

The 3-loop anomalous dimensions from off-shell operator matrix elements*

Johannes Blümlein,^{a,*} Peter Marquard,^a Carsten Schneider^b and Kay Schönwald^c

^a*Deutsches Elektronen-Synchrotron DESY, Platanenallee 6, 15738 Zeuthen, Germany*

^b*Johannes Kepler University Linz, Research Institute for Symbolic Computation (RISC), Altenberger Straße 69, 4040 Linz, Austria*

^c*Institut für Theoretische Teilchenphysik, Karlsruher Institut für Technologie (KIT) D-76128 Karlsruhe, Germany*

E-mail: Johannes.Bluemlein@desy.de, Peter.Marquard@desy.de,
Carsten.Schneider@risc.jku.at, kay.schoenwald@kit.edu

We report on the calculation of the three-loop polarized and unpolarized flavor non-singlet and the polarized singlet anomalous dimensions using massless off-shell operator matrix elements in a gauge-variant framework. We also reconsider the unpolarized two-loop singlet anomalous dimensions and correct errors in the foregoing literature.

*Loops and Legs in Quantum Field Theory - LL2022,
25-30 April, 2022
Ettal, Germany*

*This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 764850, SAGEX and from the Austrian Science Fund (FWF) grants SFB F50 (F5009-N15) and P33530.

*Speaker

1. Introduction

In these proceedings we report on recent results of the calculation of unpolarized and polarized three-loop anomalous dimensions based on massless off-shell operator matrix elements (OMEs) [1, 2] and the unpolarized two-loop case [3] correcting previous results in Refs. [4, 5].

The unpolarized and polarized non-singlet and singlet anomalous dimensions have been calculated at one- [6], two- [3, 7, 8], and three-loop order [1, 2, 9–15]. Here different techniques as off-shell massless OMEs, the forward Compton amplitude, in part also with scalar and gravitational currents, and massive on-shell OMEs have been used. In the latter case one obtains the contributions $\propto T_F$, which are the complete anomalous dimensions in the cases $(\Delta)\gamma_{qq}^{(2),PS}$ and $(\Delta)\gamma_{qg}^{(2)}$.

The method of massless off-shell OMEs is the traditional way to calculate the anomalous dimensions, cf. [6a-c]. However, it is a gauge-dependent environment implying in the unpolarized singlet case new operator mixings [3, 8, 16]. It is our goal to set up a program chain allowing for a fully automated calculation of the three-loop anomalous dimensions without making structural assumptions motivated by QCD or the expected representation of the final results in terms of harmonic sums [17, 18]. The paper is organized as follows. We discuss first the basic formalism and describe then the calculation method, before we present some examples for the three-loop anomalous dimensions.

2. Basic Formalism

We form the expectation values of the local twist-2 operators in the unpolarized and polarized case, cf. e.g. [19], between off-shell quark and gluon states and consider the physical projections, which are calculated to three-loop order. There are also other contributions due to the violation of the equation of motion (EOM), which, however, are not related to the anomalous dimensions. From the physical projection we extract the three-loop anomalous dimensions from the $O(1/\varepsilon)$ terms, where $\varepsilon = D - 4$ denotes the dimensional parameter. The corresponding amplitudes are gauge-variant, i.e. they depend on the gauge parameter ξ in the R_ξ gauges. The structure of the pole terms are fully predicted by the renormalization group and do depend on lower order expansion coefficients to higher orders in ε . In the unpolarized case, the mixing of more local operators has to be considered [3–5, 8, 16], which we studied up to two-loop order in [3].

In the polarized singlet case the anomalous dimensions are first calculated in the Larin scheme [20], which is a consistent scheme. Finally, we transform the anomalous dimensions to the M-scheme [5, 11].

3. Details of the calculation

The calculation of the unrenormalized massless off-shell OMEs is performed in the following way. Diagram generation, the performance of Lorentz/Dirac and color algebra are performed by using the packages QGRAF, FORM and color [21–24]. The local operators are resummed into propagators

by observing the current crossing relations, cf. [25, 26], as has been described in Ref. [27],

$$\sum_{N=0}^{\infty} (\Delta.k)^N \left(t^N \pm (-t)^N \right) \rightarrow \left[\frac{1}{1 - \Delta.k t} \pm \frac{1}{1 + \Delta.k t} \right]. \quad (1)$$

The three-loop anomalous dimensions are obtained from the contributions of $O(1/\varepsilon)$, determining the N th moment analytically.

