

ITP-UH-18/93



June 1993

# $N = 2$ Super-Weyl Symmetry, Super-Liouville Theory and Super-Riemannian Surfaces <sup>1</sup>

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## Abstract

The finite form of the  $N = 2$  super-Weyl transformations in the chiral and twisted-chiral irreducible formulations of the two-dimensional  $N = 2$  superfield supergravity are found in  $N = 2$  superspace. The super-Weyl anomaly of the  $N = 2$  extended fermionic string theory is computed in terms of the  $N = 2$  superfields, by using a short time expansion of the  $N = 2$  chiral heat kernel. The super-Weyl invariant  $N = 2$  superconformal structure is introduced, and a new definition of the  $N = 2$  super-Riemannian surfaces is proposed.

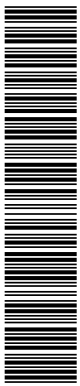
*Intern. Physics Classification #’s* 0465, 1117, 1130

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<sup>1</sup>Supported in part by the 'Deutsche Forschungsgemeinschaft'

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<sup>3</sup>Supported in part by the 'Studienstiftung des deutschen Volkes'



# 1 Introduction

Since the work of Polyakov [1], much attention has been paid to the two-dimensional quantum gravity and its supersymmetric extensions. Their better understanding is crucial for getting more insights into the structure of (super)-conformal field theories, and critical or non-critical (super)-string models formulated on the (super)-Riemannian surfaces.

The two-dimensional supergravities can be formulated and investigated either in components or in superfields. Each approach has its own obvious advantages and disadvantages, and they are always complementary to each other. The  $N = 1$  supergravity in two dimensions has been investigated in detail, both in components and in superfields [2], and its applications to the  $N = 1$  fermionic string theory (also called the NSR model) are well-known [3].<sup>4</sup> As for the  $N = 2$  or  $(2, 2)$  two-dimensional supergravity, most of its applications (see, e.g., refs. [5, 6, 7] for the critical  $N = 2$  strings and refs. [8, 9] for the non-critical  $N = 2$  strings) have been carried out in components or in the so-called  $N = 2$  superconformal gauge, despite of the known formulations of this theory in the full-fledged  $N = 2$  curved superspace [10, 11, 12, 13]. This seems to be related to the fact that some of the relevant elements of the  $N = 2$  superspace description of  $N = 2$  supergravity are yet to be completed. In particular, we believe it to be certainly true for the *finite* form of the relevant  $N = 2$  super-Weyl transformations and a calculation of the  $N = 2$  super-Weyl anomaly in terms of the  $N = 2$  superfields. To the best of our knowledge, it is apparently missing in the literature. Although the full  $N = 2$  superspace description of the  $N = 2$  supergravity in the  $N = 2$  (curved) superspace is highly redundant, knowing the finite form of the  $N = 2$  super-Weyl transformations is important to set up the invariant definition of the  $N = 2$  super-Riemannian surfaces and the  $N = 2$  superconformal gauge. It also matters in establishing the  $N = 2$  generalisation of the uniformisation theorem playing the crucial role in the theory of the (super)-Riemannian surfaces. The use of the  $N = 2$  superspace is the best way to uncover the existence of several different versions of the  $N = 2$  supergravity [10, 11, 12, 13], which is very obscure in the component approach.

Our paper is organised as follows. In sect. 2 we formulate the chiral version of  $N = 2$  supergravity in superspace and calculate the finite  $N = 2$  superfield form of its  $N = 2$  super-Weyl symmetry transformations. In sect. 3, the similar results are obtained for the twisted-chiral formulation of the theory. The  $N = 2$  fermionic string

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<sup>4</sup>See ref. [4] for the 'heterotic' case of the  $(1, 0)$  supergravity.

action in terms of the  $N = 2$  superfields is discussed in sect. 4. The covariant  $N = 2$  superconformal structure is used to define the  $N = 2$  super-Riemannian surfaces. The related definitions of the  $N = 2$  super-Teichmüller and super-moduli spaces are given in that section too. Sect. 5 is devoted to the computation of the  $N = 2$  super-Weyl anomaly in  $N = 2$  superspace. It results in the  $N = 2$  super-Liouville effective theory, as it should. Sect. 6 comprises our conclusion.

## 2 Complex Supergeometry in Superspace

When dealing with spinors on a string world-sheet, one should take into account the delicate relation which exists between their descriptions on the Minkowski and Euclidean world-sheets, and the associated spin structure. The Majorana-Weyl (MW) spinors can only be introduced in Minkowski space, while defining the super-Riemann surfaces (SRS) is based on the Euclidean formulation. That's why we find appropriate to start with the  $N = 2$  supergeometry by using the Minkowski signature, and formally stick to the Euclidean formulation when introducing the  $N = 2$  SRS. Keeping in use both formulations is also important for holomorphic factorisation [3]. There is in general a topological obstruction to introduce spinors on a given (Euclidean) manifold  $\Xi$ , in order to make possible a consistent choice of spin structure. Namely, one should have  $w_2(\Xi) = 0$ , where  $w_2$  is the second Stiefel-Whitney class [14, 15]. Since the  $w_2(\Xi)$  vanishes for the *oriented* surfaces  $\Xi$ , they will be the only ones we are going to consider.

Supersymmetric theories can be handled either in components or in superfields. To construct the correct quantum measure and analyse the anomaly structure of the  $N = 2$  superstrings, it is quite appropriate to use the superfield formalism where the  $N = 2$  supersymmetry on the world-sheet is manifest. The natural setting is provided by the  $N = 2$  (left-right symmetric) curved superspace of the  $(2, 2)$  supergravity in two dimensions [10, 11]. The formal construction of the corresponding  $N = 2$  supermoduli space and  $N = 2$  SRS goes along the lines of the conventional  $N = 1$  case [3, 16], the important differences being emphasized below. As for discussing global (or topological) issues, the component approach seems to be more appropriate.

The  $N = 2$  (flat) superspace coordinates in two dimensions are

$$z^A = (x^a, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) , \quad a = 0, 1, \quad \alpha = +, - , \quad (2.1)$$

where  $\theta$ 's represent two Grassmannian (anticommuting) complex spinor coordinates,

$(\theta^\alpha)^\dagger = \bar{\theta}^{\dot{\alpha}}$ . The spinorial covariant derivatives in the flat  $N = 2$  superspace are

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\bar{\theta}^{\dot{\alpha}}(\gamma^a)_{\alpha\dot{\alpha}} \partial_a , \\ \bar{D}_{\dot{\alpha}} &= \bar{\partial}_{\dot{\alpha}} + i\theta^\alpha(\gamma^a)_{\alpha\dot{\alpha}} \partial_a , \end{aligned} \quad (2.2)$$

and their conjugates, where  $\partial_a = \partial/\partial x^a$ ,  $\partial_\alpha = \partial/\partial\theta^\alpha$  and  $\bar{\partial}_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$ . They satisfy an algebra

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i(\gamma^a)_{\alpha\dot{\alpha}} \partial_a , \quad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 . \quad (2.3)$$

The *curved*  $N = 2$  superspace can be described by the superzweibein  $E_M^A$  with its inverse  $E_A^M$ , the spin superconnection  $\omega_A$  and the  $U_V(1)$  superconnection  $\varrho_A$ . Such formulation is based on the structure group  $SO(1,1) \times U_V(1)$  in the tangent space,<sup>5</sup> and leads to the *chiral* (irreducible) version of the  $N = 2$  supergravity with  $4 + 4$  (off-shell) components corresponding to a chiral  $N = 2$  scalar superfield as a compensator [11, 12]. A very natural but naive choice of the  $N = 2$  superspace with an  $SO(1,1) \times U_V(1) \times U_A(1)$  tangent space group leads to a *reducible* supergravity multiplet with  $8 + 8$  off-shell degrees of freedom comprising a real scalar  $N = 2$  superfield, as was shown in ref. [10]. Taking  $SO(1,1) \times U_A(1)$  as a structure group is another option, which also results in the irreducible  $4 + 4$  off-shell supergravity multiplet, corresponding to the two-dimensional *twisted-chiral*  $N = 2$  superfield as a compensator [10, 12, 13]. However, the anomaly structure of this version of the  $N = 2$  superfield supergravity is expected to be more complicated.

Given the superzweibein and superconnections, the curved superspace covariant derivative can be defined as

$$\nabla_A = E_A^M D_M + \omega_A \mathcal{X} + i\varrho_A \mathcal{Y} \equiv E_A + \Omega_A , \quad (2.4)$$

where  $D_M$  is the rigid (flat) superspace covariant derivative  $D_M = (\partial_m, \partial_\mu, \bar{\partial}_{\dot{\mu}})$  introduced above,  $\mathcal{X}$  and  $\mathcal{Y}$  are the Lorentz and  $U_V(1)$  symmetry generators, respectively,

$$\begin{aligned} [\mathcal{X}, O_a] &= \varepsilon_a^b O_b , \quad [\mathcal{X}, O_\alpha] = \frac{1}{2}(\gamma_3)_\alpha^\beta O_\beta , \\ [\mathcal{Y}, O_a] &= 0 , \quad [\mathcal{Y}, O_\alpha] = \frac{1}{2}i(\gamma_3)_\alpha^\beta O_\beta . \end{aligned} \quad (2.5)$$

The supertorsion  $T_{AB}^C$ , supercurvature  $R_{AB}$  and superfield strength  $F_{AB}$  are defined by the (graded) commutations of the covariant derivatives,

$$[\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + R_{AB} \mathcal{X} + iF_{AB} \mathcal{Y} , \quad (2.6)$$

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<sup>5</sup> In two dimensions a general  $(p, q)$ -supersymmetry algebra may have  $p(q)$  left(right)-handed Majorana supersymmetry charges, so that the internal symmetry group is  $SO(p) \times SO(q)$ , in general. When  $p = q = N = 2$ , one has  $U_V(1) \times U_A(1)$ .

and they are all tensors with respect to the reparametrisations of the  $N = 2$  superspace coordinates.

