2)}_{n,2}}_{\check{r}}. \tag{5.6}$$

The characters for $B^{(1)}_m$ and $\tilde{B}^{(2)}_m$ can be written down explicitly by choosing an orthonormal basis $e_1, \ldots, e_m$ in the weight space of the horizontal B_m Lie algebra. In this basis the roots of B_m are $\pm e_i \pm e_j$ with $1 \leq i < j \leq m$ and $\pm e_i$ with $1 \leq i \leq m$. The simple roots of the horizontal algebra are $e_1 - e_2$, $e_2 - e_3$, $\ldots$, $e_{m-1} - e_m$ and e_m, and the extended simple root is $-e_1 - e_2$ for $B^{(1)}_m$ and $-2e_1$ for $\tilde{B}^{(2)}_m$. The Weyl–Kac character formula yields in this basis

$$\chi_{\vec{w},k} = q^{\delta(\vec{w},k)} \frac{\Sigma(\vec{w},k)}{\Sigma(0,0)}, \tag{5.7}$$

where δ is the 'modular anomaly' (which equals $h - c/24$ in a conformal field theory), and

$$\Sigma(\vec{w},k) := \sum_{\pi \in S_m} \epsilon(\pi) \sum_{\substack{n_i \in \mathbf{Z} \\ \sum n_i = 0 \bmod 2}} \prod_i \sin\left([w_i + \rho_i + n_i(k+g^\vee)]\theta_{\pi(i)}\right) q^{\frac{1}{2}(k+g^\vee)\vec{n}^2 + (\vec{w}+\vec{\rho})\cdot\vec{n}} \tag{5.8}$$

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for $B_{m,k}$. Here $\vec{w}$ is a highest weight of the Lie algebra, w_i are its components in the orthonormal basis, k is the level, $\vec{\rho}$ the Weyl vector (whose components in the horizontal basis read $\vec{\rho} = \left(\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, \frac{1}{2}\right)$) and $g^\vee$ the dual Coxeter number, $g^\vee = 2m - 1$. The variables θ_i correspond to the generators of the Cartan subalgebra of $B_m^{(1)}$. As we are only interested in Virasoro characters here, we will eventually set them to zero, but this can only be done after canceling common factors between the denominator and the numerator of the character formula. The branching function is in fact independent of the variables θ_i.

For the twisted affine Lie algebra $(\tilde{B}_m^{(2)})_k$ almost the same formula as (5.8) holds, namely

$$\tilde{\Sigma}(\vec{w}, k) := \sum_{\pi \in S_m} \epsilon(\pi) \sum_{n_i \in \mathbf{Z}} \prod_i \sin\left([w_i + \rho_i + n_i(k + \tilde{g}^\vee)]\theta_\pi(i)\right) q^{\frac{1}{2}(k+\tilde{g}^\vee)\vec{n}^2 + (\vec{w}+\vec{\rho})\cdot\vec{n}} , \qquad (5.9)$$

the only modifications being that now $\tilde{g}^\vee = 2m + 1$ and that the second sum is over *all* integer-valued vectors $\vec{n}$. The vector $\vec{w}$ is obtained in this case by writing the $\tilde{B}_m^{(2)}$ Dynkin labels as $(b_0, b_1 \ldots, b_m)$ and regarding $b_1, \ldots, b_m$ as B_m Dynkin labels. (To apply the formula (5.9), the resulting highest weight of B_m must then be converted to the orthogonal basis.)

The modular anomaly for $B_m^{(1)}$ is

$$\delta_B(\vec{w}, k) = \frac{\frac{1}{2}(\Lambda + 2\rho, \Lambda)_B}{k + 2m - 1} - \frac{1}{24}\frac{m(2m+1)k}{k + 2m - 1} . \qquad (5.10)$$

For $\tilde{B}_m^{(2)}$ we find

$$\delta_{\tilde{B}}(\vec{w}, k) = \frac{\frac{1}{2}(\Lambda + 2\rho, \Lambda)_B}{k + 2m + 1} - \frac{1}{24}\frac{m(2m-1)k}{k + 2m + 1} - \frac{km}{12} . \qquad (5.11)$$

Using these results one may check that for any fixed point the branching functions of $B(n + 1, k, l)$ have the same modular anomaly as the corresponding ones of $\tilde{B}(n, k, l)$. This implies that for any fixed point the branching function and the character modifications have the same modular anomaly, so that all characters receive corrections to their leading term. It also implies that we may drop the modular anomaly terms when checking (5.6), since they cancel.

