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Kinetic vs. thermal-field-theory approach to cosmological perturbations

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Abstract

A closed set of equations for the evolution of linear perturbations of homogeneous, isotropic cosmological models can be obtained in various ways. The simplest approach is to assume a macroscopic equation of state, e.g. that of a perfect fluid. For a more refined description of the early universe, a microscopic treatment is required. The purpose of this paper is to compare the approach based on classical kinetic theory to the more recent thermal-field-theory approach. It is shown that in the high-temperature limit the latter describes cosmological perturbations supported by collisionless, massless matter, wherein it is equivalent to the kinetic theory approach. The dependence of the perturbations in a system of a collisionless gas and a perfect fluid on the initial data is discussed in some detail. All singular and regular solutions are found analytically.

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I. INTRODUCTION

Early progenitors of large-scale structure of the universe are usually provided for by small (linear) perturbations of otherwise homogeneous and isotropic cosmological models [1]. After they have come within the Hubble horizon of the growing universe and their substratum has turned non-relativistic, they eventually become large, leading to gravitational collaps — the corresponding mass scale being set by the Jeans mass [2].

The theory of linear perturbations in a Friedmann-Lemaître-Robertson-Walker (FLRW) model dates back to Lifshitz' work in 1946 [3]. The metric perturbations and correspondingly the perturbed energy-momentum tensor are decomposed into scalar, vector and tensor parts according to their behavior under spatial coordinate transformations. In linear theory these parts evolve independently. Any tensor $t^\mu{}_\nu$ is divided into a background part $\tilde{t}^\mu{}_\nu$ and a perturbation $\delta t^\mu{}_\nu := t^\mu{}_\nu - \tilde{t}^\mu{}_\nu$. The background is given by a FLRW model. However, the mapping of points x^μ on the physical manifold to points $\tilde{x}^\mu$ on the background is not unique. Unphysical (gauge) modes can be avoided by using Bardeen's [4] gauge-invariant variables for the metric and matter components. The evolution of the metric and the matter perturbations is governed by the Einstein equation

$$\tilde{G}^\mu{}_\nu + \delta G^\mu{}_\nu = -8\pi G \left(\tilde{T}^\mu{}_\nu + \delta T^\mu{}_\nu \right) , \tag{1.1}$$

where $G^\mu{}_\nu$ is the Einstein tensor and $T^\mu{}_\nu$ the energy-momentum tensor. Useful reviews of gauge-invariant cosmological perturbation theory are, e.g., Refs. [5] and [6].

In this paper we are mainly interested in comparing the methods of determining self-consistently the matter perturbations $\delta T^\mu{}_\nu$, needed to solve (1.1). The Einstein equation describes the evolution of the metric perturbations for given matter perturbations. The covariant conservation of the perturbed energy-momentum tensor follows from the Bianchi identity. This does not fix $\delta T^\mu{}_\nu$ completely; additional input is needed. The simplest possibility is to assume that both the perturbed and the unperturbed medium satisfy a simple macroscopic equation of state, e.g., that of a perfect fluid [3,6,7].

More complicated forms of matter in the early universe require of course a microscopic description. Such a description was first developed by Peebles and Yu [8] on the basis of kinetic theory [9]. There the Boltzmann equation in seven dimensional phase space $(\tau, \mathbf{x}, \mathbf{p})$ with an ansatz for the collision term is used. τ denotes the conformal time, $\mathbf{x}$ the spatial coordinate and $\mathbf{p}$ the spatial momentum. The timelike component of the momentum is not independent, since the particles are on the mass-shell $p^0 = \sqrt{p^2 + m^2}$ ($p = |\mathbf{p}|$). The Boltzmann equation together with Eq. (1.1) has then usually to be solved numerically. A lot of work has been accomplished within this framework [5,10–15].

A different approach has been formulated more recently by Kraemmer and one of the present authors [16]. In a completely field-theoretical framework, the matter perturbation $\delta T^\mu{}_\nu$ is determined by the (thermal) graviton self-energy [17], evaluated on the given background. In contrast to kinetic theory, this framework is fully quantum theoretical from the beginning, and might turn out to be an alternative to the still to be formulated quantum kinetic theory. Up to now, this approach was used to incorporate collisionless, massless matter [16,18] possibly mixed with a perfect fluid component [19,20], which can be described perturbatively by the high-temperature limit of the underlying field theory. In this limit one may expect equivalence with classical kinetic theory applied to the quanta of the field theory. A detailed discussion of the relation between the thermal-field-theory approach and the one based on kinetic theory is however still missing and will be the main purpose of the present paper. The two approaches are compared for the case of collisionless, massless matter, with an arbitrary admixture of a perfect fluid. We will show that both approaches coincide in the high-temperature limit of thermal field theory. The exact solutions first found in the field-theoretical approach [16,19] in the form of power series with infinite radius of convergence will be discussed in some detail with regard to their dependence on the initial data. Moreover, by a generalized power-series ansatz further exact solutions are obtained that are singular as the initial big-bang singularity is approached. (The existence of such solutions has been established first by Zakharov [12] and discussed further by Vishniac [13].)

The paper is organized as follows. In Sect. II we introduce the Einstein-de Sitter background and the gauge-invariant metric potentials and matter perturbations. The Einstein equations are written down for these variables and their solutions for a perfect fluid universe are recapitulated. The kinetic theory is explained in Sect. III. We follow and add on to the approach of Kasai and Tomita [15], which defines gauge-invariant distribution functions. A set of integro-differential equations similar to that already existing in the literature [10] is derived for the most general initial conditions. In Sect. IV we prove that the same equations emerge from the thermal-field-theory approach in the high-temperature limit. The gauge-invariant equations describing a mixture of a collisionless gas and an isentropic perfect fluid are derived in Sect. V, and their exact solutions in terms of (generalized) power series (details are given in the Appendix) are discussed in Sect. VI. A summary is given in Sect. VII.

Throughout the paper we use units $\hbar = c = k_B = 1$. Our notation is similar to that of [4] and [15], but we do not normalize energies and masses to $\sqrt{8\pi G}$. It differs only in the definition of the metric potentials for scalar perturbations and in that we expand all functions into planar waves instead of spherical harmonics, since we consider only a spatially flat FLRW model. Greek indices take their values in the set $\{0,1,2,3\}$ and Latin ones in $\{1,2,3\}$.

II. GAUGE-INVARIANT COSMOLOGICAL PERTURBATIONS

Before we define the gauge-invariant variables we introduce the Einstein-de Sitter background. The invariant line element is given by

$$ds^2 = S^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right) \ , \tag{2.1}$$

with τ the conformal time measuring the size of the horizon in comoving coordinates R_H. The background energy-momentum tensor has perfect-fluid form,

$$\tilde{T}^\mu{}_\nu = u^\mu u_\nu(\tilde{E} + \tilde{P}) + \tilde{P}\delta^\mu_\nu \ , \tag{2.2}$$

where $u^\mu u_\mu = -1$. We shall concentrate on the radiation dominated epoch, where $\tilde{E} = 3\tilde{P}$. The scale dependence of the mean energy density is given by $\tilde{E}(S) = \tilde{E}(S=1)S^{-4}$, and the evolution of the cosmic scale factor $S(\tau)$ follows from the Friedmann equation

$$S(\tau) = \sqrt{\frac{8\pi G\tilde{E}(S=1)}{3}}\tau \ . \tag{2.3}$$

As already explained in the introduction, the perturbed tensors are divided into scalar, vector and tensor parts:

$$\begin{aligned} \delta g_{\mu\nu} &= \delta g^S_{\mu\nu} + \delta g^V_{\mu\nu} + \delta g^T_{\mu\nu} \ , \\ \delta T^\mu{}_\nu &= \delta T^{S\mu}{}_\nu + \delta T^{V\mu}{}_\nu + \delta T^{T\mu}{}_\nu \ . \end{aligned} \tag{2.4}$$

Both tensors have ten independent components, four of which belong to the scalar, four to the vector and two to the tensor part (the latter correspond to the two polarizations of a gravitational wave).

Since the Einstein-de Sitter background is spatially flat and depends only on the cosmic scale factor $S(\tau)$, any function $F(\tau, \mathbf{x})$ can be expanded into plane waves $e^{\imath \mathbf{kx}}$. A vector and a tensor valued plane wave is given by $\mathbf{w}e^{\imath \mathbf{kx}}$ and $\mathsf{t}e^{\imath \mathbf{kx}}$, respectively, where we specify the constant (but $\mathbf{k}$-dependent) vector $\mathbf{w}$ to be transverse ($\mathbf{w}\cdot\mathbf{k} = 0$) and the constant symmetric tensor t to be transverse-traceless ($\mathsf{t}\cdot\mathbf{k} = 0$ and tr $\mathsf{t} = 0$). With this any symmetric tensor can be uniquely decomposed into its scalar, vector and tensor part. This is done for each individual mode proportional to $e^{\imath \mathbf{kx}}$ of Eqs. (2.4) as follows $[\{\mu, \nu\} = \{(0, i), (0, j)\}]$:

$$\delta g^S_{\mu\nu} = S^2 \begin{pmatrix} -2A & \imath \hat{k}_j B^{(0)} \\ \imath \hat{k}_i B^{(0)} & 2\left[\delta_{ij} H_L + \left(\frac{1}{3}\delta_{ij} - \hat{k}_i \hat{k}_j\right) H_T^{(0)}\right] \end{pmatrix} e^{\imath \mathbf{kx}} \ , \tag{2.5a}$$

$$\delta g^V_{\mu\nu} = S^2 \begin{pmatrix} 0 & -w_j B^{(1)} \\ -w_i B^{(1)} & -2\imath w_{(i}\hat{k}_{j)} H_T^{(1)} \end{pmatrix} e^{\imath \mathbf{kx}} \ , \tag{2.5b}$$

$$\delta g^T_{\mu\nu} = S^2 \begin{pmatrix} 0 & 0 \\ 0 & 2t_{ij} H_T^{(2)} \end{pmatrix} e^{\imath \mathbf{kx}} \tag{2.5c}$$

and

$$\delta T^{S\mu}{}_\nu = \frac{\tilde{E}}{3} \begin{pmatrix} -3\delta & -4\imath \hat{k}_j \left(v^{(0)} - B^{(0)}\right) \\ 4\imath \hat{k}^i v^{(0)} & \delta^i_j \pi_L + \left(\frac{1}{3}\delta^i_j - \hat{k}^i \hat{k}_j\right) \pi_T^{(0)} \end{pmatrix} e^{\imath \mathbf{kx}} \ , \tag{2.6a}$$

$$\delta T^{V\mu}{}_\nu = \frac{\tilde{E}}{3} \begin{pmatrix} 0 & 4w_j \left(v^{(1)} - B^{(1)}\right) \\ -4w^i v^{(1)} & \imath w^{(i}\hat{k}_{j)} \pi_T^{(1)} \end{pmatrix} e^{\imath \mathbf{kx}} \ , \tag{2.6b}$$

$$\delta T^{T\mu}{}_\nu = \frac{\tilde{E}}{3} \begin{pmatrix} 0 & 0 \\ 0 & t^i{}_j \pi_T^{(2)} \end{pmatrix} e^{\imath \mathbf{kx}} \ , \tag{2.6c}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$, with $k = |\mathbf{k}|$.

