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by

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Gauge Invariance and Current Algebra in Two Dimensional Non-abelian Chiral Theories

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Abstract:

We study two dimensional massless chiral QCD in a gauge invariant formulation. The Faddeev-Popov-procedure is used to obtain a gauge invariant effective action after bosonization which is then quantized canonically using Dirac's prescription for constrained systems. The gauge current algebra is deduced and covariant current conservation is shown by means of the equations of motion of the bosonic matter fields.

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1.) Introduction

Symmetries of a classical theory may be spoiled at the quantum level. In the case of gauge invariance this shows itself dynamically in the fact that the gauge current J^μ is not conserved covariantly:

$$D_\mu J^\mu \neq 0, \qquad (1.1)$$

which is a contradiction to the classical equations of motion. Since the existence of internal symmetries is one of the motivations for considering gauge theories, it should be demanded that they survive quantization. Furthermore, the existence of an anomaly $D_\mu J^\mu \neq 0$ is suspected to spoil unitarity and renormalizability [1] , which implies that the theory is inconsistent and hence useless.

For the chiral Schwinger model [2-4] , which seems to be anomalous, it has been shown that a consistent quantum theory is possible [5]. Integration in the path integral approach over equivalent gauge field configurations [6-8] yields an effective action with a Wess-Zumino-term [9] which renders the theory gauge invariant and anomaly-free. In the abelian case [10,11] the effective action has been quantized using Dirac's prescription for constrained systems [12-14]. It turned out that the current algebra is canonical, i.e. there is no Schwinger term in the $[J_0,J_0]$-commutator. In the non-abelian case in the Hamiltonian formulation, however, there has only been an approach using the gauge non-invariant formulation [15].

In this paper we want to perform the canonical quantization in a gauge invariant framework. We consider the Lagrangean

$$\mathcal{L} = -\frac{1}{4}\, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left(i\partial\!\!\!/ - eA\!\!\!/\, \frac{1-\gamma^5}{2}\right)\Psi \qquad (1.2)$$

of chiral QCD in two dimensions. We use the Faddeev-Popov-procedure [16] to get rid of superfluous gauge field configurations in the path integral. After bosonization we obtain a gauge invariant effective action which will be quantized using Dirac's prescription. The constraint algebra shows that the system is first class, thus reflecting the gauge symmetry. We fix a gauge to obtain a second class system and calculate the Dirac brackets, which are turned into quantum commutators. The gauge currents are determined, their algebra is deduced, and finally current conservation is shown by means of the equations of motion.

2.) The Chiral Model with Wess - Zumino - Term

We briefly review the chiral model to show how a Wess-Zumino-term may be treated in the framework of canonical quantization [17-19]. The action is given by

$$S = S_0 + S_{WZ} = -\int_M dx \ \mathrm{tr}\left(\omega_\mu \omega^\mu\right) + \lambda \int_Q \mathrm{tr}\ \omega^3 , \qquad (2.1)$$

where

$$\omega_\mu = h^{-1}\partial_\mu h, \ h = \exp\left(\Phi^a \frac{i}{\sqrt{2}} t^a \right). \qquad (2.2)$$

The Φ^a parametrize an SU(N)-valued field h, the t^a are normalized hermitean generators of the SU(N) gauge group, and we assume that $M = \partial Q$ is the two dimensional Minkowski space (for additional conventions see appendices A,B).

In components, the action may be written as

$$S = \int_M dx \left(\frac{1}{2} g_{ij}\ \partial_\mu\Phi^i\ \partial^\mu\Phi^j + \lambda\ D_{ij}\ \partial_0\Phi^i\ \partial_1\Phi^j\right) = \int_M dx\ \mathcal{L}(x)\ ; \qquad (2.3)$$

here D is a local two form such that $dD = \mathrm{tr}\ \omega^3$.

Since the $\Phi^i(x)$ are independent configuration space variables, canonical quantization may be carried out by requiring that the canonical momenta

$$P_i := \frac{\partial \mathcal{L}}{\partial\partial_0\Phi^i} = p_i + \lambda\ D_{ij}\ \partial_1\Phi^j,\ p_i := g_{ij}\ \partial_0\Phi^j \qquad (2.4)$$

and the coordinates Φ^j satisfy canonical commutation relations

$$\left[P_i(x), \Phi^j(y)\right]_{ETC} = -\ i\ \delta_{ij}\ \delta(\vec{x}-\vec{y}). \qquad (2.5)$$

It is easy to see that

$$\left[p_i(x), \partial_1 \Phi^j(y)\right]_{ETC} = i\, \delta_{ij}\, \partial^x \delta(\vec{x}-\vec{y}),$$

$$\left[p_i(x), f(\Phi)(y)\right]_{ETC} = -\, i\, \partial_i f(\Phi)\, \delta(\vec{x}-\vec{y}). \qquad (2.6)$$

We will drop the index "ETC" in the sequel. After some algebra using (B.12) it is possible to prove that

$$\left[p_i(x), p_j(y)\right] = \frac{3}{2\sqrt{2}}\; i\, \lambda\, g_{abc}\, \omega_{ia}\, \omega_{jb}\, \omega_{kc}\, \partial_1 \Phi^k\, \delta(\vec{x}-\vec{y}), \qquad (2.7)$$

this result is necessary to calculate the current algebra later on.