The curved  $N = 2$  superspace geometry is highly reducible even off-shell, and it is too general to describe  $N = 2$  SRS. The off-shell supergravity in superspace is actually described by some constraints on the supertorsion, which reduce a number of the independent components to the minimal one [17]. In this respect, the two-dimensional  $N = 2$  supergravity is quite similar to its four-dimensional  $N = 1$  supergravity counterpart, where the relevant constraints on the superspace torsion comprise the so-called 'conventional' constraints, the '(chiral) representation-preserving' constraints and the 'conformal' ones [17]. The latter are just necessary to reduce the super-Weyl parameter (see below) to the irreducible  $N = 2$  superfield [10]. As a result of the constraints and the subsequent corollaries from the superspace Bianchi identities, all the torsion, curvature and field-strength tensor superfields can be expressed in terms of a smaller number of (generically constrained) superfields. As for the *chiral* version of the  $N = 2$  supergravity theory,<sup>6</sup> the relevant outcome for the (anti)-commutation relations (2.6) is given by [11, 13]

$$\begin{aligned}
\{\nabla_\alpha, \nabla_\beta\} &= 2(\gamma_3)_{\alpha\beta} \bar{R}(\mathcal{X} + i\mathcal{Y}) , \\
\{\nabla_\alpha, \bar{\nabla}_{\dot{\beta}}\} &= 2i(\gamma^c)_{\alpha\dot{\beta}} \dot{\nabla}_c , \\
[\nabla_\alpha, \nabla_b] &= \frac{1}{2}i(\gamma_b)_\alpha^{\dot{\beta}} \left[ \bar{R} \bar{\nabla}_{\dot{\beta}} - (\gamma_3)_{\dot{\beta}}^{\dot{\gamma}} (\bar{\nabla}_{\dot{\gamma}} \bar{R})(\mathcal{X} + i\mathcal{Y}) \right] , \\
[\nabla_a, \nabla_b] &= \frac{1}{4}\varepsilon_{ab} \left\{ 2(\gamma_3)^{\alpha\beta} (\nabla_\alpha R) \nabla_\beta + 2(\gamma_3)^{\dot{\alpha}\dot{\beta}} (\bar{\nabla}_{\dot{\alpha}} \bar{R}) \bar{\nabla}_{\dot{\beta}} \right. \\
&\quad \left. + [\nabla^2 R + \bar{\nabla}^2 \bar{R} - 4R\bar{R}] \mathcal{X} - i[\nabla^2 R - \bar{\nabla}^2 \bar{R}] \mathcal{Y} \right\} . \tag{2.7}
\end{aligned}$$

In eq. (2.7) all the tensor-component  $N = 2$  superfields of the torsion, curvature and field-strength are expressed in terms of a single scalar  $N = 2$  complex superfield  $R$ . The first line of eq. (2.7) implies that this superfield is (covariantly) chiral,

$$\bar{\nabla}_{\dot{\alpha}} R = \nabla_\alpha \bar{R} = 0 . \tag{2.8}$$

Having imposed the supertorsion constraints, one can express, as usual, the superconnections in terms of the superzweibein. For instance, the constraint  $T_{\alpha\beta}^{\dot{\gamma}} = E_\beta^M E_\alpha^N (\nabla_N \bar{E}_M^{\dot{\gamma}} - \nabla_M \bar{E}_N^{\dot{\gamma}}) = 0$  implies

$$(\omega_M \pm \varrho_M) = (\gamma_3 \pm 1)_{\dot{\gamma}}^{\dot{\beta}} \bar{E}_{\dot{\beta}}^N \left( D_N \bar{E}_M^{\dot{\gamma}} - D_M \bar{E}_N^{\dot{\gamma}} \right) . \tag{2.9}$$

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<sup>6</sup>The different *twisted-chiral* version of the  $N = 2$  supergravity is considered in the next section.

The complex supergeometry of the  $N = 2$  supergravity is invariant under those transformations which preserve the constraints. One usually finds convenient to look at the infinitesimal variations first, in the form  $H_A{}^B = E_A{}^M \delta E_M{}^B$  and  $\phi_{A,B}{}^C = E_A{}^M \delta \Omega_{M,B}{}^C$ , where  $\Omega$  is the total connection,  $\Omega_A = \omega_A \mathcal{X} + i \varrho_A \mathcal{Y}$ . The corresponding variation of the supertorsion components reads [10]

$$\begin{aligned} \delta T_{AB}{}^C &= \nabla_A H_B{}^C - (-1)^{AB} \nabla_B H_A{}^C + T_{AB}{}^D H_D{}^C \\ &- H_A{}^D T_{DB}{}^C + (-1)^{AB} H_B{}^D T_{DA}{}^C + \phi_{A,B}{}^C - (-1)^{AB} \phi_{B,A}{}^C . \end{aligned} \quad (2.10)$$

Since the supertorsion constraints, not all of the  $H$ 's are actually independent. The  $H_\alpha{}^b$ ,  $H \equiv H_\alpha{}^\alpha$ ,  $(\gamma_3 H) \equiv (\gamma_3)_\alpha{}^\beta H_\beta{}^\alpha$  and their complex conjugates can be chosen to represent a complete set of the independent ones.

By construction, the super-reparametrisational, super-Lorentz and super-phase  $U_V(1)$  local transformations in the  $N = 2$  superspace are always among the symmetries of the theory, and we are not going to discuss them in any detail (see, however, refs. [10, 13]). Instead, we want to concentrate on another local symmetry which is closely related to two bosonic dimensions we are working in, and is crucial for the  $N = 2$  fermionic strings, namely the  $N = 2$  *super-Weyl* invariance. Taken together, the local symmetries are enough to gauge away all the  $N = 2$  supergravity fields in each given coordinate patch, but not globally. The fact that the complex supergeometry of the  $N = 2$  supergravity is *superconformally flat* was first noticed by Howe and Papadopoulos [10], but, to the best of our knowledge, the *finite*  $N = 2$  super-Weyl transformations in the irreducible formulations of the  $N = 2$  supergravity in  $N = 2$  superspace were not calculated.

In their infinitesimal form, the  $N = 2$  super-Weyl transformations are [10]

$$H_a{}^b = \delta_a{}^b (\lambda + \bar{\lambda}) , \quad H_\alpha{}^\beta = \delta_\alpha{}^\beta \bar{\lambda} , \quad H_a{}^{\dot{\beta}} = \frac{i}{2} (\gamma_a)^{\dot{\beta}\gamma} \nabla_\gamma \lambda , \quad (2.11)$$

where  $\lambda$  is an infinitesimal *chiral*  $N = 2$  superfield parameter. It follows from eq. (2.10) that the symmetry transformations (2.11) are the invariance of the constraints (2.7).

It is convenient to use  $N = 2$  super-differential forms here, in terms of which the superzweibein takes the form  $E^A = dz^M E_M{}^A = (E^a, E^\alpha, \bar{E}^{\dot{\alpha}})$ , and similarly for the derivatives. The supertorsion and the superspace Bianchi identities can then be conveniently represented as [10]

$$T^C = \frac{1}{2} E^B E^A T_{AB}{}^C = D E^C + E^B \Omega_B{}^C \equiv \nabla E^A , \quad (2.12)$$

and

$$\nabla T^A = E^B R_B{}^A , \quad D F = 0 , \quad (2.13)$$

respectively, where  $R_A{}^B = D\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B$ ,  $R_a{}^b = \varepsilon_a{}^b F$  and  $R_\alpha{}^\beta = \frac{1}{2}(\gamma_3)_\alpha{}^\beta F$ .

The form of eq. (2.11) suggests the following ansatz for the finite  $N = 2$  super-Weyl transformations with the chiral  $N = 2$  superfield parameter  $\Lambda$

$$\begin{aligned}\hat{E}^a &= S E^a, \quad \text{where } S \equiv \Lambda \bar{\Lambda}, \\ \hat{E}^\alpha &= \bar{\Lambda} E^\alpha + E^c (\gamma_c)^{\alpha\dot{\beta}} \bar{\rho}_{\dot{\beta}}, \\ \hat{\bar{E}}^{\dot{\alpha}} &= \Lambda \bar{E}^{\dot{\alpha}} + E^c (\gamma_c)^{\dot{\alpha}\beta} \rho_\beta,\end{aligned}\tag{2.14}$$

which some superfields  $\rho_\beta, \bar{\rho}_{\dot{\beta}}$  to be determined by evaluating the supertorsion components  $\hat{T}^c$  in the two different ways. First, their definition according to eq. (2.12) in terms of the  $\hat{E}^A$  and the (yet unknown)  $\hat{\omega}$  and  $\hat{\varrho}$  implies

$$\begin{aligned}\hat{T}^c &= \frac{1}{2} \hat{E}^B \hat{E}^A T_{AB}{}^c = S T^c + S [(\hat{\omega} - \omega)\mathcal{X} + i(\hat{\varrho} - \varrho)\mathcal{Y}] E^c + (\nabla S) E^c \\ &= i S \bar{E}^{\dot{\beta}} E^\alpha (\gamma^c)_{\alpha\dot{\beta}} + S E^b \varepsilon_b{}^c \Upsilon + (\nabla S) E^c,\end{aligned}\tag{2.15a}$$

where  $[(\hat{\omega} - \omega)\mathcal{X} + i(\hat{\varrho} - \varrho)\mathcal{Y}] E^c \equiv E^b \varepsilon_b{}^c \Upsilon$ . Second, the ansatz (2.14) yields another equation for the same tensor  $\hat{T}^c$ , and it is consistent with eq. (2.15a) since it appears to have the same structure, namely

$$\hat{T}^c = i(\gamma^c)_{\alpha\dot{\beta}} \left[ S \bar{E}^{\dot{\beta}} E^\alpha + \Lambda \bar{E}^{\dot{\beta}} E^a (\gamma_a)^{\alpha\dot{\delta}} \bar{\rho}_{\dot{\delta}} + \bar{\Lambda} E^\alpha E^a (\gamma_a)^{\dot{\beta}\gamma} \rho_\gamma \right].\tag{2.15b}$$

