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The 3-loop anomalous dimensions from off-shell operator matrix elements*

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We report on the calculation of the three-loop polarized and unpolarized flavor non-singlet and the polarized singlet anomalous dimensions using massless off-shell operator matrix elements in a gauge-variant framework. We also reconsider the unpolarized two-loop singlet anomalous dimensions and correct errors in the foregoing literature.

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1. Introduction

In these proceedings we report on recent results of the calculation of unpolarized and polarized three-loop anomalous dimensions based on massless off-shell operator matrix elements (OMEs) [1, 2] and the unpolarized two-loop case [3] correcting previous results in Refs. [4, 5].

The unpolarized and polarized non-singlet and singlet anomalous dimensions have been calculated at one- [6], two- [3, 7, 8], and three-loop order [1, 2, 9–15]. Here different techniques as off-shell massless OMEs, the forward Compton amplitude, in part also with scalar and gravitational currents, and massive on-shell OMEs have been used. In the latter case one obtains the contributions $\propto T_F$, which are the complete anomalous dimensions in the cases $(\Delta)\gamma_{qq}^{(2),PS}$ and $(\Delta)\gamma_{qg}^{(2)}$.

The method of massless off-shell OMEs is the traditional way to calculate the anomalous dimensions, cf. [1a-c]. However, it is a gauge-dependent environment implying in the unpolarized singlet case new operator mixings [3, 8, 16]. It is our goal to set up a program chain allowing for a fully automated calculation of the three-loop anomalous dimensions without making structural assumptions motivated by QCD or the expected representation of the final results in terms of harmonic sums [17, 18]. The paper is organized as follows. We discuss first the basic formalism and describe then the calculation method, before we present some examples for the three-loop anomalous dimensions.

2. Basic Formalism

We form the expectation values of the local twist-2 operators in the unpolarized and polarized case, cf. e.g. [19], between off-shell quark and gluon states and consider the physical projections, which are calculated to three-loop order. There are also other contributions due to the violation of the equation of motion (EOM), which, however, are not related to the anomalous dimensions. From the physical projection we extract the three-loop anomalous dimensions from the $O(1/\varepsilon)$ terms, where $\varepsilon = D - 4$ denotes the dimensional parameter. The corresponding amplitudes are gauge-variant, i.e. they depend on the gauge parameter ξ in the R_ξ gauges. The structure of the pole terms are fully predicted by the renormalization group and do depend on lower order expansion coefficients to higher orders in ε . In the unpolarized case, the mixing of more local operators has to be considered [3–5, 8, 16], which we studied up to two-loop order in [3].

In the polarized singlet case the anomalous dimensions are first calculated in the Larin scheme [20], which is a consistent scheme. Finally, we transform the anomalous dimensions to the M-scheme [5, 11].

3. Details of the calculation

The calculation of the unrenormalized massless off-shell OMEs is performed in the following way. Diagram generation, the performance of Lorentz/Dirac and color algebra are performed by using the packages QGRAF, FORM and color [21–24]. The local operators are resummed into propagators

by observing the current crossing relations, cf. [25, 26], as has been described in Ref. [27],

$$\sum_{N=0}^{\infty} (\Delta.k)^N \left(t^N \pm (-t)^N \right) \rightarrow \left[\frac{1}{1 - \Delta.k t} \pm \frac{1}{1 + \Delta.k t} \right]. \quad (1)$$

The three-loop anomalous dimensions are obtained from the contributions of $O(1/\varepsilon)$, determining the N th moment analytically.

In the individual channels there are up to $O(1600)$ Feynman diagrams. They are reduced to up to $O(250)$ master integrals using the code `Crusher` [28] by applying the integration-by-parts relations [29, 30]. Coupling constant and wave function renormalizations are performed [31, 32] and results from lower order factorizing diagrams [3] are accounted for. The method of arbitrary high Mellin moments [33], implemented in the package `SolveCoupledSystem` [34], is used to provide a necessary input set, here of 3000 moments. The necessary initial values for the difference equations can be obtained from [30, 35]. We then used the method of guessing [36, 37] and its implementation in Sage [38, 39] to obtain the recurrences for the different color and multiple zeta value factors [40]. It turned out that $O(1600)$ moments suffice and the largest difference equations was of order $o = 16$ and degree $d = 304$. These difference equations are solved using methods from difference ring theory [41] implemented in the package `Sigma` [42, 43]. Functions of the package `HarmonicSums` [17, 18, 44–47] are used to compactify the final results. The automated calculation at Intel(R) Xeon(R) CPU E5-2643 v4 processors amounted to about 40 days of CPU time for the projects [1, 2].