The action S is invariant with respect to the global gauge transformation

$$h \longrightarrow h^{\vartheta} = \vartheta \cdot h. \qquad (2.8)$$

By application of Noether's theorem we obtain conserved currents [20]

$$J^{\mu}_{a} = j^{\mu}_{a} + \frac{3}{2}\,\lambda\, j_{\nu a}\, \varepsilon_{\mu\nu}\ , \text{ where } \quad j^{\mu}_{a} := \omega_{ia}\, \partial^{\mu}\Phi^{i}\ . \qquad (2.9)$$

Current conservation $\partial_\mu J^{\mu}_{a} = 0$ is equivalent to the equations of motion. We note that

$$j_{0a} = p_i\, \xi_{ai},\ j_{1a} = g_{ij}\, \partial_1 \Phi^i\, \xi_{aj}, \qquad (2.10)$$

so the algebra of the currents $j_{\mu a}$ may be calculated by using the canonical commutation relations and the Killing equation (B.10):

$$\left[j_{0a}(x), j_{0b}(y)\right] = \frac{i}{\sqrt{2}}\, g_{abc}\left(j_{0c}(x) + \frac{3}{2}\,\lambda\, j_{1c}(x)\right)\delta(\vec{x}-\vec{y}),$$

$$\left[j_{0a}(x), j_{1b}(y)\right] = \frac{i}{\sqrt{2}}\, g_{abc}\, j_{1c}(x)\,\delta(\vec{x}-\vec{y}) \;+ i\, \delta_{ab}\, \partial^x \delta(\vec{x}-\vec{y}),$$

$$\left[j_{1a}(x), j_{1b}(y)\right] = 0. \qquad (2.11)$$

These commutators imply the algebra of the symmetry-generating currents $J_{\mu a}$:

$$\left[J_{0a}(x), J_{1b}(y)\right] = \frac{1}{\sqrt{2}} g_{abc}\, J_{0c}(x)\, \delta(\vec{x}-\vec{y}) - 3i\lambda\, \delta_{ab}\, \partial^x \delta(\vec{x}-\vec{y}),$$

$$\left[J_{0a}(x), J_{1b}(y)\right] = \frac{1}{\sqrt{2}} g_{abc}\, J_{1c}(x)\, \delta(\vec{x}-\vec{y}) + i\left(1+\frac{9}{4}\lambda^2\right) \delta_{ab}\, \partial^x \delta(\vec{x}-\vec{y}),$$

$$\left[J_{1a}(x), J_{1b}(y)\right] = \frac{1}{\sqrt{2}} g_{abc}\left(-\frac{9}{4}\lambda^2 J_{0c}(x) - 3\lambda J_{1c}(x)\right) \delta(\vec{x}-\vec{y}) - 3i\lambda\, \delta_{ab}\, \partial^x \delta(\vec{x}-\vec{y}). \qquad (2.12)$$

This current algebra coincides with the result of ref. [19], eqs. (3.8), (3.9).

3.) Chiral QCD in Two Dimensions

Chiral coupling of fermions to gauge fields is realized in nature in the weak interactions. Since under gauge transformations only one helicity of the fermions is affected, the functional path integral measure $\mathcal{D}\Psi\, \mathcal{D}\overline{\Psi}$ is not gauge invariant [21]. Hence anomalies may appear spoiling renormalizability and unitarity of the theory [1]. In two dimensions it is possible to study these anomalies explicitly since the fermions can be integrated out in the path integral leading to an effective action which can then be quantized canonically.

3.1.) The action of chiral QCD

The classical action of chiral QCD is [15]

$$S = S_{ym}(A) + S_f(\Psi, \overline{\Psi}, A) = \int_M d^2x\, \mathcal{L}_{cl}(x),$$

$$S_{ym}(A) = \int_M d^2x\, \mathcal{L}_{ym}(A) = \int_M d^2x \left[-\frac{1}{4} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)\right],$$

$$S_f(\Psi, \overline{\Psi}, A) = \int_M d^2x\, \mathcal{L}_f(A) = \int_M d^2x \left[\overline{\Psi}\left(i\partial\!\!\!/ + eA\!\!\!/\, \frac{1-\gamma^5}{2}\right)\Psi\right]. \qquad (3.1.1)$$

The gauge field and the field strength tensor may be decomposed as

$$A_\mu = A_{\mu a}\, t^a, \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie\left[A_\mu, A_\nu\right] = F_{\mu\nu\, a}\, t^a. \qquad (3.1.2)$$

The covariant derivative is given by

$$D_\mu = \partial_\mu - ie\, A_\mu. \qquad (3.1.3)$$

A short calculation yields

$$\mathcal{L}_{cl}(x) = -\frac{1}{4}\operatorname{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \bar{\Psi}\, i\not{\partial}\Psi + \Psi_L\, e A_- \gamma^0 \Psi_L, \qquad (3.1.4)$$

showing that only the "-" - component (see (A.1)) of the gauge field couples to the left-handed fermion helicity. The action is invariant under a gauge transformation $\vartheta(x)$, if

$$A_\mu \longrightarrow A_\mu^\vartheta = \vartheta\, A_\mu\, \vartheta^{-1} - \frac{1}{e}\partial_\mu\vartheta\, \vartheta^{-1},$$

$$\Psi_R \longrightarrow \Psi_R^\vartheta = \Psi_R,$$

$$\Psi_L \longrightarrow \Psi_L^\vartheta = \vartheta\, \Psi_L,$$

$$D_\mu \longrightarrow D_\mu^\vartheta = \partial_\mu - ie\, A_\mu^\vartheta. \qquad (3.1.5)$$

3.2.) The Faddeev - Popov - procedure

It is a well known fact that naive canonical quantization of gauge fields does no lead to a sensible quantum theory since the gauge freedom makes the definition of a propagator impossible. The Faddeev-Popov-method solves this problem by eliminating superfluous degrees of freedom in the path integral approach. Therefore we consider the path integral

$$Z = \int \mathcal{D}A\, \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi}\; e^{iS(\Psi,\bar{\Psi},A)} = \int \mathcal{D}A\; e^{iS_{ym}(A)} \int \mathcal{D}\Psi\, \mathcal{D}\bar{\Psi}\; e^{iS_f(\Psi,\bar{\Psi},A)}. \qquad (3.2.1)$$

In principle there are now two possibilities to proceed. One could express

$$\int \mathcal{D}\Psi\, \mathcal{D}\overline{\Psi}\; e^{iS_f(\Psi,\overline{\Psi},A)} \text{ as } \int \mathcal{D}h\; e^{iS_{eff}(h,A)} \tag{3.2.2}$$

as in [15] by simply integrating out the fermions, where $S_{eff}(h,A)$ is an effective action expressed in terms of a Lie group valued field h and the gauge field. It turns out, however, that this effective action is no longer gauge invariant. The reason for this is that the functional measure $\mathcal{D}\Psi\, \mathcal{D}\overline{\Psi}$ is not gauge invariant, yielding a non-trivial Jacobian:

$$\mathcal{D}\Psi\, \mathcal{D}\overline{\Psi} \neq \mathcal{D}\Psi^{g}\, \mathcal{D}\overline{\Psi}^{g}. \tag{3.2.3}$$