Comparing the coefficients at  $E^c$  and  $E^d \varepsilon_d{}^c$  in eq. (2.15) gives rise to the equations

$$\begin{aligned}\nabla S &= i \Lambda \bar{E}^{\dot{\beta}} \bar{\rho}_{\dot{\beta}} + i \bar{\Lambda} E^\alpha \rho_\alpha, \\ S \Upsilon &= -i \Lambda \bar{E}^{\dot{\beta}} (\gamma_3 \bar{\rho})_{\dot{\beta}} - i \bar{\Lambda} E^\alpha (\gamma_3 \rho)_\alpha.\end{aligned}\tag{2.16}$$

The first line of eq. (2.16) fixes the  $\rho_\alpha$  and  $\bar{\rho}_{\dot{\alpha}}$ . Hence, the  $N = 2$  super-Weyl transformation of the superzweibein is given by

$$\begin{aligned}\hat{E}^a &= S E^a, \quad S = \Lambda \bar{\Lambda}, \\ \hat{E}^\alpha &= \bar{\Lambda} E^\alpha - i E^b (\gamma_b)^{\alpha\dot{\beta}} \bar{\nabla}_{\dot{\beta}} \bar{\Lambda}, \\ \hat{\bar{E}}^{\dot{\alpha}} &= \Lambda \bar{E}^{\dot{\alpha}} - i E^b (\gamma_b)^{\dot{\alpha}\beta} \nabla_\beta \Lambda.\end{aligned}\tag{2.17}$$

Since  $E_A{}^M E_M{}^B = \delta_A{}^B$  and similarly for the superzweibein components with hats, it follows from eq. (2.17) that the inverse superzweibein transforms as

$$\hat{E}_a = S^{-1} E_a + i S^{-1} \bar{\Lambda}^{-1} (\gamma_a)^{\beta\dot{\gamma}} (\bar{\nabla}_{\dot{\gamma}} \bar{\Lambda}) E_\beta + i S^{-1} \Lambda^{-1} (\gamma_a)^{\dot{\beta}\gamma} (\nabla_\gamma \Lambda) \bar{E}_{\dot{\beta}},$$

$$\begin{aligned}\hat{E}_\alpha &= \bar{\Lambda}^{-1} E_\alpha , \\ \hat{\bar{E}}_{\dot{\alpha}} &= \Lambda^{-1} \bar{E}_{\dot{\alpha}} .\end{aligned}\tag{2.18}$$

Notably, the  $N = 2$  super-Weyl transformation of the spinor components of the inverse superzweibein in eq. (2.18) is very simple.

The super-Weyl transformation of the spinor components of the Lorentz superconnection follows from the second line of eq. (2.16), and it takes the form

$$\hat{\omega}_\alpha = \bar{\Lambda}^{-1} \omega_\alpha - \bar{\Lambda}^{-1} \Lambda^{-1} (\gamma_3)_\alpha{}^\gamma \nabla_\gamma \Lambda .\tag{2.19}$$

We have checked that the same result also follows by exploiting the explicit form of dependence of the Lorentz superconnection upon the superzweibein, i.e. by using eqs. (2.9), (2.17) and (2.18). Similarly, one finds

$$\hat{\varrho}_\alpha = \bar{\Lambda}^{-1} \varrho_\alpha - \bar{\Lambda}^{-1} \Lambda^{-1} (\gamma_3)_\alpha{}^\gamma \nabla_\gamma \Lambda .\tag{2.20}$$

The super-Weyl transformations of the vector components of the superconnections are more complicated, but, fortunately, we don't need them in what follows.

The  $N = 2$  super-Weyl transformations of the spinorial covariant derivatives in the  $N = 2$  superspace are straightforward to calculate, since they are the direct corollaries of the transformation rules given above. We find

$$\hat{\nabla}_\alpha = \bar{\Lambda}^{-1} \nabla_\alpha - \bar{\Lambda}^{-1} \Lambda^{-1} (\gamma_3)_\alpha{}^\gamma (\nabla_\gamma \Lambda) (\mathcal{X} + i\mathcal{Y}) .\tag{2.21}$$

The first line of eq. (2.7) then implies the transformation law for the anti-chiral superfield  $\bar{R}$  in the form

$$\hat{\bar{R}} = \bar{\Lambda}^{-2} \left[ \bar{R} - 2\Lambda^{-1} \nabla^2 \Lambda + 2\Lambda^{-2} (\nabla^\alpha \Lambda) (\nabla_\alpha \Lambda) \right] ,\tag{2.22a}$$

or, equivalently,

$$\hat{\bar{R}} = e^{-2\bar{\Sigma}} \left( \bar{R} - 2\bar{\nabla}^2 \Sigma \right) , \quad \hat{R} = e^{-2\Sigma} \left( R - 2\nabla^2 \bar{\Sigma} \right) ,\tag{2.22b}$$

where the chiral  $N = 2$  superfield  $\Sigma$  has been introduced,  $\Lambda \equiv \exp(\Sigma)$ . Eqs. (2.18) and (2.22) comprise the main results of this section. It should also be noticed that the superdeterminant <sup>7</sup>  $E \equiv \text{sdet}(E_M{}^A)$  is  $N = 2$  super-Weyl invariant,  $\hat{E} = E$  or  $\delta E = 0$ , which is easily verified. Finally, as for the  $N = 2$  chiral density  $\mathcal{E}$  to be defined as  $\mathcal{E} \equiv -\frac{1}{2} (\bar{\nabla}^2 - 4R) E$ , we get

$$\hat{\mathcal{E}} = e^{-2\Sigma} \left[ \mathcal{E} - 4(\bar{\nabla}^2 \bar{\Sigma}) E \right] , \quad \mathcal{E} = e^{2\Sigma} \hat{\mathcal{E}} + 4(\bar{\nabla}^2 \bar{\Sigma}) \hat{E} .\tag{2.23}$$

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<sup>7</sup>As for the definition of the superdeterminant, see eqs. (A.9) and (A.10) of the Appendix.



### 3 Twisted $N = 2$ Superfield Supergravity

The different set of the two-dimensional  $N = 2$  supergravity constraints in  $N = 2$  superspace can be obtained by truncating the four-dimensional  $N = 1$  superfield supergravity down to two dimensions. The alternative set of the two-dimensional  $N = 2$  supergravity constraints takes the form [11, 13]

$$\begin{aligned}
\{\nabla_\alpha, \nabla_\beta\} &= 0 , \\
\{\nabla_\alpha, \bar{\nabla}_\beta\} &= i(\gamma^c)_{\alpha\beta} \cdot \nabla_c + \left( iC_{\alpha\beta} \cdot H + (\gamma_3)_{\alpha\beta} \cdot G \right) \mathcal{X} - \left( C_{\alpha\beta} \cdot G + i(\gamma_3)_{\alpha\beta} \cdot H \right) \mathcal{Y}_t , \\
[\nabla_\alpha, \nabla_b] &= \frac{1}{2} \left[ i(\gamma_b)_\alpha{}^\beta G - (\gamma_3 \gamma_b)_\alpha{}^\beta H \right] \nabla_\beta + i(\gamma_3 \gamma_b)_\alpha{}^\beta (\nabla_\beta G) \mathcal{X} - i(\gamma_b)_\alpha{}^\beta (\nabla_\beta G) \mathcal{Y}_t , \\
[\nabla_a, \nabla_b] &= -\varepsilon_{ab} \left[ (\bar{\nabla}^\gamma G)(\gamma_3)_\gamma{}^\lambda \nabla_\lambda + (\nabla^\lambda G)(\gamma_3)_\lambda{}^\gamma \bar{\nabla}_\gamma \right. \\
&\quad \left. - \left( C^{\alpha\beta} \nabla_\alpha \bar{\nabla}_\beta \cdot G - G^2 - H^2 \right) \mathcal{X} - (\gamma_3)^{\alpha\beta} \left( \nabla_\alpha \bar{\nabla}_\beta \cdot G \right) \mathcal{Y}_t \right] , \tag{3.1}
\end{aligned}$$

where the new  $U(1)$  generator  $\mathcal{Y}_t$  has been introduced,

$$[\mathcal{Y}, O_a] = 0 , \quad [\mathcal{Y}, O_\alpha] = \frac{1}{2} i O_\alpha . \tag{3.2}$$

In this new  $N = 2$  supergravity theory all the supertorsion, supercurvature and superconnection components depend on the two real scalar superfields  $G$  and  $H$ , which can be combined into the single complex  $N = 2$  superfield to be equivalent to a twisted chiral  $N = 2$  scalar superfield [18]. It should be noticed that both (chiral and twisted-chiral) versions of  $N = 2$  supergravity are derivable by truncating the off-shell formulation of the two-dimensional  $N = 4$  supergravity theory of ref. [11].

The infinitesimal  $N = 2$  super-Weyl transformations in the twisted superfield formulation of the  $N = 2$  supergravity theory take the form

$$\begin{aligned}
H_a{}^b &= \delta_a{}^b \left( \Lambda + \bar{\Lambda} \right) , \\
H_\alpha{}^\beta &= \frac{1}{2} \left( \delta_\alpha{}^\beta - (\gamma_3)_\alpha{}^\beta \right) \Lambda + \frac{1}{2} \left( \delta_\alpha{}^\beta + (\gamma_3)_\alpha{}^\beta \right) \bar{\Lambda} , \\
H_a{}^\beta &= i(\gamma_a)^\beta{}_\gamma \nabla^\gamma \left[ \frac{1}{2} \left( \delta_\delta{}^\beta + (\gamma_3)_\delta{}^\beta \right) \Lambda + \frac{1}{2} \left( \delta_\delta{}^\beta - (\gamma_3)_\delta{}^\beta \right) \bar{\Lambda} \right] , \tag{3.3}
\end{aligned}$$

which is different from that given in ref. [10]. We have checked that the supertorsion constraints in eq. (3.1) are invariant with respect to our  $N = 2$  super-Weyl transformation laws in eq. (3.3). The  $N = 2$  super-Weyl parameter in eq. (3.3) is supposed to be a twisted-chiral superfield,  $\frac{1}{2}(1 + \gamma_3)_\alpha{}^\beta \nabla_\beta \Lambda = \frac{1}{2}(1 - \gamma_3)_\alpha{}^\beta \bar{\nabla}_\beta \bar{\Lambda} = 0$ .