Although we have no general proof of the identity (5.6), the character formulæ are explicit enough to allow partial verification. In particular we have checked that for $n = 1$ and $n = 2$ the first 40 terms in the expansion in q agree, and we have verified the identity for all n in leading and next-to-leading order in q. The leading order check is straightforward. Note that the $\tilde{B}_n^{(2)}$ character formula has as its leading term the dimension of the representation of the horizontal B_n algebra (as it does for the untwisted algebra). Since on the left-hand side of (5.6) the ground state is a spinor and on the right-hand side they are antisymmetric tensors of rank $r - 1$, the leading order of (5.6) amounts to the condition

$$[2^n]^2 = \sum_{r=0}^n \binom{2n+1}{r} , \qquad (5.12)$$

which is indeed satisfied.

6   Summary and outlook

In this article, we have solved the problem of resolving field identification fixed points of diagonal coset conformal field theories, and of fixed points in the even larger class of generalized diagonal

cosets. To this end we have implemented both field identification and fixed point resolution directly on the branching spaces. Technically, this was achieved by the use of automorphisms of the Dynkin diagram which fulfill the factorization property (2.16). They give rise to ω-twining maps $\dot{\tau}_\omega$ on the branching spaces.

One important insight that we gain from this construction is that the coset chiral algebra $\mathcal{W}_{g/h}$ is *not* the commutant of the subalgebra g' in (some closure of) the universal enveloping algebra of $U(g)$, but rather a subalgebra of it, namely the subalgebra that is fixed by all simple current automorphisms of the Dynkin diagram that fulfill the factorization property.

We have seen that precisely this property enforces fixed point resolution: on the fixed points, i.e. those branching spaces on which the identification group acts non-freely, the action of the chiral algebra cannot change the eigenvalue of eigenvectors with respect to the action of the stabilizer. Therefore we must split these branching spaces into the various eigenspaces with respect to that action. The recently developed theory of twining characters and orbit Lie algebras shows that this splitting is governed by a natural mathematical structure which allows for explicit calculations. In particular, we have derived expressions for both the characters and the S-matrix of the coset theory in terms of the corresponding data of the orbit Lie algebras and of group characters of the identification group. Finally, we have made this resolution procedure explicit for a specific example of cosets describing the $\mathbf{Z}_2$ orbifolds of a compactified free boson.

Our results can be extended in several directions. First of all, one would also like to describe more general cosets than the generalized diagonal ones. However, we do not know any other examples of coset conformal field theories where the factorization property holds. This implies that in the general case, one cannot restrict oneself to strictly outer automorphisms ω, ω' on g and g'. Rather one has to allow for more general outer automorphisms of g and g', which are the product of a strictly outer automorphism and some compensating inner automorphism corresponding to an element of the Weyl group. A candidate for such automorphisms is the spectral flow described in [4]. However, we expect that most features we encountered in the case of generalized diagonal cosets will also be present in the general case of arbitrary embeddings $g' \hookrightarrow g$; in particular, we conjecture that the formula (4.34) for the S-matrix of the coset conformal field theory will still be valid.

Taking into account inner automorphisms will definitely be necessary for the description of those identification currents which act non-trivially only in the subalgebra g'; these '$(1; J)$-currents' are expected to present particular problems. In this context, it is interesting to remark that the so-called maverick cosets [18], i.e. those cosets which possess further selection rules in addition to the conjugacy class selection rules, always have such identification currents, and that these currents have fixed points. There are, however, examples of coset conformal field theories which also fulfill these criteria, but which are nevertheless not maverick.

As a side remark, we mention that in addition to the maverick cosets found in [19], there is at least one other maverick coset, given by $g = E_8$ and $g' = A_1 \oplus E_7$, with all algebras at level 2. [8] (The analogous embedding at level 1 is a conformal embedding, just as for the other maverick cosets). It has as an identification current $(1; J, J)$, where J stands for the unique non-trivial simple current of A_1 respectively E_7; this identification current has fixed points. The branching rules for this coset (see [20, §4.3, example (j)]) show that there is an additional non-vanishing branching function, corresponding to $(\Lambda_{(7)}; \Lambda_{(0)} + \Lambda_{(1)}, \Lambda_{(7)})$, that has conformal

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[8] Thus the A-D-E classification of maverick cosets that was conjectured in [19] is not complete.

weight zero and therefore should be identified with the vacuum of the (putative) coset theory. However, the weights involved are not on any simple current orbit of the vacuum.