Therefore the effective action does not possess the same symmetries as the original one. In contrast, we will use the Faddeev-Popov-procedure [16]. A clever unity

$$1 = \Delta(A) \int \mathcal{D}g\; \delta\Big(F(A^{g}) \Big) \tag{3.2.4}$$

is inserted into the path integral, where F is some gauge fixing functional. The gauge invariance of $\Delta(A)$ may easily be shown by means of the right invariance of the (functional) Haar measure. The gauge field measure $\mathcal{D}A$ is gauge invariant, furthermore, we define

$$\tilde{\mathcal{D}}A := \mathcal{D}A\; \Delta(A)\; \delta\Big(F(A) \Big). \tag{3.2.5}$$

In the case of vector coupling of the fermions, the integration over A factorizes if the integration variables of the fermions are gauge shifted. An infinite constant may then be absorbed into an overall normalization constant. Here, however, the chiral coupling does not allow this shift, so the Lie group valued field g survives the Faddeev-Popov-procedure as a degree of freedom [6,7,8].

Using the gauge invariance of $S_{ym}(A)$, we obtain after a shift of the integration variable A [22]

$$Z = \int \tilde{\mathfrak{D}}A\, \mathfrak{D}g\, e^{iS_{ym}(A)} \int \mathfrak{D}\Psi\, \mathfrak{D}\overline{\Psi}\, e^{iS_f(\Psi,\overline{\Psi},A^{g^{-1}})} . \tag{3.2.6}$$

Thus we have restricted the integration in Z to those gauge fields which are inequivalent with respect to gauge transformations.

3.3.) Bosonization

Now we use the well known fact [15,23,24] that the functional integral

$$\int \mathfrak{D}\Psi\, \mathfrak{D}\overline{\Psi}\, e^{iS_f(\Psi,\overline{\Psi},A)} \tag{3.3.1}$$

may be replaced by an integral over a group valued field h

$$\int \mathfrak{D}h\, e^{-i\beta(A,h) + i\zeta(A)} . \tag{3.3.2}$$

Here β possesses the cocycle property and $\zeta(A)$ has the form of a mass term for the gauge field with an arbitrary parameter $a \in \mathbb{R}$ which arises upon regularization of the fermionic determinant [25,26]. Replacing g^{-1} by g and inserting the explicit parametrizations

$$g = \exp\left(\Theta^a \frac{i}{\sqrt{2}} t^a\right),\ h = \exp\left(\Phi^a \frac{i}{\sqrt{2}} t^a\right) \tag{3.3.3}$$

we obtain the path integral

$$Z = \int \tilde{\mathfrak{D}}A\, \mathfrak{D}g\, \mathfrak{D}h\, e^{iS_{eff}(A,g,h)} , \tag{3.3.4}$$

where

$$S_{eff}(A,g,h) = \int_M d^2x\, \mathfrak{L}(x),$$

$$\mathcal{L}(x) = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{ae^2}{8\pi} A^a_\mu A^\mu_a \tag{3.3.5}$$
$$+ \frac{1}{16\pi} g_{ij}\, \partial_\mu\Phi^i \partial^\mu\Phi^j - \frac{1}{12\pi} D_{ij} \partial_0\Phi^i \partial_1\Phi^j$$
$$+ (a-1)\frac{1}{16\pi} h_{ij}\, \partial_\mu\Theta^i \partial^\mu\Theta^j + \frac{1}{12\pi} C_{ij} \partial_0\Theta^i \partial_1\Theta^j$$
$$+ \frac{e}{\sqrt{2}4\pi} g_{ij}\xi_{aj}\left(\partial_0\Phi^i + \partial_1\Phi^i\right)\left(A_{0a} - A_{1a}\right)$$
$$+ \frac{e}{\sqrt{2}4\pi} h_{ij}\eta_{aj}\left(\partial_0\Theta^i\left((a-1)A_{0a}+A_{1a}\right) - \partial_1\Theta^j\left(A_{0a}+(a-1)A_{1a}\right)\right).$$

Here we have assumed that $(g, \Theta^i, h_{ij}, \eta_{ai}, \sigma_{ia})$ are the analogues of $(h, \Phi^i, g_{ij}, \xi_{ai}, \omega_{ia})$ in terms of the parameters Θ^i. Of course, locally there is a two-form C with $dC = \mathrm{tr}\, \sigma^3$.

The effective action is invariant under the gauge transformation

$$g \longrightarrow g^\vartheta = g\, \vartheta^{-1},$$

$$h \longrightarrow h^\vartheta = h\, \vartheta^{-1},$$

$$A_\mu \longrightarrow A^\vartheta_\mu = \vartheta\, A_\mu\, \vartheta^{-1} - \frac{1}{e}\partial_\mu\vartheta\, \vartheta^{-1}. \tag{3.3.6}$$

3.4.) Quantization

We are going to quantize the theory canonically. First of all, we define the momenta conjugate to $A_{\mu a}$, Φ^i and Θ^i:

$$\Pi^\mu_a := \frac{\partial\mathcal{L}}{\partial\partial_0 A_{\mu a}} = F^a_{0\mu}, \text{ so}$$

$\Pi^0_a = 0$, which is a primary constraint, and

$$\Pi^1_a = \partial_0 A_{1a} - \partial_1 A_{0a} + e\, g_{bca} A_{0b} A_{1c},$$
$$p_i := \frac{\partial\mathcal{L}}{\partial\partial_0\Phi^i} = \frac{1}{8\pi} g_{ij}\, \partial_0\Phi^j - \frac{1}{12\pi} D_{ij}\, \partial_1\Phi^j + \frac{e}{\sqrt{2}4\pi}\omega_{ia} A_{-a},$$

$$q_i := \frac{\partial \mathfrak{X}}{\partial \partial_0 \Theta^i} = \frac{1}{8\pi} h_{ij} \partial_0 \Theta^j + \frac{1}{12\pi} C_{ij} \partial_1 \Theta^j + \frac{e}{\sqrt{2}4\pi} \sigma_{ia}\left((a-1) A_{0a} + A_{1a}\right). \tag{3.4.1}$$

