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by

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Abstract:

In this paper some new exact path integral treatments are presented. These are the path integrals for the symmetric top, the "smooth step", the Rosen-Morse- and the Manning-Rosen-potential. By appropriate space-time transformations these path integral problems can be reduced to the path integral problem of the Pöschl-Teller potential.

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I. Introduction

In the last ten years there has been remarkable progress in calculating path integrals explicitly. The starting shot was given by Duru and Kleinert [6] with their treatment of the hydrogen atom. The clue for solving this path integral problem successfully was to find simultaneously a coordinate- and time-transformation ("space-time transformation") to reformulate the hydrogen problem in terms of the harmonic oscillator which was a well-known and already solved problem. This idea of attacking path integral problems by performing space-time transformations has since been marvelously fruitful and much of the following work is based on that idea.

Here we want to add some further examples. As we shall see, we must perform in three out of four cases a space-time transformation and we need in all the examples the results of the path integrals of the (modified) Pöschl-Teller potential. Let us summarize in short the results of these two path integrals and the results of the technique of space-time transformations in path integrals:

1) The Pöschl-Teller (PT) potential with some numbers κ and λ is defined as

$$V^{PT}(x) = \frac{1}{2m}\left[\frac{\kappa(\kappa-1)}{\sin^2 x} + \frac{\lambda(\lambda-1)}{\cos^2 x}\right], \qquad (0 < x < \tfrac{\pi}{2}). \tag{1}$$

This class of potentials was first discussed by Pöschl and Teller [26]. A detailed discussion can be e.g. found in Constantinescu and Magyari [4] and Nieto [22]. The path integral solution for this potential can be achieved by means of the path integral over the $SU(2)$ manifold, and it was successfully studied by Duru [7], Inomata and Kayed [17] and Böhm and Junker [3]. Alternatively, a simple form can be studied with the help of the rigid rotator [11,23].
Some care is needed in the path integral formulation for the Pöschl-Teller potential. Looking carefully at the lattice derivation [3,7] for the path integral we see that we must use a functional measure formulation similar to the one used in the lattice formulation for the radial harmonic oscillator [11,25,29]. This has the consequence that the following interpretation scheme must be used, namely (we use units $\hbar = 1$):

$$\begin{aligned} K^{PT}(x'',x';T) &= \int Dx(t)\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - \frac{\kappa(\kappa-1)}{2m\sin^2 x} - \frac{\lambda(\lambda-1)}{2m\cos^2 x}\right)dt\right] \\ &:= \int Dx(t)\mu_{\lambda,\kappa}[\sin x,\cos x]\exp\left(\frac{im}{2}\int_{t'}^{t''}\dot{x}^2 dt\right) \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}}\prod_{j=1}^{N-1}\int_0^{\frac{\pi}{2}} dx^{(j)}\prod_{j=1}^{N}\mu_{\lambda,\kappa}[\sin x^{(j)},\cos x^{(j)}]\exp\left[\frac{im}{2\epsilon}(x^{(j)}-x^{(j-1)})^2\right], \end{aligned} \tag{2}$$

where the functional measure $\mu_{\lambda,\kappa}$ is given by (we use the notation $\widehat{\sin^2\theta^{(j)}} \equiv \sin\theta^{(j)}\sin\theta^{(j-1)}$ etc.):

$$
\begin{aligned}
\mu_{\lambda,\kappa}[\sin x,\cos x] &:= \lim_{N\to\infty}\prod_{j=1}^{N}\mu_{\lambda,\kappa}[\sin x^{(j)},\cos x^{(j)}] \\
&= \lim_{N\to\infty}\left(\frac{2\pi m}{i\epsilon}\right)^{N}\prod_{j=1}^{N}\widehat{\sin x^{(j)}}\,\widehat{\cos x^{(j)}} \\
&\qquad\times\exp\left[\frac{m}{i\epsilon}\left(\widehat{\sin^2 x^{(j)}}+\widehat{\cos^2 x^{(j)}}\right)\right]I_{\lambda-\frac{1}{2}}\left(\frac{m}{i\epsilon}\widehat{\sin^2 x^{(j)}}\right)I_{\kappa-\frac{1}{2}}\left(\frac{m}{i\epsilon}\widehat{\cos^2 x^{(j)}}\right).
\end{aligned}
\tag{3}
$$

The first line in Eq.(2) has only the symbolical meaning that formally the potential appearing in the Schrödinger equation translates into $\int Dx\exp(i\times \mathit{Action})$. We want to emphasize that only the functional measure formulation has a well-defined lattice formulation. The usual expansion of the modified Bessel function $I_\nu(z)\simeq(2\pi z)^{-\frac{1}{2}}\exp\left[z-\frac{\nu^2-1/4}{2z}\right]$ $[z\to\infty$, $\arg(z)\neq 0]$ (or Eq.(3.15) in Ref.[18], respectively) seems very suggestive but gives in the lattice formulation the wrong boundary behaviour of the corresponding short-time kernels and wave-functions because the condition $\arg(z)\neq 0$ is violated. Instead of the correct behaviour we would get a highly singular one. But it is not the scope of this paper to discuss these features in detail; this will be done elsewhere [13].

The path integral solution for the Pöschl-Teller potential now reads:

$$
K^{PT}(x'',x';T)=\sum_{n=0}^{\infty}\exp\left[-\frac{iT}{2m}(\kappa+\lambda+2n)^2\right]\Psi_n^{PT}(x')\Psi_n^{PT}(x''),
\tag{4}
$$

where

$$
\begin{aligned}
\Psi_n^{PT}(x) &= \left[2(\kappa+\lambda+2n)\frac{n!\Gamma(\kappa+\lambda+n)}{\Gamma(\kappa+n+\frac{1}{2})\Gamma(\lambda+n+\frac{1}{2})}\right]^{\frac{1}{2}} \\
&\qquad\times(\sin x)^{\kappa}(\cos x)^{\lambda}P_n^{(\kappa-\frac{1}{2},\lambda-\frac{1}{2})}(1-2\sin^2 x).
\end{aligned}
\tag{5}
$$

Here the $P_n^{(\alpha,\beta)}(z)$ denote Jacobi-polynomials.