Finally, we want to point out that there are still more open conceptual problems in the description of the coset construction, which are largely independent of the problem of field identification and fixed point resolution. First of all, one should show that there is a suitable extension $\tilde{\mathsf{U}}(\mathsf{g})$ of the universal enveloping algebra $\mathsf{U}(\mathsf{g})$ such that the candidate coset chiral algebra $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$ fulfills all requirements for a chiral algebra of a conformal field theory. Moreover, in order to prove that $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$ is indeed the – maximally extended – chiral algebra of the coset theory, one must, in the absence of field identification fixed points, also show that the branching spaces are *irreducible* modules of $\mathcal{W}_{\mathsf{g}/\mathsf{h}}$, and that each isomorphism class of irreducible modules appears precisely once. (Note that for $\overset{\bullet}{c} \geq 1$ the branching spaces are highly reducible as modules over the coset Virasoro algebra.) In the presence of field identification fixed points, one has to prove the same statements for the eigenspaces of the maps τ_ω.

A The factorization property and the uniqueness of modular anomalies

In this appendix we drop the assumption that the embedding $\mathbf{g}' \hookrightarrow \mathbf{g}$ describes a generalized diagonal coset. Whenever the embedding $\mathbf{g}' \hookrightarrow \mathbf{g}$ preserves the triangular decomposition, the restriction ω' of a strictly outer automorphism of $\mathbf{g}$ which satisfies the factorization property is again a strictly outer automorphism of $\mathbf{g}'$. Namely, first, as a restriction of a homomorphism, ω' is a homomorphism of $\mathbf{g}'$. Second, from $\omega^N = id$ it follows that also $\omega'^N = id$; this implies that ω' is surjective and injective, and hence an automorphism of $\mathbf{g}'$; moreover, the order N' of ω' divides the order N of ω. Since both the embedding $\mathbf{g}' \hookrightarrow \mathbf{g}$ and the automorphism ω (2.12) respect the triangular decomposition of $\mathbf{g}$, we have $\omega'(\mathbf{g}'_\#) = \omega(\mathbf{g}'_\#) \subseteq \omega(\mathbf{g}_\#) \subseteq \mathbf{g}_\#$ for $\# \in \{+, \circ, -\}$, while $\omega'(\mathbf{g}'_\#) \subseteq \mathbf{g}'$ by the factorization property of ω'. Together it follows that $\omega'(\mathbf{g}'_\#) \subseteq \mathbf{g}_\# \cap \mathbf{g}' = \mathbf{g}'_\#$, and hence ω' is a strictly outer automorphism of $\mathbf{g}'$.

It can also be shown that any strictly outer automorphism satisfying the factorization property leads to a selection rule of the form (3.11) and leaves the generators of the coset Virasoro algebra invariant. The proof of this assertion has some interesting by-products; it proceeds as follows. From the definition (3.8) of the branching function it is apparent that (up to an overall prefactor $q^{h_\Lambda - h_{\Lambda'} - \bullet \bar{c}/24}$, where h denotes the conformal weight) the coefficient d_n of the power q^n of the branching function counts how many highest weight vectors of $\mathbf{g}'$ with highest weight Λ' occur at grade n in the $\mathbf{g}$-module with highest weight Λ. Now the highest weight vectors of $\mathbf{g}'$ in $\mathcal{H}_\Lambda$ with highest weight Λ' are in one-to-one correspondence to highest weight vectors of $\mathbf{g}'$ in $\mathcal{H}_{\omega\star\Lambda}$ with highest weight $\omega^\star(\Lambda)$ at the same grade of the module. This implies that the associated branching functions are related by

$$b_{\omega\star\Lambda, \omega'\star\Lambda'}(\tau) = \mathrm{e}^{2\pi i d_{\Lambda,\Lambda'}\tau} b_{\Lambda,\Lambda'}(\tau), \tag{A.1}$$

where $d_{\Lambda,\Lambda'} := h_{\omega\star\Lambda} - h_\Lambda - h_{\omega'\star\Lambda'} + h_{\Lambda'}$. In particular, $b_{\omega\star\Lambda, \omega'\star\Lambda'}$ vanishes if and only if $b_{\Lambda,\Lambda'}$ vanishes. Also, from (3.10) we learn that

$$b_{\omega\star\Lambda, \omega'\star\Lambda'}\left(-\tfrac{1}{\tau}\right) = \sum_{M,M'} S_{\omega\star\Lambda, M} S^*_{\omega'\star\Lambda', M'} b_{M,M'}(\tau). \tag{A.2}$$