For convenience, we define

$$\begin{aligned} \bar{p}_a &:= \xi_{ai} \left(p_i + \frac{1}{12\pi} D_{ij}\, \partial_1 \Phi^j\right), \\ \bar{q}_a &:= \eta_{ai} \left(q_i - \frac{1}{12\pi} C_{ij}\, \partial_1 \Theta^j\right), \end{aligned} \tag{3.4.2}$$

and obtain

$$\begin{aligned} \partial_0 \Phi^k &= \xi_{ak} \left(8\pi \bar{p}_a - \sqrt{2}e\, A_{-a}\right), \\ \partial_0 \Theta^k &= \frac{1}{a-1} \eta_{ak} \left(8\pi\, \bar{q}_a - \sqrt{2}e \left((a-1) A_{0a} + A_{1a}\right)\right), \end{aligned} \tag{3.4.3}$$

if we restrict ourselves to the case $a \neq 1$.

Now we introduce the covariant derivative

$$D_\mu^{ab} := \delta_{ab} \partial_\mu - e\, g_{abc} A_{\mu c} \tag{3.4.4}$$

and finally obtain the Hamiltonian density after a partial integration:

$$\begin{aligned} \mathfrak{X} = {} & \frac{1}{2} \Pi_a^1 \Pi_a^1 - A_{0a} D_1^{ab} \Pi_b^1 + \frac{e^2}{8\pi} \frac{a^2}{a-1} A_{1a} A_{1a} \\ & + 4\pi\, \bar{p}_a \bar{p}_a + \frac{1}{16\pi} \omega_{1a} \omega_{1a} + \frac{4\pi}{a-1} \bar{q}_a \bar{q}_a + \frac{a-1}{16\pi} \sigma_{1a} \sigma_{1a} \\ & - \sqrt{2}e\, \bar{p}_a \left(A_{0a} - A_{1a}\right) - \frac{e}{\sqrt{2}4\pi} \omega_{1a} \left(A_{0a} - A_{1a}\right) \\ & - \frac{\sqrt{2}e}{a-1} \bar{q}_a \left((a-1) A_{0a} + A_{1a}\right) + \frac{e}{\sqrt{2}4\pi} \sigma_{1a} \left(A_{0a} + (a-1) A_{1a}\right). \end{aligned} \tag{3.4.5}$$

The terms linear in $\partial_0 \Phi^i$ and $\partial_0 \Theta^i$ have vanished, furthermore it is obvious that $\mathfrak{X}$ can be written in terms of $\bar{p}_a$, $\bar{q}_a$ in a simple form since these variables contain the contributions from the forms C and D.

As mentioned above, Π_a^0 is not an independent dynamical variable and has to be treated as a primary constraint

$$\Omega_{1a} := \Pi_a^0 \approx 0 \tag{3.4.6}$$

in the sense of Dirac [13] . To obtain a consistent quantum theory which is not ill from the very beginning, one has to use Dirac's prescription for quantizing constrained systems, since canonical commutators might lead to contradictions. The basic Poisson brackets are

$$\begin{aligned}
&\left\{\Phi^i(\vec{x}), p_j(\vec{y})\right\} = \delta_{ij}\, \delta(\vec{x}-\vec{y}), \\
&\left\{\Theta^i(\vec{x}), q_j(\vec{y})\right\} = \delta_{ij}\, \delta(\vec{x}-\vec{y}), \\
&\left\{A_{0a}(\vec{x}), \Pi_b^0(\vec{y})\right\} = \delta_{ab}\, \delta(\vec{x}-\vec{y}), \\
&\left\{A_{1a}(\vec{x}), \Pi_b^1(\vec{y})\right\} = \delta_{ab}\, \delta(\vec{x}-\vec{y}),
\end{aligned} \tag{3.4.7}$$

the other brackets vanish.

By careful use of identities involving derivatives of δ-distributions and of (B.12) one obtains the useful Poisson brackets

$$\begin{aligned}
&\left\{\omega_{1a}(\vec{x}), \bar{p}_b(\vec{y})\right\} = \frac{1}{\sqrt{2}}\, g_{abc}\, \omega_{1c}\, \delta(\vec{x}-\vec{y}) + \delta_{ab}\, \partial^x \delta(\vec{x}-\vec{y}), \\
&\left\{\bar{p}_a(\vec{x}), \bar{p}_b(\vec{y})\right\} = \frac{1}{\sqrt{2}}\, g_{abc}\left(\bar{p}_c - \frac{1}{8\pi}\, \omega_{1c}\right) \delta(\vec{x}-\vec{y}), \\
&\left\{\sigma_{1a}(\vec{x}), \bar{q}_b(\vec{y})\right\} = \frac{1}{\sqrt{2}}\, g_{abc}\, \sigma_{1c}\, \delta(\vec{x}-\vec{y}) + \delta_{ab}\, \partial^x \delta(\vec{x}-\vec{y}), \\
&\left\{\bar{q}_a(\vec{x}), \bar{q}_b(\vec{y})\right\} = \frac{1}{\sqrt{2}}\, g_{abc}\left(\bar{q}_c + \frac{1}{8\pi}\, \sigma_{1c}\right) \delta(\vec{x}-\vec{y}).
\end{aligned} \tag{3.4.8}$$

Following Dirac, we have to look for further constraints which arise as consistency conditions

$$\partial_0 \Omega_1 \approx 0. \tag{3.4.9}$$

Since

$$\left\{\Omega_{1a}(\vec{x}),\Omega_{1b}(\vec{y})\right\} = 0, \tag{3.4.10}$$

this is equivalent to

$$0 \approx \left\{\Omega_{1a}(\vec{x}),H\right\} = \left\{\Omega_{1a}(\vec{x}),\int d\vec{y}\,\mathcal{H}(\vec{y})\right\}, \tag{3.4.11}$$

which leads to

$$\Omega_{2a} := D_1^{ab}\Pi_b^1 + \sqrt{2}e\bar{p}_a + \frac{e}{\sqrt{24\pi}}\,\omega_{1a} + \sqrt{2}e\bar{q}_a - \frac{e}{\sqrt{24\pi}}\,\sigma_{1a} \approx 0. \tag{3.4.12}$$