In the individual channels there are up to $O(1600)$ Feynman diagrams. They are reduced to up to $O(250)$ master integrals using the code `Crusher` [28] by applying the integration-by-parts relations [29, 30]. Coupling constant and wave function renormalizations are performed [31, 32] and results from lower order factorizing diagrams [3] are accounted for. The method of arbitrary high Mellin moments [33], implemented in the package `SolveCoupledSystem` [34], is used to provide a necessary input set, here of 3000 moments. The necessary initial values for the difference equations can be obtained from [30, 35]. We then used the method of guessing [36, 37] and its implementation in Sage [38, 39] to obtain the recurrences for the different color and multiple zeta value factors [40]. It turned out that $O(1600)$ moments suffice and the largest difference equations was of order $o = 16$ and degree $d = 304$. These difference equations are solved using methods from difference ring theory [41] implemented in the package `Sigma` [42, 43]. Functions of the package `HarmonicSums` [17, 18, 44–47] are used to compactify the final results. The automated calculation at Intel(R) Xeon(R) CPU E5-2643 v4 processors amounted to about 40 days of CPU time for the projects [1, 2].

All anomalous dimensions can be expressed in terms of harmonic sums [17, 18]

$$S_{b,\bar{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\bar{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}, N \in \mathbb{N} \setminus \{0\}. \quad (2)$$

Accordingly, the corresponding z -space expressions are given by harmonic polylogarithms $H_{\bar{a}}(z)$, [45]. Here $z \in [0, 1]$ denotes the momentum fraction w.r.t. the incoming nucleon momentum in the deep-inelastic process.

4. Anomalous Dimensions

We have calculated all the non-singlet anomalous dimensions $\gamma_{qq}^{\pm, \text{NS}}$, $\gamma_{qq}^{s, \text{NS}}$, $\Delta\gamma_{qq}^{s, \text{NS}}$ for unpolarized and polarized deep-inelastic scattering and also for transversity $\gamma_{qq}^{\pm, \text{tr}, \text{NS}}$, as well as the singlet polarized anomalous dimensions, Ref. [1, 2]. They can all be expressed in terms of harmonic sums, for which we apply the algebraic relations [46]. If also the structural relations are applied [47] only 10 harmonic sums contribute.

$$\{S_1, S_{2,1}, S_{-2,1}, S_{-3,1}, S_{-4,1}, S_{2,1,1}, S_{-2,1,1}, S_{2,1,-2}, S_{-3,1,1}, S_{-2,1,1,1}\}. \quad (3)$$

As examples we show one of the transversity anomalous dimensions

$$\gamma_{\text{NS}}^{(2), \text{tr}, +} = \frac{1}{2} \left[1 + (-1)^N \right] \times \left\{ C_F \left[T_F^2 N_F^2 \left[\frac{8(-8 + 17N + 17N^2)}{9N(1+N)} - \frac{128}{27} S_1 - \frac{640}{27} S_2 + \frac{128}{9} S_3 \right] \right. \right.$$