On the other hand, using (A.1), we see that

$$b_{\omega\star\Lambda, \omega'\star\Lambda'}\left(-\tfrac{1}{\tau}\right) = \mathrm{e}^{-2\pi i d_{\Lambda,\Lambda'}/\tau} \sum_{M,M'} S_{\Lambda, M} S^*_{\Lambda', M'} b_{M,M'}(\tau). \tag{A.3}$$

Let us assume that $b_{\Lambda,\Lambda'}$ does not vanish; then by comparing the two results we obtain the identity

$$\mathrm{e}^{-2\pi i d_{\Lambda,\Lambda'}/\tau} = \frac{\sum_{M,M'} S_{\omega\star\Lambda, M} S^*_{\omega'\star\Lambda', M'} b_{M,M'}(\tau)}{\sum_{M,M'} S_{\Lambda, M} S^*_{\Lambda', M'} b_{M,M'}(\tau)} \tag{A.4}$$

between two functions on the complex upper half plane H_+ ($\mathrm{Im}\,\tau > 0$).

Since both $\mathbf{g}$ and $\mathbf{g}'$ have only a finite number of primary fields, we can define M as the smallest common denominator of the conformal weights of all branching functions and rewrite (A.4) in terms of the variable $x := \mathrm{e}^{2\pi i \tau/M}$. Branching functions are holomorphic functions on H_+ [20, §1.6], and hence the right hand side of (A.4) converges to a meromorphic function on

H_+. Manifestly, we can also interpret it as a meromorphic function in x for $|x| < 1$. After multiplying both numerator and denominator of the right hand side with an appropriate power of x to get rid of negative powers of x which stem from terms of the form $q^{-c/24}$ in the characters, we can assume that both contain only positive powers of x.

The left hand side of (A.4) is a priori defined not on x-space, but rather on an infinite covering of it. If both sides of (A.4) are identical, then also this side has to give rise to a function in x and we do not have to worry about branch cuts any more. Since the left hand side of (A.4) tends to 1 for $\operatorname{Im}\tau \to \infty$, $x = 0$ has to be a regular point for both sides, and hence also the derivatives of the left side with respect to x have to exist at $x = 0$. However, the first derivative of the left side diverges for $x \to 1$, unless $d_{\Lambda,\Lambda'}$ vanishes.

Thus we have $d_{\Lambda,\Lambda'} = 0$. From (A.1) we then learn that

$$b_{\omega \star \Lambda, \omega \star' \Lambda'} \equiv b_{\Lambda,\Lambda'} \tag{A.5}$$

for all pairs (Λ, Λ') of integrable highest weights. In other words, the eigenvalue equation $\overset{\bullet}{L}_0 v_\lambda = h v_\lambda$, where v_λ is an element of any weight space in any irreducible module, implies that $\overset{\bullet}{L}_0 \tau_\omega v_\lambda = h \tau_\omega v_\lambda$ as well.

The argument we just presented also gives some further important insight. Suppose one is given a set of functions of the modular parameter τ which transform under the transformation $\tau \mapsto -1/\tau$ to linear combinations of these same functions, like e.g. the characters of some rational conformal field theory or, as in the case of our interest, the branching functions. Our argument then shows that, whenever two such functions differ only by a factor which is some power of q, they must in fact be identical. In the case of a rational conformal field theory, this implies that if the modules of two primary fields have the same number of states at any degree, then the two primary fields must have the same conformal weight.

With the help of the relation (A.5) the promised results are recovered as follows. Taking into account the transformation (2.14) of the Virasoro algebra under strictly outer automorphisms and its analogue for $\mathbf{g}'$, we can make the most general ansatz

$$\omega^{-1}(\overset{\bullet}{L}_0) = \overset{\bullet}{L}_0 - (v, H_0) + \sum_p \xi_p K_{(p)} \tag{A.6}$$

for $\omega^{-1}(\overset{\bullet}{L}_0)$, where $K_{(p)}$ are the central elements for the various ideals of $\mathbf{g}$ and the sum is over these ideals. Using the ω-twining property (3.4) of τ_ω, we then see that for any vector v_λ with weight λ in any highest weight module of $\mathbf{g}$ we have

$$\begin{aligned}
h \, \tau_\omega v_\lambda &= \overset{\bullet}{L}_0 \tau_\omega v_\lambda = \tau_\omega \omega^{-1}(\overset{\bullet}{L}_0) v_\lambda = \tau_\omega (\overset{\bullet}{L}_0 - (v, H_0) + \sum_p \xi_p K_{(p)}) \, v_\lambda \\
&= (h - (v, \lambda) + \sum_p \xi_p k_{(p)}) \, \tau_\omega v_\lambda \, .
\end{aligned} \tag{A.7}$$