The metric components $A(\tau, k)$, $B^{(a)}(\tau, k)$, $H_L(\tau, k)$, $H_T^{(a)}(\tau, k)$ and the matter variables $\delta(\tau, k)$, $v^{(a)}(\tau, k)$, $\pi_L(\tau, k)$, $\pi_T^{(a)}(\tau, k)$ with $a = \{0, 1\}$ transform non-trivially under gauge transformations, i.e., coordinate transformations of the perturbed manifold while keeping the background fixed. Only the tensor components ($a = 2$) are gauge-invariant by themselves. The gauge transformations for scalar modes are

$$\begin{aligned} \bar{\tau} &= \tau + T(\tau, k) e^{\imath \mathbf{kx}} \ , \\ \bar{x}^i &= x^i - L^{(0)}(\tau, k) \imath \hat{k}^i e^{\imath \mathbf{kx}} \ , \end{aligned} \tag{2.7a}$$

and

$$\bar{x}^i = x^i + L^{(1)}(\tau, k) w^i e^{\imath \mathbf{kx}} \tag{2.7b}$$

for the vector perturbations. From (2.7a) and (2.7b) follows that only two scalar "potentials" and one vector "potential" of the metric perturbations are gauge-invariant. A set of such metric potentials is given by

$$\begin{aligned} \Phi &:= 2\left[H_L + \tfrac{1}{3}H_T^{(0)} + \tfrac{1}{x}(B^{(0)} - H_T'^{(0)})\right] , \\ \Pi &:= -\left[A + H_L + \tfrac{1}{3}H_T^{(0)} + (B^{(0)} - H_T'^{(0)})' + \tfrac{2}{x}(B^{(0)} - H_T'^{(0)})\right] \end{aligned} \tag{2.8a}$$

for the scalar part and by

$$\Psi := B^{(1)} - H_T'^{(1)} \tag{2.8b}$$

for the vector perturbations. The prime denotes a derivative with respect to

$$x := \tau k. \tag{2.9}$$

The variable x is related to the number of (half-) wavelengths inside the Hubble horizon

$$\frac{x}{\pi} = \frac{2}{\lambda} R_H,$$

and we shall later adopt it as a normalized time variable.

Φ and Π are translated into the variables used by Bardeen [4] by

$$\begin{aligned} \Phi_H &= \frac{1}{2}\Phi \\ \Phi_A &= -\Pi - \frac{1}{2}\Phi \ . \end{aligned}$$

In the same manner gauge-invariant matter variables can be defined. For the scalar perturbations we introduce:

$$\begin{aligned} \eta &:= \pi_L - \delta \\ v_s^{(0)} &:= v^{(0)} - H_T'^{(0)} \\ \epsilon_m &:= \delta + \tfrac{4}{x}\left(v^{(0)} - B^{(0)}\right) \\ \epsilon_g &:= \epsilon_m - \tfrac{4}{x} v_s^{(0)} \ . \end{aligned} \tag{2.10a}$$

η is called the entropy perturbation, although it may not coincide with the perturbation in the true physical entropy. $v_s^{(0)}$ is the amplitude of the matter velocity, and is related to the shear σ by $\sigma = v_s^{(0)} k/S$ [4]. ϵ_m is the density contrast on a spacelike hypersurface which represents the local rest-frame of matter everywhere, whereelse ϵ_g is the density perturbation on a hypersurface whose normal vector has no shear. The anisotropic pressure $\pi_T^{(0)}$ is gauge-invariant by itself.

For vector perturbations the invariant matter variables are

$$v_s^{(1)} := v^{(1)} - H_T'^{(1)} =: v_c + \Psi \ , \tag{2.10b}$$

where $v_s^{(1)}$ can be understood as in the scalar case and v_c is the velocity relative to the normal to the constant-time hypersurface. $v_c k/S$ is the intrinsic angular velocity (vorticity)

[4]. The anisotropic pressure $\pi_T^{(1)}$ is gauge-invariant as above. For tensor perturbations the only physical variables are given by the metric potential $H_T^{(2)} =: H$ and the anisotropic pressure $\pi_T^{(2)}$, both already gauge-invariant. (Here we follow the conventions of Ref. [4], which differ from the definitions of Ψ and H in Ref. [19] by a factor of 2.)

From the Einstein equations (1.1) relations between the metric potentials Φ, Π, Ψ, and H and the matter variables $\eta, \epsilon_m, \epsilon_g, \pi_T^{(a)}, v_s^{(a)}$, and v_c follow. The details are given in Ref. [4]. For the scalar components

$$\frac{x^2}{3}\Phi = \epsilon_m \tag{2.11a}$$

$$x^2\Pi = \pi_T^{(0)} \ , \tag{2.11b}$$

which shows the reason for our choice of the metric potentials: Φ is a potential for the density perturbation and Π the potential for the anisotropic pressure perturbation. Another set of equations is useful:

$$x^2\left(\Phi'' + \frac{4}{x}\Phi' + \frac{1}{3}\Phi + \frac{2}{x}\Pi' - \frac{2}{3}\Pi\right) = -\eta \tag{2.12a}$$

$$(x^2+3)\Phi + 3x\Phi' + 6\Pi = 3\epsilon_g \ . \tag{2.12b}$$

Covariant conservation of the perturbed energy-momentum tensor implies

$$v_s'^{(0)} + \frac{1}{x}v_s^{(0)} = \frac{1}{4}\left(\epsilon_m + \eta - \frac{2}{3}\pi_T^{(0)}\right) - \frac{1}{2}\Phi - \Pi. \tag{2.13}$$

For vector perturbations the Einstein equations read

$$\frac{x^2}{8}\Psi = v_c \tag{2.14}$$

and conservation of the energy-momentum tensor yields

$$v_c' = -\frac{1}{8}\pi_T^{(1)}. \tag{2.15}$$

Tensor perturbations fulfill

$$x^2\left(H'' + \frac{2}{x}H' + H\right) = \pi_T^{(2)} \ . \tag{2.16}$$

Our task in the next Sec. (kinetic theory) and in Sec. IV (thermal field theory) is to self-consistently determine the matter perturbations from a microscopic theory. But the simplest case is to close the above equations by assuming macroscopic equations of state. We end this Section by briefly recapitulating the solutions obtained with a perfect fluid.

In the case of a perfect fluid the anisotropic pressure vanishes ($\pi_T^{(a)} \equiv 0$ for all a). Additionally we assume adiabatic (isentropic) perturbations, i.e., $\eta = 0$. Eq. (2.11) implies $\Pi \equiv 0$, and Eq. (2.12a) reduces to an ordinary differential equation for the metric potential

Φ. The solution is given by a spherical Bessel function for regular behavior at $x = 0$ and a spherical von Neumann function for singular behavior in the origin:

$$\Phi = \frac{1}{x}\left(c^{(0)}j_1(\frac{x}{\sqrt{3}}) + d^{(0)}y_1(\frac{x}{\sqrt{3}})\right) ,$$

where $c^{(0)}$ and $d^{(0)}$ are constants. The regular solution approaches a constant for small x (superhorizon scales), which leads to a growing $\epsilon_m \sim x^2$ because of Eq. (2.11). Vector perturbations in a perfect fluid have constant v_c from (2.15), which is a consequence of the Helmholz-Kelvin circulation theorem [21]. Regular behavior of Ψ at $x = 0$ however implies $v_c = 0$, for otherwise, Ψ would be proportional to $1/x^2$. The tensor perturbations are freely propagating gravity waves; the solution of (2.16) is given by:

$$H = c^{(2)}j_0(x) + d^{(2)}y_0(x) ,$$

which can be regular or singular as $x \to 0$. Their amplitude decays ($\sim x^{-1}$) for large x in accordance with energy conservation [21].

III. KINETIC THEORY APPROACH

The framework for kinetic theory within general relativity was set up long ago, see, *e.g.*, Refs. [9,22]; a recent book on this subject is Ref. [23]. In the context of cosmological perturbations kinetic theory was first used by Peebels and Yu [8]. They discussed the coupled Boltzmann and Einstein equations for photons interacting with electrons via Thomson-scattering. For collisionless, massless neutrinos Stewart [10] derived the full set of Einstein-Vlasov equations. An extensive investigation of density perturbations with massive and massless neutrinos, radiation and other matter was performed numerically by Bond and Szalay [14].

Kinetic theory is applicable for particles whose de Broglie wavelength is smaller than their mean free path. For collisionless matter this is certainly the case. Moreover, the de Broglie wavelength $2\pi/p$ has to be smaller than the Hubble horizon $R_H = \tau \sim (GT^4)^{-1/2}$, *i.e.*, with

$$p^{-1} \sim T^{-1} \ll m_{PL}T^{-2} , \tag{3.1}$$

the temperature T has to be below the Planck scale [23].

In this Section we shall concentrate on the case of collisionless, massless matter; the results for the more general and more realistic case of a two-component system including a perfect fluid will be given later on. The massless case is somewhat exceptional in that it is rigorously consistent with a thermal equilibrium situation. A thermal equilibrium in an expanding universe can only be obtained in the case of massless particles, since the FLRW models provide a conformal timelike Killing-vector only [9], or in the nonrelativistic limit.

In setting up the kinetic approach to cosmological perturbations, we follow the paper by Kasai and Tomita [15]. They use gauge-invariant distribution functions [24], split up into scalar, vector and tensor modes. Other authors are usually working in a specific gauge [10–14]. An infinite set of gauge-invariant equations for the fluid variables describing density perturbations has been obtained in Ref. [25]. The authors of Ref. [15] use kinetic theory within the tetrad description of general relativity [22]. The tetrad is defined by $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. It spans an orthonormal basis in the tangent space.