Ω_{2a} is a secondary constraint corresponding to Gauss' law. Additional Poisson brackets are

$$\left\{\Omega_{1a}(\vec{x}),\Omega_{2b}(\vec{y})\right\} = 0,$$

$$\left\{\Omega_{2a}(\vec{x}),\Omega_{2b}(\vec{y})\right\} = e\,g_{abc}\,\Omega_{2c}(\vec{x})\,\delta(\vec{x}-\vec{y}). \tag{3.4.13}$$

Now we claim that there are no further constraints. A lengthy calculation yields

$$\partial_0\Omega_{2a}(\vec{x}) = \left\{\Omega_{2a}(\vec{x}),H\right\} = e\,g_{abc}\,\Omega_{2b}(\vec{x})\,A_{0c}(\vec{x}) \approx 0, \tag{3.4.14}$$

since $\Omega_{2b} \approx 0$. Therefore the consistency relation $\partial_0\Omega_2 \approx 0$ can be expressed in terms of the constraints which are already known.

The matrix

$$\left\{\Omega_{\alpha a}(\vec{x}),\Omega_{\beta b}(\vec{y})\right\}$$

is not invertible, so we are dealing with a first class system. Dirac proves that first class constraints are generators of symmetries of the theory. In our particular case the gauge symmetry of the effective action is reflected. Furthermore, Ω_2 fulfills the correct algebraic relation (3.4.13) for generators of symmetries.

There are now two possibilities to obtain a sensible quantum theory.

a) We may assume canonical commutation relations equivalent to the Poisson brackets. The constraint equations $\Omega_{\alpha a} \; |\Psi\rangle = 0$ are fulfilled for physical states $|\Psi\rangle$ only. To obtain further information, we would have to use the restricted physical Hilbert space which is the kernel of the constraints $\Omega_{\alpha a}$.

b) We can extract information from the operators and their algebra without considering states if we could assume the constraints to be strong operator equations for all states. This may be achieved by using further constraints which force the system to be second class.

We will proceed as indicated in (b). We have to determine constraints which restrict the symmetry of the theory, so we have to fix a gauge.

3.5.) Gauge fixing and Dirac brackets

We will work in the unitary gauge $g = \mathbf{1}$, since in this gauge the Wess-Zumino-term which leads to the gauge invariance of the theory is gauged away. Then it is possible to compare the results with those obtained in the gauge non-invariant formulation.

In order to invert the antisymmetric constraint matrix

$$\{\Omega_{\alpha a}, \Omega_{\beta b}\}$$

we have to add an even number of constraints. We express $\Theta^i = 0$, which implies $g = \mathbf{1}$, as $\partial_0\Theta^i = \partial_1\Theta^i = 0$ [10]. To simplify calculations, we use as gauge fixing constraints

$$\Omega_{3a} := \frac{e}{\sqrt{2}4\pi}\, \sigma_{1a} \approx 0,$$

$$\Omega_{4a} := \sqrt{2}e\, \bar{q}_a - \frac{e^2}{4\pi}\Big((a-1)A_{0a} + A_{1a}\Big) \approx 0. \qquad (3.5.1)$$

We remark that

$$\Omega_{3a} = \frac{e}{\sqrt{2}4\pi}\, \epsilon_{ka}\partial_1\Theta^k, \quad \Omega_{4a} = \frac{\sqrt{2}4\pi}{e(a-1)}\, \epsilon_{ka}\, \partial_0\Theta^k. \tag{3.5.2}$$

The matrix

$$C := \left\{\Omega_{\alpha a}, \Omega_{\beta b}\right\}$$

may easily be calculated by using (3.4.7),(3.4.8), the result is

$$\begin{aligned}
\left\{\Omega_{1a}(\vec{x}), \Omega_{3b}(\vec{y})\right\} &= 0, \\
\left\{\Omega_{1a}(\vec{x}), \Omega_{4b}(\vec{y})\right\} &= \frac{e^2}{4\pi}(a-1)\delta_{ab}\delta(\vec{x}-\vec{y}), \\
\left\{\Omega_{2a}(\vec{x}), \Omega_{3b}(\vec{y})\right\} &= \frac{e^2}{\sqrt{2}4\pi} g_{abc}\epsilon_{1c}\delta(\vec{x}-\vec{y}) + \frac{e^2}{4\pi}\delta_{ab}\partial^x\delta(\vec{x}-\vec{y}), \\
\left\{\Omega_{2a}(\vec{x}), \Omega_{4b}(\vec{y})\right\} &= g_{abc}\left(\sqrt{2}e^2\overline{q}_c - \frac{e^3}{4\pi}A_{1c}\right)\delta(\vec{x}-\vec{y}), \\
\left\{\Omega_{3a}(\vec{x}), \Omega_{3b}(\vec{y})\right\} &= 0, \\
\left\{\Omega_{3a}(\vec{x}), \Omega_{4b}(\vec{y})\right\} &= \frac{e^2}{\sqrt{2}4\pi} g_{abc}\epsilon_{1c}\delta(\vec{x}-\vec{y}) + \frac{e^2}{4\pi}\delta_{ab}\partial^x\delta(\vec{x}-\vec{y}), \\
\left\{\Omega_{4a}(\vec{x}), \Omega_{4b}(\vec{y})\right\} &= \sqrt{2}e^2\, g_{abc}\left(\overline{q}_c + \frac{1}{8\pi}\epsilon_{1c}\right)\delta(\vec{x}-\vec{y}).
\end{aligned} \tag{3.5.3}$$

