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Electromagnetic Couplings of Axions

Anton V. Sokolov, Andreas Ringwald

Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, 22607 Hamburg, Germany

E-mail: anton.sokolov@desy.de, andreas.ringwald@desy.de

Abstract: We show that, contrary to assertions in the literature, the main contribution to the axion-photon coupling need not be quantized in the units proportional to e^2. In particular, we discuss a loophole in the argument for this quantization and then provide explicit counterexamples. Hence, we construct a generic axion-photon effective Lagrangian and find that the axion-photon coupling may be dominated by previously unknown Wilson coefficients. We show that this result implies a significant modification of conventional axion electrodynamics and sets new targets for axion experiments. At the core of our theoretical analysis lies a critical reexamination of the interactions between axions and magnetic monopoles. We show that, contrary to claims in the literature, magnetic monopoles need not give mass to axions. Moreover, we find that a future detection of an axion or axion-like particle with certain parameters can serve as evidence for the existence of magnetically charged matter.

Contents

1 Introduction

Axions are very well motivated candidates for physics beyond the Standard model, since they provide an elegant solution to the strong CP problem [1–4] and can naturally explain the abundance and properties of dark matter [5–8]. Even more, particles which have axion-like interactions with photons can explain the astrophysical anomaly of TeV transparency of the Universe associated with the propagation of high energy photons from distant TeV blazars and BL Lac type objects [9–11] [1]. The axion-photon couplings indicated by the latter anomaly are however much larger than the ones predicted in the relevant mass region in conventional axion models of KSVZ [13, 14] or DFSZ [15, 16] type. This tension has been reconciled recently by the authors of this article who found that the anomalous TeV transparency of the Universe, as well as the axion hint from the cooling of horizontal branch stars in globular clusters [17], can be accounted for in the general hadronic model for the axion [18, 19]. The axion-photon coupling in the latter model, however, seems to contradict the lore established by previous works which discuss the possible range of axion-photon couplings in minimal models[2] – namely that the main contribution to these couplings must be quantized in some units proportional to the electron charge squared e^2 [20, 22, 23]. In this work, we show that there is really no contradiction, by extending the results obtained in Ref. [20] and finding that the main contribution to the axion-photon coupling need not be quantized in the above-mentioned units even in minimal axion models.

Our central result which allows us to go beyond the conventional argument about the quantization of the axion-photon coupling stems from a critical reexamination of the interactions between axions and magnetic monopoles. It has been long believed that these interactions are necessarily induced by the Witten effect [24, 25]. What we show is that there are more possibilities, so that the shift of the axion field need not induce electric charge on monopoles. The corresponding non-conventional electromagnetic couplings of axions enter the axion-photon effective field theory (EFT) whenever one admits the "no global symmetries" [26–30] and "completeness of the charge spectrum" [28, 29, 31] conjectures of quantum gravity, since these conjectures imply the existence of magnetic monopoles with any charge allowed by the Dirac-Schwinger-Zwanziger (DSZ) quantization condition [32–34]. The same new interaction terms enter the low energy EFT of axion-like particles (ALPs) – Nambu-Goldstone bosons of any spontaneously broken global $U(1)$ symmetry. We derive phenomenological consequences of the new electromagnetic couplings of the axion and ALPs, showing that they would represent new, distinctive features, which are possible to detect in various axion experiments. We argue that the detection of axions or ALPs with such features would provide circumstantial experimental evidence for the existence of magnetically charged matter.

The article is structured as follows: in sec. 2, we give a brief introduction to the physics of magnetic monopoles with emphasis on the overwhelming theoretical evidence for their existence; in sec. 3, we review the existing argument about the quantization of the axion-photon coupling and indicate a loophole in it; in sec. 4, we review a particular QFT with magnetic charges, develop an EFT approach to it and discuss CP-violation therein; in sec. 5, we build a generic axion-photon EFT and classify different axion-photon couplings; in sec. 6, we discuss the phenomenology of the new axion-photon couplings and their implications for axion search experiments; finally, in sec. 7, we conclude.

[1] Note however that a recent astrophysical study [12] of the polarization of light from magnetic white dwarfs excludes most of the parameter region relevant for the TeV transparency hint.

[2] Minimal models exclude constructions with mixing of multiple axions [20] or clockwork mechanism [21].

2 Magnetic and electric charges in quantum theory

In 1894, Pierre Curie noted that the existence of magnetic monopoles would be perfectly consistent with the classical theory of electromagnetism [35]. His motivation for considering magnetic charges stemmed from the desire to give magnetism an analogous status to the one of electricity. Now we know that this desire was never to come true: all magnetic phenomena observed so far can be perfectly described by the motion and interactions of electrically charged particles. However, the fundamental quantum theory which provided for such a successful electric description of the magnetic phenomena has also revolutionized our views on magnetic monopoles. Whereas Pierre Curie and his contemporaries regarded magnetic charges as a purely phenomenological construct, Paul Dirac argued in 1931 that the quantum theoretical formalism itself suggests their existence [32]. In particular, he pointed out that, in a consistent quantum theory, the change in the phase of a wave function around any closed curve must be the same modulo $2\pi n$ for all the wave functions, where n is an integer. In case $n = 0$ one recovers the standard gauge principle, introduced earlier by Hermann Weyl [36]. This is the case where there are only electrically but no magnetically charged particles in the theory and which is known to be realized in the Standard model (SM) of particle physics. The quantum theory itself however does not tell us any reason for why only the case $n = 0$ should be realized in nature, so that for a generic quantum mechanical model of particle physics one would expect that any n is allowed. As Dirac showed, the $n \neq 0$ case corresponds to a model with both electric and magnetic monopoles involved. Moreover, he found that the corresponding electric (q) and magnetic (g) gauge charges are necessarily related via the quantization condition $qg = 2\pi n$, which conveniently explains quantization of the electric charge observed in nature. Note that this is still the simplest explanation for the latter phenomenon to date, since it follows directly from the formalism of quantum mechanics (QM) given one does not put any additional restrictions on the quantum states of the theory.

One can wonder if the results which Dirac obtained in the framework of QM can be derived in the more fundamental formalism of quantum field theory (QFT). Does a generic QFT of gauge interactions predict quantization of the electric gauge charge? The answer is positive and it was found by Daniel Zwanziger [37, 38]. In order to address this question, Zwanziger had to revisit the very basis of any Poincaré-invariant QFT – irreducible unitary representations of the Poincaré group under which particles of the theory can transform. One-particle irreducible representations were studied in 1939 by Eugene Wigner [39], however what Zwanziger found is that there exist also two-particle irreducible representations. The latter are parameterized by an angular momentum variable, which is quantized, and correspond to pairs of particles, each pair containing both an electric and a magnetic monopole. The quantized angular momentum variable for a given pair is proportional to the product of the corresponding electric and magnetic charges, hence one automatically recovers charge quantization. Even more, a QFT of gauge interactions which allows for the two-particle irreducible representations was explicitly constructed in the case of the Abelian gauge group, both in Hamiltonian and Lagrangian formulations, in the works by Julian Schwinger and Daniel Zwanziger [40, 41]. The latter authors also showed that in an Abelian gauge theory a particle can have both electric and magnetic charges, i.e. it can be a dyon, in which case the quantization condition is generalized: $q_i g_j - q_j g_i = 2\pi n$ for any pair of particles (i, j) [33, 34]. This means the right statement is not that the known charged particles have no magnetic charge, as it is usually claimed, but rather that their magnetic charges happen to be proportional to their electric charges. One of the essential features of both Schwinger and Zwanziger formulations is the introduction of two four-potentials for the description of the photon field, one of which couples to the electric and another to the magnetic currents. The advantage of the Zwanziger formulation is that it is based on a Lagrangian which preserves locality and which treats electric and magnetic variables symmetrically. We will review Zwanziger's Lagrangian formulation and take advantage of it later in this article.

So far we have been discussing generic magnetic monopoles, i.e. essentially the possibility of having two different kinds of gauge charges deeply rooted in the formalism of both QM and QFT. Let us now consider 't Hooft-Polyakov magnetic monopoles – topological solitons arising in the spontaneously broken symmetry phase of purely electric non-Abelian gauge theories, discovered by Gerardus 't Hooft and Alexander Polyakov in 1974 [42, 43]. These topological solitons were called magnetic monopoles because they create a monopole-like magnetic field far from their cores. Note however that they are more complicated constructs compared to their fundamental counterparts which we discussed earlier. It is important that the difference can reveal itself even at energies much lower than the inverse monopole core size, for which one would expect 't Hooft-Polyakov monopoles to behave similarly to fundamental ones due to the identical monopole-like configuration of the long-range magnetic field. The reason for this is that the instanton effects of the full non-Abelian theory are not suppressed on monopoles [44], thus contributing an extra rotor degree of freedom to the EFT describing infrared (IR) physics [45]. The best known phenomenological implication of this extra degree of freedom is the Rubakov-Callan effect [44, 46]: as Valery Rubakov and Curtis Callan showed in the beginning of 1980s, the Grand Unified Theory (GUT) magnetic monopole can induce proton decay at a strong interaction rate. If one is interested only in the processes which do not involve the rotor degree of freedom, the 't Hooft-Polyakov monopoles behave similarly to Dirac monopoles in the IR, i.e. their EFT is given by the above-mentioned Zwanziger theory [47]. Since 't Hooft-Polyakov monopoles (more generally, Julia-Zee dyons [48]) are an inevitable prediction of GUTs, they represent a well motivated case for the existence of magnetic monopoles (more generally, dyons). Explicit constructions show that such dyons can be bosonic as well as fermionic [49–51]. In this work, we will not adhere to any particular GUT, keeping our discussion as generic as possible.