Since this equation has to be valid at any combinations of levels $k_{(p)}$ and for any weight λ, it follows that all the coefficients ξ_p vanish and that also $v = 0$. This implies that

$$\sum_p (\bar{\Lambda}_{(\omega 0)_{(p)}}, \bar{H}_{(p)}) = \sum_{p'} (\bar{\Lambda}'_{(\omega' 0')_{(p')}}, \bar{H}'_{(p')}) \, , \tag{A.8}$$

which is the desired generalization of (3.11). The sums in (A.8) are over the ideals of $\bar{\mathbf{g}}$ and $\bar{\mathbf{g}}'$, respectively. From this result both the conjugacy class selection rules and the invariance of the coset Virasoro algebra follow in the same manner as in the case of diagonal cosets.

B Properties of the S-matrix of the coset theory

In this appendix we describe several checks which show that the S-matrix for the coset conformal field theory, i.e.

$$\mathcal{S}_{([\Lambda;\Lambda'],\Psi),([M;M'],\tilde{\Psi})} = \frac{|G_{\mathrm{id}}|}{|G_{\Lambda,\Lambda'}|\cdot|G_{M,M'}|} \sum_{\omega\in G_{\mathrm{id}}} \Psi^*(\omega)\, \mathcal{S}^{[\omega]}_{[\Lambda;\Lambda'],[M;M']}\, \tilde{\Psi}(\omega) \tag{B.1}$$

as defined in equation (4.34) is a unitary symmetric matrix, that its square gives a permutation of order 2 which leaves the vacuum fixed, and that together with the T-matrix $\mathcal{T}$ of (4.35) it obeys $(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2$, i.e. that it possesses all the properties required for a proper S-matrix.

As a first step we derive some identities for the S-matrices of twining branching functions which will be helpful in proving these assertions. Let us first derive some consequences of the unitarity of the tensor product $S^{\mathbf{g}} \otimes (S^{\mathbf{g}'})^*$ of the S-matrix of $\mathbf{g}$ and the complex conjugate of the S-matrix of $\mathbf{g}'$.[9] If $(\Lambda;\Lambda')$, $(N;N')$ are the labels of two non-vanishing branching functions, then when summing over all elements of the identification group G_{id} and over all combinations $(M;M')$ of integrable weights of $\mathbf{g}$ and $\mathbf{g}'$, the fact that $S^{\mathbf{g}} \otimes (S^{\mathbf{g}'})^*$ is unitary implies that

$$\sum_{\omega\in G_{\mathrm{id}}} \sum_{(M;M')} \mathcal{S}_{\omega\star(\Lambda;\Lambda'),(M;M')}\, \mathcal{S}^*_{(M;M'),(N;N')} = \sum_{\omega\in G_{\mathrm{id}}} \delta_{\omega\star(\Lambda;\Lambda'),(N;N')} = |G_{\Lambda,\Lambda'}|\cdot\delta_{[\Lambda;\Lambda'],[N;N']}\,, \tag{B.2}$$

where the last Kronecker delta refers to G_{id}-orbits. On the other hand, we can compute this sum also as

$$
\begin{aligned}
\sum_{\omega\in G_{\mathrm{id}}} & \sum_{(M;M')} \mathcal{S}_{\omega\star(\Lambda;\Lambda'),(M;M')}\, \mathcal{S}^*_{(M;M'),(N;N')} \\
&= \sum_{\omega\in G_{\mathrm{id}}} \sum_{(M;M')} \mathrm{e}^{2\pi\mathrm{i}Q_\omega((M;M'))}\, \mathcal{S}_{(\Lambda;\Lambda'),(M;M')}\, \mathcal{S}^*_{(M;M'),(N;N')} \\
&= |G_{\mathrm{id}}|\cdot \sum_{\substack{(M;M') \\ Q((M;M'))=0}} \mathcal{S}_{(\Lambda;\Lambda'),(M;M')}\, \mathcal{S}^*_{(M;M'),(N;N')} \\
&= \sum_{\substack{[M;M'] \\ Q((M;M'))=0}} \frac{|G_{\mathrm{id}}|^2}{|G_{M,M'}|}\, \mathcal{S}_{[\Lambda;\Lambda'],[M;M']}\, \mathcal{S}^*_{[M;M'],[N;N']}\,.
\end{aligned}
\tag{B.3}
$$