2) The modified Pöschl-Teller (mPT) potential with some numbers η and ν is defined as

$$
V^{mPT}(r)=\frac{1}{2m}\left[\frac{\eta(\eta-1)}{\sinh^2 r}-\frac{\nu(\nu-1)}{\cosh^2 r}\right],\qquad (r>0).
\tag{6}
$$

A classical study of this problem is due to Frank and Wolf [9], whereas the path integral treatment by means of the path integral over the $SU(1,1)$ manifold is due to Böhm and Junker [2,3]. The special case $\nu(\nu-1)=0$ can be studied with the help of the path integral on the pseudosphere [12]. Again we must use a functional measure formulation similarly

to the previous one and the following interpretation scheme for the modified Pöschl-Teller potential must be used:

$$
\begin{aligned}
K^{mPT}(r'',r';T) &= \int Dr(t)\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2-\frac{\eta(\eta-1)}{2m\sinh^2 x}+\frac{\nu(\nu-1)}{2m\cosh^2 x}\right)dt\right] \\
&:= \int Dr(t)\mu_{\eta,\nu}[\sinh r,\cosh r]\exp\left(\frac{im}{2}\int_{t'}^{t''}\dot r^2\,dt\right) \\
&= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}}\prod_{j=1}^{N-1}\int_0^\infty dr^{(j)}\prod_{j=1}^{N}\mu_{\eta,\nu}[\sinh r^{(j)},\cosh r^{(j)}]\exp\left[\frac{im}{2\epsilon}(r^{(j)}-r^{(j-1)})^2\right],
\end{aligned}
\tag{7}
$$

where the functional measure $\mu_{\eta,\nu}$ is given by

$$
\begin{aligned}
\mu_{\eta,\nu}[\sinh r,\cosh r] &= \lim_{N\to\infty}\prod_{j=1}^{N}\mu_{\eta,\nu}[\sinh r^{(j)},\cosh r^{(j)}] \\
&= \lim_{N\to\infty}\left(\frac{2\pi m}{\epsilon}\right)^{N}\prod_{j=1}^{N}\widehat{\sinh r^{(j)}}\,\widehat{\cosh r^{(j)}} \\
&\qquad\times\exp\left[-\frac{m}{i\epsilon}\left(\widehat{\sinh^2 r^{(j)}}-\widehat{\cosh^2 r^{(j)}}\right)\right]I_{\eta-\frac{1}{2}}\left(\frac{m}{i\epsilon}\widehat{\sinh^2 r}(j)\right)I_{\nu-\frac{1}{2}}\left(\frac{im}{\epsilon}\widehat{\cosh^2 r^{(j)}}\right).
\end{aligned}
\tag{8}
$$

Of course, the same line of reasoning as before is also valid for K^{mPT} as far as the functional measure is concerned. Adopting the notation of Frank and Wolf the path integral solution reads [define $2s=\eta(\eta-1)$, $-2c=\nu(\nu-1)$ and introduce the numbers k_1,k_2 which are defined in terms of c and s as $k_1=\frac{1}{2}(1\pm\sqrt{1/4-2c})$, $k_2=\frac{1}{2}(1\pm\sqrt{1/4+2s})$]:

$$
\begin{aligned}
K^{mPT}(r'',r';T) &= \sum_{n=1}^{N_M}e^{-iTE_n}\Psi_n^{(k_1,k_2)}(r')\Psi_n^{(k_1,k_2)*}(r'') \\
&\qquad+\int_0^\infty dk\,e^{-\frac{iT}{2m}k^2}\Psi_k^{(k_1,k_2)}(r')\Psi_k^{(k_1,k_2)*}(r'').
\end{aligned}
\tag{9}
$$

Here N_M denotes the maximal number of states with $0,1,\ldots,n\leq N_M<k_1-k_2-\frac{1}{2}$. The correct signs depend on the boundary conditions for $r\to 0$ and $r\to\infty$, respectively. In particular one gets for $s=0$ an even and an odd wave function corresponding to $k_1=\frac{1}{4};\ \frac{3}{4}$. The bound states are explicitly given by:

$$
\begin{aligned}
\Psi_n^{(k_1,k_2)}(r) &= N_n^{(k_1,k_2)}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{-2k_1-\frac{3}{2}} \\
&\qquad\times{}_2F_1(-k_1+k_2+\kappa,-k_1+k_2-\kappa+1;2k_2;-\sinh^2 r),
\end{aligned}
\tag{10a}
$$

$$
\begin{aligned}
&= \left[\frac{2n!(2k_1-1)\Gamma(2k_1-n-1)}{\Gamma(2k_2+n)\Gamma(2k_1-2k_2-n)}\right]^{\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{2n-2k_1+\frac{3}{2}} \\
&\qquad\times P_n^{[2k_2-1,2(k_1-k_2-n)-1]}\left(\frac{1-\sinh^2 r}{\cosh^2 r}\right),
\end{aligned}
\tag{10b}
$$

$$
N_n^{(k_1,k_2)}=\frac{1}{\Gamma(2k_2)}\left[\frac{(2k_1-1)\Gamma(k_1+k_2-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{\frac{1}{2}}
\tag{10c}
$$

and $E_n=-\frac{1}{2m}(2\kappa-1)^2=-\frac{1}{2m}[2(k_1-k_2-n)-1]^2$. The continuous states read:

$$\Psi_k^{(k_1,k_2)}(r) = N_k^{(k_1,k_2)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}} \times {}_2F_1(k_1+k_2-\kappa, k_1+k_2+\kappa-1; 2k_2; -\sinh^2 r), \tag{11a}$$

$$N_k^{(k_1,k_2)} = \frac{1}{\pi\Gamma(2k_2)}\sqrt{\frac{k\sinh\pi k}{2}} \times \Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{\frac{1}{2}}, \tag{11b}$$

where $\kappa = \frac{1}{2}(1+ik)$ $(k>0)$ and $E = \frac{k^2}{2m}$.