A derivation of the finite form of the  $N = 2$  super-Weyl transformations follows the lines of the chiral case already considered in the previous section. It results in

$$\hat{E}_M{}^a = E_M{}^a S, \quad \text{where } S \equiv \Lambda \bar{\Lambda}, \quad (3.4)$$

$$\begin{aligned} \hat{E}_M{}^\alpha &= \frac{1}{2} \left( E_M{}^\beta - 2i E_M{}^a (\gamma_a)^\beta{}_\gamma S^{-1} \bar{\nabla}_\gamma S \right) \left[ (\delta_\beta{}^\alpha - (\gamma_3)_\beta{}^\alpha) \Lambda + (\delta_\beta{}^\alpha + (\gamma_3)_\beta{}^\alpha) \bar{\Lambda} \right], \\ \hat{E}_M{}^{\dot{\alpha}} &= \frac{1}{2} \left( \bar{E}_M{}^{\dot{\beta}} - 2i E_M{}^a (\gamma_a)^{\dot{\beta}}{}_\gamma S^{-1} \nabla_\gamma S \right) \left[ \left( \delta_\beta{}^{\dot{\alpha}} + (\gamma_3)_\beta{}^{\dot{\alpha}} \right) \Lambda + \left( \delta_\beta{}^{\dot{\alpha}} - (\gamma_3)_\beta{}^{\dot{\alpha}} \right) \bar{\Lambda} \right], \end{aligned}$$

where  $M$  is a curved superspace index,  $M = (m, \mu, \dot{\mu})$ . Eq. (3.4) implies the inverse superzweibein to transform as

$$\begin{aligned} \hat{E}_a{}^M &= S^{-1} \left[ E_a{}^M - 2i S^{-1} (\gamma_a)^{\dot{\delta}}{}_\gamma (\bar{\nabla}_\gamma S) E_\delta{}^M - 2i S^{-1} (\gamma_a)^{\dot{\delta}}{}_\gamma (\nabla_\gamma S) \bar{E}_{\dot{\delta}}{}^M \right], \\ \hat{E}_\alpha{}^M &= \frac{1}{2} \left[ (\delta_\alpha{}^\beta - (\gamma_3)_\alpha{}^\beta) \Lambda^{-1} + (\delta_\alpha{}^\beta + (\gamma_3)_\alpha{}^\beta) \bar{\Lambda}^{-1} \right] E_\beta{}^M, \\ \hat{E}_{\dot{\alpha}}{}^M &= \frac{1}{2} \left[ \left( \delta_\alpha{}^{\dot{\beta}} + (\gamma_3)_\alpha{}^{\dot{\beta}} \right) \Lambda^{-1} + \left( \delta_\alpha{}^{\dot{\beta}} - (\gamma_3)_\alpha{}^{\dot{\beta}} \right) \bar{\Lambda}^{-1} \right] \bar{E}_{\dot{\beta}}{}^M. \end{aligned} \quad (3.5)$$

The super-Weyl transformation rules for the fermionic parts of the superconnections,

$$\begin{aligned} \omega_M E_\alpha{}^M &\equiv \bar{E}_{\dot{\gamma}}{}^N (\gamma_3)_\beta{}^{\dot{\gamma}} \left( D_N \bar{E}_M{}^{\dot{\beta}} - D_M \bar{E}_N{}^{\dot{\beta}} \right) E_\alpha{}^M, \\ \varrho_M E_\alpha{}^M &\equiv \bar{E}_{\dot{\beta}}{}^N \left( D_N \bar{E}_M{}^{\dot{\beta}} - D_M \bar{E}_N{}^{\dot{\beta}} \right) E_\alpha{}^M, \end{aligned} \quad (3.6)$$

are now straightforward to calculate. We find

$$\begin{aligned} \hat{\omega}_\alpha &= \frac{1}{2} \left[ (\delta_\alpha{}^\beta - (\gamma_3)_\alpha{}^\beta) \Lambda^{-1} + (\delta_\alpha{}^\beta + (\gamma_3)_\alpha{}^\beta) \bar{\Lambda}^{-1} \right] \left[ \omega_\beta - 4S^{-1} (\gamma_3)_\beta{}^\gamma (\nabla_\gamma S) \right], \\ \hat{\varrho}_\alpha &= \frac{1}{2} \left[ (\delta_\alpha{}^\beta - (\gamma_3)_\alpha{}^\beta) \Lambda^{-1} + (\delta_\alpha{}^\beta + (\gamma_3)_\alpha{}^\beta) \bar{\Lambda}^{-1} \right] \left[ \varrho_\beta + 4S^{-1} (\nabla_\beta S) \right]. \end{aligned} \quad (3.7)$$

Eq. (3.7) fixes the Weyl transformations of the superspace covariant derivatives to the form

$$\begin{aligned} \hat{\nabla}_\alpha &= \frac{1}{2} \left[ (\delta_\alpha{}^\beta - (\gamma_3)_\alpha{}^\beta) \Lambda^{-1} + (\delta_\alpha{}^\beta + (\gamma_3)_\alpha{}^\beta) \bar{\Lambda}^{-1} \right] \\ &\quad \times \left[ \nabla_\beta - 4S^{-1} \{ (\gamma_3)_\beta{}^\gamma (\nabla_\gamma S) \mathcal{X} - i (\nabla_\beta S) \mathcal{Y} \} \right], \\ \hat{\bar{\nabla}}_{\dot{\alpha}} &= \frac{1}{2} \left[ \left( \delta_\alpha{}^{\dot{\beta}} + (\gamma_3)_\alpha{}^{\dot{\beta}} \right) \Lambda^{-1} + \left( \delta_\alpha{}^{\dot{\beta}} - (\gamma_3)_\alpha{}^{\dot{\beta}} \right) \bar{\Lambda}^{-1} \right] \\ &\quad \times \left[ \bar{\nabla}_{\dot{\beta}} - 4S^{-1} \left\{ (\gamma_3)_\beta{}^{\dot{\gamma}} (\bar{\nabla}_{\dot{\gamma}} S) \mathcal{X} + i (\bar{\nabla}_{\dot{\beta}} S) \mathcal{Y} \right\} \right]. \end{aligned} \quad (3.8)$$

The transformation rules of the superfields  $G$  and  $H$  follow from the second line of eq. (3.1) to be contracted with the charge conjugation matrix  $C^{\alpha\dot{\beta}}$ , and eq. (3.8).

The finite  $N = 2$  super-Weyl transformations laws turn out to be surprisingly simple, namely <sup>8</sup>

$$\begin{aligned}\hat{H} &= e^{-\Sigma} e^{-\bar{\Sigma}} \left\{ H + 2\nabla_\alpha (\gamma_3)^{\alpha\dot{\beta}} \bar{\nabla}_{\dot{\beta}} (\Sigma + \bar{\Sigma}) \right\} , \\ \hat{G} &= e^{-\Sigma} e^{-\bar{\Sigma}} \left\{ G + 2i\nabla_\alpha C^{\alpha\dot{\beta}} \bar{\nabla}_{\dot{\beta}} (\Sigma + \bar{\Sigma}) \right\} ,\end{aligned}\tag{3.9}$$

where the twisted chiral superfield  $\Sigma$  has been introduced, <sup>9</sup>

$$\Lambda \equiv e^\Sigma , \quad \bar{\Lambda} \equiv e^{\bar{\Sigma}} .\tag{3.10}$$

Eq. (3.9) represents the main result of this section.

## 4 Superconformal Gauge and $N = 2$ SRS

Eqs. (2.22) and (3.9) play the crucial role in the complex  $N = 2$  supergeometry. First, they mean that a curved  $N = 2$  superspace of the  $N = 2$  supergravity is superconformally flat, namely the relevant superfields of the  $N = 2$  supergravity (the chiral ( $R$ ) or the twisted chiral ( $G + iH$ )  $N = 2$  superfield) can be made constants by the  $N = 2$  super-Weyl transformations.

The component gauge fields of the  $N = 2$  supergravity are naturally defined via the expansion of the  $N = 2$  superspace vector covariant derivative,

$$\nabla_a| = e_a^m \partial_m + \psi_a^\mu \partial_\mu + \bar{\psi}_a^{\dot{\mu}} \bar{\partial}_{\dot{\mu}} + \omega_a \mathcal{X} + iA_a \mathcal{Y} ,\tag{4.1}$$

where  $|$  denotes the  $\theta = 0$  projection. In eq. (4.1),  $e_a^m(x)$  is the two-dimensional zweibein,  $\psi_a^\mu(x)$  is the two-dimensional complex 'gravitino' field, and  $A_a(x)$  is an Abelian gauge field. In the Wess-Zumino gauge, where most of the auxiliary fields required by  $N = 2$  superfields are eliminated, the off-shell field contents of the  $N = 2$  supergravity in components are  $(e_m^a, \psi_a^\mu, A_a, R|)$  in the chiral formulation, and  $(e_m^a, \psi_a^\mu, A_a, G|, H|)$  in the twisted chiral one.

The  $N = 2$  superconformal flatness is due to the existence of the (unique) solution to the  $N = 2$  chiral Liouville equation for the finite  $N = 2$  super-Weyl parameters  $\Sigma$  and  $\bar{\Sigma}$ ,

$$2\bar{\nabla}^2 \bar{\Sigma} + e^{2\Sigma} \hat{R} = R ,\tag{4.2}$$

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<sup>8</sup>Eq. (3.1) and the obvious identities  $\text{tr}(\gamma_3 \gamma^a) = \text{tr}(\gamma^a) = 0$  have been used to derive eq. (3.9).