$$\begin{aligned}
& +C_{ATFN_F} \left[-\frac{16(-22+45N+45N^2)}{9N(1+N)} + \left(-\frac{16(9+209N+209N^2)}{27N(1+N)} + 64S_3 \right. \right. \\
& + \frac{256}{3}S_{-2,1} - 128\zeta_3 \Big) S_1 + \frac{5344}{27}S_2 - \frac{448}{3}S_3 + \frac{320}{3}S_4 + \left(-\frac{1280}{9}S_1 + \frac{128}{3}S_2 \right) S_{-2} \\
& + \left(-\frac{640}{9} + \frac{128}{3}S_1 \right) S_{-3} + \frac{128}{3}S_{-4} - \frac{256}{3}S_{3,1} + \frac{1280}{9}S_{-2,1} + \frac{128}{3}S_{-2,2} - \frac{512}{3}S_{-2,1,1} \\
& + 96\zeta_3 \Big] + C_A^2 \left[\frac{-968+1657N+1657N^2}{18N(1+N)} + \left(\frac{4P_{14}}{3(-1+N)N(1+N)(2+N)} - 176S_3 \right. \right. \\
& - 256S_4 + 512S_{3,1} - \frac{704}{3}S_{-2,1} - 1024S_{-2,2} - 1024S_{-3,1} + 2048S_{-2,1,1} \Big) S_1 + \left(-128S_3 \right. \\
& - 512S_{-2,1} \Big) S_1^2 + \left(-\frac{8344}{27} + 384S_3 + 1536S_{-2,1} \right) S_2 + \frac{3112}{9}S_3 - \frac{880}{3}S_4 + 64S_5 \\
& + \left(\frac{16P_2}{(-1+N)N(1+N)(2+N)} + \frac{32(-241+134N+134N^2)S_1}{9(-1+N)(2+N)} - \frac{352}{3}S_2 - 64S_3 \right. \\
& - 1536S_{2,1} + 128S_{-2,1} - 192\zeta_3 \Big) S_{-2} + \left(48 - 192S_1 \right) S_{-2}^2 + \left(256S_1^2 - 768S_2 - 320S_{-2} \right. \\
& + \frac{32(-107+67N+67N^2)}{9(-1+N)(2+N)} - \frac{352}{3}S_1 \Big) S_{-3} + \left(-\frac{208}{3} + 320S_1 \right) S_{-4} - 704S_{-5} - 384S_{2,3} \\
& - 768S_{2,-3} + \frac{704}{3}S_{3,1} + 384S_{4,1} - \frac{64(-107+67N+67N^2)S_{-2,1}}{9(-1+N)(2+N)} - \frac{352}{3}S_{-2,2} \\
& + 1088S_{-2,3} - 448S_{-4,1} + 1536[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] - 768S_{3,1,1} \\
& \left. + \frac{1408}{3}S_{-2,1,1} + 512S_{-2,1,-2} - 3072S_{-2,1,1,1} - \frac{24(-6+5N+5N^2)\zeta_3}{(-1+N)(2+N)} \right] \Big\} \\
& + C_F^2 \left\{ T_F N_F \left[92 + \left(-\frac{8(-8+55N+55N^2)}{3N(1+N)} + \frac{1280}{9}S_2 - \frac{512}{3}S_3 - \frac{512}{3}S_{-2,1} + 128\zeta_3 \right) \right. \right. \\
& \times S_1 - \frac{80}{3}S_2 - \frac{128}{3}S_2^2 + \frac{1856}{9}S_3 - \frac{512}{3}S_4 + \left(\frac{2560}{9}S_1 - \frac{256}{3}S_2 \right) S_{-2} + \left(\frac{1280}{9} \right. \\
& \left. \left. - \frac{256}{3}S_1 \right) S_{-3} - \frac{256}{3}S_{-4} + \frac{256}{3}S_{3,1} - \frac{2560}{9}S_{-2,1} - \frac{256}{3}S_{-2,2} + \frac{1024}{3}S_{-2,1,1} - 96\zeta_3 \right] \\
& + C_A \left[-\frac{151}{2} + \left(-\frac{8(-206+211N+211N^2)}{3(-1+N)N(1+N)(2+N)} - \frac{4288}{9}S_2 + \frac{1984}{3}S_3 + 320S_4 - 1024S_{3,1} \right. \right. \\
& + \frac{1984}{3}S_{-2,1} + 3712S_{-2,2} + 3840S_{-3,1} - 7168S_{-2,1,1} \Big) S_1 + \left(256S_3 + 1792S_{-2,1} \right) S_1^2 \\
& \left. + \left(\frac{604}{3} - 832S_3 - 5248S_{-2,1} \right) S_2 + \frac{352}{3}S_2^2 - \frac{6160}{9}S_3 + \frac{2416}{3}S_4 \right]
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{48P_2}{(-1+N)N(1+N)(2+N)} + \left(-\frac{64P_7}{9(-1+N)N(1+N)(2+N)} - 256S_2 \right) S_1 \right. \\
& + \frac{992}{3}S_2 + 64S_3 + 5376S_{2,1} - 384S_{-2,1} + 576\zeta_3 \left. \right) S_{-2} + \left(-96 + 512S_1 \right) S_{-2}^2 \\
& + \left(-\frac{32(-187 + 134N + 134N^2)}{9(-1+N)(2+N)} + \frac{992}{3}S_1 - 1152S_1^2 + 2624S_2 + 960S_{-2} \right) S_{-3} \\
& + \left(\frac{560}{3} - 1472S_1 \right) S_{-4} + 2304S_{-5} + 768S_{2,3} + 2688S_{2,-3} - \frac{1856}{3}S_{3,1} - 768S_{4,1} \\
& + \frac{64(-187 + 134N + 134N^2)S_{-2,1}}{9(-1+N)(2+N)} + \frac{992}{3}S_{-2,2} - 3648S_{-2,3} + 1728S_{-4,1} \\
& - 5376[S_{2,1,-2} + S_{-2,2,1} + S_{-3,1,1}] + 1536[S_{3,1,1} - S_{-2,1,-2}] - \frac{3968}{3}S_{-2,1,1} \\
& + 10752S_{-2,1,1,1} + \frac{72(-6 + 5N + 5N^2)\zeta_3}{(-1+N)(2+N)} \left. \right\} \\
& + C_F^3 \left\{ -29 + \left(\frac{384(-1+N+N^2)}{(-1+N)N(1+N)(2+N)} + 128S_2^2 - 384S_3 + 128S_4 + 512S_{3,1} \right. \right. \\
& - 384S_{-2,1} - 3328S_{-2,2} - 3584S_{-3,1} + 6144S_{-2,1,1} \left. \right) S_1 - 256S_{-2}^2 S_1 + \left(12 + 512S_3 \right. \\
& + 4352S_{-2,1} \left. \right) S_2 - 96S_2^2 + 104S_3 - 480S_4 + \left(\frac{32P_2}{(-1+N)N(1+N)(2+N)} \right. \\
& + \left(\frac{384}{(-1+N)N(1+N)(2+N)} + 512S_2 \right) S_1 - 192S_2 + 128S_3 - 4608S_{2,1} + 256S_{-2,1} \\
& - 384\zeta_3 \left. \right) S_{-2} + \left(\frac{192}{(-1+N)(2+N)} - 192S_1 + 1280S_1^2 - 2176S_2 - 640S_{-2} \right) S_{-3} \\
& + \left(-96 + 1664S_1 \right) S_{-4} - 1792S_{-5} + 384[-S_{2,3} + S_{3,1} + S_{4,1}] - 2304S_{2,-3} \\
& - \frac{384S_{-2,1}}{(-1+N)(2+N)} - 1536S_1^2 S_{-2,1} - 192S_{-2,2} + 2944S_{-2,3} - 1664S_{-4,1} + 4608S_{2,1,-2} \\
& - 768S_{3,1,1} + 768S_{-2,1,1} + 1024S_{-2,1,-2} + 4608[S_{-2,2,1} + S_{-3,1,1}] - 9216S_{-2,1,1,1} \\
& - \frac{48(-6 + 5N + 5N^2)\zeta_3}{(-1+N)(2+N)} \left. \right\} \tag{4}
\end{aligned}$$