3) Let us now discuss the general method for a space-time transformation [11]. It works as follows: One starts with the path integral

$$K(x'', x'; T) = \int Dx(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x) - V_{qu}(x)\right]dt\right\} \tag{12}$$

$[V_{qu} = \frac{1}{8m}(\Gamma^2 + 2\Gamma')$ is a quantum potential due to a non-trivial metric, see below], where it is assumed that the effective potential $V + V_{qu}$ is so complicated that a direct evaluation is not possible. One then defines a new "time" s together with a coordinate transformation $x(t) \to q(s)$, $s(t) = \int_{t'}^{t}(d\sigma/f[x(\sigma)])$ and $x = F(q)$ with some well-defined positive functions f and F, where we further assume that the relation $f(F(q)) = [F'(q)]^2$ holds. Let us consider the Legendre-transformed general one-dimensional Hamiltonian

$$H_E = -\frac{1}{2m}\left(\frac{d^2}{dx^2} + \Gamma(x)\frac{d}{dx}\right) + V(x) - E \tag{13}$$

which is hermitian with respect to the inner product $(f_1, f_2) = \int f_1^*(x)f_2(x)J(x)dx$, where $J(x) = e^{\int \Gamma(x)dx}$. Let us define the quantities $G(q) = \Gamma(F(q))$, $\sqrt{g} = J(q) = e^{\int^q \tilde{\Gamma}(q')dq'}$, $\tilde{\Gamma}(q) = G(q)F'(q) - F''(q)/F'(q)$ and $p_q = \frac{1}{i}[\frac{d}{dq} + \frac{1}{2}\tilde{\Gamma}(q)]$, where p_q is, of course, a momentum operator which is hermitian with respect to the scalar product $(f_1, f_2) = \int f_1^*(q)f_2(q)\sqrt{g(q)}dq$. With the constraint $f(F(q)) = F'^2(q)$ we thus obtain the **space-time transformed Hamiltonian** $\tilde{H} = fH_E$:

$$\tilde{H}(p_q, q) = \frac{1}{2m}p_q^2 + f(F(q))\big[V(F(q)) - E\big] + \tilde{V}_{qu}(q) \tag{14}$$

with the well-defined quantum correction

$$\tilde{V}_{qu}(q) = \frac{1}{8m}\left[3\left(\frac{F''(q)}{F'(q)}\right)^2 - 2\frac{F'''(q)}{F'(q)} + \big(G(q)F'(q)\big)^2 + 2G'(q)F'(q)\right]. \tag{15}$$

Now assume that the constraint $\int_0^{s''} ds f[F(q(s))] = T$ $[s(t'') = s'']$ has for all admissible paths a unique solution $s'' \geq 0$. Of course, since T is fixed, the "time" s'' will be path-dependent.

We must incorporate the constraint into the path integral and define the energy-dependent Feynman-kernel $G(x'', x'; E)$ (Green's function) via the Fourier transformation

$$K(x'', x'; T) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-iTE}G(x'', x'; E)dE \tag{16}$$

and obtain the **transformation formula**

$$G(x'', x'; E) = i[f(x'')f(x')]^{1/4}\int_0^{\infty}\tilde{K}(q'', q'; s'')ds'' \tag{17}$$

which gives the energy-dependent kernel G as a time integral over the transformed Feynman path integral $\tilde{K}(q'', q, s'') = \langle q''|e^{-is''\tilde{H}}|q'\rangle$:

$$\tilde{K}(q'', q'; s'') = \int Dq(s)\exp\left(i\int_0^{s''}\left\{\frac{m}{2}\dot{q}^2 - f(F(q))\Big[V(F(q)) - E\Big] - \tilde{V}_{qu}(q)\right\}ds\right) \tag{18}$$

$[\dot{q} = dq(s)/ds, q' = F^{-1}(x'), q'' = F^{-1}(x'')]$. Here the measure $Dq(s)$ is defined in the same way as $Dx(t)$ in the path integral (12). This technique of space-time transformation was orginally developed by Duru and Kleinert [6]. It was further developed by Steiner [28], Pak and Sökmen [24], Inomata [16], Kleinert [19] and Grosche and Steiner [11]. The lattice derivation of the equations (16-18) is far from beeing trivial but can be rigorously performed [e.g. by appropriate symmetrization rules and the well-known midpoint-prescription (which must be used in path integrals on curved manifolds)].

The further content of this paper reads as follows. In section II the path integral solution for the symmetric top is presented, a problem closely related to the $SU(2)$ path integral.

In section III the path integral solution for the "smooth step" function $V(x) = -V_0/(1+e^{x/R})$ $(x \in \mathbf{R}, V_0, R > 0$, constants) is calculated, a potential important in solid state physics.

Section III contains finally the path integrals for the Rosen-Morse potential $V(x) = A\tanh\frac{x}{R} - B/\cosh^2\frac{x}{R}$ $(x \in \mathbf{R})$ and for the Manning-Rosen potential $V(r) = A\coth\frac{r}{R} - B/\sinh^2\frac{r}{R}$ $(r > 0, A, B, R > 0$ denote constants).

Section V contains some summarizing remarks.

II. The Symmetric Top

A quantum mechanical treatment of the symmetric top (ST) is due to Dennison [5]. He used it for describing the spectra of molecules with rotational degrees of freedom. There is also a close relationship to the problem of an electron moving in the field of a magnetic monopole [30]. The simplest top is, of course, where all three principle momenta of inertia are equal. This, in fact, is the $SU(2)$ problem [7]. Here we want to discuss the symmetric top, where only two principle momenta of inertia are equal. We introduce the Eulerian angles θ, ψ, ϕ with $\theta \in [0, \pi]$, $\psi \in [-2\pi, 2\pi]$ and $\phi \in [0, 2\pi]$. The kinetic energy giving the rotational energy (= classical Hamiltonian = classical Lagrangian) of a rigid body with the two principle momenta of inertia A and B in these coordinates is then given by (e.g.[20]):

$$
\begin{aligned}
L_{Cl} &= \frac{A}{2}(\dot\theta^2 + \sin^2\theta\,\dot\psi^2) + \frac{B}{2}(\dot\phi + \cos\theta\,\dot\psi)^2 \\
&= \frac{A}{2}\dot\theta^2 + \frac{1}{2}(A\,\sin^2\theta + B\,\cos^2\theta)\dot\psi^2 + \frac{B}{2}\dot\phi^2 + B\cos\theta\,\dot\psi\dot\phi \\
&= \frac{A}{2} g_{ab}\dot q^a \dot q^b,
\end{aligned}
\tag{19}
$$

where q^a $(a = 1,2,3)$ denote the three Eulerian angles and the metric tensor g_{ab} and its inverse g^{ab} are given by ($\tilde B = B/A$):

$$
(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta + \tilde B\cos^2\theta & \tilde B\cos\theta \\ 0 & \tilde B\cos\theta & \tilde B \end{pmatrix}, \qquad
(g^{ab}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sin^2\theta} & -\frac{\cos\theta}{\sin^2\theta} \\ 0 & -\frac{\cos\theta}{\sin^2\theta} & \frac{1}{\tilde B} + \cot^2\theta \end{pmatrix}
\tag{20}
$$

and the determinant g of g_{ab} reads: $g = \det(g_{ab}) = \tilde B\sin^2\theta$. The Hamiltonian reads as:

$$
H_{Cl} = \frac{1}{2A}\left[p_\theta^2 + \frac{p_\psi^2}{\sin^2\theta} + \left(\frac{A}{B} + \cot^2\theta\right)p_\phi^2 - 2\frac{\cos\theta}{\sin^2\theta}p_\psi p_\phi\right].
\tag{21}
$$