To proceed, we define linear combinations of the constraints:

$$\chi_1 := \Omega_3, \; \chi_2 := \Omega_2, \; \chi_3 := \Omega_1, \; \chi_4 := \Omega_4 - \Omega_2. \tag{3.5.4}$$

Then it is easy to see that the matrix

$$C'(\vec{x},\vec{y}) := \left\{\chi_{\alpha a}(\vec{x}), \chi_{\beta b}(\vec{y})\right\}$$

is invertible. Therefore, the system is now second class, and we may write down Dirac brackets

$$\left\{A,B\right\}_D = \left\{A,B\right\} - \sum_{\substack{\alpha,a \\ \beta,b}} \int d\vec{x}\, d\vec{y}\, \left\{A, \chi_{\alpha a}(\vec{x})\right\} C'^{-1}_{\alpha a,\beta b}(\vec{x},\vec{y}) \left\{\chi_{\beta b}(\vec{y}), B\right\} \tag{3.5.5}$$

which are turned into quantum commutators by the prescription

$$\{\ ,\ \}_D \longrightarrow \frac{1}{i}[\ ,\ ].$$

It is a tedious bit of algebra to do the calculation, so we simply state the result here. Some commutators are simplified by using the fact that constraints are now strong operator equations.

$$\frac{1}{i}\left[A_{0a}(x),A_{0b}(y)\right] = \frac{1}{(a-1)^2}\frac{4\pi}{e}\, g_{abc}\left(A_{1c} - (a-1)A_{0c}\right)\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[A_{0a}(x),A_{1b}(y)\right] = -\frac{1}{a-1}\frac{4\pi}{e^2}\left(e\, g_{abc}A_{1c}\,\delta(\vec{x}-\vec{y}) - \delta_{ab}\partial^x\delta(\vec{x}-\vec{y})\right),$$

$$\frac{1}{i}\left[A_{0a}(x),\Pi^1_b(y)\right] = -\frac{1}{a-1}\left(\frac{4\pi}{e}\, g_{abc}\,\Pi^1_c + \delta_{ab}\right)\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[A_{0a}(x),\omega_{1b}(y)\right] = -\frac{1}{a-1}\left(\frac{4\pi}{e} g_{abc}\,\omega_{1c}\,\delta(\vec{x}-\vec{y}) + \frac{\sqrt{2}4\pi}{e}\delta_{ab}\partial^x\delta(\vec{x}-\vec{y})\right),$$

$$\frac{1}{i}\left[A_{0a}(x),\bar{p}_b(y)\right] = -\frac{1}{a-1}\left(\frac{4\pi}{e} g_{abc}\,\bar{p}_c\,\delta(\vec{x}-\vec{y}) + \frac{1}{\sqrt{2}e}\delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y})\right),$$

$$\frac{1}{i}\left[A_{0a}(x),\bar{q}_b(y)\right] = -\frac{1}{\sqrt{2}}\, g_{abc}A_{0c}\,\delta(\vec{x}-\vec{y}) + \frac{1}{a-1}\frac{1}{\sqrt{2}e}\,\delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[A_{1a}(x),\Pi^1_b(y)\right] = \delta_{ab}\,\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[A_{1a}(x),\bar{q}_b(y)\right] = -\frac{1}{\sqrt{2}e}\left(e\, g_{abc}\,A_{1c}\,\delta(\vec{x}-\vec{y}) - \delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y})\right),$$

$$\frac{1}{i}\left[\Pi^1_a(x),\bar{q}_b(y)\right] = -\frac{1}{\sqrt{2}}\, g_{abc}\,\Pi^1_c\,\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[\omega_{1a}(x),\bar{p}_b(y)\right] = \frac{1}{\sqrt{2}}\, g_{abc}\,\omega_{1c}\,\delta(\vec{x}-\vec{y}) + \delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[\omega_{1a}(x),\bar{q}_b(y)\right] = -\left(\frac{1}{\sqrt{2}} g_{abc}\,\omega_{1c}\,\delta(\vec{x}-\vec{y}) + \delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y})\right),$$

$$\frac{1}{i}\left[\bar{p}_a(x),\bar{p}_b(y)\right] = \frac{1}{\sqrt{2}} g_{abc}\left(\bar{p}_c - \frac{1}{8\pi}\,\omega_{1c}\right)\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[\bar{p}_a(x),\bar{q}_b(y)\right] = -\left(\frac{1}{\sqrt{2}}\, g_{abc}\,\bar{p}_c\,\delta(\vec{x}-\vec{y}) + \frac{1}{8\pi}\,\delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y})\right),$$

$$\frac{1}{i}\left[\bar{q}_a(x),\bar{q}_b(y)\right] = -\frac{e}{8\pi}\left((a-1)A_{0c}+A_{1c}\right) + \frac{1}{4\pi}\,\delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[D_1^{ac}\,\Pi^1_c(x),\, A_{0b}(y)\right] = -\frac{1}{a-1} g_{abc}\left(\frac{4\pi}{e} D_1^{cd}\,\Pi^1_d + e A_{1c}\right)\delta(\vec{x}-\vec{y})$$
$$+\frac{1}{a-1}\,\delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[D_1^{ac}\Pi_c^1(x), A_{1b}(y)\right] = e\, g_{abc} A_{1c}\,\delta(\vec{x}-\vec{y}) - \delta_{ab}\,\partial^x \delta(\vec{x}-\vec{y}),$$

$$\frac{1}{i}\left[D_1^{ac}\Pi_c^1(x), D_1^{bd}\Pi_d^1(y)\right] = e\, g_{abc}\, D_1^{ce}\,\Pi_e^1\,\delta(\vec{x}-\vec{y}), \tag{3.5.6}$$

where the last three comutators will prove to be useful to calculate the current algebra.