and one of the polarized singlet anomalous dimensions,

$$\begin{aligned}
\Delta\gamma_{gg}^{(2)} = & C_A T_F^2 N_F^2 \left[-\frac{16P_8}{27N^2(1+N)^2} S_1 - \frac{4P_{48}}{27N^3(1+N)^3} \right] + C_F \left[T_F^2 N_F^2 \left[-\frac{8P_{59}}{27N^4(1+N)^4} \right. \right. \\
& \left. \left. + \frac{64(N-1)(2+N)(-6-8N+N^2)}{9N^3(1+N)^3} S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^2 \right] \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 \Big] + C_{ATFN_F} \left[\frac{8P_6}{N^3(1+N)^3}S_2 - \frac{8P_9}{3N^3(1+N)^3}S_1^2 \right. \\
& + \frac{2P_{77}}{27(N-1)N^5(1+N)^5(2+N)} + \left(-\frac{8P_{67}}{9(-1+N)N^4(1+N)^4(2+N)} \right. \\
& \left. - \frac{32(N-1)(2+N)}{N^2(1+N)^2}S_2 + 128\zeta_3 \right) S_1 + \frac{32(N-1)(2+N)}{3N^2(1+N)^2}S_1^3 - \frac{32(34+N+N^2)}{3N^2(1+N)^2} \\
& \times S_3 + \left(\frac{128P_2}{(N-1)N^2(1+N)^2(2+N)}S_1 - \frac{32P_{23}}{(N-1)N^2(1+N)^3(2+N)} \right) S_{-2} \\
& - \frac{192(4-N-N^2)}{N^2(1+N)^2}S_{-3} + \frac{64(N-1)(2+N)}{N^2(1+N)^2}S_{2,1} - \frac{128(-8+N+N^2)}{N^2(1+N)^2}S_{-2,1} \\
& \left. - \frac{64(-3+N)(4+N)}{N^2(1+N)^2}\zeta_3 \right] + C_A^3 \left[\frac{64P_{16}}{9N^2(1+N)^2}S_{-2,1} - \frac{32P_{18}}{9N^2(1+N)^2}S_3 \right. \\
& + \frac{P_{74}}{27(N-1)N^5(1+N)^5(2+N)} + \left(\frac{4P_{69}}{9(N-1)N^4(1+N)^4(2+N)} \right. \\
& \left. - \frac{64P_{17}}{9N^2(1+N)^2}S_2 + 128S_2^2 + \frac{16(-96+11N+11N^2)}{3N(1+N)}S_3 + 192S_4 \right. \\
& \left. + \frac{1024}{N(1+N)}S_{-2,1} - 640S_{-2,2} - 768S_{-3,1} + 1024S_{-2,1,1} \right) S_1 \\
& + \left(-\frac{256(1+3N+3N^2)}{N^3(1+N)^3} + 128S_3 - 256S_{-2,1} \right) S_1^2 + \left(-\frac{16P_{41}}{9N^3(1+N)^3} \right. \\
& \left. + 64S_3 + 640S_{-2,1} \right) S_2 - \frac{256}{N(1+N)}S_2^2 - \frac{384}{N(1+N)}S_4 + 64S_5 \\
& + \left(\frac{32P_{52}}{9(N-1)N^3(1+N)^3(2+N)} + \left(-\frac{64P_{32}}{9(-1+N)N(1+N)^2(2+N)} + 256S_2 \right) \right. \\
& \left. \times S_1 - \frac{512}{N(1+N)}S_2 + 128S_3 - 768S_{2,1} \right) S_{-2} + \left(-\frac{16(24+11N+11N^2)}{3N(1+N)} \right. \\
& \left. + 64S_1 \right) S_{-2}^2 + \left(-\frac{32P_{15}}{9N^2(1+N)^2} - \frac{1536}{N(1+N)}S_1 + 384S_1^2 - 320S_2 \right) S_{-3} \\
& + \left(-\frac{1024}{N(1+N)} + 512S_1 \right) S_{-4} - 192S_{-5} - 384S_{2,-3} + \frac{1280}{N(1+N)}S_{-2,2} \\
& + 384S_{-2,3} + \frac{1536}{N(1+N)}S_{-3,1} - 384S_{-4,1} + 768S_{2,1,-2} - \frac{2048}{N(1+N)}S_{-2,1,1} \\
& \left. + 768[S_{-2,2,1} + S_{-3,1,1}] - 1536S_{-2,1,1,1} \right] \\
& + C_F^2 T_F N_F \left[-\frac{4P_{75}}{(N-1)N^5(1+N)^5(2+N)} + \left(\frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16P_{42}}{N^4(1+N)^4} \Big) S_1 + \frac{8(N-1)(2+N)(2+3N+3N^2)}{N^3(1+N)^3} S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} \\
& \times S_1^3 - \frac{8(2+N)(2-11N-16N^2+9N^3)}{N^3(1+N)^3} S_2 + \frac{32(10+7N+7N^2)}{3N^2(1+N)^2} S_3 \\
& + \left(-\frac{64(10+N+N^2)}{(N-1)N(1+N)(2+N)} + \frac{512}{N^2(1+N)^2} S_1 \right) S_{-2} + \frac{256}{N^2(1+N)^2} S_{-3} \\
& - \frac{64(N-1)(2+N)}{N^2(1+N)^2} S_{2,1} - \frac{512}{N^2(1+N)^2} S_{-2,1} + \frac{192(-2-N-N^2)}{N^2(1+N)^2} \zeta_3 \Big] \\
& + C_A^2 T_F N_F \left[\frac{32P_4}{9N^2(1+N)^2} S_2 + \frac{32P_{11}}{9N^2(1+N)^2} S_{-3} - \frac{64P_{11}}{9N^2(1+N)^2} S_{-2,1} \right. \\
& + \frac{16P_{13}}{9N^2(1+N)^2} S_3 + \frac{2P_{76}}{27(N-1)N^5(1+N)^5(2+N)} + \left(\frac{1280}{9} S_2 - \frac{64}{3} S_3 \right. \\
& \left. - \frac{8P_{68}}{27(-1+N)N^4(1+N)^4(2+N)} - 128\zeta_3 \right) S_1 + \frac{64}{3} S_{-2}^2 \\
& + \left(\frac{64P_{45}}{9(N-1)N^2(1+N)^2(2+N)} S_1 - \frac{32P_{50}}{9(N-1)N^3(1+N)^3(2+N)} \right) S_{-2} \\
& \left. + \frac{128(-3+2N+2N^2)}{N^2(1+N)^2} \zeta_3 \right]. \tag{5}
\end{aligned}$$