<sup>9</sup>We use the notation similar to that of the previous section, since the chiral and twisted chiral formulations of the  $N = 2$  supergravity can never simultaneously appear in one theory.

where  $\hat{R} = \text{const.}$  The value of the complex constant is constrained by topology, since one has (in the Wess-Zumino gauge) the relation

$$\frac{1}{\pi} \int d^2x d^2\theta \mathcal{E} R = \frac{1}{2\pi} \int d^2x e R^{(2)} + \frac{i}{2\pi} \int d^2x e F = \chi(\Sigma) + ic, \quad (4.3)$$

where the two-dimensional scalar curvature  $R^{(2)}(e)$  and the Abelian field strength  $F(A)$ , as well as the corresponding Euler characteristic  $\chi(\Sigma) = 2 - 2h$  and the first Chern class  $c$ , have been introduced,  $h, c \in \mathbf{Z}$ ,  $h \geq 0$ . The  $\mathcal{E}$  in eq. (4.3) means the chiral density,  $\mathcal{E} \equiv -\frac{1}{2}(\bar{\nabla}^2 - 4R)E$ ,  $E \equiv \text{sdet}(E_M^A)$  and  $\bar{\nabla}_{\dot{\alpha}} \mathcal{E} = 0$ .

Clearly, the  $N = 2$  *flat* superspace is characterized by  $\hat{R}_{\text{flat}} = 0$  and  $\nabla_{\text{flat}} = D$ , so that

$$R_{\text{flat}} = 2\bar{D}^2\bar{\Sigma}. \quad (4.4)$$

There are, of course, the topological obstructions (moduli!) when  $h > 0$ . In addition, eq. (4.4) is only valid in classical theory, because the  $N = 2$  super-Weyl invariance is anomalous after quantisation. Eq. (2.22) also implies that the superspace constraints of the  $N = 2$  supergravity can be locally solved by setting the superzweibein to be equal to the  $N = 2$  super-Weyl-transformed *flat* superzweibein. Such choice constitutes the  $N = 2$  *superconformal gauge*. This gauge is very convenient for quantum calculations, just like the conformal gauge is, since the redundant (super)gravity fields disappear in that gauge, being fixed by the non-anomalous local symmetries.

The  $N = 2$  fermionic string (Polyakov-type) action on the  $N = 2$  supersymmetric (curved) 'world-sheet'  $\Xi$  is written in terms of the  $N = 2$  (covariantly) chiral superfields  $X^{\underline{a}}$ ,  $\underline{a} = 0, 1, \dots, d-1$ ,  $\bar{\nabla}_{\dot{\alpha}} X = \nabla_{\alpha} \bar{X} = 0$ , as <sup>10</sup>

$$S_0 = \int d^2x d^2\theta d^2\bar{\theta} E \bar{X}^{\underline{a}} X^{\underline{b}} \eta_{\underline{ab}}, \quad (4.5)$$

with some flat 'space-time' metric  $\eta_{\underline{ab}}$ . The most general renormalisable  $N = 2$  fermionic string action  $S_{\text{str}}$  also includes the 'topological' term of eq. (4.3) and the 'cosmological' term

$$S_c = \mu_0 \int d^2x d^2\theta \mathcal{E} + \text{h.c.}, \quad (4.6)$$

where  $\mu_0$  is a constant.

Going along the lines of the conventional bosonic and  $N = 1$  supersymmetric cases [19], we now perform the Wick rotation and switch to the Euclidean formulation

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<sup>10</sup>For definiteness, we use the  $N = 2$  chiral superfields to represent the  $N = 2$  scalar matter, or the  $N = 2$  superstring coordinates. They could equally be represented by the  $N = 2$  twisted chiral superfields.

of the  $N = 2$  fermionic string theory (4.5) characterized by the partition function  $Z = \sum_{h,c} Z_{h,c}$ , where

$$Z_{h,c} = \int [dE_M^A][d\Omega_N]\delta(T)[d\bar{X}][dX]e^{-S_{\text{str}}[E,\Omega,X]} . \quad (4.7)$$

In eq. (4.7) the delta-function symbolizes the  $N = 2$  supergravity constraints, which effectively remove the integration over the superconnection, in particular. The functional integration measure is determined by the generalised ultra-locality principle [20] and the  $N = 2$  super-reparametrisational invariance, and it is based on the following definitions of the norm for the superfields  $X$  and  $E_M^A$ ,

$$\begin{aligned} ||\delta X^a||^2 &= \int d^2x d^2\theta d^2\bar{\theta} E \delta X^a \delta X_{\underline{a}} , \\ ||\delta E_M^A||^2 &= \int d^2x d^2\theta d^2\bar{\theta} E \left[ \varepsilon^{\alpha\beta} H_{\alpha}^a H_{\beta}^a + c_1 H^2 + c_2 (\gamma_3 H)^2 \right] + \text{h.c.} , \end{aligned} \quad (4.8)$$

where some arbitrary constants  $c_1$  and  $c_2$  have been introduced. Since both norms in eq. (4.8) fail to be  $N = 2$  super-Weyl invariant, this symmetry is expected to be anomalous, which is the case when  $d \neq 2$  indeed, as is well-known from the component considerations [5, 21]. The detailed form of the  $N = 2$  super-Weyl anomaly in the  $N = 2$  superspace will be determined in the next section. To the end of this section, we are working in the critical dimension  $d = 2$ , where there is no super-Weyl anomaly.

It is quite natural to assume that the bosonic part ('body') of our supermanifold (or supersurface)  $\Xi$  forms a Riemann surface  $\Xi|$ . We can then introduce the  $N = 2$  almost supercomplex structure on  $\Xi$  as follows

$$J_M^N = E_M^a \varepsilon_a^b E_b^N + i E_M^{\alpha} (\gamma_3)_{\alpha}^{\beta} E_{\beta}^N + i E_M^{\dot{\alpha}} (\gamma_3)_{\dot{\alpha}}^{\dot{\beta}} E_{\dot{\beta}}^N , \quad (4.9)$$

which satisfies

$$J_M^N J_N^P = -\delta_M^P , \quad (4.10)$$

and is *invariant* under the  $N = 2$  super-Weyl transformations, as we explicitly verified. Similarly to the conventional  $N = 1$  supersymmetric case [3], it is not difficult to show that the  $N = 2$  almost supercomplex structure defined above is integrable, and, in particular, this allows us to globally define  $N = 2$  superholomorphic coordinates. It can actually be done by introducing the 1-(super)form  $\zeta^M = dz^M - i dz^M J_N^M$ , having only two independent components because of eq. (4.10), and checking that  $d\zeta^M = 0 \pmod{\zeta^N}$  indeed. It happens to be the case just because of the  $N = 2$  supergravity constraints. Hence, we are in a position to define  $N = 2$  super-holomorphic and super-antiholomorphic functions  $\Phi$  and  $\bar{\Phi}$  as solutions to the equations

$$J_M^N \nabla_N \Phi = i \nabla_M \Phi , \quad J_M^N \nabla_N \bar{\Phi} = -i \nabla_M \bar{\Phi} . \quad (4.11a)$$

or, equivalently,

$$\nabla_- \Phi = 0, \quad \nabla_+ \bar{\Phi} = 0, \quad (4.11b)$$

where  $\nabla_{\pm}$  are the corresponding 'chiral' (with respect to the  $N = 2$  supercomplex structure  $J$ ) covariant derivatives. The  $N = 2$  superspace coordinates associated to the  $N = 2$  supercomplex structure are termed the  $N = 2$  superconformal coordinates. It is now natural *to define the  $N = 2$  SRS as an  $N = 2$  supersurface equipped with an  $N = 2$  supercomplex structure*, in a complete analogy with the  $N = 0$  and  $N = 1$  cases [3].<sup>11</sup> On the  $N = 2$  SRS, the (local) coordinate patches should exist, whose transition functions (instructing how to put those patches together) are superholomorphic. This would establish the contact with the alternative (presumably, equivalent) description of the  $N = 2$  SRS introduced earlier [23, 24] as the 1|2-(complex)dimensional superconformal manifolds with  $N = 2$  superconformal transition functions in overlapping regions.<sup>12</sup>

The local symmetries of the theory (4.5) in the curved  $N = 2$  superspace comprise (i)  $N = 2$  super-diffeomorphisms  $2s\text{Diff}(\Xi)$ , (ii)  $N = 2$  supersymmetric Lorentz transformations  $2sL$ , (iii)  $N = 2$  supersymmetric Abelian (phase) transformations  $2sU(1)$ , and (iv)  $N = 2$  super-Weyl transformations  $2s\text{Weyl}(\Xi)$ . Let  $\delta V_A(z)$ ,  $\delta L(z)$ ,  $\delta M(z)$  and  $\delta \Sigma(z)$  be the corresponding infinitesimal  $N = 2$  superfield local parameters, respectively. Using the symmetries (iii) and (iv), the  $H$  and  $(\gamma_3 H)$  can be eliminated from eq. (4.8) without topological obstructions, which explains the redundancy of the coefficients  $c_1$  and  $c_2$  in this equation. The variation  $H_{\alpha}^a$  under the infinitesimal  $N = 2$  super-diffeomorphisms, *orthogonal* to the action of the  $N = 2$  super-Weyl symmetry, is governed by the  $N = 2$  super-differential operator  $\mathcal{P}_1$  of the form

$$(\mathcal{P}_1 \delta V)_{\alpha}{}^b = \frac{1}{2} (\gamma^c \gamma^b)_{\alpha}{}^{\beta} \nabla_{\beta} \delta V_c, \quad (4.12)$$

in a complete analogy with the bosonic and  $N = 1$  supersymmetric cases [3], where the two-dimensional identity  $\gamma_a \gamma^b \gamma^a = 0$  has been used.

We are actually interested in the different  $N = 2$  supergeometries which are not related by the  $N = 2$  super-diffeomorphisms,  $N = 2$  local Lorentz,  $U(1)$  or Weyl transformations. So, let's consider an arbitrary total variation  $\{H_A{}^B\}$ , which can be decomposed as

$$\{H_A{}^B\} = \{\delta \Sigma\} \oplus \{\delta L\} \oplus \{\delta M\} \oplus \{\text{Range } \mathcal{P}_1\} \oplus \{\text{Ker } \mathcal{P}_1^{\dagger}\}. \quad (4.13)$$

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<sup>11</sup>We always assume here that all the supermanifolds we consider are the supermanifolds in the conventional sense [22], with all the non-trivial topology due to the bosonic 'body' only.