From what has been already discussed, we see that the existence of magnetically charged matter would fit very well both in the structure of QM, completing the gauge principle, and in the structure of relativistic quantum theory, completing the irreducible unitary representations of the Poincaré group realized in nature. The observed quantization of the electric charge would be explained. Moreover, the existence of magnetic monopoles would be a natural consequence of the unification of fundamental interactions, if the latter unification takes place. This is however not an exhaustive list of arguments in support of the existence of magnetically charged particles. Another strong motivation comes from our understanding of gravity: the consistency of a quantum theory of the latter was shown to imply a number of restrictions on the structure of admissible field theories. In particular, it was argued that there can be no global symmetries in a consistent theory which includes quantum gravity [26–30] and that in such a theory, the charge spectrum is complete [28, 29, 31]. These conjectures were shown to imply the existence of magnetic monopoles with any magnetic charge allowed by the DSZ quantization condition [28, 31].

What are the ways to probe magnetic monopoles experimentally? Many direct search techniques have been proposed [52], such as searches for monopoles bound in matter, searches in cosmic rays, searches at colliders and, in the case of some GUT monopoles, searches via the catalysis of nucleon decay. None of the direct detection experiments have however yielded a conclusive signal so far. Moreover, it is quite difficult to derive accurate exclusion limits on the monopole mass due to the large theoretical uncertainty. In fact, the QFT of magnetic monopoles is an essentially non-perturbative theory and there is still no reliable method to calculate cross-sections of the QFT processes involving magnetic charges. The interpretation of the indirect searches for virtual monopoles at colliders [53] suffers from the same problem. In this work, we will point out a new possible signature for virtual magnetic monopoles, which has the advantage of being independent of any non-rigorous statements within the non-perturbative theory of magnetic charges. In particular, we will show that there is a certain modification of free electrodynamics which, if experimentally detected, would favor the existence of magnetic monopoles. A complication is that such a modification must involve a new hypothetical particle – the axion or, more generally, an ALP.

3 Structure of the axion-photon coupling – need for revision

3.1 Previous arguments supporting quantization

The axion is the pseudo Nambu-Goldstone boson of the spontaneously broken $U(1)_{\mathrm{PQ}}$ Peccei-Quinn (PQ) symmetry [1–4]. Since the PQ symmetry is anomalous, the low energy axion Lagrangian generally contains non-derivative couplings of the axion to CP-odd combinations of the gauge fields of the low energy SM:

$$\mathcal{L}_a \supset -\frac{1}{4}\, g_{a\gamma\gamma}\, a\, F^{\mu\nu}F^d_{\mu\nu} + \frac{a g_s^2}{32\pi^2 f_a}\, G^{a\,\mu\nu}G^{d\,a}_{\mu\nu}\,, \tag{3.1}$$

where a is the axion field, $F_{\mu\nu}$ ($G_{\mu\nu}$) is the field strength tensor of the QED (QCD) gauge field, $g_{a\gamma\gamma}$ is the axion-photon coupling, g_s is the coupling constant of QCD, f_a is the axion decay constant; summation over the index $a = 1\ldots 8$ for gluons is implied and for any tensor $A_{\mu\nu}$ its Hodge dual is defined as $A^d_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho}A^{\lambda\rho}/2$, where $\epsilon_{0123} = 1$.

Both axion-photon and axion-gluon couplings are probed in various experiments. Interactions of the axion with photons are particularly well constrained. In fact, a large parameter region on the $(g_{a\gamma\gamma}, f_a)$ plane has been excluded already and there are a lot of new different experiments planned which are going to explore the axion-photon coupling further in the nearest future. A natural question then arises: where should we look in the first place? What are the best motivated values for $g_{a\gamma\gamma}$ and f_a from the theoretical viewpoint? In the last years, it has been claimed by many authors that this question can be answered by considering a consistency condition for the axion EFT [20, 22, 23]. The latter condition takes advantage of the fact that the axion is essentially an angular variable with a period $2\pi v_a$, where v_a is the PQ symmetry breaking scale, so that the effective low energy action must be symmetric under the following shifts:

$$a \to a + 2\pi v_a n,\ n \in \mathbb{Z}\,. \tag{3.2}$$

Let us review the argument of Refs. [20, 22]. First, since the topological charge of QCD,

$$Q_t \,=\, \frac{g_s^2}{32\pi^2}\int d^4x\; G^{a\,\mu\nu}G^{d\,a}_{\mu\nu}\,, \tag{3.3}$$

is an integer, symmetry of the axion-gluon interaction under the transformation (3.2) requires

$$f_a = v_a/N_{\rm DW},\ \ N_{\rm DW} \in \mathbb{Z}\,, \tag{3.4}$$

in which case under the transformation (3.2), the action changes by $2\pi k$, $k \in \mathbb{Z}$, and the path integral is unchanged. Second, since one cannot generically exclude the presence of magnetic monopoles at high energies, it has been claimed that the Witten effect [24] makes the term

$$\theta_{\rm em}\cdot\frac{e^2}{16\pi^2}\int d^4x\; F^{\mu\nu}F^d_{\mu\nu}\,, \tag{3.5}$$

which enters the QED action, physically relevant. The parameter $\theta_{\rm em}$ is cyclic with the period that depends on the global structure of the SM group, see Ref. [54]. For the following argument, it is important that the period of $\theta_{\rm em}$ is always an integer multiple of 2π. Identification of the two similar structures in Eqs. (3.1) and (3.5) restricts the values of the axion-photon coupling $g_{a\gamma\gamma}$ due to the periodicity of the axion field Eq. (3.2):

$$g_{a\gamma\gamma} \,=\, \frac{E}{N}\cdot\frac{e^2}{8\pi^2 f_a}\,, \tag{3.6}$$

where $N = N_{\rm DW}/2$, $E \in \mathbb{Z}$, and we used Eq. (3.4) in order to relate $g_{a\gamma\gamma}$ to f_a. The authors of Refs. [20, 22] then proceed to argue that any contribution to $g_{a\gamma\gamma}$ that is not quantized, i.e. which does not satisfy Eq. (3.6), must be proportional to the mass of the axion squared and can be significantly larger than the order of magnitude of the quantized contribution $e^2/(8\pi^2 f_a)$ only in non-minimal models which introduce new unnecessary energy scales and/or particles.

Let us highlight the step in the derivation where one identifies the two similar $F^{\mu\nu}F^d_{\mu\nu}$ structures in Eqs. (3.1) and (3.5) in the presence of magnetic monopoles. Physically, it is equivalent to stating that electromagnetic interactions between axions and magnetic monopoles are necessarily induced by the Witten effect. What we found is that the latter statement has actually never been consistently derived; moreover, it does not necessarily hold. Before we explain the loophole that has been overlooked, let us briefly review the Witten effect and its low energy description since they are central to the following discussion.

3.2 Witten effect and its low energy description

The Witten effect is an effect in a theory with spontaneously broken non-Abelian gauge symmetry derived in 1979 by Edward Witten [24]. The latter author showed that if the full non-Abelian $SO(3)$ theory with coupling constant $\bar{g}$ and field strength $G_{\mu\nu}$ has a CP-violating parameter θ in the Lagrangian:

$$\mathcal{L}_\theta = -\frac{\theta \bar{g}^2}{32\pi^2} G^{a\,\mu\nu} G^{d\,a}_{\mu\nu}\,, \;\; \theta \neq 2\pi n\,, \tag{3.7}$$

where $n \in \mathbb{Z}$, then 't Hooft-Polyakov monopoles of the broken phase of such a theory get an additional contribution δq to their electric charges $q = m\bar{g} + \delta q$, $m \in \mathbb{Z}$:

$$\delta q = -\frac{\theta \bar{g}}{2\pi}\, k, \;\; k \in \mathbb{Z}\,, \tag{3.8}$$

which is not quantized in units of $\bar{g}$, but which is proportional to the CP-violating parameter θ. One can wonder if there exists a low energy Lagrangian, which would account for the Witten effect, i.e. which would ensure that any monopole-like magnetic field comes together with the monopole-like electric field of strength corresponding to the non-quantized electric charge δq from Eq. (3.8). One would normally write such a Lagrangian in the following form[3]:

$$\mathcal{L} \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\theta e^2}{32\pi^2} F^{\mu\nu} F^{d}_{\mu\nu}\,, \tag{3.9}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, A_ν is the electromagnetic four-potential and e is the low energy electric gauge coupling. The non-trivial topology of the A_ν field in the vicinity of the magnetic monopole ensures that the second term of Eq. (3.9) has physical importance. Indeed, if one imposes

$$\partial_\mu F^{d\,\mu\nu} = \frac{4\pi k}{e} j^\nu_m\,, \;\; k \in \mathbb{Z}\,, \tag{3.10}$$

where j^ν_m is the current of magnetic monopoles and $4\pi/e$ is the minimal allowed charge of the $SO(3)$ 't Hooft-Polyakov monopole, then one obtains the following equations of motion corresponding to the Lagrangian (3.9):

$$\partial_\mu F^{\mu\nu} = -\frac{\theta e}{2\pi}\, k\, j^\nu_m\,. \tag{3.11}$$

Comparing the latter equation with the expression (3.8) for the non-quantized contribution to the electric charge of the 't Hooft-Polyakov monopole, we see that the low energy Lagrangian (3.9) does allow us to account for the Witten effect.