To construct Hamiltonians in quantum mechanics we must respect the ordering ambiguity of position- and momentum-operators. Following our prescriptions given in [10,11] we start by considering the hermitian momenta $p_q = \frac{1}{i}(\frac{\partial}{\partial q} + \frac{1}{2}\Gamma_q)$ with $\Gamma_q = \partial_q \ln\sqrt{g}$. Thus

$$
p_\theta = \frac{1}{i}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right), \qquad p_\psi = \frac{1}{i}\frac{\partial}{\partial\psi}, \qquad p_\phi = \frac{1}{i}\frac{\partial}{\partial\phi}.
\tag{22}
$$

For the Hamiltonian we can use the well-known Weyl-ordering prescription

$$
H = \frac{1}{8A}\left[g^{ab}p_a p_b + 2p_a g^{ab} p_b + p_a p_b g^{ab}\right] + \Delta V_{Weyl}
\tag{23}
$$

with the well-defined quantum potential

$$
\Delta V_{Weyl} = \frac{1}{8A}\left[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_b)_{,a} + g^{ab}{}_{,ab}\right],
\tag{24}
$$

or, respectively, the "product-ordering" prescription [10]

$$
H = \frac{1}{2A}h^{ac}p_a p_b h^{cb} + \Delta V_{Weyl} + \frac{1}{8A}\left(2h^{ac}h^{bc}{}_{,ab} - h^{ac}{}_{,a}h^{bc}{}_{,b} - h^{ac}{}_{,b}\,h^{bc}{}_{,a}\right)
\tag{25}
$$

with the decomposition $g^{ab} = h^{ac}h^{cb}$. Due to the special nature of g^{ab} and g_{ab}, respectively, the quantum potentials turn out to be equal for both prescriptions and we find for the quantum Hamiltonian of the symmetric top:

$$
\begin{aligned}
H &= \frac{1}{2A}\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\psi^2} + \left(\frac{A}{B} + \cot^2\theta\right)\frac{\partial^2}{\partial\phi^2} - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\psi}\frac{\partial}{\partial\phi}\right] \\
&= \frac{1}{2A}\left[p_\theta^2 + \frac{p_\psi^2}{\sin^2\theta} + \left(\frac{A}{B} + \cot^2\theta\right)p_\phi^2 - 2\frac{\cos\theta}{\sin^2\theta}p_\psi p_\phi\right] - \frac{1}{8A}\left(1 + \frac{1}{\sin^2\theta}\right).
\end{aligned}
\tag{26}
$$

The path integral for the symmetric top can now be constructed in the usual way [10,11] and it reads in the "product-form" definition [$\Delta q^{(j)} = q^{(j)} - q^{(j-1)}$, $q^{(j)} = q(t^{(j)}) = q(t' + j\epsilon)$ ($j = 0, 1, \ldots, N$), $\epsilon = T/N = (t'' - t')/N$ $(N \to \infty)$]:

$$
\begin{aligned}
K^{ST}&(\theta'', \theta', \psi'', \psi', \phi'', \phi'; T) = \int \sqrt{g}\,D\theta(t)D\psi(t)D\phi(t) \\
&\times \exp\left\{i\int_{t'}^{t''}\left[\frac{A}{2}\dot\theta^2 + \frac{1}{2}(A\sin^2\theta + B\cos^2\theta)\dot\psi^2 + \frac{B}{2}\dot\phi^2 + B\cos\theta\,\dot\psi\dot\phi + \frac{1}{8A}\left(1 + \frac{1}{\sin^2\theta}\right)\right]dt\right\} \\
&= \lim_{N\to\infty}\left(\frac{A}{2\pi i\epsilon}\right)^{\frac{3}{2}N}\prod_{j=1}^{N-1}\sqrt{\frac{A}{B}}\int_0^\pi \sin\theta^{(j)}d\theta^{(j)}\int_{-2\pi}^{2\pi}d\psi^{(j)}\int_0^{2\pi}d\phi^{(j)} \\
&\times \exp\left\{i\sum_{j=1}^N\left[\frac{A}{2\epsilon}\Delta^2\theta^{(j)} + \frac{1}{2\epsilon}\left(A\sin^2\widehat{\theta^{(j)}} + B\cos^2\widehat{\theta^{(j)}}\right)\Delta^2\psi^{(j)} + \frac{B}{2\epsilon}\Delta^2\phi^{(j)}\right.\right. \\
&\qquad\left.\left. + \frac{B}{\epsilon}\cos\widehat{\theta^{(j)}}\Delta\psi^{(j)}\Delta\phi^{(j)} + \frac{\epsilon}{8A}\left(1 + \frac{1}{\sin^2\theta^{(j)}}\right)\right]\right\}.
\end{aligned}
\tag{27}
$$

We calculate this path integral by starting with a Fourier expansion:

$$
\begin{aligned}
K^{ST}(\theta'', \theta', \psi'', \psi', \phi'', \phi'; T) &= \sum_{L=-\infty}^{\infty}\sum_{M=-\infty}^{\infty}K_{LM}^{ST}(\theta'', \theta'; T)e^{iL(\phi''-\phi')}e^{iM(\psi''-\psi')} \\
K_{LM}^{ST}(\theta'', \theta'; T) &= \frac{1}{8\pi^2}\int_{-2\pi}^{2\pi}d\psi''\int_0^{2\pi}d\phi''K^{ST}(\theta'', \theta', \psi'', \psi', \phi'', \phi'; T)e^{-iL(\phi''-\phi')-iM(\psi''-\psi')}.
\end{aligned}
\tag{28}
$$