All anomalous dimensions can be expressed in terms of harmonic sums [17, 18]

$$S_{b,\bar{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\bar{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}, N \in \mathbb{N} \setminus \{0\}. \quad (2)$$

Accordingly, the corresponding z -space expressions are given by harmonic polylogarithms $H_{\bar{a}}(z)$, [45]. Here $z \in [0, 1]$ denotes the momentum fraction w.r.t. the incoming nucleon momentum in the deep-inelastic process.

4. Anomalous Dimensions

We have calculated all the non-singlet anomalous dimensions $\gamma_{qq}^{\pm, \text{NS}}$, $\gamma_{qq}^{s, \text{NS}}$, $\Delta\gamma_{qq}^{s, \text{NS}}$ for unpolarized and polarized deep-inelastic scattering and also for transversity $\gamma_{qq}^{\pm, \text{tr}, \text{NS}}$, as well as the singlet polarized anomalous dimensions, Ref. [1, 2]. They can all be expressed in terms of harmonic sums, for which we apply the algebraic relations [46]. If also the structural relations are applied [47] only 10 harmonic sums contribute.

$$\{S_1, S_{2,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{2,1,1}, S_{-2,1,1}, S_{2,1,-2}, S_{-3,1,1}, S_{-2,1,1,1}\}. \quad (3)$$

As examples we show one of the transversity anomalous dimensions

$$\begin{aligned} \gamma_{\text{NS}}^{(2), \text{tr}, +} &= \frac{1}{2} \left[1 + (-1)^N \right] \\ &\times \left\{ C_F \left[T_F^2 N_F^2 \left[\frac{8(-8 + 17N + 17N^2)}{9N(1+N)} - \frac{128}{27} S_1 - \frac{640}{27} S_2 + \frac{128}{9} S_3 \right] \right] \right\} \end{aligned}$$