The phase space of an ensemble of particles is described by the four space-time coordinates x^μ and the momentum components of the particles $p^\mu := dx^\mu/ds$, s being an affine parameter. Since the particles' energy is on the mass-shell $p^0 = |\mathbf{p}| =: p$ (for $m = 0$), the dimension of the phase space is seven. An invariant volume of momentum 3-space is given by $dp^1 dp^2 dp^3/p =: d^3p/p$, where the indices refer to the tetrad basis. The distribution function $F(\tau, \mathbf{x}, \mathbf{p})$ fulfills the Boltzmann equation

$$L(F) := \left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\nu\rho} p^\nu p^\rho \frac{\partial}{\partial p^\mu} \right) F(\tau, \mathbf{x}, \mathbf{p}) = C[F] \ . \tag{3.2}$$

The collision term C is a functional of F and has usually to be put in by hand. It is identically zero for collisionless matter. $\Gamma^\mu_{\nu\rho}$ denotes the usual Christoffel symbol.

The energy-momentum tensor is defined by

$$T^\mu{}_\nu := \int \frac{d^3p}{p} p^\mu p_\nu F \ . \tag{3.3}$$

In the collisionless case covariant conservation of $T^\mu{}_\nu$ follows from the Vlasov equation $L(F) = 0$. The perturbations of the energy-momentum tensor are given by

$$\delta T^\mu{}_\nu = \int \frac{d^3p}{p} \left(\tilde{p}^\mu \tilde{p}_\nu \delta F + (\delta p^\mu \tilde{p}_\nu + \tilde{p}^\mu \delta p_\nu) \tilde{F} \right) \ . \tag{3.4}$$

The perturbations of the particle momenta δp^μ are calculated with help of the inverse tetrad, which reads

$$e^\mu_a = \frac{1}{S} \left(\delta^\mu_a - \frac{1}{2} \delta g^\mu{}_a \right)$$

in linear theory. $\delta p^\mu = \delta e^\mu_a p^a$ relates δp^μ to the metric perturbations. In (3.4) the perturbed distribution function is $\delta F = F - \tilde{F}$, where $\tilde{F} = \tilde{F}(Sp)$ is the background distribution function. In thermal equilibrium, the latter is a Bose-Einstein or a Fermi-Dirac distribution depending on $S(\tau)p$, the energy of the particles in comoving coordinates, only. Thus

$$\tilde{F} = \frac{1}{e^{\beta S p} \pm 1} \ ,$$

with $\beta S(\tau)$ being the inverse temperature.

The decomposition into scalar, vector and tensor perturbations in the plane wave expansion is given by

$$\delta F(\tau, \mathbf{x}, \mathbf{p}) = \sum_{\mathbf{k}} \left(f^{(0)} + f^{(1)} \frac{\mathbf{p} \cdot \mathbf{w}}{p} + f^{(2)} \frac{\mathbf{p}^T \cdot \mathbf{t} \cdot \mathbf{p}}{p^2} \right) e^{\imath \mathbf{k}\mathbf{x}} \ . \tag{3.5}$$

With an isotropic background distribution, the $f^{(a)}$ depend on τ, the absolute values k and p, and the cosine $\mu := \mathbf{k} \cdot \mathbf{p}/kp$. For a given mode with wave vector $\mathbf{k}$, we write $f^{(a)} = f^{(a)}(x, p, \mu)$. Like the matter variables introduced in Eq. (2.6) the distribution function is not invariant under the gauge transformations (2.7).

A. Scalar perturbations

For scalar perturbations a gauge-invariant distribution function is given by

$$I^{(0)}(x,p,\mu) := f^{(0)} + p\frac{\partial \tilde{F}}{\partial p}\left(\frac{1}{x}(B^{(0)} - H_T'^{(0)}) + \imath\mu(\frac{1}{2}B^{(0)} - H_T'^{(0)})\right) , \tag{3.6}$$

or by

$$J(x,p,\mu) := I^{(0)} - \frac{p}{2}\frac{\partial \tilde{F}}{\partial p}\Phi . \tag{3.7}$$

According to the expansion theorem for Legendre functions,

$$I^{(0)}(x,p,\mu) = \sum_{n=0}^{\infty} a_n^{(0)}(x,p)\imath^n P_n(\mu) \tag{3.8}$$

for every $I^{(0)}(\mu) \in L^2[-1,1]$. The utility of (3.8) becomes apparent by calculating the gauge invariant matter variables $\epsilon_g, v_s^{(0)}$ and $\pi_T^{(0)}$ from (2.10a) and (3.4) (a more detailed derivation is given in [15]):

$$\epsilon_g = \frac{4\pi}{\tilde{E}}\int_0^\infty dp\, p^3 a_0^{(0)}(x,p) \tag{3.9a}$$

$$v_s^{(0)} = -\frac{\pi}{\tilde{E}}\int_0^\infty dp\, p^3 a_1^{(0)}(x,p) \tag{3.9b}$$

$$\pi_T^{(0)} = \frac{12\pi}{5\tilde{E}}\int_0^\infty dp\, p^3 a_2^{(0)}(x,p) . \tag{3.9c}$$

Here the orthogonality of the Legendre polynomials has been used. ϵ_m follows from ϵ_g and $v_s^{(0)}$. η vanishes, since $p^\mu p_\mu = 0$ in (3.4) and $\delta(p^\mu p_\mu) = 0$, as well. Inserting J into (3.2) for scalar perturbations leads to the equation

$$J' - \frac{1}{x}p^a\frac{\partial J}{\partial p^a} + \imath\mu J = -\imath\mu p\frac{\partial \tilde{F}}{\partial p}(\Phi + \Pi) , \tag{3.10}$$

where a takes the values $1,2,3$ and refers to the tetrad basis. Its solution is

$$J(x,p,\mu) = e^{-\imath\mu(x-x_0)}J(x_0,p,\mu) - \imath\mu p\frac{\partial \tilde{F}}{\partial p}\int_{x_0}^{x} dx'\,(\Phi + \Pi)(x')e^{-\imath\mu(x-x')} . \tag{3.11}$$

With

$$\epsilon_g = \frac{1}{\tilde{E}}\int\frac{d^3p}{p}p^2 J(x,p,\mu) - 2\Phi ,$$

and

$$\frac{1}{\tilde{E}}\int_0^\infty dp\, p^4 \frac{\partial \tilde{F}}{\partial p} = -\frac{1}{\pi}$$

the Einstein equations (2.12) read:

$$\Phi'' + \frac{2}{x}\Phi' + \Phi = \frac{2}{3}(\Phi + \Pi) - \frac{2}{x}(\Phi + \Pi)' \tag{3.12a}$$

$$(x^2+3)\Phi + 3x\Phi' = 6(\Phi+\Pi) - 12\int_{x_0}^x dx' j_0(x-x')(\Phi+\Pi)'(x') + 12\sum_{n=0}^\infty \beta_n^{(0)} j_n(x-x_0)\ . \tag{3.12b}$$

The spherical Bessel functions $j_n(x-x_0)$ are the Fourier transforms of the Legendre polynomials in Eq. (3.8). The last term on the r.h.s. determines the initial conditions. The coefficients $\beta_n^{(0)}$ are defined by

$$\begin{aligned} \beta_0^{(0)} &:= \tfrac{\pi}{\tilde{E}}\textstyle\int_0^\infty dp\, p^3 a_0^{(0)}(x_0,p) - \Pi(x_0) - \tfrac{1}{2}\Phi(x_0)\ , \\ \beta_n^{(0)} &:= \tfrac{\pi}{\tilde{E}}\textstyle\int_0^\infty dp\, p^3 a_n^{(0)}(x_0,p)\ , \qquad n \geq 1\ . \end{aligned} \tag{3.13}$$

B. Vector perturbations

Here we use the gauge-invariant distribution function

$$I^{(1)}(x,p,\mu) := f^{(1)} + \frac{p}{2}\frac{\partial \tilde{F}}{\partial p} B^{(1)} \tag{3.14}$$

and expand it, as above, into Legendre polynomials in μ

$$I^{(1)}(x,p,\mu) = \sum_{n=0}^\infty a_n^{(1)}(x,p)\imath^n P_n(\mu)\ . \tag{3.15}$$

In terms of the coefficients $a_n^{(1)}$, the matter variables are

$$v_c = \frac{\pi}{\tilde{E}}\int_0^\infty dp\, p^3 \left(a_0^{(1)}(x,p) + \frac{1}{5}a_2^{(1)}(x,p)\right)\ , \tag{3.16a}$$

$$\pi_T^{(1)} = -\frac{24\pi}{5\tilde{E}}\int_0^\infty dp\, p^3 \left(\frac{1}{3}a_1^{(1)}(x,p) + \frac{1}{7}a_3^{(1)}(x,p)\right)\ . \tag{3.16b}$$

For vector perturbations Eq. (3.2) reads:

$$I'^{(1)} - \frac{1}{x}p^a \frac{\partial I^{(1)}}{\partial p^a} + \imath\mu I^{(1)} = \imath\mu p \frac{\partial \tilde{F}}{p}\Psi\ . \tag{3.17}$$

Its solution is:

$$I^{(1)}(x,p,\mu) = e^{-\imath\mu(x-x_0)} I^{(1)}(x_0,p,\mu) + \imath\mu p \frac{\partial \tilde{F}}{\partial p} \int_{x_0}^{x} dx' \Psi(x') e^{-\imath\mu(x-x')} \ . \tag{3.18}$$

With this solution inserted into v_c, the Einstein eq. (2.14) becomes

$$x^2\Psi = -24 \int_{x_0}^{x} dx' \frac{j_2(x-x')}{x-x'} \Psi(x') + 12 \sum_{n=0}^{\infty} \beta_n^{(1)} \left(j_n(x-x_0) + j_n''(x-x_0)\right) \ . \tag{3.19}$$

The initial conditions for the Volterra type integral equation (3.19) are given by the coefficients

$$\beta_n^{(1)} = \frac{\pi}{\tilde{E}} \int_0^{\infty} dp\, p^3 a_n^{(1)}(x_0,p) \ . \tag{3.20}$$

C. Tensor perturbations

For tensor perturbations the distribution function $f^{(2)}$ is gauge-invariant itself. With its expansion into Legendre polynomials

$$f^{(2)}(x,p,\mu) = \sum_{n=0}^{\infty} a_n^{(2)}(x,p) \imath^n P_n(\mu) \tag{3.21}$$

the anisotropic pressure reads:

$$\pi_T^{(2)} = \frac{24\pi}{15\tilde{E}} \int_0^{\infty} dp\, p^3 \left(a_0^{(2)}(x,p) + \frac{2}{7} a_2^{(2)}(x,p) + \frac{1}{21} a_4^{(2)}(x,p) \right) \ . \tag{3.22}$$

The Vlasov equation for the tensor perturbations reads:

$$f'^{(2)} - \frac{1}{x} p^a \frac{\partial f^{(2)}}{\partial p^a} + \imath\mu f^{(2)} = p \frac{\partial \tilde{F}}{\partial p} H' \ . \tag{3.23}$$

It is solved by

$$f^{(2)}(x,p,\mu) = e^{-\imath\mu(x-x_0)} f^{(2)}(x_0,p,\mu) + p \frac{\partial \tilde{F}}{\partial p} \int_{x_0}^{x} dx' H'(x') e^{-\imath\mu(x-x')} \ . \tag{3.24}$$

Inserting this into (2.16) yields

$$\begin{aligned} x^2 \left(H'' + \frac{2}{x} H' + H \right) &= -24 \int_{x_0}^{x} dx' \frac{j_2(x-x')}{(x-x')^2} H'(x') \\ &\quad + 3 \sum_{n=0}^{\infty} \beta_n^{(2)} \left(j_n(x-x_0) + 2 j_n''(x-x_0) + j_n^{(IV)}(x-x_0) \right) . \end{aligned} \tag{3.25}$$

The coefficients encoding the initial conditions are

$$\beta_n^{(2)} = \frac{\pi}{\tilde{E}} \int_0^{\infty} dp\, p^3 a_n^{(2)}(x_0,p) \ . \tag{3.26}$$

IV. THERMAL-FIELD-THEORY APPROACH

Apart from some notable exceptions [24,26,25] all results given in the literature within kinetic theory have been derived by choosing a specific gauge. As was shown in the preceding section, this dependence on the gauge may be circumvented by completing the program lined out by Kasai and Tomita [15]. As we have seen, this requires a skillful redefinition of the basic distribution function.