This yields

$$
\begin{aligned}
K_{LM}^{ST}&(\theta'', \theta'; T) = \frac{1}{8\pi^2}\exp\left(\frac{iT}{8A}\right) \\
&\times \lim_{N\to\infty}\left(\frac{A}{2\pi i\epsilon}\right)^{\frac{3}{2}N}\prod_{j=1}^{N-1}\sqrt{\frac{A}{B}}\int_0^\pi \sin\theta^{(j)}d\theta^{(j)}\exp\left[i\sum_{j=1}^N\left(\frac{A}{2\epsilon}\Delta^2\theta^{(j)} + \frac{\epsilon}{8A\sin^2\theta^{(j)}}\right)\right] \\
&\times \prod_{j=1}^N\int_{-2\pi}^{2\pi}d\psi^{(j)}\int_0^{2\pi}d\phi^{(j)}\exp\left\{\frac{i}{2\epsilon}\left[\left(A\sin^2\widehat{\theta^{(j)}} + B\cos^2\widehat{\theta^{(j)}}\right)\Delta^2\psi^{(j)} + B\Delta^2\phi^{(j)}\right.\right. \\
&\qquad\left.\left. + 2B\cos\widehat{\theta^{(j)}}\Delta\psi^{(j)}\Delta\phi^{(j)}\right] - iL\Delta\phi^{(j)} - iM\Delta\psi^{(j)}\right\} \\
&= \frac{1}{8\pi^2}\sqrt{\frac{A}{B}}\exp\left[-iT\left(\frac{L^2}{2B} - \frac{1}{8A}\right)\right] \\
&\times \lim_{N\to\infty}\left(\frac{A}{2\pi i\epsilon}\right)^{N}\prod_{j=1}^{N-1}\int_0^\pi \sin\theta^{(j)}d\theta^{(j)}\exp\left[i\sum_{j=1}^N\left(\frac{A}{2\epsilon}\Delta^2\theta^{(j)} + \frac{\epsilon}{8A\sin^2\theta^{(j)}}\right)\right] \\
&\times \prod_{j=1}^N\int_{-2\pi}^{2\pi}d\psi^{(j)}\exp\left[\frac{iA}{2\epsilon}\sin^2\widehat{\theta^{(j)}}\Delta\psi^{(j)} + i(L\cos\widehat{\theta^{(j)}} - M)\Delta\psi^{(j)}\right] \\
&= \sqrt{\frac{A}{B}}\frac{(\sin\theta'\sin\theta'')^{-\frac{1}{2}}}{8\pi^2}\exp\left[-iT\left(\frac{L^2}{2B} - \frac{1}{8A}\right)\right]\tilde K_{LM}^{ST}(\theta'', \theta'; T),
\end{aligned}
\tag{29}
$$

where the kernel $\tilde{K}_{LM}^{ST}$ is given by:

$$\begin{aligned}\tilde{K}_{LM}^{ST}(\theta'',\theta';T) &= \int D\theta(t)\exp\left[i\int_{t'}^{t''}\left(\frac{A}{2}\dot{\theta}^2 - \frac{(M-L\cos\theta)^2-\frac{1}{4}}{2A\sin^2\theta}\right)dt\right] \\ &= \lim_{N\to\infty}\left(\frac{A}{2\pi i\epsilon}\right)^{\frac{N}{2}}\prod_{j=1}^{N-1}\int_0^\pi d\theta^{(j)}\exp\left[i\sum_{j=1}^{N}\left(\frac{A}{2\epsilon}\Delta^2\theta^{(j)} - \epsilon\frac{(M-L\cos\theta^{(j)})^2-\frac{1}{4}}{2A\sin^2\theta^{(j)}}\right)\right].\end{aligned} \quad (30)$$

This potential problem related to the symmetric top has also its own right as a path integral, especially if one analytically continues in L and M to, say, arbitrary real numbers. The path integral (30) can now be solved by a simple coordinate transformation by means of the Pöschl-Teller potential or $SU(2)$ path integral (see also [7] for a similar calculation), respectively. We consider the Hamiltonian

$$H = -\frac{1}{2A}\frac{d^2}{d\theta^2} + \frac{(M-L\cos\theta)^2-\frac{1}{4}}{2A\sin^2\theta}, \quad (31)$$

perform the transformation $(1-\cos\theta) = 2\sin^2 x$, i.e. $x = \theta/2$, and arrive at the transformed Hamiltonian

$$\hat{H} = -\frac{1}{8A}\frac{d^2}{dx^2} + \frac{(M-L)^2-\frac{1}{4}}{8A\sin^2 x} + \frac{(M+L)^2-\frac{1}{4}}{8A\cos^2 x} - \frac{L^2}{2A}. \quad (32)$$

Thus we get the transformed path integral corresponding to the Hamiltonian $\hat{H}$:

$$\begin{aligned}\tilde{K}_{LM}^{ST}(\theta'',\theta';T) &\equiv \frac{1}{2}e^{iTL^2/2A}\hat{K}(x'',x';T) \\ &= \frac{1}{2}e^{iTL^2/2A}\int Dx(t)\mu_{|M-L|,|M+L|}[\sin x,\cos x]\exp\left(2iA\int_{t'}^{t''}\dot{x}^2 dt\right) \\ &= \sum_{n=0}^{\infty}\exp\left[-iT\left(\frac{(d+s+2n+1)^2}{8A} - \frac{L^2}{2A}\right)\right]\Psi_n^{PT}(x')\Psi_n^{PT}(x'').\end{aligned} \quad (33)$$

Here we used the notation of Eq.(4), where $m = 4A$, $\kappa = d + \frac{1}{2}$, $\lambda = s + \frac{1}{2}$ with $d = |M-L|$ and $s = |M+L|$. Thus we get finally the path integral for the symmetric top:

$$K^{ST}(\theta'',\theta',\psi'',\psi',\phi'',\phi';T) = \sum_{n=0}^{\infty}\sum_{M=-\infty}^{\infty}\sum_{L=-\infty}^{\infty} e^{-iTE_n^{ST}}\Psi_n^{ST}(\theta',\psi',\phi')\Psi_n^{ST*}(\theta'',\psi'',\phi''), \quad (34)$$

where the wave-functions and the energy-spectrum are given by:

$$\Psi_n^{ST}(\theta,\psi,\phi) = N_n^{ST}e^{-iL\phi}e^{-iM\phi}(1-\cos\theta)^{\frac{|M-L|}{2}}(1+\cos\theta)^{\frac{|M+L|}{2}}P_n^{(|M-L|,|M+L|)}(\cos\theta)$$

$$N_n^{ST} = \left[\sqrt{\frac{A}{B}}\frac{(d+s+2n+1)n!\,\Gamma(d+s+n+1)}{8\pi^2 2^{d+s+1}(|M-L|+n)!(M+L|+n)!}\right]^{\frac{1}{2}} \quad (35)$$

$$E_n^{ST} = \frac{\left(n+\frac{d+s+1}{2}\right)^2-\frac{1}{4}}{2A} + \frac{L^2}{2}\left(\frac{1}{B}-\frac{1}{A}\right). \quad (36)$$

We can introduce the quantity $J = n + \frac{d+s}{2} = 0,1,2,\ldots$ and can interpret the energy as

$$E^{ST} = \frac{J(J+1)}{2A} + \frac{L^2}{2}\left(\frac{1}{B}-\frac{1}{A}\right). \quad (37)$$

For $A = B$ we recover the $SU(2)$- and $SO(3)$-spectrum, respectively. Note that

$$\begin{aligned}\frac{d+s}{2} = \frac{1}{2}|L+M| + \frac{1}{2}|L-M| &= |L|, \qquad (|L| \geq |M|) \\ &= |M|, \qquad (|M| \geq |L|).\end{aligned} \quad (38)$$

This suggests that J, L and M may be interpreted as representing angular momentum and it can be in fact shown that, e.g. the total angular momentum P can be described by $P^2 = J(J+1)$. For a detailed discussion of these features see [5].