The polynomials are given in Refs. [1, 2]. The small z and large z limits of the anomalous dimensions have also been considered in explicit form. The latter are related to the cusp anomalous dimensions. There are various partial predictions on the small z behaviour, however, not specifying the factorization scheme used. These are theoretically interesting, but are not of quantitative importance since it is known for long that subleading terms do strongly modify the leading order effects over about three additional subleading orders [48].

We have also recalculated the unpolarized two-loop anomalous dimensions using the method of massless off-shell OMEs, [3]. Here new local operators contribute [3, 8, 16] to cancel the gauge-variant contributions to obtain the anomalous dimensions. We corrected a series of errors contained in [4, 5] and provided all expansion coefficients emerging at two-loop order [3], which are relevant for upcoming four-loop calculations.

5. Conclusions

We have calculated the three-loop polarized and unpolarized non-singlet anomalous dimensions, including transversity, and as well the polarized three-loop singlet anomalous dimensions applying the traditional method of massless off-shell operator matrix elements, a gauge-variant method, which requires special projections. The calculations have been performed without assuming special conditions concerning the structure of the final result. The method of arbitrary high Mellin moments plays a central role in these computations in establishing the corresponding difference equations

through the method of guessing, after having exploited the equations for the master integrals. These equations are solved subsequently using algorithms of difference ring theory. We agree with all the results previously obtained in the literature. The polarized non-singlet anomalous dimension $\Delta\gamma_{qq}^{s,NS}$, related to the polarized structure function g_3^- , [25], has been calculated using the associated forward Compton amplitude. In the unpolarized singlet case also new OMEs contribute. Here we have calculated all contributions emerging at the two-loop level and corrected results in the literature. The present method is suited to be expanded to the four-loop level. The knowledge of the higher-loop anomalous dimensions form one important asset for the precise description of the scaling violations of the deep-inelastic structure functions, which provide an important way to measure the QCD coupling constant $\alpha_s(M_Z^2)$ [49] at highest precision possible.