[3] Note the difference in the coefficients of the θ-terms in Eqs. (3.5) and (3.9). The reason is that defining the Abelian theory of Eq. (3.9) we made a definite choice for the underlying non-Abelian gauge group $SO(3) = SU(2)/\mathbb{Z}_2$. The spectrum of line operators in the non-Abelian theory with this gauge group fixes the period of θ to be 4π [54], so that an extra factor of 2 accumulates in the denominator of the corresponding θ-term.

3.3 Loophole in the previous arguments

Suppose now that we know nothing about the high energy non-Abelian theory and want to justify the Witten effect merely by means of the low energy EFT. In this EFT, we introduce the topological term, which coincides with the second term of the low energy Lagrangian Eq. (3.9). The coefficient $\theta e^2/(32\pi^2)$ is fixed by topology. Derivation of the Witten effect then proceeds on the lines reviewed in the previous section. Let us however stress an essential assumption that enters such considerations. Namely, one assumes that the electromagnetic field is quantized in terms of the four-potential A_μ in the presence of magnetic monopoles. The latter quantization procedure is only valid if all the magnetic monopoles of the theory are treated as merely quasi-classical external sources. Only then is the second term in Eq. (3.9) topological. This description of nature, where the magnetic monopoles are non-dynamical, is incomplete. An exhaustive quantization of the system with magnetic charges has to follow the works by Schwinger and Zwanziger [40, 41]. Note that the particular case of GUT monopoles is not an exception [47].

The interactions between axions and magnetic monopoles have never been derived from a high energy theory, such as a GUT, but only from the low energy axion EFT, namely from the first term in the Lagrangian (3.1) – see Ref. [25], which is followed by all other works on the subject. Such derivation suffers from the same supposition as that discussed in the previous paragraph. Quantum dynamical effects of the monopoles are neglected. This means for example that the loop effects of magnetic charges cannot be reliably accounted for. Such truly QFT effects involving magnetic monopoles can however play a very important role in low energy physics: in particular, as we show in the following sections, they dominate the axion-photon coupling and can lead to novel experimental signatures for axions. Thus, the conventional derivation of the axion-monopole Witten effect induced interactions from the low energy EFT (3.1) has a limited range of applicability.

What are the ultraviolet (UV) models in which it is possible to prove that the interactions between axions and magnetic monopoles are the ones induced by the Witten effect? We can find the answer to this question by deliberately preserving the full analogy to the case of constant θ. Note that the second term on the right-hand side of the Eq. (3.9) is derived from the θ-term (3.7) of the corresponding high energy Lagrangian, either by requiring that the Witten effect is reproduced, as we did in sec. 3.2, or by simply neglecting the contribution from the heavy gauge fields in the symmetry broken phase. The latter consideration allows us to substitute θ with the axion field, but only *both* at low and high energy scales:

$$\mathcal{L}_a^{\mathrm{UV}} = \frac{a\bar{g}^2}{32\pi^2 f_a}\, G^{a\,\mu\nu} G^{d\,a}_{\mu\nu} \quad \Rightarrow \quad \mathcal{L}_a^{\mathrm{IR}} = \frac{ae^2}{32\pi^2 f_a} F^{\mu\nu} F^d_{\mu\nu}\,, \tag{3.12}$$

where the notations are adopted from sec. 3.2. The logic outlined in (3.12) is possible only in a theory where the PQ scale v_a is much larger than the GUT scale: $v_a \gg v_{\mathrm{GUT}}$. It is thus not applicable to the vast majority of axion GUT models (starting from the model of Ref. [55]), where one normally identifies some part of the PQ field with the scalar Higgs fields which govern the breaking of the unifying gauge group.

In closing, we see that there is really no robust theoretical argument showing that the electromagnetic interactions between axions and magnetic monopoles are necessarily induced by the Witten effect. Thus, the argument for the axion-photon coupling quantization from sec. 3.1 is inconclusive. But then, how can one infer the structure of the axion-photon coupling relevant for experimental searches? The answer to this question follows straightforwardly from the nature of the loophole discussed: we have to consider a proper QFT with magnetic charges, e.g. Zwanziger theory. Let us proceed to the next sections, where we first briefly review Zwanziger theory and its classical limit, then develop an EFT approach to the latter theory and discuss CP-violation therein, and finally construct a generic low energy axion-photon EFT.

4 Quantum electromagnetodynamics

4.1 Zwanziger theory

Quantum electromagnetodynamics (QEMD) is the QFT describing interactions of electric charges, magnetic charges and photons. Local-Lagrangian QEMD was constructed by Zwanziger [34]. In the latter theory, the photon is described by two four-potentials A_μ and B_μ, which are regular everywhere. The gauge group $U(1)$ of electrodynamics is substituted with the new one $U(1)_{\mathrm{E}} \times U(1)_{\mathrm{M}}$, where the electric (E) and magnetic (M) factors act in the standard way on A_μ and B_μ, respectively. One fixes the gauge freedom and restricts the physical states by requiring that they be vacuum states with respect to the free scalar fields[4] $(n\cdot A)$ and $(n\cdot B)$, where $n^\mu = (0, \vec{n})$ is an arbitrary fixed spatial vector. The right number of degrees of freedom of the photon is preserved due to the special form of the equal-time commutators between the potentials:

$$[A^\mu(t,\vec{x}), B^\nu(t,\vec{y})] = i\epsilon^{\mu\nu}{}_{\rho 0}\, n^\rho \,(n\cdot\partial)^{-1}(\vec{x}-\vec{y}) \,, \tag{4.1}$$

$$[A^\mu(t,\vec{x}), A^\nu(t,\vec{y})] = [B^\mu(t,\vec{x})\,, B^\nu(t,\vec{y})] = -i\,({g_0}^\mu n^\nu + {g_0}^\nu n^\mu)\,(n\cdot\partial)^{-1}(\vec{x}-\vec{y}) \,, \tag{4.2}$$

where $(n\cdot\partial)^{-1}(\vec{x}-\vec{y})$ is the kernel of the integral operator $(n\cdot\partial)^{-1}$ satisfying $n\cdot\partial\,(n\cdot\partial)^{-1}(\vec{x}) = \delta(\vec{x})$:

$$(n\cdot\partial)^{-1}(\vec{x}) = \frac{1}{2}\int_{-\infty}^{\infty} \delta^3\left(\vec{x}-\vec{n}s\right)\varepsilon(s)ds \,, \tag{4.3}$$

$\varepsilon(s)$ is the signum function. The commutation relations Eqs. (4.1), (4.2) thus make the theory essentially different from the simple case of the gauge theory with two electric $U(1)$ gauge groups, used e.g. in models with a hidden photon. The two four-potentials are not independent and their relation absorbs the non-locality which is inherent to any QFT with both electric and magnetic charges. The Lagrangian of the Zwanziger theory is local and is given by the expression[5]:

$$\begin{aligned}\mathcal{L} \;=\; \frac{1}{2n^2}\Big\{&[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d] \;-\; [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] \;-\\ &[n\cdot(\partial\wedge A)]^2 \;-\; [n\cdot(\partial\wedge B)]^2\Big\} \;-\; j_e\cdot A \;-\; j_m\cdot B \;+\; \mathcal{L}_G \,,\end{aligned} \tag{4.4}$$

[4] We use the following simplified notations: $a\cdot b = a_\mu b^\mu$.

[5] The notations are further simplified: $(a\wedge b)^{\mu\nu} = a^\mu b^\nu - a^\nu b^\mu$, $(a\cdot G)^\nu = a_\mu G^{\mu\nu}$.

where j_e and j_m are electric and magnetic currents, respectively, and $\mathcal{L}_G$ is the gauge-fixing part:

$$\mathcal{L}_G \;=\; \frac{1}{2n^2}\left\{[\partial\,(n\cdot A)]^2 + [\partial\,(n\cdot B)]^2\right\} . \tag{4.5}$$

The Lagrangian (4.4) is invariant under those $SO(2)$ transformations which rotate the two-vectors (A, B) and (j_e, j_m) simultaneously. This symmetry ensures that the absolute directions in the gauge charge space (q, g) are not observable. Another important symmetry is however not manifest in the Lagrangian (4.4) – the Lorentz-invariance seems to be lost. This appearance is in fact deceptive. The reason is intimately connected to the non-perturbativity of the theory and to the DSZ quantization condition. It was shown in Refs. [56, 57] that, after all the quantum corrections are properly accounted for, the dependence on the vector n_μ in the action S factorizes into a linking number L_n, which is an integer, multiplied by the combination of charges entering the quantization condition $q_i g_j - q_j g_i$, which is 2π times an integer. Since S contributes to the generating functional as $\exp(iS)$, this Lorentz-violating part does not play any role in physical processes. The same result has been obtained directly at the level of amplitudes in the toy model where the magnetic charge is made perturbative [58].