As a final result we consider the kernel $\tilde{K}_{LM}^{ST}$, interpret the numbers L and M as, say, arbitrary real numbers and obtain the path integral identity:

$$\begin{aligned}&\int D\theta(t)\exp\left[i\int_{t'}^{t''}\left(\frac{A}{2}\dot{\theta}^2 - \frac{(M-L\cos\theta)^2-\frac{1}{4}}{2A\sin^2\theta}\right)dt\right] \\ &\qquad = \sum_{n=0}^{\infty}\exp\left[-\frac{iT}{2A}\left(\frac{(2n+d+s+1)^2}{4} - L^2\right)\right]\Psi_n^{PT}(\cos\theta')\Psi_n^{PT}(\cos\theta'')\end{aligned} \quad (39)$$

in the notation of Eqs.(4,5,34). In Eq.(39) a functional measure formulation must be used if needed.

III. The Smooth Step

Let us consider now the path integral for the "smooth step" potential ($x \in \mathbf{R}$) $V(x) = -V_0/(1+e^{x/R})$:

$$K(x'',x';T) = \int Dx(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 + \frac{V_0}{1+e^{\frac{x}{R}}}\right]dt\right\}. \quad (40)$$

Potentials like this are important in solid state physics. They can smear out the potential step at metal-vacuum interfaces. Following our prescription for a general space-time transformation in path integrals we start by considering the Legendre-transformed Hamiltonian H_E:

$$H_E = -\frac{1}{2m}\frac{d^2}{dx^2} - \frac{V_0}{1+e^{\frac{x}{R}}} - E. \quad (41)$$

We perform the transformation $(1+e^{\frac{x}{R}})^{-1} = \tanh^2 r$, introduce the momentum operator $p_r = \frac{1}{i}(\frac{d}{dr} + \frac{1}{2}\Gamma_r)$ $[\Gamma_r = 1/(\sinh r\cosh r)]$ and define a new "time" s by $dt = f[q(s)]ds$, with

$f(r) = 4R^2\coth^2 r$. Thus we have $\Delta V = -\frac{1}{8m}(3/\cosh^2 r + 1/\sinh^2 r)$ and arrive at the **space-time transformed Hamiltonian** $\tilde{H} = f(r)\hat{H}_E$:

$$\tilde{H} = \frac{p_r^2}{2m} - 4R^2(E+V_0) - \frac{8mR^2E + \frac{1}{4}}{2m\sinh^2 r} - \frac{3}{8m\cosh^2 r}. \tag{42}$$

Therefore the **space-time transformed path integral** $\tilde{K}$ reads $[\nu = -\frac{1}{2}, \frac{3}{2};\ \eta = \frac{1}{2} \pm \sqrt{-8mR^2E}]$:

$$\tilde{K}(r'', r'; s'') = e^{is''4R^2(E+V_0)} \int Dr(s)\mu_{\eta,\nu}[\sinh r, \cosh r] \exp\left(\frac{im}{2}\int_0^{s''} \dot{r}^2 ds\right) \tag{43}$$

This path integral is a special case of the modified Pöschl-Teller potential as discussed in the introduction. Equations (9) and (11) give immediately the solution yielding (there are no bound states)

$$\tilde{K}(r'', r'; s'') = e^{is''4R^2(E+V_0)} \int_0^\infty dk\, e^{-\frac{is''}{2m}k^2} \Psi_k^{(k_1,k_2)}(r')\Psi_k^{(k_1,k_2)*}(r''). \tag{44}$$

Due to the boundary conditions $\Psi_k^{(k_1,k_2)} \to 0$ for $r \to 0$ we get $k_1 = 0$ and $k_2 = \frac{1}{2}(1 + \sqrt{-8mR^2E})$. Performing the s''-integration gives therefore

$$G(x'', x'; E) = \int_0^\infty dk \frac{\Psi_{Step}(x')\Psi^*_{Step}(x'')}{\frac{k^2}{8mR^2} - V_0 - E}. \tag{45}$$

The energy-spectrum reads $E_k = \frac{k^2}{8mR^2} - V_0$ and has a constant shift V_0. The wave functions are given by $[y \equiv (1 + e^{\frac{x}{R}})^{-1},\ p \equiv \sqrt{2mER^2},\ k \equiv \sqrt{2mR^2(E+V_0)}]$:

$$\begin{aligned} \Psi_{Step}(x) &= Ny^{ip}(1-y)^{ik}{}_2F_1[i(p+k), i(p+k)+1; 1+2ip; y] \\ N &= \left[\frac{k}{8p\pi R}\frac{\sinh \pi k \sinh 2\pi p}{\sinh \pi(p+k)\sinh \pi(p-k)}\right]^{\frac{1}{2}}. \end{aligned} \tag{46}$$

This is the correct solution [4].

IV. The Rosen-Morse and Manning-Rosen Potential

1) Let us first consider the path integral for the Rosen-Morse (RM) potential $V^{RM}(x) = A\tanh\frac{x}{R} - B/\cosh^2\frac{x}{R}$, $(x \in \mathbf{R}, A, B, R$ positive constants):

$$K^{RM}(x'', x,; T) = \int Dx(t)\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - A\tanh\frac{x}{R} + \frac{B}{\cosh^2\frac{x}{R}}\right)dt\right]. \tag{47}$$

The potential V^{RM} was introduced by Rosen and Morse [27] to discuss the spectra of polyatomic molecules. With this potential one can also study the penetration of electrons through

a potential barrier [8]. The Rosen-Morse potential has a discrete and continuous spectrum and thus a "hidden" $SU(1,1)$ symmetry. There is already a path integral solution for V^{MR} by Junker and Inomata [18], but their treatment is based on the $SU(2)$ path integral instead of the $SU(1,1)$ path integral. Therefore these authors did not get the continuous part of the spectrum.