<sup>12</sup>The supercoordinate transformation is called superconformal provided the flat supercovariant derivative  $D_+$  transforms as a superconformal tensor.

The elements of  $\text{Ker}\mathcal{P}_1$  are natural to term the  $N = 2$  superconformal Killing vectors, while the elements of  $\text{Ker}\mathcal{P}_1^\dagger$  should be termed the  $N = 2$  supersymmetric Teichmüller deformations or the  $N = 2$  super-quadratic differentials. The  $N = 2$  supersymmetric Teichmüller space  $\mathcal{T}_{h,c}^{N=2} = \text{Ker}\mathcal{P}_1^\dagger$  can be naturally introduced in terms of the original quantities by setting

$$\mathcal{T}_{h,c}^{N=2} = \frac{\{E_M^A, \Omega_N; \delta(T)\}}{\{2\text{sDiff}_0(\Xi) \otimes 2\text{sWeyl}(\Xi) \otimes 2\text{sL} \otimes 2\text{sU}(1)\}} , \quad (4.14)$$

where the  $N = 2$  supergravity constraints ' $T = 0$ ' are supposed to be satisfied, and  $2\text{sDiff}_0(\Xi)$  means the group of topologically trivial diffeomorphisms connected to the identity. Since any non-trivial topology of the  $N = 2$  SRS is due to its 'body' (which is an ordinary Riemann surface), the quotient  $2\text{sDiff}(\Xi)/2\text{sDiff}_0(\Xi)$  is the *ordinary* mapping class group  $\text{MCG}_h$ ,  $\text{MCG}_h = \text{Diff}(\Xi)/\text{Diff}_0(\Xi)$ .

The  $N = 2$  super-moduli space is defined by

$$\mathcal{M}_{h,c}^{N=2} = \frac{\mathcal{T}_{h,c}^{N=2}}{\text{MCG}_h} . \quad (4.15)$$

Since the  $N = 2$  superconformal structure (4.9) on  $N = 2$  SRS is already invariant with respect to the local symmetries (ii), (iii) and (iv), the equivalent definition of the  $N = 2$  super-moduli space is given by

$$\mathcal{M}_{h,c}^{N=2} = \frac{\{J\}}{2\text{sDiff}(\Xi)} . \quad (4.16)$$

In analysing the  $N = 2$  super-moduli space of  $N = 2$  SRS, it is sometimes useful to consider the  $N = 2$  supergeometries characterised by a constant supercurvature,  $R = \text{const}$ . The 'constant-curvature'  $N = 2$  supergeometries comprise the globally defined slice with respect to the  $N = 2$  super-Weyl transformations in the space of all  $N = 2$  supergeometries, when the  $N = 2$  super-Liouville equation (4.2) shows an action of these transformations along the orbits. One gets, in particular, from eqs. (4.14) and (4.15) that

$$\mathcal{M}_{h,c}^{N=2} = \frac{\{E_M^A, \Omega_N; \delta(T)\}_{R=\text{const}}}{\{2\text{sDiff}(\Xi) \otimes 2\text{sL} \otimes 2\text{sU}(1)\}} . \quad (4.17)$$

The  $N = 2$  super-moduli space  $\mathcal{M}_{h,c}^{N=2}$  is a supermanifold, whose tangent space is the space spanned by all  $N = 2$  supersymmetric Teichmüller deformations  $\text{Ker}\mathcal{P}_1^\dagger$ . Unfortunately, it is not clear for us at the moment, how to get more insights into the complicated structure of the  $N = 2$  super-moduli space, while keeping in use the  $N = 2$  superspace approach and not going to the components.

## 5 $N = 2$ Super-Weyl Anomaly

Given the action (4.5), the Green's functions or the propagators of the  $N = 2$  chiral scalar superfields  $X$  and  $\bar{X}$  satisfy the equations

$$\begin{pmatrix} 0 & -\frac{1}{2}(\bar{\nabla}^2 - 4R) \\ -\frac{1}{2}(\nabla^2 - 4\bar{R}) & 0 \end{pmatrix} \begin{pmatrix} 0 & G_{\text{ca}} \\ G_{\text{ac}} & 0 \end{pmatrix} = \begin{pmatrix} \delta_{\text{c}}^4 & 0 \\ 0 & \delta_{\text{a}}^4 \end{pmatrix}, \quad (5.1)$$

where the chiral delta-functions have been introduced,

$$\begin{aligned} \delta_{\text{c}}^4 &= -\frac{1}{2}(\bar{\nabla}^2 - 4R)E^{-1}\delta^6(z, z'), \\ \delta_{\text{a}}^4 &= -\frac{1}{2}(\nabla^2 - 4\bar{R})E^{-1}\delta^6(z, z'), \end{aligned} \quad (5.2)$$

and

$$\delta^6(z, z') \equiv \delta^2(x, x')\delta^2(\theta - \theta')\delta^2(\bar{\theta} - \bar{\theta}'). \quad (5.3)$$

The integration over the  $N = 2$  matter fields in the partition function (4.7) yields

$$\int [d\bar{X}][dX] e^{-S_0} = e^{-W}, \quad (5.4)$$

where

$$W = -\frac{1}{2}\text{sTr} \ln G + \text{h.c.}, \quad (5.5)$$

and

$$G \equiv G_{\text{ca}}G_{\text{ac}}, \quad \bar{G} \equiv G_{\text{ac}}G_{\text{ca}}. \quad (5.6)$$

The Green's functions  $G_{\text{ac}}$  and  $G_{\text{ca}}$  can now be written down in the form

$$\begin{aligned} G_{\text{ac}} &= -\frac{1}{2}(\nabla^2 - 4\bar{R})G, \\ G_{\text{ca}} &= -\frac{1}{2}(\bar{\nabla}^2 - 4R)\bar{G}, \end{aligned} \quad (5.7)$$

where the new ones  $G$  and  $\bar{G}$  satisfy the equations

$$\begin{aligned} \mathcal{H}G &\equiv \frac{1}{4}(\bar{\nabla}^2 - 4R)(\nabla^2 - 4\bar{R})G = \delta_{\text{c}}^4, \\ \bar{\mathcal{H}}\bar{G} &\equiv \frac{1}{4}(\nabla^2 - 4\bar{R})(\bar{\nabla}^2 - 4R)\bar{G} = \delta_{\text{a}}^4. \end{aligned} \quad (5.8)$$

The  $\mathcal{H}$ -operators can be thought of as the  $N = 2$  (chiral) covariant scalar 'Laplacians' squared [25], *viz.*

$$\begin{aligned} \mathcal{H} &= \frac{1}{4}(\bar{\nabla}^2 - 4R)(\nabla^2 - 4\bar{R}) = \frac{1}{4}[\bar{\nabla}^2\nabla^2 - 4\bar{\nabla}^2\bar{R} - 4R\nabla^2 + 16R\bar{R}], \\ \bar{\mathcal{H}} &= \frac{1}{4}(\nabla^2 - 4\bar{R})(\bar{\nabla}^2 - 4R) = \frac{1}{4}[\nabla^2\bar{\nabla}^2 - 4\nabla^2R - 4\bar{R}\bar{\nabla}^2 + 16\bar{R}R]. \end{aligned} \quad (5.9)$$



The formal expression  $\text{sTr} \ln G = \ln \text{sdet} G = -\ln \text{sdet} \mathcal{H}$  needs to be regularised, and it has to be carried over the space orthogonal to the zero modes of  $\mathcal{H}$ :  $\text{sdet} \rightarrow \text{sdet}'$ . The natural definition is [3, 26]

$$\ln \text{sdet}'(\mathcal{H} + s) = -\int_{\varepsilon}^{\infty} \frac{dt}{t} e^{-ts} \text{sTr}' [e^{-t\mathcal{H}}] , \quad (5.10)$$

where the real UV cutoff  $\varepsilon$  and the complex parameter  $s$  have been introduced. The integral in eq. (5.10) absolutely converges for sufficiently large  $\text{Re}(s)$  and  $\varepsilon > 0$ . The definition can then be extended throughout the complex  $s$ -plane by analytic continuation. The limit  $s \rightarrow 0$  determines the regularised superdeterminant we are interested in.

The infinitesimal variation

$$\delta \ln \text{sdet}'(\mathcal{H} + s) = \int_{\varepsilon}^{\infty} dt e^{-ts} \text{sTr}' [\delta \mathcal{H} e^{-t\mathcal{H}}] \quad (5.11)$$

under the  $N = 2$  super-Weyl transformations with the infinitesimal  $N = 2$  chiral superfield local parameter  $\delta \Sigma$  can be explicitly computed since eqs. (2.21), (2.22) and (5.9). The finite  $N = 2$  super-Weyl transformation of the  $\mathcal{H}$ -operator takes the form

$$\hat{\mathcal{H}} = e^{-2\Sigma} e^{-2\bar{\Sigma}} \left\{ \mathcal{H} + 8(\bar{\nabla}^2 - 4R)\nabla^2 \Sigma - [(\bar{\nabla}^{\dot{\alpha}} \bar{\Sigma}) \bar{\nabla}_{\dot{\alpha}} - 6(\bar{\nabla}^2 \bar{\Sigma}) + 12(\bar{\nabla} \bar{\Sigma})^2](\nabla^2 - 4\bar{R} + 8\nabla^2 \Sigma) \right\} . \quad (5.12)$$