4.2 Classical limit and its peculiarities

Let us now show that the classical limit of the theory with the Lagrangian (4.4) indeed corresponds to classical electromagnetism with magnetic currents. The classical equations of motion for the potentials corresponding to the Lagrangian (4.4) are:

$$\frac{n\cdot\partial}{n^2}\left(n\cdot\partial A^\mu \;-\; \partial^\mu n\cdot A \;-\; n^\mu\partial\cdot A \;-\; \epsilon^\mu{}_{\nu\rho\sigma}n^\nu\partial^\rho B^\sigma\right) \;=\; j_e^\mu \;, \tag{4.6}$$

$$\frac{n\cdot\partial}{n^2}\left(n\cdot\partial B^\mu \;-\; \partial^\mu n\cdot B \;-\; n^\mu\partial\cdot B \;-\; \epsilon^\mu{}_{\nu\rho\sigma}n^\nu\partial^\rho A^\sigma\right) \;=\; j_m^\mu \;. \tag{4.7}$$

They are first-order equations in the time derivative, which allows the two different four-potentials to describe a sole particle – the photon. To transform these equations, it is convenient to use the identity

$$X = \frac{1}{n^2}\left\{[n\wedge(n\cdot X)] \;-\; [n\wedge(n\cdot X^d)]^d\right\} , \tag{4.8}$$

which holds for any antisymmetric tensor X. Namely, assume $X = F$, where F is the field strength tensor introduced such that $n\cdot F = n\cdot(\partial\wedge A)$ and $n\cdot F^d = n\cdot(\partial\wedge B)$. Then, recalling that the scalar expressions $n\cdot A$ and $n\cdot B$ are free fields by definition, one can transform Eqs. (4.6), (4.7) into the Maxwell equations with magnetic currents:

$$\partial_\mu F^{\mu\nu} = j_e^\nu \,, \tag{4.9}$$

$$\partial_\mu F^{d\,\mu\nu} = j_m^\nu \,. \tag{4.10}$$

Thus the Lagrangian (4.4) gives us the correct classical equations of motion for the electromagnetic field.

What remains to be seen is whether the classical equations of motion for the charged particles are recovered. Classical expressions for the electric and magnetic currents are:

$$j_e^\nu(x) = \sum_i q_i \int \delta^4(x - x_i(\tau_i))\, dx_i^\nu \,, \tag{4.11}$$

$$j_m^\nu(x) = \sum_i g_i \int \delta^4(x - x_i(\tau_i))\, dx_i^\nu \,, \tag{4.12}$$

where $x_i(\tau_i)$ is the trajectory of the i-th particle. Supplementing the Lagrangian (4.4) with the standard kinetic terms for the particles, one obtains the following classical equations of motion for the i-th particle:

$$\frac{d}{d\tau_i}\left(\frac{m_i u_i}{(u_i^2)^{1/2}}\right) = \left(q_i\,[\partial\wedge A(x_i)] + g_i\,[\partial\wedge B(x_i)]\,\right)\cdot u_i \,, \tag{4.13}$$

where $u_i^\mu = dx_i^\mu/d\tau_i$. The way the electromagnetic field strength tensor was introduced above ($n\cdot F = n\cdot(\partial\wedge A)$ and $n\cdot F^d = n\cdot(\partial\wedge B)$) and Eqs. (4.9), (4.10) suggest that

$$\partial\wedge A = F + (n\cdot\partial)^{-1}(n\wedge j_m)^d \,, \tag{4.14}$$

$$\partial\wedge B = F^d - (n\cdot\partial)^{-1}(n\wedge j_e)^d \,, \tag{4.15}$$

so that the final expression describing the classical force exerted on the i-th particle by the electromagnetic field is:

$$\begin{aligned}\frac{d}{d\tau_i}\left(\frac{m_i u_i}{(u_i^2)^{1/2}}\right) &= \Big(q_i F(x_i) + g_i F^d(x_i)\Big)\cdot u_i \\ &\quad - \sum_j (q_i g_j - g_i q_j)\, n\cdot\!\int (n\cdot\partial)^{-1}(x_i - x_j)\ (u_i\wedge u_j)^d\, d\tau_j \,.\end{aligned} \tag{4.16}$$

This expression correctly accounts for the Lorentz force law only if the non-local term in the second row does not contribute. It is easy to see that the latter term indeed cannot play any role in classical dynamics, since the support of the kernel $(n\cdot\partial)^{-1}(x_i - x_j)$ is restricted by the condition

$$\vec{x}_i(\tau) - \vec{x}_j(\tau) = \vec{n}s \,, \tag{4.17}$$

which contains three equations, but only two independent variables and is thus satisfied only for exceptional trajectories. At the points of these trajectories where Eq. (4.17) is satisfied, Eq. (4.13) should be solved by continuity, which makes it basically equivalent to the conventional equation for the Lorentz force given by the first row of Eq. (4.16). As it was mentioned before, the full quantum dynamics does not depend on the choice of $\vec{n}$, so that the appearance of the non-local $\vec{n}$-dependent term in Eq. (4.16) is a mere artifact of the classical approximation. For instance, in the path integral formulation, exceptional trajectories form a measure zero subset of all trajectories and thus do not contribute to physical amplitudes. The practical prescription which one can use for deriving the classical equations of motion is simple: in the resulting equations, one should omit any singular terms proportional to $(n\cdot\partial)^{-1}G$, where G is some function of the currents.

4.3 EFT approach to QEMD

Let us consider the QEMD Lagrangian (4.4) from the EFT perspective. In particular, we want to find all independent marginal operators respecting the gauge invariance of the theory and preserving the number of degrees of freedom of QEMD. Such operators can be constructed from the gauge invariant tensors and the vector n_μ. For now, we will not consider the operators containing the gauge currents j_e and j_m, which will be discussed in the next section. We find six classes of dimension four operators, each class containing operators of the form $\mathrm{tr}(X\cdot Y)$ and $(n\cdot X)(n\cdot Y)$, where X and Y can stand for any of the two tensors $\partial\wedge A$ and $\partial\wedge B$. From the identity (4.8), one can find the relation between the operators pertaining to the same class:

$$\mathrm{tr}(X\cdot Y) = \frac{2}{n^2}\Big[(n\cdot X^d)(n\cdot Y^d) - (n\cdot X)(n\cdot Y)\Big]\,. \tag{4.18}$$

Let us name the classes depending on the pair (X, Y):

$$\begin{aligned}
&x \quad \text{for} \quad (\partial\wedge A,\, \partial\wedge B)\,, \qquad y \quad \text{for} \quad (\partial\wedge A,\, [\partial\wedge B]^d\,)\,,\\
&\alpha \quad \text{for} \quad (\partial\wedge A,\, \partial\wedge A)\,, \qquad \beta \quad \text{for} \quad (\partial\wedge B,\, \partial\wedge B)\,,\\
&a \quad \text{for} \quad (\partial\wedge A,\, [\partial\wedge A]^d\,)\,, \qquad b \quad \text{for} \quad (\partial\wedge B,\, [\partial\wedge B]^d\,)\,.
\end{aligned}$$

The members of the same class are distinguished by indices:

$$\begin{aligned}
&x_1 \equiv \frac{2}{n^2}\,(n\cdot(\partial\wedge A))\,(n\cdot(\partial\wedge B))\,,\\
&x_2 \equiv \frac{2}{n^2}\,(n\cdot(\partial\wedge A)^d\,)\,(n\cdot(\partial\wedge B)^d\,)\,,\\
&x_+ \equiv x_1 + x_2 = \frac{2}{n^2}\Big\{(n\cdot(\partial\wedge A))\,(n\cdot(\partial\wedge B)) + (n\cdot(\partial\wedge A)^d\,)\,(n\cdot(\partial\wedge B)^d\,)\Big\}\,,\\
&x_- \equiv x_1 - x_2 = -\mathrm{tr}\,((\partial\wedge A)\,(\partial\wedge B))\,,
\end{aligned}$$

where we used Eq. (4.18); indices are assigned analogously for the operators in the other five classes. In each of the classes x, y, α or β, the basis is formed by any two members. The classes a and b each contain only one operator, since $a_1 = -a_2 = a_-/2$, $a_+ = 0$ and $b_1 = -b_2 = b_-/2$, $b_+ = 0$. Disregarding the source terms, there are thus 10 independent gauge-invariant dimension four operators in the Zwanziger theory, which we choose to be x_1, x_-, y_+, y_-, α_1, α_-, β_1, β_-, a_-, b_-. From these, only three enter the Lagrangian (4.4), the free part of which can be rewritten as follows:

$$\mathcal{L}_\gamma \;=\; -\frac{1}{4}\,(y_+ + \alpha_1 + \beta_1)\;. \tag{4.19}$$

Let us see which operators can be added to this Lagrangian without conflicting with the structure of the theory. The inclusion of the terms $x_- = -\mathrm{tr}\,((\partial\wedge A)\,(\partial\wedge B))$, $\alpha_- = -\mathrm{tr}\,((\partial\wedge A)\,(\partial\wedge A))$ and $\beta_- = -\mathrm{tr}\,((\partial\wedge B)\,(\partial\wedge B))$ is incompatible with the number of degrees of freedom in QEMD, since these operators give rise to second order time derivatives of the four-potentials A_μ or B_μ in the classical equations of motion. There is no such problem with the four remaining independent operators, three of which correspond to the total derivative terms in the Lagrangian:

$$a_- = -\mathrm{tr}\left\{(\partial\wedge A)\,(\partial\wedge A)^d\right\}\,, \tag{4.20}$$

$$b_- = -\mathrm{tr}\left\{(\partial\wedge B)\,(\partial\wedge B)^d\right\}\,, \tag{4.21}$$

$$y_- = -\mathrm{tr}\left\{(\partial\wedge A)\,(\partial\wedge B)^d\right\}\,, \tag{4.22}$$

and thus do not contribute to the equations of motion. The last operator from our basis is:

$$x_1 = \frac{2}{n^2}\,\left(n\cdot(\partial\wedge A)\right)\left(n\cdot(\partial\wedge B)\right)\,, \tag{4.23}$$

which does modify the equations of motion and should be added to the Zwanziger Lagrangian (4.4) in the EFT approach. Note that since the two four-potentials A_μ and B_μ have different parities[6], the operators a_-, b_- and x_1 are CP-odd, while the operator y_- is CP-even. This means that one can expect the operator x_1 to be responsible for CP-violation in QEMD. Let us proceed to the next section to see that x_1 is directly related to the Witten effect.