We proceed similarly to the previous section. Let us start with the Legendre transformed Hamiltonian H_E:

$$H_E = -\frac{1}{2m}\frac{d^2}{dx^2} + A\tanh\frac{x}{R} - \frac{B}{\cosh^2\frac{x}{R}} - E. \tag{48}$$

We perform the transformation $\frac{1}{2}(1+\tanh\frac{x}{R}) = \tanh^2 r$, introduce the momentum operator $p_r = \frac{1}{i}(\frac{d}{dr} + \frac{1}{2}\Gamma_r)$ with $\Gamma_r = 1/(\sinh r\cosh r)$. We define a "time" s together with the coordinate transformation $x(t) \to r(s)$ with $F(r) = R\,\mathrm{artanh}(2\tanh^2 r - 1)$, $f(r) = R^2\coth^2 r$ and $\Delta V = -\frac{1}{8m}(3/\cosh^2 r + 1/\sinh^2 r)$. Thus we get the **space-time transformed** Hamiltonian $\tilde{H} = f(r)H_E$:

$$\tilde{H} = \frac{p_r^2}{2m} - \frac{2mR^2(A+E) + \frac{1}{4}}{2m\sinh^2 r} - \frac{8mBR^2 + \frac{3}{4}}{2m\cosh^2 r} + R^2(A-E) \tag{49}$$

and the space-time transformed path integral $\tilde{K}$ reads $[\eta = \frac{1}{2} \pm \sqrt{-2mR^2(A+E)},\ \nu = \frac{1}{2} \pm \sqrt{1+8mBR^2}]$:

$$\begin{aligned} &\tilde{K}(r'', r'; s'') \\ &= e^{is''R^2(A-E)}\int Dr(s)\mu_{\eta,\nu}[\sinh r, \cosh r]\exp\left(\frac{im}{2}\int_0^{s''}\dot{r}^2 ds\right) \\ &= \sum_{n=0}^{N_M} e^{is''[(E-A)R^2 - E_n]}\Psi_n^{(k_1,k_2)}(r')\Psi_n^{(k_1,k_2)*}(r'') \\ &\qquad + \int_0^\infty dk\, e^{is''[(E-A)R^2 - \frac{k^2}{2m}]}\Psi_k^{(k_1,k_2)}(r')\Psi_k^{(k_1,k_2)*}(r'') \end{aligned} \tag{50}$$

in the notation of Eq.(9). Performing the s''-integration [c.f.Eq.(17)] and respecting the correct boundary conditions for $x \to \pm\infty$ yields:

$$G(x'', x'; E) = \sum_{n=0}^{N_M}\frac{\Psi_n^{RM}(x')\Psi_n^{RM*}(x'')}{E_n^{RM} - E} + \int_0^\infty dk\frac{\Psi_k^{RM}(x')\Psi_k^{RM}(x'')}{A + \frac{k^2}{2mR^2} - E}. \tag{51}$$

The wave-functions and the energy spectrum are given by $[s \equiv \sqrt{1+8mBR^2},\ 0, 1, \ldots, n \le N_M < \frac{1}{2}(s-1) - \sqrt{m|A|R^2},\ k_1 = \frac{1}{2}(1+s),\ k_2 = \frac{1}{2}[1 + \frac{1}{2}(s-2n-1) - \frac{2mAR^2}{s-2n-1}],\ u = \frac{1}{2}(1+\tanh\frac{x}{R})$, note $k_2 - \frac{1}{2} > 0]$:

$$\Psi_n^{RM} = \frac{N_n^{(k_1,k_2)}}{\sqrt{R}}u^{k_2-\frac{1}{2}}(1-u)^{\frac{1}{2}s-k_2-n}{}_2F_1(-n, s-n; 2k_2; u) \tag{52a}$$

$$\Psi_n^{RM} = \left[\frac{s\,n!\,\Gamma(s-n)}{R\,2^{s-2n-1}\Gamma(2k_2-n)\Gamma(s-2k_2-n+1)}\right]^{\frac{1}{2}}$$
$$\times\left(1-\tanh\tfrac{x}{R}\right)^{\frac{1}{2}s-k_2-n}\left(1+\tanh\tfrac{x}{R}\right)^{k_2-\frac{1}{2}}P_n^{(s-2k_2-2n,2k_2-1)}\left(\tanh\tfrac{x}{R}\right) \quad (52b)$$

$$E_n^{RM} = -\left[\frac{(s-2n-1)^2}{8mR^2}+\frac{2mA^2R^2}{(s-2n-1)^2}\right]. \quad (53)$$

The wave-functions and the energy spectrum of the continuous states are given by $[k_2 \equiv \frac{1}{2}(1+i\tilde{k}),\ \tilde{k}\equiv\sqrt{2mR^2(A+E_k)}>0]$:

$$\Psi_k^{RM}(x) = \frac{N_k^{(k_1,k_2)}}{\sqrt{R}}(1-u)^{-\frac{i}{2}k}u^{\frac{i}{2}\tilde{k}}{}_2F_1\{\tfrac{1}{2}[1+s+i(\tilde{k}-k)],\tfrac{1}{2}[1-s+i(\tilde{k}-k)];1+i\tilde{k};u\}$$
$$E_k = A+\frac{k^2}{2mR^2}. \quad (54)$$

2) Second we discuss the Manning-Rosen (MR) potential $V^{MR}(x) = B/\sinh^2\frac{x}{R}-A\coth\frac{x}{R}$, $(x>0,\ A,B,R$ positive constants). V^{MR} was introduced by Manning and Rosen [21] to study vibrations of diatomic molecules. It can furthermore be used for describing the Kepler problem in a space of constant negative curvature [15]. The path integral for the Kepler problem in a space of constant positive curvature was discussed by Barut, Inomata and Junker [1]. The path integral for V^{MR} reads:

$$K^{MR}(x'',x';T) = \int Dx(t)\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+A\coth\frac{x}{R}-\frac{B}{\sinh^2\frac{x}{R}}\right)dt\right]. \quad (55)$$

Again we consider the Legendre-transformed Hamiltonian H_E:

$$H_E = -\frac{1}{2m}\frac{d^2}{dx^2}-A\coth\frac{x}{R}+\frac{B}{\sinh^2\frac{x}{R}}-E. \quad (56)$$