A manifestly gauge-invariant approach is provided by thermal field theory. For a system containing gravity as well as matter at a temperature T below the Planck scale ($T \ll m_{PL}$), the effective action (*i.e.* including all radiative contributions) $\Gamma[g]$ can be split in a part describing classical gravity, the Einstein-Hilbert action S^G itself, and an effective action $\Gamma[g]^M$ induced by the thermal matter with classical action S^M,

$$\Gamma|_{T\ll m_{PL}} = S^G + \Gamma^M \ . \tag{4.1}$$

This is automatically gauge-invariant provided any gauge fields are subject to background-covariant gauge conditions.

Γ^M does not depend on curvature effects in the leading-order temperature contribution. In the radiation dominated regime, the Ricci tensor is proportional to GT^4. Thus for temperatures below the Planck scale ($T \ll m_{PL}$) curvature effects are lower order in temperature, i.e. $\ll T^2$.

The energy-momentum tensor is given by

$$T^{\mu\nu}(x) := \frac{2}{\sqrt{-g}} \frac{\delta \Gamma^M}{\delta g_{\mu\nu}} \ , \tag{4.2}$$

and self-consistent perturbations thereof have to fulfill

$$\delta T^\mu{}_\nu(x) = \int_{x'} \frac{\delta T^\mu{}_\nu(x)}{\delta g_{\rho\sigma}(x')} \delta g_{\rho\sigma}(x') \ . \tag{4.3}$$

This relates the perturbed matter variables to the thermal graviton self-energy which is defined by

$$\sqrt{-g}\Pi^{\mu\nu\rho\sigma}(x,x') := \frac{1}{2} \frac{\delta(\sqrt{-g}T^{\mu\nu}(x))}{\delta g_{\rho\sigma}(x')} \ . \tag{4.4}$$

A perturbative expansion in Feynman diagrams is appropriate for weakly interacting matter. Higher loop orders due to internal graviton propagators are suppressed by powers of $GT^2 \ll 1$ for $T \ll m_{PL}$. A high-temperature expansion of Γ^M in the sense of $k \ll T$ is appropriate for the study of cosmological perturbations, because the external scale is set by the Hubble horizon $R_H = \tau \sim (GT^4)^{-1/2}$, which is $\gg T^{-1}$ for temperatures well below the Planck scale.

First attempts to calculate the thermal gravity self-energy have been undertaken in Refs. [27]. A complete calculation of the leading high-temperature contribution was given first by one of the present authors in Ref. [17]. For this only one-loop diagrams without internal graviton lines need to be computed. Similar issues have been studied in Refs. [28].

Recently, de Almeida et al. [29] have computed the next-to-leading order contributions for radiation and bosonic matter in the massive, but still collisionless case.

By calculating the graviton self-energy from thermal field theory on the cosmological background, one can obtain the r.h.s. of the Einstein equations (1.1) from (4.3) using the definition (4.4). In the kinetic theory approach the Boltzmann-Einstein system of equations has to be solved simultaneously. This can be done analytically for collisionless, massless matter, but in general only numerical solutions can be obtained. In the thermal field theory approach, the corresponding problem is to calculate the graviton self-energy on a curved background space-time. An explicit reference to perturbations of distribution functions is completely obviated, which might turn out to be useful when more than the classical limit is of interest. A complete evaluation of the thermal graviton self-energy in curved space is certainly a formidable task. However, restricting our attention to collisionless, massless matter again, one can take advantage of the fact that the high-temperature limit of the effective action possesses, in addition to diffeomorphism invariance, an invariance under conformal transformations, which is expressed by the Ward identity [16,19]

$$\Pi^{\alpha\beta\gamma}{}_{\gamma} = 2T^{\alpha\beta} \ .$$

Since the Einstein-de Sitter background is conformally flat, the graviton self-energy can be simply calculated by evaluating $\Pi_{\alpha\beta\gamma\delta}$ on flat space-time and transferring it to the curved space by multiplication with cosmic scale factors S. This calculation was done by one of the authors in Refs. [17,19]. We will not repeat it here, but sketch the most important steps of its derivation.

The graviton self-energy $\Pi_{\alpha\beta\gamma\delta}$ is related to the Fourier transformed self-energy in flat space-time $\bar{\Pi}_{\alpha\beta\gamma\delta}$ by

$$\Pi^{\mu\nu\rho\sigma}(x,x')\big|_{g=S^2\eta} = S^{-2}(\tau)\int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')}\, \bar{\Pi}^{\mu\nu\rho\sigma}(k)\Big|_{\eta} S^{-2}(\tau') \ . \tag{4.5}$$

In the high-temperature limit, its tensorial structure turns out to be the same for any field theory and is given by

$$\bar{\Pi}^{\mu\nu\rho\sigma}\Big|_{g=\eta} = \frac{1}{2} I^{\mu\nu\rho\sigma} - \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{(\rho}_{\beta}\eta^{\sigma)\nu} + \delta^{\nu}_{\alpha}\delta^{(\rho}_{\beta}\eta^{\sigma)\mu}\right)\eta_{\gamma\delta} I^{\alpha\beta\gamma\delta} \ , \tag{4.6}$$

which follows in a straightforward calculation from the Feynman rules of the imaginary time formulation (see e.g. [30]) of thermal field theory. The totally symmetric quantity $I^{\alpha\beta\gamma\delta}$ is given by

$$I^{\alpha\beta\gamma\delta}(k) = T \sum_{n=\frac{p_0}{2\pi i T}} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{p^\alpha p^\beta p^\gamma p^\delta}{p^2 (p-k)^2} \ . \tag{4.7}$$

Sum and integral are conveniently performed with help of the so-called Dzyaloshinski algorithm [31,30]. Its values are tabulated in the appendix of Ref. [17]. They are polynomials in k_0/k times

$$Q_0(\frac{k_0}{k}) = \frac{1}{2}\ln\left(\frac{k_0+k}{k_0-k}\right) \ ,$$

the zeroth Legendre function of second kind. These may be rewritten with help of the formula

$$Q_n(z) = P_n(z)Q_0(z) - \sum_{m=1}^{n} \frac{1}{m} P_{m-1}(z) P_{n-m}(z)$$

in terms of higher Legendre functions of second kind plus polynomials.

Like all other quantities, the graviton self-energy is expanded into plane waves. Therefore, only the Fourier transformation of k_0 into τ in (4.5) has to be performed. The integrals involved are

$$\lim_{\gamma\to 0^+} \frac{1}{2\pi} \int_{-\infty+\imath\gamma}^{\infty+\imath\gamma} d\omega e^{-\imath\omega(x-x')} Q_n(\omega) = (-\imath)^{n+1} \Theta(x-x') j_n(x-x') \ ,$$

where $\omega = k_0/k$ and retarded boundary conditions are imposed.

In Eq. (4.3) this leads to convolution integrals of the form

$$\sum_n \alpha_n \int^{\infty} dx' \Theta(x-x') j_n(x-x') \{\Phi, \Pi, \Psi, H\}'(x').$$

There has to be a finite non-negative integration bound x_0 because $\tau \geq 0$. On the other hand, the functional derivative in equation (4.3) is usually defined without restriction. For that reason we have to put in the initial conditions for x_0 by hand. We do this by the replacement of the metric potentials by

$$\{\Phi, \Pi, \Psi, H\}'(x) \to \{\Phi, \Pi, \Psi, H\}'(x) \Theta(x-x_0) + \sum_{n=0}^{\infty} \gamma_n^{(a)} \delta^{(n)}(x-x_0) \ .$$

The infinite sum provides the initial conditions for an formally infinite-order differential equation, which is obtained differentiating the convolution integral above (see [19]). The infinite sum yields derivatives of spherical Bessel functions, which can be rewritten into spherical Bessel functions.

A. Scalar perturbations

To arrive at the expressions for the matter perturbations in thermal field theory, we have to relate them to the graviton self-energy through Eq. (4.3). The matter variables on the r.h.s. of Eqs. (2.12) read:

$$\eta = 0$$

and

$$\epsilon_g = 2\Phi + 4\Pi - 4 \int_{x_0}^{x} dx' j_0(x-x') \left(\Phi + \Pi\right)'(x') - 4 \sum_{n=0}^{\infty} \gamma_n^{(0)} j_0^{(n)}(x-x_0) \ . \qquad (4.8)$$

Inserting this into (2.12), the same equations as in the kinetic approach, Eq. (3.12), are obtained, except that the initial conditions are parametrized differently. The coefficients $\gamma_n^{(0)}$ can be related to the $\beta_n^{(0)}$ appearing in Eq. (3.12) through the formula

$$j_n' = \frac{1}{2n+1}\left(n j_{n-1} - (n+1) j_{n+1}\right), \tag{4.9}$$

by a linear transformation which is non-singular and of upper-triangular form (albeit infinite-dimensional). If $\beta_n^{(0)} = 0$ for all $n \geq 3$, which is a frequently adopted simplifying assumption in the kinetic approach (the so-called 14-moment approximation, see, e.g., Ref. [32]), then this relationship is given by

$$\begin{aligned} \gamma_0^{(0)} &= -\beta_0^{(0)} - \frac{1}{2}\beta_2^{(0)}, \\ \gamma_1^{(0)} &= \beta_1^{(0)}, \\ \gamma_2^{(0)} &= -\frac{3}{2}\beta_2^{(0)}. \end{aligned} \tag{4.10}$$