4.4 CP-violation in QEMD

Contrary to QED, the theory of QEMD has an intrinsic source of CP-violation. The reason is that the magnetic charge changes its sign under any of the discrete transformations C, P or T [59], so that a dyon with charges (q,g) is mapped into a dyon with charges $(-q,g)$ under a CP-transformation. The spectrum of charges is not CP-invariant if there exists a state (q,g) while its CP-conjugate state $(-q,g)$ is missing. In this case, it is impossible to define a CP transformation in such a way that the theory is invariant under it [60]. Note that due to the DSZ quantization condition,

$$q_i g_j - q_j g_i = 2\pi n, \quad n\in\mathbb{Z}, \tag{4.24}$$

and our choice for the gauge charges carried by the electron $(e,0)$, any magnetic charge must be quantized in the units of the minimal magnetic charge $g_0 = 2\pi/e$:

$$g_i = n_i^m g_0\,, \quad n_i^m \in \mathbb{Z}\,. \tag{4.25}$$

The case of electric charges is however different: what one can infer from the quantization condition (4.24) applied to dyons with charges (q_1,g_1) and (q_2,g_2) is that only the difference of some multiples of the electric charges of dyons is quantized: $n_2^m q_1 - n_1^m q_2 = ne$, $n\in\mathbb{Z}$. The latter condition leads to the quantization of the electric charges themselves only if $q_1 = -q_2$ and $g_1 = g_2$, i.e. only if the theory is CP-invariant. Thus, absolute values of the electric charges introduce a CP-violating parameter θ into the theory:

$$q_i = \left(n_i^e + \frac{\theta}{2\pi}n_i^m\right)\cdot e\,, \quad n_i^e\in\mathbb{Z}\,. \tag{4.26}$$

[6]Parities of A_μ and B_μ can be inferred for instance from Eqs. (4.14) and (4.15).

Since only the total value of the charge, and not any separate contribution, is physical, the parameter θ introduced in this way is defined on the unit circle $\theta \in [0, 2\pi)$. The additional contribution to the electric charge which is proportional to θ is in perfect consistency with Eq. (3.8) derived from the Witten effect, which means that in the particular case of 't Hooft-Polyakov monopoles the parameter θ is the vacuum angle of the full non-Abelian theory.

Let us now find the connection between the CP-violation in QEMD discussed in the previous paragraph and the CP-violating operator x_1 introduced in the previous section. We will show that it is possible to remove θ from the definition of charges (4.26) at the cost of adding the operator x_1 with an appropriate coefficient to the kinetic part of the Lagrangian as well as modifying the coefficient in front of the $[n\cdot(\partial\wedge A)]^2$ term. First, we redefine the electric current $j_e \to \bar{j}_e$ so that it contains only the contribution proportional to $n_i^e\, e$. The QEMD Lagrangian becomes:

$$\begin{aligned}\mathcal{L} \;=\; \frac{1}{2n^2}\Big\{&[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d] \;-\; [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] \;-\\ &[n\cdot(\partial\wedge A)]^2 \;-\; [n\cdot(\partial\wedge B)]^2\Big\} \;-\; \left(\bar{j}_e + \frac{e^2\theta}{4\pi^2}\,j_m\right)\cdot A \;-\; j_m\cdot B\,. \end{aligned} \tag{4.27}$$

Next, we make the following $SL(2,\mathbb{R})$ transformation in the space of four-potentials:

$$\begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ -\dfrac{e^2\theta}{4\pi^2} & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{4.28}$$

The first row of the Lagrangian (4.27), which corresponds to the operator y_+ from the previous section, is not affected by this transformation. The second row is transformed yielding the conventional source terms and an extra x_1 term as promised:

$$\begin{aligned}\mathcal{L} \;=\; \frac{1}{2n^2}\Big\{&[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d] \;-\; [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] \;-\\ &\left(1+\frac{e^4\theta^2}{16\pi^4}\right)[n\cdot(\partial\wedge A)]^2 \;-\; [n\cdot(\partial\wedge B)]^2 + \frac{e^2\theta}{2\pi^2}\,(n\cdot(\partial\wedge A))\,(n\cdot(\partial\wedge B))\Big\} \;-\\ &\bar{j}_e\cdot A \;-\; j_m\cdot B\,, \end{aligned} \tag{4.29}$$

which can be rewritten more compactly in our operator notation:

$$\mathcal{L} \;=\; -\frac{1}{4}\left(y_+ + \left(1+\frac{e^4\theta^2}{16\pi^4}\right)\alpha_1 + \beta_1 - \frac{e^2\theta}{2\pi^2}\,x_1\right) - \bar{j}_e\cdot A \;-\; j_m\cdot B\,. \tag{4.30}$$

Several important comments are in order. First, note that the periodicity of θ is no longer explicit in the Lagrangian (4.30). In fact, to see the symmetry under $\theta \to \theta + 2\pi$ transformation, we have to account for the implicit dependence of the four-potential B_μ on θ arising from the transformation (4.28). The term

$$\begin{aligned}\frac{1}{4}\cdot\frac{e^2\theta}{2\pi^2}\,x_1 \;&=\; \frac{e^2\theta}{4\pi^2 n^2}\,(n\cdot(\partial\wedge A))\,(n\cdot(\partial\wedge B))\\ &=\frac{e^2\theta}{4\pi^2 n^2}\,(n\cdot F)\,(n\cdot F^d) \;=\; -\frac{e^2\theta}{16\pi^2}\mathrm{tr}\left(FF^d\right)\,, \end{aligned} \tag{4.31}$$

is similar to the conventional QED θ-term (3.5), but is by no means symmetric under the transformation $\theta \to \theta + 2\pi$ by itself. We see that in the theory where magnetic currents are properly included in the Lagrangian of the theory, not only does the term (3.5) lose its total derivative structure, but it is also no longer topological.

The second comment which we would like to make is about Lorentz-invariance of QEMD with CP-violation. Although the Lagrangian (4.30) contains an extra term with n_μ-dependence, added to the Zwanziger Lagrangian (4.4), and a change in the coefficient in front of the α_1 term, it is clear that the theory is Lorentz-invariant, since one can get rid of the unusual n_μ-dependence by performing a $SL(2, \mathbb{R})$ transformation of the potentials. Since it is always possible to get rid of the x_1 term in this way, we see that the three operators y_+, α_1 and β_1 entering the Zwanziger Lagrangian (4.19) are indeed the only independent gauge-invariant four-dimensional operators which are relevant for the kinetic part of QEMD. The possible CP-violation is most elegantly accounted for in the expression (4.27) for the QEMD Lagrangian, since in this form the periodicity of the θ-parameter is made obvious. The latter form of the QEMD Lagrangian is also convenient for finding the extension of QEMD which incorporates axions – the endeavor we accomplish in sec. 5.

4.5 QEMD of 't Hooft-Polyakov monopoles

Let us now consider the QEMD of 't Hooft-Polyakov monopoles. After introducing the CP-violating parameter θ into the Zwanziger theory, we identified it with the instanton angle of the UV non-Abelian theory through the Witten effect. Still, the modified Zwanziger Lagrangian (4.27) misses some of the effects associated with the 't Hooft-Polyakov monopoles, since, as we discussed in sec. 2, the latter monopoles cannot be modeled by simple point-like magnetic field sources even in the IR.

Consider instanton effects of the UV non-Abelian theory. At low energies, in the symmetry-broken phase, they are known to be suppressed everywhere, but on 't Hooft-Polyakov monopoles [44]. As a result, some of the good symmetries of the low energy EFT can be violated by unsuppressed instanton-induced effects on the monopole. The most famous example is the Rubakov-Callan effect [44, 46]: the decay of a proton catalyzed by a monopole. A consistent QEMD of 't Hooft-Polyakov monopoles has to account for such instanton effects. To satisfy this requirement, we introduce an extra degree of freedom $\phi\,(x^\mu)$ into QEMD, which interacts with the electric current j_e^μ via the following Lagrangian: $\mathcal{L} = (j_e \cdot \partial)\,\phi$. The field ϕ does not contribute to the classical equations of motion, as it should be for a variable describing instanton effects. As the latter effects are localized on the monopole, we require that the interaction Hamiltonian $\mathcal{H} = -\left(\mathbf{j}_e \cdot \boldsymbol{\nabla}\right)\phi$ vanishes outside the monopole core, so that $\boldsymbol{\nabla}\phi$ is zero everywhere but on the monopole. The latter localization property also means that in our low energy EFT, only s-wave fermions can interact with ϕ, because wave-functions of scalars and higher partial wave fermions vanish on the monopole due to the centrifugal barrier [61, 62] [7].