Performing the transformation $\frac{1}{2}(1-\coth\frac{x}{R}) = -1/\sinh^2 r$ yields:

$$\tilde{H} = \frac{p_r^2}{2m}+\frac{2mR^2(E-A)+\frac{1}{4}}{2m\cosh^2 r}+\frac{8mBR^2+\frac{3}{4}}{2m\sinh^2 r}-(E+A)R^2. \quad (57)$$

Here we have introduced the momentum operator $p_r = \frac{1}{i}(\frac{d}{dr}+\frac{1}{2}\Gamma_r)$ $[\Gamma_r = -1/(\sinh r\cosh r)]$ together with the time transformation $dt = f(r)ds$, where $f(r) = R^2\tanh^2 r$. With $\Delta V = \frac{1}{8m}(1/\cosh^2 r+3/\sinh^2 r)$ we thus arrive at the **space-time transformed** path integral $[\eta = \frac{1}{2}\pm\sqrt{1+8mBR^2},\ \nu = \frac{1}{2}\pm\sqrt{-2mR^2(E-A)}]$:

$$\begin{aligned}&\tilde{K}(r'',r,;s'')\\ &= e^{is''(E+A)R^2}\int Dr(s)\mu_{\eta,\nu}[\sinh r,\cosh r]\exp\left(i\int_0^{s''}\frac{m}{2}\dot{r}^2ds\right)\\ &= \sum_{n=0}^{N_M}e^{is''[(E+A)R^2-E_n]}\Psi_n^{(k_1,k_2)}(r')\Psi_n^{(k_1,k_2)*}(r'')\\ &\qquad+\int_0^\infty dk\,e^{is''[(E+A)R^2-\frac{k^2}{2m}]}\Psi_k^{(k_1,k_2)}(r')\Psi_k^{(k_1,k_2)*}(r''), \quad (58)\end{aligned}$$

again with the notation of Eq.(9). Performing the s''-integration [c.f.Eq.(17)] yields:

$$G(x'',x';E) = \sum_{n=0}^{N_M}\frac{\Psi_n^{MR}(x')\Psi_n^{MR}(x'')}{E_n^{MR}-E}+\int_0^\infty dk\frac{\Psi_k^{MR}(x')\Psi_k^{MR*}(x'')}{\frac{k^2}{2mR^2}-A-E}. \quad (59)$$

Respecting the correct boundary conditions for $x\to 0$ and $x\to\infty$ gives $k_1 = \frac{1}{2}[(1+\frac{1}{2}(s+2n+1)+\frac{2mAR^2}{s+2n+1}]$ and $k_2 = \frac{1}{2}(1+\sqrt{1+8mBR^2})\equiv\frac{1}{2}(1+s)$. The wave-functions and the energy-spectrum of the bound states read $[0,1,\dots,n\le N_M<\sqrt{mAR^2}-\frac{1}{2}(s+1)$, $u = \frac{1}{2}(1-\coth\frac{x}{R})$, note $n+\frac{1}{2}-k_1<0]$:

$$\begin{aligned}\Psi_n^{MR}(x) &= \frac{N_n^{(k_1,k_2)}}{\sqrt{R}}(u-1)^{\frac{1}{2}-k_1+n}u^{k_1-1-\frac{1}{2}s-n}{}_2F_1\left(-n,2k_1-n-1;s+1;\frac{1}{1-u}\right)\\ &= \left[\frac{(2k_1-1)n!\,\Gamma(2k_1-n-1)}{R\,\Gamma(n+s+1)\Gamma(2k_1-s-n-1)}\right]^{\frac{1}{2}}\\ &\qquad\times\left(1+e^{-2x}\right)^{k_2}e^{-2x(k_1-\frac{1}{2}s-n-1)}P_n^{(2k_1-2n-s-2,s)}(1-2e^{-2x}) \quad (60)\end{aligned}$$

$$E_n^{RM} = -\left[\frac{(s+2n+1)^2}{8mR^2}+\frac{2mA^2R^2}{(s+2n+1)^2}\right]. \quad (61)$$

The wave-functions and the energy-spectrum of the continuous states are given by $[k_1 \equiv \frac{1}{2}(1+i\tilde{k}),\ \tilde{k}\equiv\sqrt{2mR^2(E_k-A)}>0]$:

$$\begin{aligned}\Psi_k^{MR}(x) &= \frac{N_k^{(k_1,k_2)}}{\sqrt{R}}u^{-\frac{i}{2}k}(u-1)^{\frac{1}{2}[i\tilde{k}-(1+s)]}\\ &\qquad\times{}_2F_1\left(\frac{1+s+i(\tilde{k}-k)}{2},\frac{1+s-i(\tilde{k}+k)}{2};s+1;\frac{1}{1-u}\right) \quad (62)\end{aligned}$$

and $E_k = \frac{k^2}{2mR^2}-A$.

V. Summary

In this paper several further examples of exact path integral treatments have been presented. These have been the path integral for the "smooth step", the Rosen-Morse-, the Manning-Rosen potential and the symmetric top. All examples have a close relation to the usual Pöschl-Teller potential (for a pure discrete spectrum as for the top) or to the modified Pöschl-Teller potential (if the spectrum is also continuous). As already noted these two path integrals are derived by means of the path integrals of free motion over the $SU(2)$ and $SU(1,1)$ manifolds, respectively. This establishes a close relationship between free motion on curved manifolds and potential problems. But, following Duru [7], "it is well-known that all of the special functions which appear as solutions of problems in theoretical physics are the matrix elements of some Lie group. Because of this fact, if one parametrizes the problems suitably with their symmetries, it may be possible to relate their path integrals to the ones written for the motions on the appropriate group spaces."

We have also discussed the problem of the correct functional measure to be used in the path integral of the the Pöschl-Teller and modified Pöschl-Teller case, respectively. The usual expansion of the modified Bessel function seems very suggestive but gives in the lattice formulation the wrong boundary behaviour of the corresponding short-time kernels and wave-functions. This is very analogous to the radial path integral which has been discussed in great detail in Refs.[11,29].

The examples solved in this paper suggest that the two Pöschl-Teller path integrals play an analogous important role for applications as the path integral for the radial harmonic oscillator [25]. In a forthcoming publication we use these three path integrals (Pöschl-Teller, modified Pöschl-Teller and radial harmonic oscillator) to give a classification scheme (similar to the classical factorization method of Schrödinger on the operator level as reviewed by Infeld and Hull [14]) of the known path integral solutions [13].

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