$$\begin{aligned}
& +C_{ATFN_F} \left[-\frac{16(-22+45N+45N^2)}{9N(1+N)} + \left(-\frac{16(9+209N+209N^2)}{27N(1+N)} + 64S_3 \right. \right. \\
& + \frac{256}{3}S_{-2,1} - 128\zeta_3 \left. \right) S_1 + \frac{5344}{27}S_2 - \frac{448}{3}S_3 + \frac{320}{3}S_4 + \left(-\frac{1280}{9}S_1 + \frac{128}{3}S_2 \right) S_{-2} \\
& + \left(-\frac{640}{9} + \frac{128}{3}S_1 \right) S_{-3} + \frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{1280}{9}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1} \\
& + 96\zeta_3 \left. \right] + C_A^2 \left[\frac{-968+1657N+1657N^2}{18N(1+N)} + \left(\frac{4P_{14}}{3(-1+N)N(1+N)(2+N)} - 176S_3 \right. \right. \\
& - 256S_4 + 512S_{3,1} - \frac{704}{3}S_{-2,1} - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \left. \right) S_1 + \left(-128S_3 \right. \\
& - 512S_{-2,1} \left. \right) S_1^2 + \left(-\frac{8344}{27} + 384S_3 + 1536S_{-2,1} \right) S_2 + \frac{3112}{9}S_3 - \frac{880}{3}S_4 + 64S_5 \\
& + \left(\frac{16P_2}{(-1+N)N(1+N)(2+N)} + \frac{32(-241+134N+134N^2)S_1}{9(-1+N)(2+N)} - \frac{352}{3}S_2 - 64S_3 \right. \\
& - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3 \left. \right) S_{-2} + \left(48 - 192S_1 \right) S_{-2}^2 + \left(256S_1^2 - 768S_2 - 320S_{-2} \right. \\
& + \frac{32(-107+67N+67N^2)}{9(-1+N)(2+N)} - \frac{352}{3}S_1 \left. \right) S_{-3} + \left(-\frac{208}{3} + 320S_1 \right) S_{-4} - 704S_{-5} - 384S_{2,3} \\
& - 768S_{2,-3} + \frac{704}{3}S_{3,1} + 384S_{4,1} - \frac{64(-107+67N+67N^2)S_{-2,1}}{9(-1+N)(2+N)} - \frac{352}{3}S_{-2,2} \\
& + 1088S_{-2,3} - 448S_{-4,1} + 1536[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] - 768S_{3,1,1} \\
& + \frac{1408}{3}S_{-2,1,1} + 512S_{-2,1,-2} - 3072S_{-2,1,1,1} - \frac{24(-6+5N+5N^2)\zeta_3}{(-1+N)(2+N)} \left. \right] \left. \right\} \\
& + C_F^2 \left\{ T_{FN_F} \left[92 + \left(-\frac{8(-8+55N+55N^2)}{3N(1+N)} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3 \right) \right. \right. \\
& \times S_1 - \frac{80}{3}S_2 - \frac{128}{3}S_2^2 + \frac{1856}{9}S_3 - \frac{512}{3}S_4 + \left(\frac{2560}{9}S_1 - \frac{256}{3}S_2 \right) S_{-2} + \left(\frac{1280}{9} \right. \\
& - \frac{256}{3}S_1 \left. \right) S_{-3} - \frac{256}{3}S_{-4} + \frac{256}{3}S_{3,1} - \frac{2560}{9}S_{-2,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - 96\zeta_3 \left. \right] \\
& + C_A \left[-\frac{151}{2} + \left(-\frac{8(-206+211N+211N^2)}{3(-1+N)N(1+N)(2+N)} - \frac{4288}{9}S_2 + \frac{1984}{3}S_3 + 320S_4 - 1024S_{3,1} \right. \right. \\
& + \frac{1984}{3}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \left. \right) S_1 + \left(256S_3 + 1792S_{-2,1} \right) S_1^2 \\
& + \left(\frac{604}{3} - 832S_3 - 5248S_{-2,1} \right) S_2 + \frac{352}{3}S_2^2 - \frac{6160}{9}S_3 + \frac{2416}{3}S_4
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{48P_2}{(-1+N)N(1+N)(2+N)} + \left(-\frac{64P_7}{9(-1+N)N(1+N)(2+N)} - 256S_2 \right) S_1 \right. \\
& + \frac{992}{3}S_2 + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3 \left. \right) S_{-2} + \left(-96 + 512S_1 \right) S_{-2}^2 \\
& + \left(-\frac{32(-187 + 134N + 134N^2)}{9(-1+N)(2+N)} + \frac{992}{3}S_1 - 1152S_1^2 + 2624S_2 + 960S_{-2} \right) S_{-3} \\
& + \left(\frac{560}{3} - 1472S_1 \right) S_{-4} + 2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{1856}{3}S_{3,1} - 768S_{4,1} \\
& + \frac{64(-187 + 134N + 134N^2)S_{-2,1}}{9(-1+N)(2+N)} + \frac{992}{3}S_{-2,2} - 3648S_{-2,3} + 1728S_{-4,1} \\
& - 5376[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] + 1536[S_{3,1,1} - S_{-2,1,-2}] - \frac{3968}{3}S_{-2,1,1} \\
& + 10752S_{-2,1,1,1} + \frac{72(-6 + 5N + 5N^2)\zeta_3}{(-1+N)(2+N)} \left. \right\} \\
& + C_F^3 \left\{ -29 + \left(\frac{384(-1+N+N^2)}{(-1+N)N(1+N)(2+N)} + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} \right. \right. \\
& - 384S_{-2,1} - 3328S_{-2,2} - 3584S_{-3,1} + 6144S_{-2,1,1} \left. \right) S_1 - 256S_{-2}^2 S_1 + \left(12 + 512S_3 \right. \\
& + 4352S_{-2,1} \left. \right) S_2 - 96S_2^2 + 104S_3 - 480S_4 + \left(\frac{32P_2}{(-1+N)N(1+N)(2+N)} \right. \\
& + \left(\frac{384}{(-1+N)N(1+N)(2+N)} + 512S_2 \right) S_1 - 192S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} \\
& - 384\zeta_3 \left. \right) S_{-2} + \left(\frac{192}{(-1+N)(2+N)} - 192S_1 + 1280S_1^2 - 2176S_2 - 640S_{-2} \right) S_{-3} \\
& + \left(-96 + 1664S_1 \right) S_{-4} - 1792S_{-5} + 384[-S_{2,3} + S_{3,1} + S_{4,1}] - 2304S_{2,-3} \\
& - \frac{384S_{-2,1}}{(-1+N)(2+N)} - 1536S_1^2 S_{-2,1} - 192S_{-2,2} + 2944S_{-2,3} - 1664S_{-4,1} + 4608S_{2,1,-2} \\
& - 768S_{3,1,1} + 768S_{-2,1,1} + 1024S_{-2,1,-2} + 4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1} \\
& - \frac{48(-6 + 5N + 5N^2)\zeta_3}{(-1+N)(2+N)} \left. \right\} \tag{4}
\end{aligned}$$