3.6.) The current algebra

We define the gauge currents by

$$J_a^\mu := \frac{\partial \mathcal{L}_I}{\partial A_{\mu a}}\ , \text{ where } \mathcal{L} = \mathcal{L}_{ym} + \mathcal{L}_I,\ \mathcal{L}_{ym} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}\ . \tag{3.6.1}$$

An explicit calculation yields

$$\begin{aligned}
J_a^0 &= \frac{ae^2}{4\pi} A_{0a} + \frac{e}{\sqrt{2}4\pi}\,\omega_{1a}\left(\partial_0\Phi^1 + \partial_1\Phi^1\right) + \frac{e}{\sqrt{2}4\pi}\,\sigma_{1a}\left((a-1)\partial_0\Theta^1 - \partial_1\Theta^1\right)\\
&= -D_1^{ab}\,\Pi_b^1,\\
J_a^1 &= -\frac{ae^2}{4\pi} A_{1a} - \frac{e}{\sqrt{2}4\pi}\,\omega_{1a}\left(\partial_0\Phi^1 + \partial_1\Phi^1\right) + \frac{e}{\sqrt{2}4\pi}\,\sigma_{1a}\left(\partial_0\Theta^1 - (a-1)\partial_1\Theta^1\right)\\
&= D_1^{ab}\,\Pi_b^1 + \frac{ae^2}{4\pi}\left(A_{0a} - A_{1a}\right),
\end{aligned} \tag{3.6.2}$$

where in the second step the constraint equations have been used.

The commutators (3.5.6) involving covariant derivatives allow the determination of the current algebra, we express the result in terms of the currents (3.6.1) and the gauge field:

$$\left[J_a^0(x), J_b^0(y)\right] = -ie\, g_{abc}\, J_c^0\,\delta(\vec{x}-\vec{y}), \tag{3.6.3}$$

$$\begin{aligned}
\left[J_a^0(x), J_b^1(y)\right] = i\Big[&e\, g_{abc}\left(-\frac{1}{a-1} J_c^0 + \frac{a^2}{a-1}\frac{e^2}{4\pi} A_{1c}\right)\delta(\vec{x}-\vec{y})\\
&-\frac{a^2}{a-1}\frac{e^2}{4\pi}\,\delta_{ab}\,\partial^x\delta(\vec{x}-\vec{y})\Big],
\end{aligned}$$

$$\left[J_a^1(x), J_b^1(y)\right] = ie\, g_{abc}\left[\frac{a+1}{a-1} J_c^0 + \frac{a^2}{(a-1)2}\frac{e^2}{4\pi}\left(A_{1c} - (a-1)A_{0c}\right)\right]\delta(\vec{x}-\vec{y}).$$

We note that the $[J^0, J^0]$-commutator is canonical, no Schwinger term arises, as required for a consistent quantum theory, as already observed in [22]. We want to make some remarks concerning the gauge non-invariant formulation. If the Faddeev-Popov-method is not used, the field g is absent and the effective action is not gauge invariant. Then Dirac's prescription leads directly to a second class system since no gauge symmetry is present. Dirac brackets and current commutators may be calculated, too, and it turns out that they coincide with the results we have given above if the constraint equations are used. Therefore we have found a gauge (namely a gauge, which eliminates the field g that is not contained in the gauge non-invariant formulation) which reproduces the result obtained in the non-invariant formulation.

3.7.) Equations of motion and current conservation

Finally, we want to show that the gauge current is covariantly conserved:

$$D_\mu^{ab} J_b^\mu = 0. \tag{3.7.1}$$

This condition has to be fulfilled, because the equation of motion $D_\mu F^{\mu\nu} = J^\nu$ for the gauge field implies $D_\nu D_\mu F^{\mu\nu} = D_\nu J^\nu = 0$.

A straightforward, but tedious calculation using the Lagrangean formalism starting from (3.3.5) yields the equation of motion of the matter fields Φ^i and Θ^i:

$$\begin{aligned}
&\frac{e}{\sqrt{2}4\pi}\,\omega_{jb}\,\partial^2\Phi^j \\
&= -\frac{e}{\sqrt{2}8\pi}\,\xi_{bi}\left(2\partial_r g_{is} - \partial_i g_{rs}\right)\left(\partial_0\Phi^r\partial_0\Phi^s - \partial_1\Phi^r\partial_1\Phi^s\right) - \frac{e}{8\pi}\,g_{bef}\,\omega_{re}\,\omega_{sf}\,\partial_0\Phi^r\partial_1\Phi^s \\
&\quad + \frac{e^2}{\sqrt{2}4\pi}\,g_{abe}\,\omega_{re}\left[\partial_0\Phi^r\left(A_{0a}-A_{1a}\right) + \partial_1\Phi^r\left(A_{0a}-A_{1a}\right)\right] \\
&\quad - \frac{e^2}{4\pi}\left(\partial_0 A_{0b} - \partial_1 A_{1b} + \partial_1 A_{0b} - \partial_1 A_{1b}\right);
\end{aligned}$$

$$\frac{e}{\sqrt{2}4\pi}\sigma_{jb}\partial^2\Theta^j$$

$$= -\frac{e(a-1)}{\sqrt{2}8\pi}\eta_{bi}\left(2\partial_r h_{is}-\partial_i h_{rs}\right)\left(\partial_0\Theta^r\partial_0\Theta^s-\partial_1\Theta^r\partial_1\Theta^s\right)-\frac{e}{8\pi}g_{dbf}\sigma_{rd}\sigma_{sf}\partial_0\Theta^r\partial_0\Theta^s$$
$$+\frac{e^2}{\sqrt{2}4\pi}g_{abe}\sigma_{re}\left[\partial_0\Theta^r\left((a-1)A_{0a}+A_{1a}\right) + \partial_1\Theta^r\left(-A_{0a}-(a-1)A_{1a}\right)\right]$$
$$-\frac{e^2}{4\pi}\left((a-1)\partial_0A_{0b} + \partial_0A_{1b} - \partial_1A_{0b} - (a-1)\partial_1A_{1b}\right). \qquad (3.7.2)$$

A simple calculation gives

$$\partial_\mu J^\mu_b = \frac{e^2}{\sqrt{2}4\pi}g_{abe}\left[\omega_{re}\left[\partial_0\Phi^r\left(A_{0a}-A_{1a}\right) + \partial_1\Phi^r\left(A_{0a}-A_{1a}\right)\right]\right. \qquad (3.7.3)$$
$$\left.+\sigma_{re}\left[\partial_0\Theta^r\left((a-1)A_{0a}+A_{1a}\right) + \partial_1\Theta^r\left(-A_{0a}-(a-1)A_{1a}\right)\right]\right],$$

where the equations of motion (3.7.2) of Φ^i and Θ^i have been used.