Let us now show that the interaction Hamiltonian $\mathcal{H}$ introduced in the previous paragraph can provide a valid description for the Rubakov-Callan effect. For this, we take

[7] In this section, we assume the magnetic monopole to be a scalar particle, since to our knowledge this is the only case which has been studied in the literature on the Rubakov-Callan effect.

advantage of the work by Joseph Polchinski [45]. In the latter work, the author showed that the Rubakov-Callan effect can be described as an interaction between s-wave fermions and a rotor coordinate $\alpha(t)$. One can show that this description is equivalent to ours as long as one identifies $\alpha(t)$ with the temporal dependence of $e\phi$. For the sake of comparison, consider the case of a left-handed Weyl fermion χ interacting with an $SU(2)$ monopole. The only part of the electric current contributing to $\mathcal{H}$ is associated to s-wave fermions $j_e^i = e\,\bar{\chi}^k_{(s)}\sigma^i\chi^k_{(s)}/\,2$, where k is a flavor index, so that the theory can be reduced to (1+1) dimensions:

$$H = -\int d^3x\,(\mathbf{j}_e\cdot\boldsymbol{\nabla})\,\phi = -\frac{e}{2}\int\limits_0^{+\infty} dr\left(\xi_+^\dagger\xi_+ - \xi_-^\dagger\xi_-\right)\partial_r\phi = \int\limits_{-\infty}^{+\infty} dr\,\psi_k^\dagger \alpha q'(r)\psi_k\,, \tag{4.32}$$

where, following the notations of Ref. [45], we define $\xi_\pm$ spinors as charge eigenstates, $\psi_k(\pm r) \equiv \xi_\pm^{(k)}(r)$; $q(r\,|\,r < -r_0) = 1/2$ and $q(r\,|\,r > r_0) = -1/2$; r_0 is the size of the monopole core; we omitted the terms which are suppressed by the high energy scale $1/r_0$. One sees that the interaction Hamiltonian (4.32) is equivalent to the one used in Ref. [45] [8]. Thus, to account for the Rubakov-Callan effect, the source term of the QEMD Lagrangian has to be modified as follows: $-j_e\cdot A \;\;\rightarrow\;\; -j_e\cdot(A - \partial\phi)$. In the case of non-zero θ, one obtains:

$$\mathcal{L}\;\supset\;-\left(\bar{j}_e + \frac{e^2\theta}{4\pi^2}\,j_m\right)\cdot(A - \partial\phi)\;. \tag{4.33}$$

The term $e^2\theta\,(j_m\cdot\partial)\,\phi/4\pi^2$ corresponds to the θ-term in the worldline action for the collective coordinate $e\phi$:

$$S_{[e\phi]} \supset \sum_i \int\limits_{\gamma_i} \frac{\theta}{2\pi}\,d(e\phi)\;, \tag{4.34}$$

where γ_i are the monopole worldlines.

There is yet another way to understand why in the case of 't Hooft-Polyakov monopoles, the θ-term of the QEMD Lagrangian (4.27) has to be modified. In particular, consider the case of a varying θ. It is clear that in the full non-Abelian theory the coupling of the new pseudoscalar field θ to GG^d is legitimate. However, varying θ in the Lagrangian (4.27) would be inconsistent with electric charge conservation: the gauge invariance of the theory would require $\partial_\mu\theta = 0$. A way to resolve this paradox is to introduce a Stückelberg field ϕ localized on the monopole, so that $j_m\cdot(A - \partial\phi)$ is gauge invariant [63], which leads us again to the coupling (4.33). Note that the dependence of the vacuum energy on θ, which was calculated in Ref. [45] using the low energy theory with the interaction Hamiltonian (4.32), agrees with the high energy non-Abelian theory result obtained in dilute instanton gas approximation $V(\theta) \propto -\cos\theta$. A similar dependence was found in Ref. [64] where the authors took advantage of the worldline action (4.34) and computed the self-energy of ϕ.

[8] The other two terms in the Hamiltonian of the model of Ref. [45] containing only the collective coordinate α and its canonical momentum Π correspond to the potential and kinetic energy terms for ϕ, respectively.

5 Generic low energy axion-photon EFT

5.1 Anomalous axion-photon interactions

Let us now find the extension of QEMD which incorporates axions. We first limit ourselves to the CP-conserving axion interactions, which means that the dimension five operators containing the axion field are obtained from the CP-odd dimension four operators of QEMD: a_-, b_- and x_1. Axion EFT must be symmetric under the transformation $a \to a+2\pi v_a n$, $n \in \mathbb{Z}$, which suggests that we use the operator $j_m A$ instead of x_1, since ax_1 would not have the discrete shift symmetry required, as outlined in sec. 4.4. The operator $j_m A$ corresponds to the Witten-effect induced axion interaction and we postpone its discussion until the next section, limiting ourselves to the pure axion-photon couplings first. Thus, the Lagrangian for a generic CP-conserving axion-photon EFT is:

$$\mathcal{L} = \frac{1}{2n^2}\Big\{[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d] - [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] - [n\cdot(\partial\wedge A)]^2 - [n\cdot(\partial\wedge B)]^2\Big\} - \frac{1}{4}\,g_{a\mathrm{AA}}\,a\,\mathrm{tr}\Big\{(\partial\wedge A)\,(\partial\wedge A)^d\Big\} - \frac{1}{4}\,g_{a\mathrm{BB}}\,a\,\mathrm{tr}\Big\{(\partial\wedge B)\,(\partial\wedge B)^d\Big\}\,, \quad (5.1)$$

or written in a more compact operator notation:

$$\mathcal{L} = -\frac{1}{4}\,(y_+ + \alpha_1 + \beta_1) + \frac{1}{4}\,g_{a\mathrm{AA}}\,a\,a_- + \frac{1}{4}\,g_{a\mathrm{BB}}\,a\,b_-\,. \quad (5.2)$$

The coefficients $g_{a\mathrm{AA}}$ and $g_{a\mathrm{BB}}$ cannot be determined by symmetry arguments, since both a_- and b_- terms are total derivatives, which ensures shift symmetry regardless of their coefficients. However, to compute the coefficients, we can take advantage of the fact that these terms arise from the anomalous divergence of the Peccei-Quinn current, so that $g_{a\mathrm{AA}}$ and $g_{a\mathrm{BB}}$ are determined by the $U(1)_{\mathrm{PQ}}\,(U(1)_{\mathrm{E}})^2$ and $U(1)_{\mathrm{PQ}}\,(U(1)_{\mathrm{M}})^2$ anomalies, respectively[9]:

$$g_{a\mathrm{AA}} = \frac{Ee^2}{4\pi^2 v_a}\,, \quad E = \sum_\psi q_\psi^2\cdot d(C_\psi)\,, \quad (5.3)$$

$$g_{a\mathrm{BB}} = \frac{Mg_0^2}{4\pi^2 v_a}\,, \quad M = \sum_\psi g_\psi^2\cdot d(C_\psi)\,, \quad (5.4)$$

where E and M are electric and magnetic anomaly coefficients, respectively; q_ψ and g_ψ are electric and magnetic charges of heavy PQ-charged fermions ψ in units of e and g_0, respectively; $d(C_\psi)$ is the dimension of the color representation of ψ. Due to the DSZ quantization condition, $g_0 \gg e$ so that the Wilson coefficient $g_{a\mathrm{BB}}$ is expected to dominate the axion-photon coupling.

Let us now consider the CP-violating axion interactions. We have not yet taken advantage of the CP-even four-dimensional operator y_-, which can be coupled to the axion since the resulting CP-odd five-dimensional operator ay_- respects the axion shift symmetry. The corresponding term in the Lagrangian is:

$$\mathcal{L}_{\not{CP}} \supset -\frac{1}{2}\,g_{a\mathrm{AB}}\,a\,\mathrm{tr}\Big\{(\partial\wedge A)\,(\partial\wedge B)^d\Big\}\,, \quad (5.5)$$

[9] For the detailed derivation, see Appendix A.

where the coefficient $g_{a\mathrm{AB}}$ is determined by the $U(1)_{\mathrm{PQ}}\,U(1)_{\mathrm{E}}\,U(1)_{\mathrm{M}}$ anomaly. Note that the latter anomaly is non-zero only in the case where the spectrum of dyons violates CP. In this case, the intrinsic CP-violation of high energy QEMD is transferred to the low energy axion-photon EFT after integrating out heavy dyons. As we show in Appendix A, the coefficient $g_{a\mathrm{AB}}$ is given by the following expression:

$$g_{a\mathrm{AB}} = \frac{Deg_0}{4\pi^2 v_a}\,, \quad D = \sum_{\psi} q_\psi g_\psi \cdot d(C_\psi)\ , \tag{5.6}$$

where D is the mixed electric-magnetic CP-violating anomaly coefficient, which depends on the spectrum of heavy PQ-charged dyons. The DSZ quantization condition ensures $g_0 \gg e$, so that the CP-violating axion-photon coupling $g_{a\mathrm{AB}}$ is naturally suppressed compared to the CP-conserving $g_{a\mathrm{BB}}$ coupling, but dominates over the CP-conserving $g_{a\mathrm{AA}}$ coupling: $g_{a\mathrm{BB}} \gg |g_{a\mathrm{AB}}| \gg g_{a\mathrm{AA}}$.