and one of the polarized singlet anomalous dimensions,

$$\begin{aligned}
\Delta\gamma_{gg}^{(2)} & = C_A T_F^2 N_F^2 \left[-\frac{16P_8}{27N^2(1+N)^2} S_1 - \frac{4P_{48}}{27N^3(1+N)^3} \right] + C_F \left[T_F^2 N_F^2 \left[-\frac{8P_{59}}{27N^4(1+N)^4} \right. \right. \\
& \left. \left. + \frac{64(N-1)(2+N)(-6-8N+N^2)}{9N^3(1+N)^3} S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^2 \right] \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 \Big] + C_{ATFN_F} \left[\frac{8P_6}{N^3(1+N)^3}S_2 - \frac{8P_9}{3N^3(1+N)^3}S_1^2 \right. \\
& + \frac{2P_{77}}{27(N-1)N^5(1+N)^5(2+N)} + \left(-\frac{8P_{67}}{9(-1+N)N^4(1+N)^4(2+N)} \right. \\
& \left. -\frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 + 128\zeta_3 \right) S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^3 - \frac{32(34+N+N^2)}{3N^2(1+N)^2} \\
& \times S_3 + \left(\frac{128P_2}{(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{23}}{(N-1)N^2(1+N)^3(2+N)} \right) S_{-2} \\
& - \frac{192(4-N-N^2)}{N^2(1+N)^2}S_{-3} + \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{128(-8+N+N^2)}{N^2(1+N)^2}S_{-2,1} \\
& \left. - \frac{64(-3+N)(4+N)}{N^2(1+N)^2}\zeta_3 \right] + C_A^3 \left[\frac{64P_{16}}{9N^2(1+N)^2}S_{-2,1} - \frac{32P_{18}}{9N^2(1+N)^2}S_3 \right. \\
& + \frac{P_{74}}{27(N-1)N^5(1+N)^5(2+N)} + \left(\frac{4P_{69}}{9(N-1)N^4(1+N)^4(2+N)} \right. \\
& \left. - \frac{64P_{17}}{9N^2(1+N)^2}S_2 + 128S_2^2 + \frac{16(-96+11N+11N^2)}{3N(1+N)}S_3 + 192S_4 \right. \\
& \left. + \frac{1024}{N(1+N)}S_{-2,1} - 640S_{-2,2} - 768S_{-3,1} + 1024S_{-2,1,1} \right) S_1 \\
& + \left(-\frac{256(1+3N+3N^2)}{N^3(1+N)^3} + 128S_3 - 256S_{-2,1} \right) S_1^2 + \left(-\frac{16P_{41}}{9N^3(1+N)^3} \right. \\
& \left. + 64S_3 + 640S_{-2,1} \right) S_2 - \frac{256}{N(1+N)}S_2^2 - \frac{384}{N(1+N)}S_4 + 64S_5 \\
& + \left(\frac{32P_{52}}{9(N-1)N^3(1+N)^3(2+N)} + \left(-\frac{64P_{32}}{9(-1+N)N(1+N)^2(2+N)} + 256S_2 \right) \right. \\
& \left. \times S_1 - \frac{512}{N(1+N)}S_2 + 128S_3 - 768S_{2,1} \right) S_{-2} + \left(-\frac{16(24+11N+11N^2)}{3N(1+N)} \right. \\
& \left. + 64S_1 \right) S_{-2}^2 + \left(-\frac{32P_{15}}{9N^2(1+N)^2} - \frac{1536}{N(1+N)}S_1 + 384S_1^2 - 320S_2 \right) S_{-3} \\
& + \left(-\frac{1024}{N(1+N)} + 512S_1 \right) S_{-4} - 192S_{-5} - 384S_{2,-3} + \frac{1280}{N(1+N)}S_{-2,2} \\
& + 384S_{-2,3} + \frac{1536}{N(1+N)}S_{-3,1} - 384S_{-4,1} + 768S_{2,1,-2} - \frac{2048}{N(1+N)}S_{-2,1,1} \\
& \left. + 768[S_{-2,2,1} + S_{-3,1,1}] - 1536S_{-2,1,1,1} \right] \\
& + C_F^2 T_F N_F \left[-\frac{4P_{75}}{(N-1)N^5(1+N)^5(2+N)} + \left(\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16P_{42}}{N^4(1+N)^4} \Big) S_1 + \frac{8(N-1)(2+N)(2+3N+3N^2)}{N^3(1+N)^3} S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} \\
& \times S_1^3 - \frac{8(2+N)(2-11N-16N^2+9N^3)}{N^3(1+N)^3} S_2 + \frac{32(10+7N+7N^2)}{3N^2(1+N)^2} S_3 \\
& + \left(-\frac{64(10+N+N^2)}{(N-1)N(1+N)(2+N)} + \frac{512}{N^2(1+N)^2} S_1 \right) S_{-2} + \frac{256}{N^2(1+N)^2} S_{-3} \\
& - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} - \frac{512}{N^2(1+N)^2} S_{-2,1} + \frac{192(-2-N-N^2)}{N^2(1+N)^2} \zeta_3 \Big] \\
& + C_A^2 T_F N_F \left[\frac{32P_4}{9N^2(1+N)^2} S_2 + \frac{32P_{11}}{9N^2(1+N)^2} S_{-3} - \frac{64P_{11}}{9N^2(1+N)^2} S_{-2,1} \right. \\
& + \frac{16P_{13}}{9N^2(1+N)^2} S_3 + \frac{2P_{76}}{27(N-1)N^5(1+N)^5(2+N)} + \left(\frac{1280}{9} S_2 - \frac{64}{3} S_3 \right. \\
& \left. - \frac{8P_{68}}{27(-1+N)N^4(1+N)^4(2+N)} - 128\zeta_3 \right) S_1 + \frac{64}{3} S_{-2}^2 \\
& + \left(\frac{64P_{45}}{9(N-1)N^2(1+N)^2(2+N)} S_1 - \frac{32P_{50}}{9(N-1)N^3(1+N)^3(2+N)} \right) S_{-2} \\
& \left. + \frac{128(-3+2N+2N^2)}{N^2(1+N)^2} \zeta_3 \right]. \tag{5}
\end{aligned}$$