The additional term in the covariant derivative is easily seen to be

$$e\, g_{bac}A_{\mu c}J^\mu_a = \partial_\mu J^\mu_b\ , \qquad (3.7.4)$$

so that indeed

$$D^{ab}_\mu\ J^\mu_b = 0, \qquad (3.7.5)$$

the gauge current J^μ is covariantly conserved.

So it turns out that the dynamics of the Wess-Zumino-field g ensures current conservation [22]. Of course, it is possible to try a similar calculation in the gauge non-invariant formulation. Then it turns out, however, that the equations of motion do not suffice to prove covariant current conservation. We suppose that in this case, as in the abelian case [10], the complete and explicit solutions of the equations of motion are necessary. Since they are not known in the non-abelian case, the use of a gauge invariant effective action seems to be the only possibility to prove gauge current conservation.

4.) Summary and Conclusions

We have studied chiral QCD in two dimensions in a gauge invariant formulation. Using the Faddeev-Popov-procedure we obtained a gauge invariant effective action after bosonization. The calculation of the constraint algebra showed that we were dealing with a first class system. As Dirac has shown, first class constraints are generators of symmetries which leave physical observables invariant. In our particular case, the constraints generate local gauge transformations. Gauge fixing leads to a second class system. Quantization is possible by calculation of Dirac brackets of the dynamical variables which are then turned into quantum commutators. We defined gauge currents and determined the current algebra. The $[J^0,J^0]$-commutator turned out to be canonical reflecting the fact that J^0 is a generator of a symmetry. No Schwinger term is present as required for the consistency of the theory.

Finally we proved covariant current conservation by means of the equations of motion of the bosonic matter fields.

The additional matter field surviving the Faddeev-Popov-procedure turned out to be essential for the two main features of this formulation, namely gauge invariance and the possibility of proving covariant current conservation explicitly.

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Appendix A: Conventions and Notation

We use 2-dimensional Minkowski space M = ∂Q with metric g = diag(1,-1) and Levi-Civita-symbol $\varepsilon_{01} = 1$. Lightcone variables are defined by

$$A_{\pm} = A^{\mp} = A_0 \pm A_1, \ \partial_{\pm} = \partial^{\mp} = \partial_0 \pm \partial_1. \tag{A.1}$$

The γ-matrices in the chiral representation read

$$\gamma^0 = \sigma^1,\ \gamma^1 = i\sigma^2,\ \gamma^5 = \gamma^0\gamma^1 = \mathrm{diag}\,(1,-1). \tag{A.2}$$

Projection operators onto left- and right-handed components of the fermions are defined by

$$\lambda = \frac{1-\gamma^5}{2}\ ,\ \varrho = \frac{1+\gamma^5}{2}\ ,\ \Psi_L = \lambda\Psi,\ \Psi_R = \varrho\Psi. \tag{A.3}$$

Appendix B: Differential Geometry of SU(N)

Let G be a Lie group [27,28] SU(N) with hermitean, normalized generators t^a,

$$\mathrm{tr}\ t^a t^b = \delta^{ab},\ \mathrm{tr}\ t^a = 0, \tag{B.1}$$

and totally antisymmetric structure constants g_{abc}, where

$$\left[t^a, t^b\right] = i\ g_{abc}\ t^c. \tag{B.2}$$

Each $h \in G$ may be parametrized by

$$h = \exp\left(\Phi^a \frac{i}{\sqrt{2}}\ t^a\right). \tag{B.3}$$

The Maurer-Cartan-form

$$\omega = h^{-1}dh = \omega_{ia} \frac{i}{\sqrt{2}}\ t^a\ d\Phi^i \tag{B.4}$$

is a canonical left-invariant 1-form on G.

On G, there is a left-invariant metric

$$g = g_{ij}\ d\Phi^i \otimes d\Phi^j,\ g_{ij} = -\ 2\ \mathrm{tr}\ (\omega_i\omega_j) = \omega_{ia}\omega_{ja}. \tag{B.5}$$

We define ξ to be the inverse of the matrix ω,

$$\omega_{ia}\ \xi_{aj} = \delta_{ij}, \tag{B.6}$$

and we note that the ξ_{ai} are the components of left-invariant vector fields

$$\tilde{X}^a = \xi_{ai} \frac{\partial}{\partial \Phi^I}, \quad g(\tilde{X}^a, \tilde{X}^b) = \delta^{ab}. \tag{B.7}$$

The structure equation

$$[\tilde{X}^a, \tilde{X}^b] = - \frac{1}{\sqrt{2}} g_{abc} \tilde{X}^c \tag{B.8}$$

may be expressed in components as

$$\xi_{ai} \partial_j \xi_{bj} - \xi_{bi} \partial_i \xi_{aj} = - \frac{1}{\sqrt{2}} g_{abc} \xi_{cj}, \tag{B.9}$$

where ∂_i denotes $\frac{\partial}{\partial \Phi^I}$.

A very useful relation is Killing's equation

$$\xi_{aj} \partial_j g_{kl} + g_{lj} \partial_k \xi_{aj} + g_{kj} \partial_l \xi_{aj} = 0. \tag{B.10}$$

Let tr ω^3 be a three-form on G which is a "$\wedge$"-product with resect to 1-forms, the matrix product and the trace are taken with respect to Lie algebra matrices.

It is easy to see that tr ω^3 is a closed form, so locally there is a two-form D with tr ω^3 = dD. We may expand D as

$$D = \frac{1}{2} D_{ij} \, d\Phi^i \wedge d\Phi^j, \quad D_{ij} = - D_{ji}, \tag{B.11}$$

and note that

$$\partial_i D_{jk} - \partial_j D_{ik} + \partial_k D_{ij} = \frac{3}{2\sqrt{2}} g_{abc} \, \omega_{ia} \, \omega_{jb} \, \omega_{kc}. \tag{B.12}$$

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