References

- [1] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, Nucl. Phys. B **971** (2021) 115542 [arXiv: 2107.06267 [hep-ph]].
- [2] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, JHEP **01** (2022) 193 [arXiv:2111.12401 [hep-ph]].
- [3] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, Nucl. Phys. B **980** (2022) 115794 [arXiv:2202.03216 [hep-ph]].
- [4] Y. Matiounine, J. Smith and W.L. van Neerven, Phys. Rev. D **57** (1998) 6701–6722 [arXiv:hep-ph/9801224 [hep-ph]];
- [5] Y. Matiounine, J. Smith and W.L. van Neerven, Phys. Rev. D **58** (1998) 076002 [hep-ph/9803439].
- [6] D.J. Gross and F. Wilczek, Phys. Rev. D **8** (1973) 3633–3652; Phys. Rev. D **9** (1974) 980–993; H. Georgi and H.D. Politzer, Phys. Rev. D **9** (1974) 416–420; K. Sasaki, Prog. Theor. Phys. **54** (1975) 1816–1827; M.A. Ahmed and G.G. Ross, Phys. Lett. B **56** (1975) 385–390.
- [7] E.G. Floratos, D. A.Ross and C.T. Sachrajda, Nucl. Phys. B **129** (1977) 66–88 Erratum: [Nucl. Phys. B **139** (1978) 545–546]; **152** (1979) 493–520; A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429–459; A. Gonzalez-Arroyo, C. Lopez and F. J. Yndurain, Nucl. Phys. B **159** (1979) 512–527; Nucl. Phys. B **153** (1979) 161–186; G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27–92; W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437–442; E.G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417–462; R. Mertig and W.L. van Neerven, Z. Phys. C **70** (1996) 637–654 [hep-ph/9506451v2]; W. Vogelsang, Phys. Rev. D **54** (1996) 2023–2029 [hep-ph/9512218]; Nucl. Phys. B **475** (1996) 47–72 [hep-ph/9603366]; R.K. Ellis and W. Vogelsang, *The Evolution of parton distributions beyond leading order:*

- The Singlet case*, arXiv:hep-ph/9602356 [hep-ph];
S. Moch and J.A.M. Vermaseren, Nucl. Phys. B **573** (2000) 853–907 [hep-ph/9912355].
- [8] R. Hamberg and W.L. van Neerven, Nucl. Phys. B **379** (1992) 143–171.
- [9] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **688** (2004) 101–134 [hep-ph/0403192].
- [10] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B **691** (2004) 129–181 [hep-ph/0404111].
- [11] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **889** (2014) 351–400 [arXiv:1409.5131 [hep-ph]].
- [12] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, Nucl. Phys. B **890** (2014) 48–151 [arXiv:1409.1135 [hep-ph]].
- [13] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, Nucl. Phys. B **922** (2017) 1–40 [arXiv:1705.01508 [hep-ph]].
- [14] A. Behring, J. Blümlein, A. De Freitas, A. Goedicke, S. Klein, A. von Manteuffel, C. Schneider and K. Schönwald, Nucl. Phys. B **948** (2019) 114753 [arXiv:1908.03779 [hep-ph]].
- [15] T. Gehrmann et al., these Proceedings and in preparation.
- [16] J.A. Dixon and J.C. Taylor, Nucl. Phys. B **78** (1974) 552–560;
H. Kluberg-Stern and J. B. Zuber, Phys. Rev. D **12** (1975) 482–488; 3159–3180;
S. Sarkar, Nucl. Phys. B **82** (1974) 447–460;
S. Sarkar and H. Strubbe, Nucl. Phys. B **90** (1975) 45–51;
S.D. Joglekar and B.W. Lee, Annals Phys. **97** (1976) 160–215;
S.D. Joglekar, Annals Phys. **108** (1977) 233–241; **109** (1977) 210–287;
R. Hamberg, *Second order gluonic contributions to physical quantities*, Thesis, Univ. Leiden (1991);
J.C. Collins and R.J. Scalise, Phys. Rev. D **50** (1994) 4117–4136 [arXiv:hep-ph/9403231 [hep-ph]];
B.W. Harris and J. Smith, Phys. Rev. D **51** (1995) 4550–4560 [arXiv:hep-ph/9409405 [hep-ph]];
G. Falcioni and F. Herzog, JHEP **05** (2022) 177 [arXiv:2203.11181 [hep-ph]] and these Proceedings.
- [17] J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037–2076 [hep-ph/9806280].
- [18] J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [hep-ph/9810241].
- [19] J. Blümlein, Prog. Part. Nucl. Phys. **69** (2013) 28–84 [arXiv:1208.6087 [hep-ph]].
- [20] S.A. Larin, Phys. Lett. B **303** (1993) 113–118 [hep-ph/9302240].
- [21] P. Nogueira, J. Comput. Phys. **105** (1993) 279–289.