Fortunately, in order to compute the  $N = 2$  super-Weyl anomaly whose local form can be fixed up to a coefficient (see below) on the symmetry grounds, knowing the infinitesimal chiral part of the *rigid*  $N = 2$  super-Weyl transformation of  $\mathcal{H}$ ,  $\delta \mathcal{H} = -2\mathcal{H}\delta \Sigma + \dots$ , is enough. Therefore, keeping only the relevant term, we find

$$\begin{aligned} \lim_{s \rightarrow 0} \delta \ln \text{sdet}'(\mathcal{H} + s) &= -2 \lim_{s \rightarrow 0} \int_{\varepsilon}^{\infty} dt e^{-ts} \text{sTr}' [\delta \Sigma \mathcal{H} e^{-t\mathcal{H}}] \\ &= 2 \lim_{s \rightarrow 0} \int_{\varepsilon}^{\infty} dt e^{-ts} \frac{\partial}{\partial t} (\text{sTr}' [\delta \Sigma e^{-t\mathcal{H}}]) = -2s \text{sTr}' [\delta \Sigma e^{-\varepsilon \mathcal{H}}] . \end{aligned} \quad (5.13)$$

In the limit  $\varepsilon \rightarrow 0$  the expression on the right-hand side of eq. (5.13) is local, and it is entirely determined by the symmetry, locality and chirality arguments up to an overall constant, namely

$$\lim_{\varepsilon \rightarrow 0} s \text{sTr}' [\delta \Sigma e^{-\varepsilon \mathcal{H}}] = \text{const.} \int d^2 x d^2 \theta \mathcal{E} R \delta \Sigma . \quad (5.14)$$

There is no  $1/\varepsilon$  (divergent) term due to world-sheet supersymmetry [3, 26]. The constant in eq. (5.14) can be computed<sup>13</sup> via a short-time expansion of the  $N = 2$

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<sup>13</sup>In fact, the actual form of the anomaly in eq. (5.14) also follows from the short-time expansion, as can be shown by a straightforward (tedious) calculation.

chiral super-heat kernel  $U(z, z'; t)$  to be defined as the solution to the equation

$$\left(\frac{\partial}{\partial t} + \mathcal{H}\right) U(z, z'; t) = \delta(t) \delta_c^4(z, z') . \quad (5.15)$$

Since we are actually interested in calculating the elements of  $U$  on the diagonal  $z = z'$  for short times  $t \sim \varepsilon \rightarrow +0$ , which is a *local* problem, we use the  $N = 2$  (non-anomalous) local symmetries to render the  $N = 2$  supergeometry to be super-conformally flat, i.e. take the reference  $N = 2$  supergeometry ('with hats') to be flat. Moreover, since the anticipated structure of the result in eq. (5.14), we keep only those terms in  $\mathcal{H}$  which are *linear* in  $\Sigma$  and have no more than *two* derivatives acting on  $\Sigma$ .<sup>14</sup> Finally, we temporarily omit the constant scaling factor  $e^{-2\Sigma}$  in the expression for the  $\mathcal{H}$ -operator, in order to restore it at the end. After all that simplifications the remaining terms of  $\mathcal{H}$  read:

$$\mathcal{H} \rightarrow \mathcal{H}_{\text{re}} = \bar{D}^2 D^2 - C^{\dot{\alpha}\dot{\beta}}(\bar{D}_{\dot{\beta}} \bar{\Sigma}) \bar{D}_{\dot{\alpha}} D^2 + 6(\bar{D}^2 \bar{\Sigma}) D^2 , \quad (5.16)$$

where we have used the conventional notation for the flat  $N = 2$  superspace covariant derivatives:

$$\bar{D}^2 = \frac{1}{2} C^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\beta}} \bar{D}_{\dot{\alpha}} , \quad D^2 = \frac{1}{2} C^{\alpha\beta} D_{\beta} D_{\alpha} , \quad (5.17)$$

as well as the identities

$$C^{\dot{\alpha}\dot{\beta}} \bar{D}_{\dot{\beta}} (\gamma_3)_{\dot{\alpha}}^{\dot{\gamma}} \bar{D}_{\dot{\gamma}} = C^{\alpha\beta} (\gamma_3)_{\beta}^{\gamma} D_{\gamma} D_{\alpha} = 0 , \quad (5.18)$$

$$D^2 \bar{D}^2 D^2 = \partial_a \partial^a D^2 \equiv \Delta D^2 .$$

The  $\mathcal{H}_{\text{re}}$  can then be rewritten to the form  $\mathcal{H}_{\text{re}} = (\Delta - \mathcal{V}_{\text{re}}) D^2$ , with  $\mathcal{V}_{\text{re}}$  to be considered as a perturbation,

$$\mathcal{V}_{\text{re}} = C^{\dot{\alpha}\dot{\beta}}(\bar{D}_{\dot{\beta}} \bar{\Sigma}) \bar{D}_{\dot{\alpha}} - 6\bar{D}^2 \bar{\Sigma} . \quad (5.19)$$

The *flat* chiral  $N = 2$  supersymmetric heat kernel equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) U(z, z'; t) = \delta^2(x, x') \delta^2(\theta - \theta') \delta(t) \quad (5.20)$$

is solved by

$$U_0(z, z'; t) = U_0(x, x'; t) \delta^2(\theta - \theta') , \quad (5.21)$$

where the usual flat space heat kernel

$$U_0(x, x'; t) = \frac{1}{4\pi t} e^{-\frac{(x-x')^2}{4t}} \vartheta(t) , \quad (5.22)$$

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<sup>14</sup>It can be shown that the other contributions vanish in the limit  $t \rightarrow 0$  [3, 26].

has been introduced.<sup>15</sup> The only term we need to consider in the iterative solution for the  $U(z, z'; t)$  is

$$\begin{aligned} U_{\text{re}}(z, z'; t) &= \int_0^t dt' \int d^2 x' d^2 \theta' U_0(z, z'; t - t') \mathcal{V}_{\text{re}}(z', \theta') U_0(z, z'; t') + O(t) \\ &= \int_0^t dt' \int d^2 x' U_0(x, x'; t - t') U_0(x, x'; t') \int d^2 \theta' \delta^2(\theta, \theta') \mathcal{V}_{\text{re}}(z', \theta') \delta^2(\theta, \theta') + O(t) . \end{aligned}$$

The integral over  $\theta'$  in the last line of this equation contributes

$$\mathcal{V}_{\text{re}}(z', \theta') \delta^2(\theta, \theta')|_{\theta'=0} = -6 \bar{D}^2 \bar{\Sigma} ,$$

whereas the remaining integral over  $t'$  gives the factor  $(2\pi)^{-1}$  in the limit  $t \rightarrow 0$ . Putting it all together, we find

$$\lim_{t \rightarrow 0} U_{\text{re}}(z, z; t) = -\frac{6}{2\pi} \bar{D}^2 \bar{\Sigma} , \quad (5.23a)$$

and, hence, after restoring the constant scaling by  $\Sigma$ , the covariant form of the solution reads

$$\lim_{t \rightarrow 0} U(z, z; t) = -\frac{3}{2\pi} R . \quad (5.23b)$$

Given  $d$  chiral scalar  $N = 2$  superfields, the anomalous contribution of eq. (5.23) should be multiplied by  $d$ . Therefore, the constant in eq. (5.14) is equal to  $-3d/(2\pi)$ .

Eqs. (5.5), (5.13), (5.14) and (5.23) imply for the anomalous part of the induced action  $W$

$$\delta W_{\text{anomalous}} = \frac{3d}{2\pi} \int d^2 x d^2 \theta \mathcal{E} R \delta \Sigma + \text{h.c.} , \quad (5.24)$$

whose integration yields

$$W_{\text{anomalous}} = -\frac{3d}{2\pi} \int d^2 x d^2 \theta d^2 \bar{\theta} \hat{E} \bar{\Sigma} \Sigma + \left\{ \frac{3d}{2\pi} \int d^2 x d^2 \theta \hat{\mathcal{E}} \hat{R} \Sigma + \text{h.c.} \right\} , \quad (5.25)$$

where all the  $N = 2$  superderivatives and supercurvatures are to be defined with respect to the reference (e.g., of constant supercurvature)  $N = 2$  superspace geometry. The complete expression for the  $W$  contains the additional anomaly-free factor

$$\frac{1}{2} \ln \frac{\text{sdet}' \hat{\mathcal{H}}}{\text{sdet} \langle \hat{\Phi}_I | \hat{\Phi}_J \rangle \text{sdet} \langle \hat{\Psi}_I | \hat{\Psi}_J \rangle} + \text{h.c.} , \quad (5.26)$$

where the (orthonormal) bases  $\{\Phi_I\}$  and  $\{\Psi_I\}$  of the finite-dimensional spaces  $\text{Ker}(\bar{\nabla}^2 - 4R)$  and  $\text{Ker}(\nabla^2 - 4\bar{R})$ , respectively, have been introduced.

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<sup>15</sup>The  $\vartheta(t)$  denotes the conventional step-function:  $\vartheta(t) = 0$  when  $t < 0$ , and  $\vartheta(t) = 1$  when  $t > 0$ .

The  $N = 2$  superfield supergravity measure yields, in its turn, the contribution to the partition function (4.7),

$$[dE_M^A][d\Omega_N]\delta(T) = \left(\text{sdet}'\mathcal{P}_1^\dagger\mathcal{P}_1\right)^{1/2} \frac{1}{\text{Vol}(\text{Ker}\mathcal{P}_1)} [d\Sigma][d\bar{\Sigma}][d'V_A][dL][dM] , \quad (5.27)$$

where some factors cancel after the appropriate normalisation of the partition function by

$$\mathcal{N}^{-1} = \text{Vol}(2\text{sDiff}_0) \times \text{Vol}(2\text{sL}) \times \text{Vol}(2\text{sU}(1)) . \quad (5.28)$$

The  $N = 2$  superghosts associated with the non-trivial factor  $(\text{sdet}'\mathcal{P}_1^\dagger\mathcal{P}_1)^{1/2}$  in the measure (5.27) contribute a local factor to the  $N = 2$  super-Weyl anomaly, which should be similar to that in eq. (5.25), although we have not yet computed it. To match with the component approach, it should give rise to a shift  $d \rightarrow (d - 2)$  in the total  $N = 2$  super-Weyl anomaly. The 'cosmological' term is also allowed to be added to the final result, since it is consistent with the symmetries of the non-critical  $N = 2$  fermionic string theory. Perhaps, it seems to be equally consistent to set the 'cosmological' term to be zero,  $\mu_0 = 0$ .