Not only do the values of the anomaly coefficients E, M and D depend on the details of the UV model, but also the value of the minimal magnetic charge g_0 does. While we used $g_0 = 2\pi/e$ for pure QEMD in sec. 4.4, the real low energy theory describing nature involves also the $SU(3)_c$ color gauge group, and the quarks charged under this group have minimal electric charge $|e_0| = e/3$. Naively, this implies that the minimal magnetic charge is $g_0 = 2\pi/|e_0| = 6\pi/e$. However, this is true only if the magnetic monopoles are Abelian, i.e. if they do not carry color magnetic charge. If the monopoles are to the contrary non-Abelian, i.e. if they carry also color magnetic charge[10], the DSZ quantization condition generalizes to include such extra magnetic charges [68–70] and allows for a minimal $U(1)_{\mathrm{M}}$ magnetic charge similar to the one of pure QEMD: $g_0 = 2\pi/e$. In Ref. [18], we built an axion model with heavy PQ-charged fermions ψ_i carrying $SU(3)_{\mathrm{M}}$ color magnetic charges and showed that it indeed solves the strong CP problem. In the explicit calculations of the next sections, we will parameterize the minimal magnetic charge g_0 by an integer ζ:

$$g_0 = \frac{2\pi\zeta}{e}\,, \quad \zeta = \begin{cases} 3\,, & \psi_i \ \in \ U(1)_{\mathrm{E}}\times U(1)_{\mathrm{M}}\times SU(3)_{\mathrm{E}} \\ 1\,, & \psi_i \ \in \ U(1)_{\mathrm{E}}\times U(1)_{\mathrm{M}}\times SU(3)_{\mathrm{M}} \end{cases}\ . \tag{5.7}$$

As we derived the axion-photon couplings (5.1) and (5.5) from general symmetry arguments, the field a entering our EFT need not be the QCD axion, but could as well correspond to a generic ALP. In this case, Eqs. (5.3), (5.4) and (5.6) need not hold. Nevertheless, the scaling of the corresponding ALP-photon couplings with electric and magnetic elementary charges e and g_0 given in Eqs. (5.3), (5.4) and (5.6) persists for any ALP, because with our normalisation of A_μ and B_μ four-potentials, the former four-potential always enters the interaction Lagrangian with a factor of e while the latter one always enters the interaction Lagrangian with a factor of g_0. This means that for a generic ALP, one still expects the above-mentioned hierarchy of couplings: $g_{a\mathrm{BB}} \gg |g_{a\mathrm{AB}}| \gg g_{a\mathrm{AA}}$.

[10] Note that contrary to the Abelian case, there are no non-Abelian dyons, i.e. particles which carry both color electric and color magnetic charges [65–67].

5.2 Witten-effect induced axion interaction

Let us return to the discussion of the CP-conserving $\mathcal{O} = a\,(j_m \cdot A)$ operator of a generic axion EFT. The coefficient in front of this operator is determined by the discrete shift symmetry requirement. The corresponding term in the Lagrangian is obtained by the substitution $\theta \to a/v_a$ in Eq. (4.27):

$$\mathcal{L} \supset -\left(\bar{j}_e + \frac{e^2 a}{4\pi^2 v_a}\, j_m\right) \cdot A\,. \tag{5.8}$$

Indeed, the results on the periodicity of θ obtained in sec. 4.4 show that the axion field has the required discrete shift symmetry $a \to a + 2\pi v_a n$, $n \in \mathbb{Z}$. Note also that, as we discussed in sec. 4.4 for the analogous case of the θ-parameter, the latter symmetry would no longer be explicit if we were to redefine the fields and move the axion dependence into the kinetic part of the Lagrangian.

The term (5.8) is not gauge invariant unless $\partial_\mu a = 0$, which tells us that our axion EFT has to be modified. A way to restore the gauge invariance is to introduce a Stückelberg field ϕ into the Lagrangian:

$$\mathcal{L} \supset -\left(\bar{j}_e + \frac{e^2 a}{4\pi^2 v_a}\, j_m\right) \cdot (A - \partial\phi)\ . \tag{5.9}$$

As we discussed in sec. 4.5, such an extra degree of freedom ϕ arises naturally while considering the case of 't Hooft-Polyakov monopoles, where it plays the role of the dyon collective coordinate. Let us then consider the case where the interaction Lagrangian (5.9) corresponds to the low energy phase of a non-Abelian gauge theory. Comparing Eq. (5.9) with the θ-term (4.33) of the low energy phase, we see that the CP-violating θ-parameter of a non-Abelian theory is simply substituted with the axion field $\theta \to a/v_a$, so that the $a\,(j_m \cdot A)$ operator corresponds to the aGG^d operator at high energies. This means that the coupling (5.9) describes Witten-effect induced axion interactions. Indeed, as we discussed in sec. 3.3, one expects the Witten-effect kind of coupling between axions and monopoles whenever one considers the spontaneously broken symmetry phase of a high energy non-Abelian theory with aGG^d term.

Contrary to the three anomalous axion couplings described in the previous section, the coupling (5.9) does not respect the continuous shift symmetry $a \to a + C$, where C is an arbitrary constant. This means that the latter coupling generates a non-flat contribution to the potential for the axion field. Since the axion coupling (5.9) arises in the low energy EFT of a non-Abelian theory with aGG^d interaction, such a contribution to the axion potential is not unexpected: in fact, it has to correspond to the potential created by instantons of the high energy non-Abelian theory. As it was discussed in the end of sec. 4.5, explicit calculations [45, 64] support the latter correspondence. Note, however, that contrary to the claim made in Ref. [64], additional contribution to the axion potential need not arise in *every* theory of an axion coupled to an Abelian gauge field whenever there are monopoles magnetically charged under the latter field. The axion mass is generated *not* by magnetic monopoles, but always by instantons, even if these instantons happen to live on the monopole worldvolume in the low energy EFT. Indeed, consider the simplest example of a QEMD theory (4.4) which has no extra degrees of freedom. In such a theory, there cannot exist a consistent Witten-effect induced axion coupling, although there can exist the anomalous axion-photon couplings discussed in the previous section. Thus, such a theory has both axions and magnetic monopoles interacting with the Abelian gauge field, but no axion mass is generated through these interactions, which contradicts the statement of Ref. [64]. In general, we see that the anomalous axion-photon couplings and the Witten-effect induced axion coupling are independent. The Witten-effect induced coupling arises only in theories which have instanton degrees of freedom, e.g. in the spontaneously broken symmetry phase of a high energy non-Abelian gauge theory.

5.3 Axion Maxwell equations

Having analyzed different axion-photon interactions in the previous two sections, we are now ready to collect them all together in a generic axion-photon EFT Lagrangian:

$$\begin{aligned}\mathcal{L} \;=\; &\frac{1}{2n^2}\Big\{[n\cdot(\partial\wedge B)]\cdot[n\cdot(\partial\wedge A)^d] \;-\; [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] \;-\; [n\cdot(\partial\wedge A)]^2 \;-\\ &[n\cdot(\partial\wedge B)]^2\Big\} \;-\; \frac{1}{4}\,g_{a\mathrm{AA}}\,a\,\mathrm{tr}\Big\{(\partial\wedge A)\,(\partial\wedge A)^d\Big\} \;-\; \frac{1}{4}\,g_{a\mathrm{BB}}\,a\,\mathrm{tr}\Big\{(\partial\wedge B)\,(\partial\wedge B)^d\Big\} \;-\\ &\frac{1}{2}\,g_{a\mathrm{AB}}\,a\,\mathrm{tr}\Big\{(\partial\wedge A)\,(\partial\wedge B)^d\Big\} - \left(\bar{j}_e + \frac{e^2 a}{4\pi^2 v_a}\,j_m^{\phi}\right)\cdot(A-\partial\phi) - j_m\cdot B + \mathcal{L}_G\;,\end{aligned} \tag{5.10}$$

or written in a more compact operator notation:

$$\begin{aligned}\mathcal{L} \;=\; &-\frac{1}{4}\Big(y_+ + \alpha_1 + \beta_1 - g_{a\mathrm{AA}}\,a\,a_- - g_{a\mathrm{BB}}\,a\,b_- - 2\,g_{a\mathrm{AB}}\,a\,y_-\Big) \;-\\ &\qquad\left(\bar{j}_e + \frac{e^2 a}{4\pi^2 v_a}\,j_m^{\phi}\right)\cdot(A-\partial\phi) - j_m\cdot B + \mathcal{L}_G\;,\end{aligned} \tag{5.11}$$

where we denoted the part of the magnetic current j_m carrying an instanton degree of freedom ϕ by j_m^{ϕ}. For instance, j_m^{ϕ} can correspond to a current of 't Hooft-Polyakov monopoles. Let us remind the reader that $\mathcal{L}_G$ is the gauge-fixing Lagrangian given by Eq. (4.5), $\bar{j}_e$ is the part of the electric current which is quantized in units of elementary electric charge and y_+, α_1, β_1, a_-, b_-, y_- are the QEMD operators defined in sec. 4.3. Note that since we derived the Lagrangian (5.11) from general symmetry arguments, the field a entering our EFT need not be the QCD axion, but could as well correspond to a generic ALP.