B. Vector perturbations

In a similar manner we obtain for vector perturbations

$$v_c = -3\int_{x_0}^{x} dx' \frac{j_2(x-x')}{x-x'}\Psi(x') + \sum_{n=0}^{\infty} \gamma_n^{(1)} \left(j_0(x-x_0) + j_2(x-x_0)\right)^{(n)} \tag{4.11}$$

which together with (2.14) reproduces (3.19), if the coefficients $\gamma_n^{(1)}$ are re-expressed in terms of the $\beta_n^{(1)}$ which parametrize the initial conditions in Eq. (3.19). Because of $j_0 + j_0'' = \frac{2}{3}(j_0 + j_2)$, the relationship between the $\gamma^{(1)}$ and the $\beta^{(1)}$ is, apart from an over-all sign, the same as in the scalar case. Hence, we have

$$\begin{aligned} \gamma_0^{(1)} &= \beta_0^{(1)} + \frac{1}{2}\beta_2^{(1)}, \\ \gamma_1^{(1)} &= -\beta_1^{(1)}, \\ \gamma_2^{(1)} &= \frac{3}{2}\beta_2^{(1)}. \end{aligned} \tag{4.12}$$

C. Tensor perturbations

In this case

$$\pi_T^{(2)} = -24\left[\int_{x_0}^{x} dx' \frac{j_2(x-x')}{(x-x')^2} H'(x') + \sum_{n=0}^{\infty} \gamma_n^{(2)} (\frac{1}{15} j_0 + \frac{2}{21} j_2 + \frac{1}{35} j_4)^{(n)}(x-x_0)\right], \tag{4.13}$$

which with (2.16) leads again to Eq. (3.25). Because of $j_0 + 2j_0'' + j_0^{(IV)} = 8(\frac{1}{15}j_0 + \frac{2}{21}j_2 + \frac{1}{35}j_4)$, the first three coefficients $\gamma_n^{(2)}$ and $\beta_n^{(2)}$ are related by

$$\begin{aligned} \gamma_0^{(2)} &= -\beta_0^{(2)} - \frac{1}{2}\beta_2^{(2)}, \\ \gamma_1^{(2)} &= \beta_1^{(2)}, \\ \gamma_2^{(2)} &= -\frac{3}{2}\beta_2^{(2)}, \end{aligned} \tag{4.14}$$

if all higher coefficients are zero, exactly as in the scalar case.

Summarizing, the kinetic theory approach in the gauge invariant formulation of Ref. [15] and the thermal-field-theory approach in the high temperatur limit lead to the same equations for cosmological perturbations of a Einstein-de Sitter universe filled with collisionless, massless particles. There is a difference, however, in the way the initial conditions are introduced, which may seem somewhat arbitrary in the field theoretical approach. In fact, the initial condition terms in Eqs. (3.12), (3.19), and (3.25) are completely arbitrary functions due to the von Neumann expansion theorem. We could therefore just have added the inhomogeneous terms by hand at the end of the calculation, but as we have seen the way in which we have implemented them there is still a close relation to the inital moments of the initial distribution function of the kinetic approach.

V. COLLISIONLESS MATTER AND A PERFECT FLUID

The universe contains, besides collisionless particles, various different forms of matter as radiation and baryons. An example for a collisionless, massless gas is provided by neutrinos after their decoupling below the electroweak scale. Another example are background gravitons, left over from the epoch of quantum gravity. In both examples all other matter may be described by a perfect fluid during a certain epoch, while collisions keep it close to thermal equilibrium.

To model such systems we derive the equations for cosmological perturbations evolving in a mixture of a collisionless, massless gas and a perfect fluid. We assume that the gas and the fluid interact only through gravitational forces, thus

$$T^{\mu}{}_{\nu} = T_{\rm cll}{}^{\mu}{}_{\nu} + T_{\rm pf}{}^{\mu}{}_{\nu} \ . \tag{5.1}$$

To describe the radiation-dominated epoch of the universe, we assume that the components of the background perfect-fluid energy-momentum tensor $\tilde{T}_{\rm pf}{}^{\mu}{}_{\nu}$ obey the same equation of state as the collisionless gas components, i.e., $\tilde{E}_{\rm pf} = 3\tilde{P}_{\rm pf}$. For both perturbed energy-momentum tensors, $\delta T_{\rm cll}{}^{\mu}{}_{\nu}$ and $\delta T_{\rm pf}{}^{\mu}{}_{\nu}$, gauge-invariant matter variables are defined as in Sec. II. We define the ratio of the collisionless background energy-density to the total background energy-density to be

$$\alpha := \frac{\tilde{E}_{\rm cll}}{\tilde{E}_{\rm cll} + \tilde{E}_{\rm pf}} \ . \tag{5.2}$$

In order to gain a closed set of gauge-invariant equations we allow only for adiabatic (isentropic) perturbations, i.e., $\eta = \alpha\eta_{\rm cll} + (1-\alpha)\eta_{\rm pf} = 0$. Entropy perturbations would arise, e.g., from phase transitions [6], which lie beyond the scope of this work.

The extension of Bardeens [4] gauge-invariant formalism to a multi-fluid universe was given in Refs. [5,33]. In connection with a collisionless gas this was done in Ref. [15]. Since we deal only with two components, fulfilling the same background equation of state, the more general formalism of these references is not necessary here. The equations we are looking for have been obtained by one of the present authors in Ref. [19], but without including the most general initial conditions. The following gauge-invariant equations are derived from (1.1) with (5.1).

A. Scalar perturbations

To arrive at a closed set of equations for scalar perturbations we define $\Phi_{\rm pf}+\Phi_{\rm cll}:=\Phi$ with help of (2.11a), i.e.,

$$\frac{x^2}{3}\Phi_{\rm cll}=\alpha\epsilon_{m\ \rm cll}\ . \tag{5.3}$$

Eq. (2.12a) does not change since we deal with adiabatic perturbations only. With (2.12a), (5.3), and (2.11b), $\Phi, \Phi_{\rm cll}$, and Π can be calculated from $\epsilon_{m\ \rm cll}$ and $\pi^{(0)}_{T\ \rm cll}$. The anisotropic pressure of the perfect fluid is zero from its definition. Within the field-theoretic approach from Eqs. (5.3) and (2.11b)

$$\frac{x^2}{3}\Phi_{\rm cll}=-2\alpha\Big[-\Phi-2\Pi+2\int_{x_0}^{x}dx'\left(j_0(x-x')+\frac{3}{x}j_1(x-x')\right)(\Phi+\Pi)'(x')$$
$$+2\sum_{n=0}^{\infty}\gamma_n^{(0)}(j_0+\frac{3}{x}j_1)^{(n)}(x-x_0)\Big] \tag{5.4a}$$

and

$$x^2\Pi=-12\alpha\Big[\int_{x_0}^{x}dx'j_2(x-x')(\Phi+\Pi)'(x')+\sum_{n=0}^{\infty}\gamma_n^{(0)}j_2^{(n)}(x-x_0)\Big] \tag{5.4b}$$

follow. In terms of $\beta_n^{(0)}$'s (cf. Sec. III) the inhomogeneous terms respectively read:

$$4\alpha\sum_{n=0}^{\infty}\beta_n^{(0)}(j_n-\frac{3}{x}j_n')(x-x_0) \tag{5.5a}$$

and

$$6\alpha\sum_{n=0}^{\infty}\beta_n^{(0)}(j_n+3j_n'')(x-x_0)\ . \tag{5.5b}$$

B. Vector perturbations

For vector perturbations (2.14) reads:

$$\frac{x^2}{8}\Psi=\alpha\left[-3\int_{x_0}^{x}dx'\frac{j_2(x-x')}{x-x'}\Psi(x')+\sum_{n=0}^{\infty}\gamma_n^{(1)}\left(j_0+j_2\right)^{(n)}(x-x_0)\right]+(1-\alpha)v_{c\ \rm pf}\ . \tag{5.6}$$

The covariant conservation of the energy-momentum tensor (2.15) is valid for the gas and the fluid seperately, i.e., $v_{c\ \rm pf}=const.$ In terms of $\beta_n^{(1)}$ the infinite sum reads:

$$\frac{3}{2}\alpha\sum_{n=0}^{\infty}\beta_n^{(1)}\left(j_n+j_n''\right)(x-x_0)\ .$$

C. Tensor perturbations

Since the anisotropic pressure for the perfect fluid component vanishes, (2.16) yields

$$x^2\left(H''+\frac{2}{x}H'+H\right) = -24\alpha\Big[\int_{x_0}^{x} dx' \frac{j_2(x-x')}{(x-x')^2}H'(x') + \sum_{n=0}^{\infty}\gamma_n^{(2)}(\frac{1}{15}j_0+\frac{2}{21}j_2+\frac{1}{35}j_4)^{(n)}(x-x_0)\Big] \; . \tag{5.7}$$

In terms of $\beta_n^{(2)}$ the inhomogeneity reads:

$$3\alpha\sum_{n=0}^{\infty}\beta_n^{(2)}\left(j_n+2j_n''+j_n^{(IV)}\right)(x-x_0) \; . \tag{5.8}$$

The relation between the coefficients $\beta^{(a)}$ and $\gamma^{(a)}$ is the same as in the purely collisionless case.

VI. SOLUTIONS

The above integro-differential equations cannot, in general, be solved by, say, a power series ansatz about x_0. Instead various methods have been employed to solve them approximately: direct numerical integration [11,14,26]; approximation by a finite sum of spherical Bessel functions [10]; or by a finite system of ordinary differential equations [25].

However, for $x_0 \to 0$, which usually is the most interesting point to define initial conditions anyway, a generalized power series ansatz can be solved recursively. Exact regular solutions were found in this way in Refs. [16,19], where it turned out that the power series involved have infinite radius of convergence, and converge faster than trigonometric functions do. In the following, this will be generalized to include singular solutions as well as the most general initial data at $x_0 = 0$. Some of the more unwieldy details are relegated to the Appendix.

If $F(x)$ represents one of the metric potentials Φ, Π, Ψ, H, the general solution with $x_0 = 0$ turns out to be of the form

$$F(x) = C_1 F_{\text{reg}}(x) + C_2 x^{\sigma} F_{\text{sing}}(x) \tag{6.1a}$$

with

$$F_{\text{reg,sing}}(x) = \sum_{n=0}^{\infty} c_n^{\text{reg,sing}} \frac{x^n}{n!} \tag{6.1b}$$

and C_1, C_2 arbitrary constants. This form follows from demanding that the coefficients $\beta_n^{(a)}$ remain finite as $x_0 \to 0$.