The polynomials are given in Refs. [1, 2]. The small z and large z limits of the anomalous dimensions have also been considered in explicit form. The latter are related to the cusp anomalous dimensions. There are various partial predictions on the small z behaviour, however, not specifying the factorization scheme used. These are theoretically interesting, but are not of quantitative importance since it is known for long that subleading terms do strongly modify the leading order effects over about three additional subleading orders [48].

We have also recalculated the unpolarized two-loop anomalous dimensions using the method of massless off-shell OMEs, [3]. Here new local operators contribute [3, 8, 16] to cancel the gauge-variant contributions to obtain the anomalous dimensions. We corrected a series of errors contained in [4, 5] and provided all expansion coefficients emerging at two-loop order [3], which are relevant for upcoming four-loop calculations.

5. Conclusions

We have calculated the three-loop polarized and unpolarized non-singlet anomalous dimensions, including transversity, and as well the polarized three-loop singlet anomalous dimensions applying the traditional method of massless off-shell operator matrix elements, a gauge-variant method, which requires special projections. The calculations have been performed without assuming special conditions concerning the structure of the final result. The method of arbitrary high Mellin moments plays a central role in these computations in establishing the corresponding difference equations through the method of guessing, after having exploited the equations for the master integrals. These

equations are solved subsequently using algorithms of difference ring theory. We agree with all the results previously obtained in the literature. The polarized non-singlet anomalous dimension $\Delta\gamma_{qq}^{s,NS}$, related to the polarized structure function g_5^- , [25], has been calculated using the associated forward Compton amplitude. In the unpolarized singlet case also new OMEs contribute. Here we have calculated all contributions emerging at the two-loop level and corrected results in the literature. The present method is suited to be expanded to the four-loop level. The knowledge of the higher-loop anomalous dimensions form one important asset for the precise description of the scaling violations of the deep-inelastic structure functions, which provide an important way to measure the QCD coupling constant $\alpha_s(M_Z^2)$ [49] at highest precision possible.

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