- [22] I. Bierenbaum, J. Blümlein and S. Klein, Nucl. Phys. B **820** (2009) 417–482 [arXiv:0904.3563 [hep-ph]].
- [23] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025;
M. Tentyukov and J. A. M. Vermaseren, Comput. Phys. Commun. **181** (2010) 1419–1427 [hep-ph/0702279].
- [24] T. van Ritbergen, A.N. Schellekens and J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 41–96 [arXiv:hep-ph/9802376 [hep-ph]].
- [25] J. Blümlein and N. Kochelev, Nucl. Phys. B **498** (1997) 285–309 [arXiv:hep-ph/9612318 [hep-ph]].
- [26] H.D. Politzer, Phys. Rept. **14** (1974) 129–180
- [27] J. Ablinger, J. Blümlein, C. Raab, C. Schneider and F. Wißbrock, Nucl. Phys. B **885** (2014) 409–447 [arXiv:1403.1137 [hep-ph]].
- [28] P. Marquard and D. Seidel, *The Crusher algorithm*, unpublished.
- [29] J. Lagrange, Nouvelles recherches sur la nature et la propagation du son, Miscellanea Taurinensis, t. II, 1760-61; Oeuvres t. I, p. 263;
C.F. Gauß, Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractate, Commentationes societatis scientiarum Gottingensis recentiores, Vol III, 1813, Werke Bd. V pp. 5-7;
G. Green, Essay on the Mathematical Theory of Electricity and Magnetism, Nottingham, 1828 [Green Papers, pp. 1-115];
M. Ostrogradski, Mem. Ac. Sci. St. Peters., **6** (1831) 39–53;
S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087–5159 [hep-ph/0102033].
- [30] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B **192** (1981) 159–204.
- [31] D.J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343–1346;
H.D. Politzer, Phys. Rev. Lett. **30** (1973) 1346–1349;
W.E. Caswell, Phys. Rev. Lett. **33** (1974) 244–246;
D.R.T. Jones, Nucl. Phys. B **75** (1974) 531–538;
O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, Phys. Lett. B **93** (1980) 429–432;
S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B **303** (1993) 334–336 [hep-ph/9302208];
T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, Phys. Lett. B **400** (1997) 379–384 [hep-ph/9701390];
M. Czakon, Nucl. Phys. B **710** (2005) 485–498 [hep-ph/0411261];
P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. **118** (2017) no.8, 082002 [arXiv:1606.08659 [hep-ph]];
P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. **118** (2017) no.8, 082002 [arXiv:1606.08659 [hep-ph]];
F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, JHEP **02** (2017) 090

- [arXiv:1701.01404 [hep-ph]);
 T. Luthe, A. Maier, P. Marquard and Y. Schröder, JHEP **10** (2017) 166 [arXiv:1709.07718 [hep-ph]);
 K.G. Chetyrkin, G. Falcioni, F. Herzog and J.A.M. Vermaseren, JHEP **10** (2017) 179 [arXiv:1709.08541 [hep-ph]].
- [32] K.G. Chetyrkin, G. Falcioni, F. Herzog and J.A.M. Vermaseren, JHEP **1710** (2017) 179 Addendum: [JHEP **1712** (2017) 006] [arXiv:1709.08541 [hep-ph)];
 K.G. Chetyrkin, Nucl. Phys. B **710** (2005) 499–510 [arXiv:hep-ph/0405193 [hep-ph]]. E. Egorian and O.V. Tarasov, Teor. Mat. Fiz. **41** (1979) 26–32 [Theor. Math. Phys. **41** (1979) 863–867];
 T. Luthe, A. Maier, P. Marquard and Y. Schröder, JHEP **1701** (2017) 081 [arXiv:1612.05512 [hep-ph]].
- [33] J. Blümlein and C. Schneider, Phys. Lett. B **771** (2017) 31–36 [arXiv:1701.04614 [hep-ph]].
- [34] J. Blümlein, P. Marquard and C. Schneider, PoS (RADCOR2019) 078 [arXiv:1912.04390 [cs.SC]].
- [35] F.V. Tkachov, Phys. Lett. B **100** (1981) 65–68; Theor. Math. Phys. **56** (1983) 866–870; Teor. Mat. Fiz. **56** (1983) 350–356;
 See also: T. Gehrmann, E.W.N. Glover, T. Huber, N. Iqbal and C. Studerus, JHEP **06** (2010) 094 [arXiv:1004.3653 [hep-ph)];
 R.N. Lee, A.V. Smirnov and V.A. Smirnov, Nucl. Phys. B Proc. Suppl. **205-206** (2010) 308–313 [arXiv:1005.0362 [hep-ph)];
 R.N. Lee and V.A. Smirnov, JHEP **02** (2011) 102 [arXiv:1010.1334 [hep-ph]].
- [36] M. Kauers, *Guessing Handbook*, JKU Linz, Technical Report RISC 09-07.
- [37] J. Blümlein, M. Kauers, S. Klein and C. Schneider, Comput. Phys. Commun. **180** (2009) 2143–2165 [arXiv:0902.4091 [hep-ph]].
- [38] Sage, <http://www.sagemath.org/>
- [39] M. Kauers, M. Jaroschek, and F. Johansson, in: *Computer Algebra and Polynomials*, Editors: J. Gutierrez, J. Schicho, Josef, M. Weimann, Eds.. Lecture Notes in Computer Science **8942** (Springer, Berlin, 2015) 105–125 [arXiv:1306.4263 [cs.SC]].
- [40] J. Blümlein, D.J. Broadhurst and J.A.M. Vermaseren, Comput. Phys. Commun. **181** (2010), 582-625 [arXiv:0907.2557 [math-ph]].
- [41] M. Karr, J. ACM **28** (1981) 305–350;
 M. Bronstein, J. Symbolic Comput. **29** (2000) no. 6 841–877;
 C. Schneider, *Symbolic Summation in Difference Fields*, Ph.D. Thesis RISC, Johannes Kepler University, Linz technical report 01–17 (2001);
 C. Schneider, An. Univ. Timisoara Ser. Mat.-Inform. **42** (2004) 163–179;
 C. Schneider, J. Differ. Equations Appl. **11** (2005) 799–821;