When inserted into the integro-differential equations (5.4,5.6,5.7) this leads to solvable recursion relations for the coefficients c_n such that $c_n = c_n(c_0, \ldots, c_{n-1}; \alpha, \{\beta\})$, because the power series representation of the spherical Bessel functions,

$$j_k(x-x') = \sum_{m=0}^{\infty} \frac{(-1)^m (x-x')^{2m+k}}{(2m)!(2m+1)(2m+3)\cdots(2m+2k+1)}, \tag{6.2}$$

leads to integrals

$$\int_0^x dx'(x-x')^m x'^\nu = \frac{m!}{(1+\nu)\cdots(1+m+\nu)} x^{1+m+\nu}, \tag{6.3}$$

where m is a natural number and ν arbitrary.

In solving these recursion relations it turns out that the coefficients $\beta_n^{(a)}$ (or, equivalently, $\gamma_n^{(a)}$) in Eqs. (5.4,5.6,5.7) either have to be all zero, which also puts to zero $F_{\rm reg}$ except for $H_{\rm reg}$, or have to satisfy certain constraints dictated by the values $F_{\rm reg}(0)$. In each case a singular part in Eq. (6.1a) exists that can be superimposed on the regular solutions by choosing $C_2 \neq 0$. With $x_0 = 0$, only the regular part depends on the initial data. The presence of singular perturbations means that the initial singularity is no longer approximately one of FLRW type, but is essentially anisotropic. Linear perturbation theory then applies only for those values of x where F has become sufficiently small.

It turns out that the singular behavior for $x \to 0$ is given by

$$\sigma_\pm^{(a)} = -\frac{5}{2} + a \pm \sqrt{\frac{5-32\alpha}{20}}, \tag{6.4}$$

with $a = 0,1,2$ for $(a) = S, V, T$. If the fraction α of energy density contained in collisionless matter exceeds $\alpha_{\rm crit.} = \frac{5}{32}$, σ becomes complex, leading to an essential singularity at $x = 0$ with F oscillating like $\cos([{\rm Im}\sigma]\ln x)$. This asymptotic behavior has been found previously by Zakharov [12] and examined further by Vishniac [13].[1] As explained by the latter, the essentially singular, oscillatory behavior for $x \to 0$ arises because the momentum flux carried by the collisionless particles depends preferentially on the expansion of the universe in their direction of travel, instead of the net expansion of the volume, which leads to an excessive feedback for $\alpha > \alpha_{\rm crit.}$. Thus in this case, the initial singularity is of a mixmaster type [34]. It has been shown by Lukash et al. [35] that collisionless matter provides an efficient means to disperse the initially strong anisotropy of such models; the singular parts of our solutions (6.1a) could therefore correspond to the later stage of such a scenario where the linear regime has finally been reached.

A. Scalar perturbations

Regular solutions for scalar perturbations, where $\Phi(x)$ and $\Pi(x)$ stay finite for $x \to 0$, are possible only when the coefficients $\beta_n^{(0)}$ in Eqs. (5.4) and (5.5) satisfy certain constraints. Evaluation of Eqs. (5.4) and their first derivatives at $x = x_0 = 0$ yields restrictions on the initial matter distribution, i.e.,

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[1] The result of Eq. (6.4) agrees with the asymptotic behavior found in Ref. [12] for scalar, vector, and tensor perturbations, but not with Ref. [13], where the vector case was reported to have a different critical value $\alpha_{\rm crit.}$.

$$\beta_1^{(0)} = \beta_2^{(0)} = \beta_3^{(0)} = 0 \ , \tag{6.5}$$

and relations between the initial values of the metric potentials

$$\Phi(0) + 2\Pi(0) = -4\beta_0^{(0)} \tag{6.6a}$$

and

$$2\Phi'(0) = -\Pi'(0) \ . \tag{6.6b}$$

The restrictions on $\beta_0^{(0)}$, $\beta_1^{(0)}$, and $\beta_2^{(0)}$ can be understand immediately from the Einstein equations written in the form of Eqs. (2.11b) and (2.12b) together with covariant conservation of the energy-momentum tensor, Eq. (2.13), by inserting the definitions in Eqs. (3.9) and (3.13); the vanishing of $\beta_3^{(0)}$ is a consequence of demanding regularity of Π in Eq. (5.4b).

The explicit recursion relations defining $\Phi_{\rm cll}$, $\Phi_{\rm pf}$, and Π follow from the Eqs. (3.12a) and (5.4). They are listed in the first part of the Appendix. The recursion relations determine the regular parts of all potentials uniquely for a given set of $\beta_n^{(0)}$'s, respecting Eq. (6.5) (or equivalently a set of $\gamma_n^{(0)}$'s, satisfying more involved conditions). The singular solutions do not depend on the inhomogeneous terms. Thus they can be added to any regular solution. Singular solutions for the metric potentials with poles of first or second order are forbidden for $x_0 = 0$ since the integral (6.3) is then divergent for all negative integers ν. For finite x_0, however such solutions exist. (An example can be found in Ref. [13], Eq. (40b).)

In Fig. 1, regular solutions for ϵ_m, which coincides with the energy-density contrast δ on comoving hypersurfaces, are given in the purely collisionless case for three different initial data: one where only $\beta_0^{(0)}$ is nonvanishing, a second where the first four coefficients have been put to zero, and a more generic one. All of them grow according to a power-law on superhorizon scales, where the second (dashed-line) solution exemplifies that an arbitrarily high power can be achieved by appropriately contrived initial date; on sub-horizon scales all solutions undergo damped oscillations whose amplitude falls off like $1/x$ eventually. The exact matching of the allowed asymptotic behaviors is however seen to depend strongly on the form of the initial data. The asymptotic regimes themselves are determined by rather different physical situations. On superhorizon scales, $x \ll 1$, everything can be viewed as being determined by the global geometry set up in accordance with the inital matter distribution. For $x \gg 1$, microphysics becomes important, and the damped oscillations there can be understood as the dispersion of density contrast carried by the collisionless particles through directional dispersion [14]. The intermediate growth of the amplitude of the dotted solution in Fig. 1 after its 5th maximum is due to a nonzero $\beta_{20}^{(0)}$. This demonstrates that higher momenta in the initial distribution function may have considerable effects even at subhorizon scales. [2]

In Fig. 2, a particular regular solution is shown in the two-component case with equal energy density in the perfect-fluid and in the collisionless component. In order that the collisionless component can be matched smoothly to a perfect-fluid behavior for $x \to 0$ (in

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[2]The sometimes used 14-moment approximation [25] therefore potentially misses important details in the evolution of cosmological perturbations.

reality there will be a finite time and therefore a finite value of $x \neq 0$, where the collisionless component decouples), the initial data have been chosen so that $\Pi(0) = 0$, which makes $\Phi_{\rm pf}(0) = \Phi_{\rm cll}(0)$. After horizon-crossing, the energy-density contrast that remains in the perfect-fluid component is seen to be undamped and to have a smaller phase velocity, namely $c/\sqrt{3}$, in contrast to c in the strongly damped collisionless component. The ratio of $\delta_{\rm cll}$ to $\delta_{\rm pf}$ at horizon crossing depends strongly on the initial data chosen. In Fig. 2 the collisionless component dominates over the perfect-fluid component up to fairly large values of x; the solutions presented in Refs. [20] show the opposite behavior. [3] This is due to having chosen $\beta_5^{(0)} \neq 0$ in the case of Fig. 2.

In Figs. 3–5, singular solutions are displayed. These do not depend on the initial matter distribution at $x_0 = 0$, which underlines their geometrical nature. In Fig. 3, the energy-density contrast δ and the anisotropic pressure π_T is plotted for the purely collisionless case $\alpha = 1$. There are now oscillations also on superhorizon scales as discussed above, which go over to damped oscillations with phase velocity equal to c on subhorizon scales, showing the same asymptotic behavior there as did the regular solutions. In Fig. 4, the singular solutions are plotted for $\alpha = 5/32$, where the perfect-fluid component is just strong enough to eliminate the superhorizon oscillations. For smaller values of α, there a two essentially different solutions with different degree of singularity. For $\alpha = 1/10$, they are rendered in Fig. 5. In the two-component cases of Figs. 4 and 5, the perfect-fluid component turns out to have a regular limit for $x \to 0$, growing until horizon crossing, after which they again turn into undamped acoustic waves with phase velocity $c/\sqrt{3}$.

B. Vector perturbations

The requirement of regularity yields the conditions

$$\frac{\alpha}{\alpha - 1}\left(\beta_0^{(1)} + \frac{1}{5}\beta_2^{(1)}\right) = v_{c\ \rm pf} \tag{6.7a}$$

and

$$\beta_1^{(1)} = -\frac{3}{7}\beta_3^{(1)} \tag{6.7b}$$

for the vector perturbations. These conditions follow from Eq. (5.6) and its first derivative at $x = x_0 = 0$. The singular solutions may be added as in the scalar case.

For the perfect-fluid case $\alpha = 0$ there are no regular solutions as mentioned at the end of Sec. II. However, with collisionless matter, vorticity can be generated on super-horizon scales, which after horizon-crossing dies out through directional dispersion. This is shown for various initial data in Fig. 6 for $\alpha = 1/2$. The growth in $v \sim x^2$ at superhorizon scales is brought about by $\Psi(x)$ approaching a constant for $x \to 0$. A more rapid growth arises when Ψ itself goes like a positive power of x, which is the case for $\beta_0^{(1)} = 0$ and nonvanishing

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[3]The author of Ref. [26] claims that $\pi_T^{(0)} < \epsilon_m$ in general, which is seen to hold true for certain initial conditions only.

higher coefficients. One such example is shown by the dashed line in Fig. 6. In all these cases there is a genuine production of vorticity, which in the perfect-fluid case is forbidden by the Helmholtz-Kelvin circulation theorem [36].

With different initial matter distributions it is also possible to arrange for a nonvanishing initial vorticity in the perfect-fluid component, which is compensated on superhorizon scales by an equal amount with opposite sign in the collisionless component. A net vorticity then survives on subhorizon scales after the dispersion of the collisionless part. In Fig. 7 such solutions are plotted for several initial conditions. This interesting possibility has been studied extensively in Refs. [19,20], to which we refer the reader for more details. Here we just mention that the vorticity in the perfect-fluid component which by the presence of collisionless matter can be reconciled with an initial FLRW-type singularity, gives rise to magnetic fields during the transition to the matter-dominated area, and the limits set by the anisotropy of the cosmic microwave background are such that astrophysically interesting amounts of primordial magnetic fields seem possible [20].

In Fig. 8, we display the singular vector solutions for $\alpha = 1$, $5/32$, and $1/10$. For $\alpha > 5/32$, there are again superhorizon oscillations, which cease at $\alpha = 5/32$, and for smaller α two different power-law asymptotics occur. While $v \to 0$ for $x \to 0$, the metric potential Ψ in fact diverges. Hence, these solutions do not correspond to a Friedmann-type initial singularity, but are essentially anisotropic.