Here in the first line we used (1.3); then we performed the summation over the identification group; and finally we wrote the result in terms of a sum over orbits of non-vanishing branching functions that have to be identified. The notation $Q([M;M']) = 0$ indicates that the monodromy charge with respect to all simple currents in the identification group G_{id} vanishes. The S-matrix elements corresponding to *orbits* $[\Lambda;\Lambda']$ and $[M;M']$ of pairs $(\Lambda;\Lambda')$ and $(M;M')$ appearing in the final form are well-defined because the original S-matrix elements are constant on the orbits provided that their monodromy charges vanish.

Comparison of the two results yields the identity

$$\sum_{\substack{[M;M'] \\ Q((M;M'))=0}} \frac{|G_{\mathrm{id}}|^2}{|G_{M,M'}|}\cdot \mathcal{S}_{[\Lambda;\Lambda'],[M;M']}\, \mathcal{S}^*_{[M;M'],[N;N']} = |G_{\Lambda,\Lambda'}|\cdot\delta_{[\Lambda;\Lambda'],[N;N']}\,. \tag{B.4}$$

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[9] Note, however, that this tensor product matrix does *not* describe the behavior of the branching functions under $\tau \mapsto -1/\tau$. This follows e.g. from the observation that this matrix has non-vanishing matrix elements between vanishing and non-vanishing branching functions.

Next we note that charge conjugation acts in a natural way on simple current orbits, so that we are given a matrix $\mathcal{C}_{[\Lambda;\Lambda'],[M;M']}$ which describes a permutation of order two of the orbits of non-vanishing branching functions. Performing a similar computation as above, one derives the identity

$$\sum_{\substack{[M;M'] \\ Q((M;M'))=0}} \frac{|G_{\mathrm{id}}|^2}{|G_{M,M'}|} \cdot \mathcal{S}_{[\Lambda;\Lambda'],[M;M']}\, \mathcal{S}_{[M;M'],[N;N']} = |G_{\Lambda,\Lambda'}| \cdot \mathcal{C}_{[\Lambda;\Lambda'],[N;N']}\,. \tag{B.5}$$

analogous to (B.4).

Furthermore, recall that we defined a T-matrix $\mathcal{T}$ as the restriction to orbits of non-vanishing branching functions of the product $T^{\mathbf{g}} \otimes T^{\mathbf{g}'}$ of T-matrices of $\mathbf{g}$ and $\mathbf{g}'$; a similar computation as above leads to the relation

$$\sum_{\substack{[M;M'] \\ Q((M;M'))=0}} \frac{|G_{\mathrm{id}}|}{|G_{M,M'}|} \mathcal{S}_{[\Lambda;\Lambda'],[M;M']}\, \mathcal{T}_{[M;M']}\, \mathcal{S}_{[M;M'],[N;N']} = \mathcal{T}_{[\Lambda;\Lambda']}^{-1}\, \mathcal{S}_{[\Lambda;\Lambda'],[N;N']}\, \mathcal{T}_{[N;N']}^{-1}\,. \tag{B.6}$$

Finally, in the derivation of the identities (B.4) – (B.6) one can substitute $\mathcal{S}_{[\Lambda;\Lambda'],[M;M']}$ by $\mathcal{S}^{[\omega]}_{[\Lambda;\Lambda'],[M;M']}$ and $\mathcal{T}_{[M;M']}$ by $\mathcal{T}^{[\omega]}_{[M;M']}$, thereby obtaining analogous relations for the matrices $\mathcal{S}^{[\omega]}$ and $\mathcal{T}^{[\omega]}$, where the sum is now over all fields in the fixed point theory for which the monodromy charge with respect to to all simple currents in the fixed point theory vanishes, $\breve{Q}((\breve{\Lambda};\breve{\Lambda}')) = 0$. However, as we have seen after equation (4.19), this monodromy charge vanishes precisely if the corresponding monodromy charge before projection vanishes. Thus the formulae (B.4) remains true if only $\mathcal{S}$ is replaced by $\mathcal{S}^{[\omega]}$. As for the analogue of equation (B.6) we remark that, since the conformal weights of the fields into which a fixed point is resolved differ only by integers, one has in fact $\mathcal{T}^{[\omega]}_{[M;M']} = \mathcal{T}_{[M;M']}$. Also, one checks by inspection that for all diagram automorphisms of affine Lie algebras $\mathcal{C}^{[\omega]}_{[\Lambda;\Lambda'],[N;N']} = \mathcal{C}_{[\Lambda;\Lambda'],[N;N']}$ is independent of ω, too. Therefore, we have formulæ analogous to (B.4) – (B.6) in which only $\mathcal{S}$ is replaced by $\mathcal{S}^{[\omega]}$.