- C. Schneider, Appl. Algebra Engrg. Comm. Comput. **16** (2005) 1–32;
 C. Schneider, J. Algebra Appl. **6** (2007) 415–441;
 C. Schneider, Clay Math. Proc. **12** (2010) 285–308 [arXiv:0904.2323 [cs.SC]];
 C. Schneider, Ann. Comb. **14** (2010) 533–552 [arXiv:0808.2596];
 C. Schneider, in: Computer Algebra and Polynomials, Applications of Algebra and Number Theory, J. Gutierrez, J. Schicho, M. Weimann (ed.), Lecture Notes in Computer Science (LNCS) 8942 (2015) 157–191 [arXiv:1307.7887 [cs.SC]];
 C. Schneider, J. Symb. Comput. **72** (2016) 82–127 [arXiv:1408.2776 [cs.SC]]. C. Schneider, J. Symb. Comput. **80** (2017) 616–664 [arXiv:1603.04285 [cs.SC]]. S.A. Abramov, M. Bronstein, M. Petkovšek, Carsten Schneider, J. Symb. Comput. **107** (2021) 23–66 [arXiv:2005.04944 [cs.SC]].
- [42] C. Schneider, Sémin. Lothar. Combin. **56** (2007) 1–36 article B56b.
- [43] C. Schneider, Simplifying Multiple Sums in Difference Fields, in: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions Texts and Monographs in Symbolic Computation eds. C. Schneider and J. Blümlein (Springer, Wien, 2013) 325–360 [arXiv:1304.4134 [cs.SC]].
- [44] J. Ablinger, J. Blümlein and C. Schneider, J. Phys. Conf. Ser. **523** (2014) 012060 [arXiv:1310.5645 [math-ph]];
 J. Ablinger, PoS (LL2014) 019 [arXiv:1407.6180[cs.SC]]; *A Computer Algebra Toolbox for Harmonic Sums Related to Particle Physics*, Diploma Thesis, JKU Linz, 2009, arXiv:1011.1176[math-ph]; *Computer Algebra Algorithms for Special Functions in Particle Physics*, Ph.D. Thesis, Linz U. (2012) arXiv:1305.0687[math-ph]; PoS (LL2016) 067; Experimental Mathematics **26** (2017) [arXiv:1507.01703 [math.CO]]; PoS (RADCOR2017) 001 [arXiv:1801.01039 [cs.SC]]; arXiv:1902.11001 [math.CO]; PoS (LL2018) 063;
 J. Ablinger, J. Blümlein and C. Schneider, J. Math. Phys. **54** (2013) 082301 [arXiv:1302.0378 [math-ph]]; J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063 [math-ph]];
 J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822 [hep-th]].
- [45] E. Remiddi and J.A.M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725–754 [hep-ph/9905237].
- [46] J. Blümlein, Comput. Phys. Commun. **159** (2004) 19–54 [arXiv:hep-ph/0311046 [hep-ph]].
- [47] J. Blümlein, Comput. Phys. Commun. **180** (2009) 2218–2249 [arXiv:0901.3106 [hep-ph]].
- [48] J. Blümlein and A. Vogt, Phys. Lett. B **370** (1996) 149–155 [hep-ph/9510410]; Phys. Lett. B **386** (1996), 350–358 [arXiv:hep-ph/9606254 [hep-ph]]; Phys. Rev. D **58** (1998) 014020 [arXiv:hep-ph/9712546 [hep-ph]];
 J. Blümlein, Lect. Notes Phys. **546** (2000) 42–57, [arXiv:hep-ph/9909449 [hep-ph]].
- [49] S. Bethke *et al.*, *Workshop on Precision Measurements of α_s* , arXiv:1110.0016 [hep-ph];
 S. Moch *et al.*, *High precision fundamental constants at the TeV scale*, arXiv:1405.4781 [hep-ph];
 S. Alekhin, J. Blümlein and S.O. Moch, Mod. Phys. Lett. A **31** (2016) no.25, 1630023.