On sub-horizon scales, where the mechanism of directional dispersion becomes operative, both the regular and the singular solutions decay like $1/x^2$ for large x.

C. Tensor perturbations

A necessary condition for the regular solutions is obtained from Eq. (5.7) at $x = x_0 = 0$, i.e.,

$$\beta_0^{(2)} + \frac{2}{7}\beta_2^{(2)} + \frac{1}{21}\beta_4^{(2)} = 0 \ . \tag{6.8}$$

In contrast to vector and scalar perturbations the initial value $H(0)$ is not fixed by the initial matter distribution. This corresponds to the existence of gravitational waves. Both polarisations evolve with the same $H(x)$, since there is no preferred direction. Again the singular solutions can be added to the regular ones.

In Fig. 9 various regular solutions for the metric potential H are given. The ones that start out as constants at superhorizon scales are similar to the ones found already in the perfect-fluid case (see above), the only difference being that the amplitude drops more strongly at horizon-crossing. On subhorizon scales, these solutions become gravitational waves and their decay $\sim 1/x$ is determined only be the expansion, exactly as in the perfect-fluid case. A novel type of solution is obtained, however, by putting $\beta_0^{(2)} = 0$. As shown by the dashed curve, this gives a growing solution on superhorizon scales.

Finally, in Fig. 10 we exhibit also the singular solutions for H for the same set of parameters as in Fig. 8. Again, the spectacular behavior is restricted to superhorizon scales, upon horizon crossing these solutions again describe ordinary, albeit primordial, gravitational waves.

VII. CONCLUSION

We have presented a detailed analytical study of cosmological perturbations of an Einstein-de Sitter universe filled with collisionless massless particles and a perfect radiation fluid in terms of the gauge-invariant variables of Bardeen. A closed set of equations for the latter has been derived within the recently proposed approach based on thermal field theory on a curved background. There the response of matter under metric perturbations is determined by the thermal graviton two-point function, its leading terms in a high-temperature expansion describing ultrarelativistic collisionless matter. We have found complete equivalence with classical kinetic theory, when the latter is recast in terms of gauge invariant quantities along the lines of Ref. [15]. A conceptual advantage of the thermal-field-theory approach appears to be its purely geometrical character — no explicit recourse to the generally gauge-variant distribution functions of kinetic theory is required. This might turn out to be useful when going beyond the case of collisionless matter, an issue that we intend to explore in a future work.

In extension of the work of Ref. [16,19], we have given the general solutions of the linear perturbation equations in terms of generalized power series expansions, the asymptotic behavior of which has been obtained previously in Ref. [12]. In a strictly FLRW setting, only the regular solutions are admitted, but the general solutions might be of interest when to be matched to earlier, inflationary epochs, where the initial perturbations may have been generated quantum-mechanically [37].

With special emphasis on the role of the initial data, we have also presented a selection of explicit results for regular and singular solutions for scalar, vector, and tensor cosmological perturbations. Typically, the regular solutions exhibit growth on superhorizon scales $\sim x^n$, with n a characteristic integer, which can be increased by special choices of the initial data. The singular solutions have a small-x behavior dictated only by the ratio α which gives the ratio of the energy-density in collisionless matter over the total one. For $\alpha > 5/32$, there is an essential singularity for $x \to 0$, giving rise to superhorizon oscillations, whereas for $\alpha \leq 5/32$, all perturbations have a (singular) power-law behavior for small x. After horizon-crossing, all types of perturbations that are carried by collisionless matter decay. In the scalar and vector case, this is due to directional dispersion, whereas tensor perturbations only decrease according to the expansion of the FRLW universe. We have found that the transition between superhorizon and subhorizon regimes depends rather strongly on the initial data. Whereas previous investigations have always considered only the simplest cases, our results indicate that the effective matching of the asymptotic regimes can vary appreciably by allowing for more complicated initial matter distributions.

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APPENDIX

To solve Eqs. (3.12a), (5.4), (5.6) and (5.7) the metric potentials $\Phi, \Phi_{\rm cll}, \Pi, \Psi$, and H are expanded into (generalized) power series. As mentioned in Sec. VI regular and singular solutions can be derived in such a way. Since the latter equations conserve the parity of the metric potentials, the regular solutions are calculated for the even and the odd part seperately.

The recursion relations have been implemented in a *Mathematica* [38] code to calculate the first 60 coefficients and to plot the solutions in Figs. 1–10.

1. Scalar perturbations

The metric potentials are expanded into power series:

$$\Phi = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\left(\phi_n^{\rm even}+\frac{x}{(2n+1)}\phi_n^{\rm odd}+x^{\sigma^{(0)}}\phi_n^{\rm sing}\right) , \tag{A1a}$$

$$\Pi = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\left(\pi_n^{\rm even}+\frac{x}{(2n+1)}\pi_n^{\rm odd}+x^{\sigma^{(0)}}\pi_n^{\rm sing}\right) , \tag{A1b}$$

$$\Phi_{\rm cll} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\left(q_n^{\rm even}+\frac{x}{(2n+1)}q_n^{\rm odd}+x^{\sigma^{(0)}}q_n^{\rm sing}\right) . \tag{A1c}$$

For simplicity $e_n := 2(\phi_n+\pi_n)$ (for regular even and odd solutions and for the singular solution) is defined. From Eq. (3.12a), the relations

$$3(2n+1)\phi_n^{\rm even} = 3\phi_0^{\rm even}-e_0^{\rm even}-3e_n^{\rm even}-\sum_{l=1}^{n-1}(2l+4)e_l^{\rm even} , \tag{A2a}$$

$$3(2n+2)\phi_n^{\rm odd} = 2(3\phi_0^{\rm odd}-e_0^{\rm odd})-3e_n^{\rm odd}-\sum_{l=1}^{n-1}(2l+5)e_l^{\rm odd} , \tag{A2b}$$

$$\begin{aligned}3(\sigma^{(0)}+2n+1)\phi_n^{\rm sing} &= \frac{(2n)!\Gamma(\sigma^{(0)}+2)}{\Gamma(\sigma^{(0)}+2n+1)}(3\phi_0^{\rm sing}-e_0^{\rm sing})-3e_n^{\rm sing}- \\ &-\frac{(2n)!}{\Gamma(\sigma^{(0)}+2n+1)}\sum_{l=1}^{n-1}\frac{\Gamma(\sigma^{(0)}+2l+1)(\sigma^{(0)}+2l+4)}{(2l)!}e_l^{\rm sing}\end{aligned} \tag{A2c}$$

determine Φ, with $n \geq 1$. For the odd regular solutions $2\phi_0^{\rm odd} = -e_0^{\rm odd}$ and the singular solutions obey $(\sigma^{(0)}+1)\phi_0^{\rm sing} = -e_0^{\rm sing}$. Together with $(n \geq 0)$

$$\begin{aligned}(2n+2)(2n+1)\pi_n^{\rm even} &= -6\alpha\sum_{l=0}^{n-1}\frac{2l+2}{(2l+3)(2l+5)}e_{n-l}^{\rm even}+ \\ &+6\alpha(-1)^n\left[(2n+2)!\sum_{l=0}^{n+1}\frac{(-1)^l\beta_{2n-2l+2}^{(0)}}{(2l)!!(4n-2l+5)!!}+\right. \\ &\left.+\,3(2n+4)!\sum_{l=0}^{n+2}\frac{(-1)^l\beta_{2n-2l+4}^{(0)}}{(2l)!!(4n-2l+9)!!}\right] ,\end{aligned} \tag{A3a}$$

$$(2n+3)(2n+2)\pi_n^{\rm odd} = -6\alpha \sum_{l=0}^{n} \frac{2l+2}{(2l+3)(2l+5)} e_{n-l}^{\rm odd} +$$

$$+6\alpha(-1)^n \left[(2n+3)! \sum_{l=0}^{n+1} \frac{(-1)^l \beta_{2n-2l+3}^{(0)}}{(2l)!!(4n-2l+7)!!} + \right.$$

$$\left. +3(2n+5)! \sum_{l=0}^{n+2} \frac{(-1)^l \beta_{2n-2l+5}^{(0)}}{(2l)!!(4n-2l+11)!!}\right] , \qquad \text{(A3b)}$$

$$(2n+2)(2n+1)\pi_n^{\rm sing} = -6\alpha \frac{(2n+2)!}{\Gamma(\sigma^{(0)}+2n+3)} \sum_{l=0}^{n} \frac{\Gamma(\sigma^{(0)}+2n-2l+1)(2l+2)}{(2n-2l)!(2l+3)(2l+5)} e_{n-l}^{\rm sing} \qquad \text{(A3c)}$$

from Eq. (5.4b), Φ and Π can be calculated. The condition (6.5) has to be satisfied to gain regular solutions. The collisionless part of Φ, $\Phi_{\rm cll}$, is determined by Eq. (5.4a). From the constant term of the latter equation

$$\phi_0^{\rm even} + 2\pi_0^{\rm even} = -4\beta_0^{(0)}$$

follows for the initial values. The higher terms give the relations ($n \geq 0$):

$$(2n+2)(2n+1)q_n^{\rm even} = 6\alpha \left[\phi_{n+1}^{\rm even} - e_{n+1}^{\rm even} + \sum_{l=0}^{n} \left(\frac{1}{2l+1} + \frac{3}{(2l+3)(2n+3)}\right) e_{n-l+1}^{\rm even}\right] +$$

$$+12\alpha(-1)^n \left[(2n+2)! \sum_{l=0}^{n+1} \frac{(-1)^l \beta_{2n-2l+2}^{(0)}}{(2l)!!(4n-2l+5)!!} - \right.$$

$$\left. -3(2n+2)!(2n+4) \sum_{l=0}^{n+2} \frac{(-1)^l \beta_{2n-2l+4}^{(0)}}{(2l)!!(4n-2l+9)!!}\right] , \qquad \text{(A4a)}$$

$$(2n+3)(2n+2)q_n^{\rm odd} = 6\alpha \left[\phi_{n+1}^{\rm odd} - e_{n+1}^{\rm odd} + \sum_{l=0}^{n+1} \left(\frac{1}{2l+1} + \frac{3}{(2l+3)(2n+4)}\right) e_{n-l+1}^{\rm odd}\right] +$$

$$+12\alpha(-1)^n \left[(2n+3)! \sum_{l=0}^{n+1} \frac{(-1)^l \beta_{2n-2l+3}^{(0)}}{(2l)!!(4n-2l+7)!!} - \right.$$

$$\left. -3(2n+3)!(2n+5) \sum_{l=0}^{n+2} \frac{(-1)^l \beta_{2n-2l+5}^{(0)}}{(2l)!!(4n-2l+11)!!}\right] , \qquad \text{(A4b)}$$

$$(2n+2)(2n+1)q_n^{\rm sing} = 6\alpha \left[\phi_{n+1}^{\rm sing} - e_{n+1}^{\rm sing} + \frac{(2n+2)!}{\Gamma(\sigma^{(0)}+2n+3)} \times \right. \qquad \text{(A4c)}$$

$$\left. \times \sum_{l=0}^{n+1} \left(\frac{1}{2l+1} + \frac{3}{(2l+3)(\sigma^{(0)}+2n+3)}\right) \frac{\Gamma(\sigma^{(0)}+2n-2l+3)}{(2n-2l+2)!} e_{n-l+1}^{\rm sing}\right] .$$

The perfect fluid component is provided by $\Phi_{\rm pf} = \Phi - \Phi_{\rm cll}$.