We now combine the equations (B.4), (B.5) and (B.6) and their analogues for $\mathcal{S}^{[\omega]}$ instead of $\mathcal{S}$ with the following two identities for the characters of an abelian group. First,

$$\sum_{\Psi \in G^\star_{\Lambda,\Lambda'}} \Psi(\omega) = |G_{\Lambda,\Lambda'}| \cdot \delta_{\omega,e}\,, \tag{B.7}$$

where e denotes the unit element of the group $G_{\Lambda,\Lambda'} \subseteq G_{\mathrm{id}}$. Second, for any $\Psi \in G^\star_{\Lambda,\Lambda'}$ and any $\tilde{\Psi} \in G^\star_{M,M'}$ one has

$$\sum_{\omega \in G_{\mathrm{id}}} \Psi^*(\omega)\, \tilde{\Psi}(\omega) = \sum_{\omega \in G_{\Lambda,\Lambda'} \cap G_{M,M'}} \Psi^*(\omega)\, \tilde{\Psi}(\omega) = |G_{\Lambda,\Lambda'} \cap G_{M,M'}| \cdot \delta_{\Psi,\tilde{\Psi}}\,. \tag{B.8}$$

To show that the S-matrix is symmetric, we take the transpose of (B.1). Observing that $\dot{\omega}$ and $\dot{\omega}^{-1}$ yield the same orbit Lie algebra so that

$$\mathcal{S}^{[\omega^{-1}]}_{(\Lambda;\Lambda'),(M;M')} = \mathcal{S}^{[\omega]}_{(\Lambda;\Lambda'),(M;M')}\,, \tag{B.9}$$

and that this is a symmetric matrix, we get

$$\mathcal{S}_{([M;M'],\Psi),([\Lambda;\Lambda'],\tilde{\Psi})} = \frac{|G_{\mathrm{id}}|}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|} \sum_{\omega \in G_{\mathrm{id}}} \tilde{\Psi}^*(\omega)\, \mathcal{S}^{[\omega^{-1}]}_{[\Lambda;\Lambda'],[M;M']}\, \Psi(\omega)\,. \tag{B.10}$$

Using the property $\Psi(\omega^{-1}) = \Psi^*(\omega)$ of group characters, the symmetry property then follows by summing over ω^{-1} instead of over ω.

To show that the S-matrix (4.34) is unitary, we calculate

$$\sum_{[M;M']} \sum_{\Psi_2 \in G^\star_{M,M'}} \mathcal{S}_{([\Lambda;\Lambda'],\Psi_1),([M;M'],\Psi_2)} \left(\mathcal{S}_{([M;M'],\Psi_2),([N;N'],\Psi_3)} \right)^*$$

$$= \sum_{[M;M']} \frac{|G_{\mathrm{id}}|^2}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|^2 \cdot |G_{N,N'}|} \sum_{\Psi_2 \in G^\star_{M,M'}} \sum_{\omega_1,\omega_2 \in G_{\mathrm{id}}} \mathcal{S}^{[\omega_1]}_{[\Lambda;\Lambda'],[M;M']} \left(\mathcal{S}^{[\omega_2]}_{[M;M'],[N;N']} \right)^*$$

$$\times\ \Psi_1^*(\omega_1)\Psi_2(\omega_1)\Psi_2(\omega_2)\Psi_3^*(\omega_2)$$

$$= \sum_{[M;M']} \frac{|G_{\mathrm{id}}|^2}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}| \cdot |G_{N,N'}|} \sum_{\omega_1 \in G_{\mathrm{id}}} \mathcal{S}^{[\omega_1]}_{[\Lambda;\Lambda'],[M;M']} \left(\mathcal{S}^{[\omega_1]}_{[M;M'],[N;N']} \right)^* \Psi_1^*(\omega_1)\Psi_3^*(\omega_1^{-1})$$