Finally, we arrive at the following expression for the  $N = 2$  super-Weyl anomaly (with the anticipated ghost contribution included)

$$W_{\text{total, anomalous}} = -\frac{3(d-2)}{2\pi} S_{N=2 \text{ Liouville}}(\Sigma, \bar{\Sigma}) , \quad (5.29)$$

where the  $N = 2$  super-Liouville action has been introduced,

$$S_{N=2 \text{ Liouville}} = \int d^2x d^2\theta d^2\bar{\theta} \hat{E} \bar{\Sigma} \Sigma + \left\{ \int d^2x d^2\theta \left( \hat{\mathcal{E}} \hat{R} \Sigma + \mu_0 \mathcal{E} \right) + \text{h.c.} \right\} \quad (5.30)$$

In the  $N = 2$  superconformal gauge, it reduces to ( $\hat{R} \equiv r^{-1} = \text{const}$ )

$$S_{N=2 \text{ Liouville}}^{\text{g.-f.}} = \int d^2x d^2\theta d^2\bar{\theta} \bar{\Sigma} \Sigma + \left\{ \int d^2x d^2\theta \left[ r^{-1} \Sigma + \mu_0 (e^{2\Sigma} - 1) \right] + \text{h.c.} \right\} \quad (5.31)$$

The main results of this section are summarized by eqs. (5.29) and (5.30). The effective theory of the quantised  $N = 2$  supergravity in two dimensions is given by the  $N = 2$  super-Liouville theory, as it should have been expected from a consistency with the component approach. The current algebra of the  $N = 2$  super-Liouville theory in the  $N = 2$  superconformal gauge was recently discussed in refs. [27, 28].

## 6 Conclusion

The results reported above are thought to be useful for a systematic covariant quantisation of the  $N = 2$  fermionic string theory in  $N = 2$  superspace, which is yet to be

completed. A covariant derivation of the  $N = 2$  superfield BRST operator, as well as an inspection of the BRST transformation properties of the  $N = 2$  superfield path integral measure would be of particular interest.

There are several indications in the literature that both critical and non-critical  $N = 2$  fermionic string theories are in fact *topological* quantum field theories in the sense of ref. [29]. Namely, the BRST super-current algebra in these theories appears to be the particular  $N = 2$  supersymmetric extension of the topological conformal algebra [29] for an arbitrary conformal matter [28]. As was noticed in ref. [28] for both critical and non-critical  $N = 2$  fermionic string theories, the vanishing of the ghost-number anomaly associated with the total divergence term in the full BRST anomaly is crucial for the apparently topological properties of these theories. There are also many reasons to believe that the two-dimensional  $N = 2$  supergravity itself is a topological quantum field theory [21, 28].

An analysis of the zero modes of various  $N = 2$  super-differential operators on the  $N = 2$  super-Riemannian surfaces turns out to be very involved in terms of superfields, compared to the standard approach in components. This currently appears to be the major technical obstruction preventing an efficient use of the covariant (not related with the  $N = 2$  superconformal gauge) definition of the  $N = 2$  super-Riemannian surfaces, which has been proposed in sect. 4. Nevertheless, nothing seems to be preventing, in principle, to formulate  $N = 2$  super-analogue of the Riemann-Roch theorem which would be based on that definition, and then explicitly construct the  $N = 2$  superfield measure at non-vanishing values of genus  $h \neq 0$  and Chern class  $c \neq 0$ . It is currently under study.

One of the authors (S.V.K.) would like to thank S. J. Gates Jr. for useful conversations.

## 7 Appendix: Notation

The two-component (complex) world-sheet spinor is labelled as

$$\psi^\alpha = (\psi^+, \psi^-) , \quad (A.1)$$

while its complex conjugate takes the form

$$\psi^{\dot{\alpha}} = (\psi^{\dot{+}}, \psi^{\dot{-}}) . \quad (A.2)$$

The lower-case Greek letters are used to represent spinor indices, the lower-case Latin letters are used for tensor indices. The early (both Greek and Latin) indices are referred to a tangent space, the middle ones being referred to a base space. The capital letters are normally used to denote  $N = 2$  superspace indices.

The two-dimensional Minkowski metric is

$$\eta_{ab} = \text{diag}(-, +) , \quad a, b = 0, 1 . \quad (A.3)$$

The two-dimensional Dirac gamma matrices satisfy an algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} . \quad (A.4)$$

Their explicit forms are

$$(\gamma^a) = (i\sigma_1, \sigma^2) , \quad (\gamma_3) = \gamma^0 \gamma^1 = \sigma^3 . \quad (A.5)$$

The Levi-Civita antisymmetric symbol in two dimensions is normalized by the condition

$$\varepsilon^{01} = 1 . \quad (A.6)$$

The spinor metrics

$$C_{\alpha\beta} = C_{\dot{\alpha}\dot{\beta}} = \sigma^2 \quad (A.7)$$

and their inverses are used to raise and lower the spinors indices:  $\psi_\alpha = C_{\beta\alpha}\psi^\beta$ ,  $\psi^\alpha = C^{\alpha\beta}\psi_\beta$ , and similarly for the dotted indices.

The obvious identities take place

$$\begin{aligned} \varepsilon^{ab}\varepsilon_{cd} &= \delta_c^{[a}\delta_d^{b]} , \\ C^{\alpha\beta}C_{\gamma\delta} &= \delta_\gamma^{[\alpha}\delta_\delta^{\beta]} , \\ \gamma^a\gamma^b &= \eta^{ab} + \varepsilon^{ab}\gamma_3 , \end{aligned}$$

$$\psi^2 = \frac{1}{2}C_{\beta\alpha}\psi^\alpha\psi^\beta = \frac{1}{2}\psi^\alpha\psi_\alpha = i\psi^+\psi^- . \quad (A.8)$$

Given a general supermatrix

$$M = \begin{pmatrix} A_a{}^b & B_a{}^{\beta,\dot{\beta}} \\ C_{\alpha,\dot{\alpha}}{}^b & D_{\alpha,\dot{\alpha}}{}^{\beta,\dot{\beta}} \end{pmatrix} , \quad (A.9)$$

its superdeterminant and supertrace are defined by

$$\begin{aligned} \text{sdet } M &= \det \left( A - BD^{-1}C \right) \det^{-1} D , \\ \text{str } M &= \text{tr } A - \text{tr } D . \end{aligned} \quad (A.10)$$

The delta-functions of the anticommuting coordinates  $\theta_\alpha$  and  $\bar{\theta}_{\dot{\alpha}}$  are defined by

$$\delta^2(\theta_1 - \theta_2) = (\theta_1 - \theta_2)^2 , \quad \delta^2(\bar{\theta}_1 - \bar{\theta}_2) = (\bar{\theta}_1 - \bar{\theta}_2)^2 , \quad (A.11)$$

so that

$$\int d^2\theta \delta^2(\theta) = \int d^2\bar{\theta} \delta^2(\bar{\theta}) = 1 . \quad (A.12)$$

## References

- [1] A. M. Polyakov, Phys. Lett. **103B** (1981) 207, *ibid.* 211.
- [2] P. Howe, J. Phys. **A12** (1979) 393.
- [3] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60** (1988) 917.
- [4] R. Brooks, F. Muhammad and S. J. Gates, Jr., Nucl. Phys. **B268** (1986) 599.
- [5] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **106B** (1981) 63.
- [6] P. Bouwknegt and P. van Nieuwenhuizen, Class. and Quantum Grav. **3** (1986) 207.
- [7] S. D. Mathur and S. Mukhi, Nucl. Phys. **B302** (1988) 130.
- [8] J. Distler, Z. Hlousek and H. Kawai, Int. J. Mod. Phys. **A5** (1990) 391.
- [9] L. Antoniadis, C. Bachas and C. Kounnas, Phys. Lett. **242B** (1990) 185.
- [10] P. Howe and G. Papadopoulos, Class. and Quantum Grav. **4** (1987) 11.
- [11] S. J. Gates Jr., L. Lu and R. Oerter, Phys. Lett. **218B** (1989) 33.
- [12] S. J. Gates Jr., S. J. Hassoun and P. van Nieuwenhuizen, Nucl. Phys. **B317** (1989) 302.
- [13] A. Alnowaiser, Class. and Quantum Grav. **7** (1990) 1033.
- [14] M. Atiyah, Ann. Sci. Ec. Norm. Sup. **4** (1971) 47.
- [15] L. Dabrowski and R. Percacci, Commun. Math. Phys. **107** (1986) 691.
- [16] K. Aoki, E. D'Hoker and D. H. Phong, Nucl. Phys. **B342** (1990) 149.
- [17] S. J. Gates Jr., M. T. Grisaru, M. Rocek and W. Siegel, *Superspace or One Thousand and One Lessons in Supersymmetry*, Benjamin-Cummings, MA, 1983.
- [18] S. G. Gates, Jr., C. M. Hull and M. Roček, Nucl. Phys. **B248** (1984) 570.
- [19] M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory*, Vol. I, Cambridge University Press, Cambridge, 1987.
- [20] J. Polchinski, Commun. Math. Phys. **104** (1986) 37.



- [21] S. V. Ketov, *Space-Time Supersymmetry of Extended Fermionic Strings in  $2+2$  Dimensions*, preprint Hannover ITP-UH-1/93 and DESY 93-050, February 1993, to appear in "Classical and Quantum Gravity".
- [22] B. De Witt, *Supermanifolds*, Cambridge University Press, Cambridge, 1983.
- [23] I. D. Cohn, Nucl. Phys. **B284** (1987) 349.
- [24] E. Melzer, J. Math. Phys. **29** (1988) 1555.
- [25] O. Alvarez, Nucl. Phys. **B216** (1983) 125.
- [26] E. Martinec, Phys. Rev. **D28** (1983) 2604.
- [27] W. A. Sabra, Nucl. Phys. **B392** (1993) 385.
- [28] J. Gomis and H. Suzuki, Nucl. Phys. **B393** (1993) 126.
- [29] E. Witten, Commun. Math. Phys. **177** (1988) 353, *ibid.* **118** (1988) 411.