2. Vector perturbations

Accordingly, the metric potential Ψ is split up:

$$\Psi = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\left(\psi_n^{\rm even} + \frac{x}{(2n+1)}\psi_n^{\rm odd} + x^{\sigma^{(1)}}\psi_n^{\rm sing}\right) \ . \tag{A5}$$

The regular even solutions are defined by

$$\begin{aligned}\left[(2n+2)(2n+1)+\frac{8\alpha}{5}\right]\psi_n^{\rm even} = {} & -24\alpha\sum_{l=1}^{n}\frac{\psi_{n-l}^{\rm even}}{(2l+3)(2l+5)} + \\ & + 12\alpha(-1)^n\left[(2n+2)!\sum_{l=0}^{n+1}\frac{(-1)^l\beta^{(1)}_{2n-2l+2}}{(2l)!!(4n-2l+5)!!} + \right. \\ & \left. + (2n+4)!\sum_{l=0}^{n+2}\frac{(-1)^l\beta^{(1)}_{2n-2l+4}}{(2l)!!(4n-2l+9)!!}\right] \ ,\end{aligned} \tag{A6a}$$

for $n \geq 0$. The solution for the regular odd part is provided for $n \geq 0$ by

$$\begin{aligned}\left[(2n+3)(2n+2)+\frac{8\alpha}{5}\right]\psi_n^{\rm odd} = {} & -24\alpha\sum_{l=1}^{n}\frac{\psi_{n-l}^{\rm odd}}{(2l+3)(2l+5)} + \\ & + 12\alpha(-1)^n\left[(2n+3)!\sum_{l=0}^{n+1}\frac{(-1)^l\beta^{(1)}_{2n-2l+3}}{(2l)!!(4n-2l+7)!!} + \right. \\ & \left. + (2n+5)!\sum_{l=0}^{n+2}\frac{(-1)^l\beta^{(1)}_{2n-2l+5}}{(2l)!!(4n-2l+11)!!}\right] \ .\end{aligned} \tag{A6b}$$

Additionaly the conditions (6.7) have to be satisfied. Singular solutions are fixed by $\psi_0^{\rm sing}$ and

$$\begin{aligned}&\left[n(2n+3)+2n\sigma^{(1)}\right]\psi_n^{\rm sing} = \\ &-12\alpha\frac{(2n)!}{\Gamma(\sigma^{(1)}+2n+1)}\sum_{l=1}^{n}\frac{\Gamma(\sigma^{(1)}+2n-2l+1)}{(2n-2l)!}\frac{\psi_{n-l}^{\rm sing}}{(2l+3)(2l+5)} \ ,\end{aligned} \tag{A6c}$$

for $n \geq 1$.

3. Tensor perturbations

For the metric potential H the ansatz

$$H = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\left(h_n^{\rm even} + \frac{x}{(2n+1)}h_n^{\rm odd} + x^{\sigma^{(2)}}h_n^{\rm sing}\right) \tag{A7}$$

is made. Regular even solutions are defined by

$$\begin{aligned}\left[(2n+1)2n+\frac{8\alpha}{5}\right]h_n^{\rm even} = {} & 2n(2n-1)h_{n-1}^{\rm even} - 24\alpha\sum_{l=1}^{n-1}\frac{h_{n-l}^{\rm even}}{(2l+1)(2l+3)(2l+5)} + \\ & + 3\alpha(-1)^n\left[(2n)!\sum_{l=0}^{n}\frac{(-1)^l\beta^{(2)}_{2n-2l}}{(2l)!!(4n-2l+1)!!} + \right.\end{aligned}$$

$$+ 2(2n+2)! \sum_{l=0}^{n+1} \frac{(-1)^l \beta^{(2)}_{2n-2l+2}}{(2l)!!(4n-2l+5)!!} +$$

$$\left. + (2n+4)! \sum_{l=0}^{n+2} \frac{(-1)^l \beta^{(2)}_{2n-2l+4}}{(2l)!!(4n-2l+9)!!} \right] , \tag{A8a}$$

for $n \geq 1$. The initial value h_0^{even} specifies the regular solution together with the inhomogeneous terms. The latter have to respect the condition (6.8). The solution for the regular odd part is provided for $n \geq 0$ by

$$\left[(2n+2)(2n+1) + \frac{8\alpha}{5}\right] h_n^{\mathrm{odd}} = (2n+1)2n h_{n-1}^{\mathrm{odd}} - 24\alpha \sum_{l=1}^{n} \frac{h_{n-l}^{\mathrm{odd}}}{(2l+1)(2l+3)(2l+5)} +$$

$$+ 3\alpha(-1)^n \left[(2n+1)! \sum_{l=0}^{n} \frac{(-1)^l \beta^{(2)}_{2n-2l+1}}{(2l)!!(4n-2l+3)!!} + \right.$$

$$+ 2(2n+3)! \sum_{l=0}^{n+1} \frac{(-1)^l \beta^{(2)}_{2n-2l+3}}{(2l)!!(4n-2l+7)!!} +$$

$$\left. + (2n+5)! \sum_{l=0}^{n+2} \frac{(-1)^l \beta^{(2)}_{2n-2l+5}}{(2l)!!(4n-2l+11)!!} \right] . \tag{A8b}$$

Singular solutions are fixed by h_0^{sing} and

$$\left[n(2n+1) + 2n\sigma^{(2)}\right] h_n^{\mathrm{sing}} = n(2n-1) h_{n-1}^{\mathrm{sing}} -$$

$$-12\alpha \frac{(2n)!}{\Gamma(\sigma^{(2)}+2n+1)} \sum_{l=1}^{n} \frac{\Gamma(\sigma^{(2)}+2n-2l+1)}{(2n-2l)!} \frac{h_{n-l}^{\mathrm{sing}}}{(2l+1)(2l+3)(2l+5)} , \tag{A8c}$$

for $n \geq 1$.

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FIGURES

FIG. 1. Regular density perturbations on comoving hypersurfaces in the purely collisionless case ($\alpha = 0$). The full line shows a regular solution for $|\delta| = |\epsilon_m|$ with $\beta_0^{(0)} = -5/28$ and all other $\beta_n^{(0)}$'s vanishing. The dashed line represents a solution with $\beta_5^{(0)} = -1309/20$ nonvanishing only. For the dotted line $\beta_0^{(0)}$ takes the above value and $\beta_5^{(0)} = -10$, $\beta_{20}^{(0)} = 60$.

FIG. 2. Regular scalar perturbations in a two-component universe ($\alpha = 1/2$). The various lines show regular solutions for $|\delta_{\rm cll}|$ (full line), $|\delta_{\rm pf}|$ (dotted line), and $|\pi_T^{(0)}|$ (dashed line). The initial conditions are specified by $\beta_0^{(0)} = -1/4, \beta_1^{(0)} = -1309/20$, and $\beta_4^{(0)} = 7/8$.

FIG. 3. Singular solutions for $|\delta|$ (full line) and $|\pi_T^{(0)}|$ (dashed line) for $\alpha = 1$ (purely collisionless case). Superhorizon oscillations show up for $x/\pi < 1$. Notice the logarithmic scale in x — the solutions are essentially singular as $x \to 0$. The second solution corresponding to $\sigma_-^{(0)}$ in Eq. (6.4), which is not plotted, differs essentially by a phase.

FIG. 4. Singular scalar perturbations for the critical value of $\alpha = 5/32$. $|\delta_{\rm cll}|$ (full line), $|\delta_{\rm pf}|$ (dotted line), and $|\pi_T^{(0)}|$ (dashed line) are plotted.

FIG. 5. Singular scalar perturbations in the subcritical case ($\alpha < 5/32$). The two different modes ($\sigma_\pm^{(0)}$) are shown by the full lines for $|\delta_{\rm cll}|$ and by the dotted ones for $|\delta_{\rm pf}|$. They have different power-law behavior on superhorizon scales.

FIG. 6. Regular vector perturbations with $v_{c\ \rm pf} = 0$. The solutions for $|v_c|$ are given with only $\beta_0^{(1)} = -21/16$ nonzero (full line), $\beta_1^{(1)} = -1197/80$ alone (dashed line), and $\beta_0^{(1)} = -21/16, \beta_1^{(1)} = -10$ and $\beta_{10}^{(1)} = 20$ for the dotted line.

FIG. 7. Regular vector perturbations with non-vanishing vorticity in the perfect-fluid component. The contribution of $v_{c\ \rm pf} = -21/16$ to $|v_c|$ is plotted dashed-dotted. The other lines show various solutions for $|v_c|$ using the same initial conditions as in Fig. 6 exept for $\beta_2^{(1)}$, which is determined by Eq. (6.7a).

FIG. 8. Singular vector perturbations in terms of $|v_c|$. The different lines correspond to different values of α and to different modes. The solutions with, e.g., $\alpha = 1$ (thin full line) show superhorizon oszillations, whereas solutions for $\alpha_{\rm crit.}$ (thick full line) and below, e.g., $\alpha = 1/10$ (dashed-dotted and dotted) do not. The latter two lines correspond to the two different modes from Eq. (6.4). In the supercritical case both modes differ by their phase only.

FIG. 9. Regular tensor perturbations. The solutions, representing gravitational waves for $x \gg 1$, are normalized to $H(0) = 1$, exept for the dashed line ($H(0) = 0, \beta_1^{(2)} = 63/2$). The full line shows the solution with all $\beta_n^{(2)}$'s vanishing. Again the dotted line represents a more general case with $\beta_1^{(2)} = 1, \beta_4^{(2)} = 5, \beta_6^{(2)} = -10$, and $\beta_7^{(2)} = 20$ nonzero.

FIG. 10. Singular tensor perturbations. $|H|$ is plotted for the same values of α as for the vector perturbations of Fig. 8.