$$= \frac{1}{|G_{N,N'}|} \sum_{\omega_1 \in G_{\mathrm{id}}} \delta_{[\Lambda;\Lambda'],[N;N']} \Psi_1^*(\omega_1)\Psi_3(\omega_1)$$

$$= \delta_{[\Lambda;\Lambda'],[N;N']}\, \delta_{\Psi_1,\Psi_3} \, .$$

$$\tag{B.11}$$

Here we used the analogue of the relation (B.4); a parallel computation using (B.5) shows that

$$\sum_{[M;M']} \sum_{\Psi_2 \in G^\star_{M,M'}} \mathcal{S}_{([\Lambda;\Lambda'],\Psi_1),([M;M'],\Psi_2)} \mathcal{S}_{([M;M'],\Psi_2),([N;N'],\Psi_3)} = \mathcal{C}_{[(\Lambda;\Lambda')],[(N;N')]}\, \delta_{\Psi_1,\Psi_3} \, . \tag{B.12}$$

Here $\mathcal{C}_{[(\Lambda;\Lambda')],[(N;N')]}$ is the charge conjugation matrix of the coset theory; the matrix $\mathcal{C}$ is manifestly a permutation of order two of the primary fields which leaves the identity primary field $[0;0]$ fixed. Thus charge conjugation acts in a trivial way on the labels which describe the fixed point resolution.

Our final check concerns the relation $(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2$, or equivalently, $\mathcal{S}\mathcal{T}\mathcal{S} = \mathcal{T}^{-1}\mathcal{S}\mathcal{T}^{-1}$. Using (B.6) we have

$$\sum_{[M;M']} \sum_{\Psi_2 \in G^\star_{M,M'}} \mathcal{S}_{([\Lambda;\Lambda'],\Psi_1),([M;M'],\Psi_2)} \mathcal{T}_{[M;M']} \mathcal{S}_{([M;M'],\Psi_2),([N;N'],\Psi_3)}$$

$$= \sum_{[M;M']} \frac{|G_{\mathrm{id}}|^2}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}|^2 \cdot |G_{N,N'}|} \sum_{\Psi_2 \in G^\star_{M,M'}}$$

$$\times \sum_{\omega_1,\omega_2 \in G_{\mathrm{id}}} \mathcal{S}^{[\omega_1]}_{[\Lambda;\Lambda'],[M;M']} \mathcal{T}_{[M;M']} \mathcal{S}^{[\omega_2]}_{[M;M'],[N;N']}\, \Psi_1^*(\omega_1)\Psi_2(\omega_1)\Psi_2^*(\omega_2)\Psi_3(\omega_2)$$

$$= \sum_{[M;M']} \frac{|G_{\mathrm{id}}|^2}{|G_{\Lambda,\Lambda'}| \cdot |G_{M,M'}| \cdot |G_{N,N'}|} \cdot \sum_{\omega_1 \in G_{\mathrm{id}}} \mathcal{S}^{[\omega_1]}_{[\Lambda;\Lambda'],[M;M']} \mathcal{T}_{[M;M']} \mathcal{S}^{[\omega_1]}_{[M;M'],[N;N']}\, \Psi_1^*(\omega_1)\Psi_3(\omega_1)$$

$$= \frac{|G_{\mathrm{id}}|}{|G_{\Lambda,\Lambda'}| \cdot |G_{N,N'}|} \sum_{\omega_1 \in G_{\mathrm{id}}} \mathcal{T}^{-1}_{(\Lambda;\Lambda')} \mathcal{S}^{[\omega_1]}_{(\Lambda;\Lambda'),(N;N')} \mathcal{T}^{-1}_{(N;N')}\, \Psi_1^*(\omega_1)\Psi_3(\omega_1)$$

$$= \mathcal{T}^{-1}_{[\Lambda;\Lambda']}\, \mathcal{S}_{([\Lambda;\Lambda'],\Psi_1),([N;N'],\Psi_3)}\, \mathcal{T}^{-1}_{[N;N']} \, .$$

$$\tag{B.13}$$

In summary, for any coset conformal field theory in which the identification group G_{id} possesses the factorization property we obtain a unitary representation of SL(2,$\mathbf{Z}$), with $\mathcal{S}$ symmetric and $\mathcal{T}$ diagonal.

Let us also mention that we do not yet have a general proof that inserting the formula (B.1) into the Verlinde formula yields non-negative integral fusion coefficients. However, we have

checked in many examples that this further condition for the consistency of the coset conformal field theory is indeed fulfilled.

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