Let us derive the classical equations of motion corresponding to the Lagrangian (5.11). For this, we follow the standard procedure outlined in sec. 4.2. Varying over the two four-potentials, we obtain:

$$\begin{aligned}&\frac{n\cdot\partial}{n^2}\left(n\cdot\partial A^{\mu} \;-\; \partial^{\mu}n\cdot A \;-\; n^{\mu}\partial\cdot A \;-\; \epsilon^{\mu}{}_{\nu\rho\sigma}n^{\nu}\partial^{\rho}B^{\sigma}\right)\\ &\qquad -g_{a\mathrm{AA}}\,\partial_{\nu}a\,\Big\{(\partial\wedge A)^d\Big\}^{\nu\mu} - g_{a\mathrm{AB}}\,\partial_{\nu}a\,\Big\{(\partial\wedge B)^d\Big\}^{\nu\mu} \;-\; \frac{e^2 a}{4\pi^2 v_a}\,j_m^{\phi\,\mu} \;=\; \bar{j}_e^{\,\mu}\;,\end{aligned} \tag{5.12}$$

$$\begin{aligned}&\frac{n\cdot\partial}{n^2}\left(n\cdot\partial B^{\mu} \;-\; \partial^{\mu}n\cdot B \;-\; n^{\mu}\partial\cdot B \;-\; \epsilon^{\mu}{}_{\nu\rho\sigma}n^{\nu}\partial^{\rho}A^{\sigma}\right)\\ &\qquad\qquad -g_{a\mathrm{BB}}\,\partial_{\nu}a\,\Big\{(\partial\wedge B)^d\Big\}^{\nu\mu} - g_{a\mathrm{AB}}\,\partial_{\nu}a\,\Big\{(\partial\wedge A)^d\Big\}^{\nu\mu} \;=\; j_m^{\mu}\;.\end{aligned} \tag{5.13}$$

Then, transitioning to the description in terms of the field strength tensor F, we find the following axion Maxwell equations:

$$\partial_\mu F^{\mu\nu} - g_{a\mathrm{AA}}\,\partial_\mu a\, F^{d\,\mu\nu} + g_{a\mathrm{AB}}\,\partial_\mu a\, F^{\mu\nu} - \frac{e^2 a}{4\pi^2 v_a}\, j_m^{\phi\,\nu} \;=\; \bar{j}_e^{\,\nu}\,, \tag{5.14}$$

$$\partial_\mu F^{d\,\mu\nu} + g_{a\mathrm{BB}}\,\partial_\mu a\, F^{\mu\nu} - g_{a\mathrm{AB}}\,\partial_\mu a\, F^{d\,\mu\nu} \;=\; j_m^{\nu}\,. \tag{5.15}$$

Note that the terms proportional to $(n\cdot\partial)^{-1}\,(n\wedge j_m)^{\mu\nu}$ and $(n\cdot\partial)^{-1}\,(n\wedge j_e)^{\mu\nu}$ do not contribute to the classical equations of motion, as it was discussed in sec. 4.2 [11]. Eqs. (5.14) and (5.15) are to be supplemented by the following equation of motion for the axion field:

$$\left(\partial^2 - m_a^2\right) a \;=\; -\frac{1}{4}\,\left(g_{a\mathrm{AA}} + g_{a\mathrm{BB}}\right) F_{\mu\nu} F^{d\,\mu\nu} - \frac{1}{2}\, g_{a\mathrm{AB}} F_{\mu\nu} F^{\mu\nu}, \tag{5.16}$$

where the right-hand side is obtained by varying the Lagrangian (5.11) with respect to the axion field and transitioning to the description in terms of the field strength tensor F. According to the discussion of the previous section, the axion mass m_a receives an additional contribution from the Witten-effect induced interaction in the case $j_m^\phi \neq 0$.

Let us now bring Eqs. (5.14), (5.15) and (5.16) into the form convenient for their experimental study. First, we set $j_m = j_m^\phi = 0$, since there are no magnetic monopoles in the laboratory. Second, we expand the electromagnetic field in powers of the anomalous axion-photon couplings $g_{a\mathrm{AA}}$, $g_{a\mathrm{BB}}$ and $g_{a\mathrm{AB}}$, keeping only the zeroth and the first orders $F = F_0 + F_a$. At zeroth order, Eqs. (5.14), (5.15) and (5.16) decouple and give the ordinary Maxwell equations as well as the homogeneous Klein-Gordon equation for the axion field. At first order, Eqs. (5.14), (5.15) and (5.16) yield:

$$\partial_\mu F_a^{\mu\nu} - g_{a\mathrm{AA}}\,\partial_\mu a\, F_0^{d\,\mu\nu} + g_{a\mathrm{AB}}\,\partial_\mu a\, F_0^{\mu\nu} \;=\; 0\,, \tag{5.17}$$

$$\partial_\mu F_a^{d\,\mu\nu} + g_{a\mathrm{BB}}\,\partial_\mu a\, F_0^{\mu\nu} - g_{a\mathrm{AB}}\,\partial_\mu a\, F_0^{d\,\mu\nu} \;=\; 0\,, \tag{5.18}$$

$$\left(\partial^2 - m_a^2\right) a \;=\; -\frac{1}{4}\,\left(g_{a\mathrm{AA}} + g_{a\mathrm{BB}}\right) F_{0\,\mu\nu} F_0^{d\,\mu\nu} - \frac{1}{2}\, g_{a\mathrm{AB}} F_{0\,\mu\nu} F_0^{\mu\nu}\,, \tag{5.19}$$

so that F_a is an axion-induced part of the electromagnetic field sourced by the following effective electric and magnetic currents:

$$j_{e,\,\mathrm{eff}}^{\nu} = g_{a\mathrm{AA}}\,\partial_\mu a\, F_0^{d\,\mu\nu} - g_{a\mathrm{AB}}\,\partial_\mu a\, F_0^{\mu\nu}\,, \tag{5.20}$$

$$j_{m,\,\mathrm{eff}}^{\nu} = -g_{a\mathrm{BB}}\,\partial_\mu a\, F_0^{\mu\nu} + g_{a\mathrm{AB}}\,\partial_\mu a\, F_0^{d\,\mu\nu}\,, \tag{5.21}$$

which depend on the external field F_0 created in the laboratory. Eqs. (5.20) and (5.21) extend the results of the axion EFT of Ref. [73], which yields $j_{e,\,\mathrm{eff}}^{\nu} = g_{a\mathrm{AA}}\,\partial_\mu a\, F_0^{d\,\mu\nu}$ and $j_{m,\,\mathrm{eff}}^{\nu} = 0$. As we discussed in sec. 5.1, in an axion model with a generic spectrum of heavy

[11] That such terms cannot contribute to the classical equations of motion is also clear from the fact that the interaction of axions with electromagnetic field cannot depend on the kind of currents sourcing the latter field: for instance, in a given setting the axion field could be causally disconnected from these currents. In fact, one can obtain the axion Maxwell equations (5.17), (5.18) and (5.19) by using an even simpler two-potential framework of Ref. [71] which does not involve currents at all. The latter framework is used in Ref. [72] for deriving the Maxwell equations in General Relativity.

PQ-charged dyons the couplings satisfy $g_{a\mathrm{BB}} \gg |g_{a\mathrm{AB}}| \gg g_{a\mathrm{AA}}$, so that the additional terms we obtain dominate the conventional contribution to the effective currents.

Finally, in terms of electric and magnetic fields, Eqs. (5.17), (5.18) and (5.19) are given by the following expressions:

$$\boldsymbol{\nabla}\times\mathbf{B}_a - \dot{\mathbf{E}}_a = g_{a\mathrm{AA}}\left(\mathbf{E}_0\times\boldsymbol{\nabla}a - \dot{a}\mathbf{B}_0\right) + g_{a\mathrm{AB}}\left(\mathbf{B}_0\times\boldsymbol{\nabla}a + \dot{a}\mathbf{E}_0\right), \tag{5.22}$$

$$\boldsymbol{\nabla}\times\mathbf{E}_a + \dot{\mathbf{B}}_a = -g_{a\mathrm{BB}}\left(\mathbf{B}_0\times\boldsymbol{\nabla}a + \dot{a}\mathbf{E}_0\right) - g_{a\mathrm{AB}}\left(\mathbf{E}_0\times\boldsymbol{\nabla}a - \dot{a}\mathbf{B}_0\right), \tag{5.23}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}_a = -g_{a\mathrm{BB}}\,\mathbf{E}_0\cdot\boldsymbol{\nabla}a + g_{a\mathrm{AB}}\,\mathbf{B}_0\cdot\boldsymbol{\nabla}a\,, \tag{5.24}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}_a = g_{a\mathrm{AA}}\,\mathbf{B}_0\cdot\boldsymbol{\nabla}a - g_{a\mathrm{AB}}\,\mathbf{E}_0\cdot\boldsymbol{\nabla}a\,, \tag{5.25}$$

$$\left(\partial^2 - m_a^2\right)a = \left(g_{a\mathrm{AA}} + g_{a\mathrm{BB}}\right)\mathbf{E}_0\cdot\mathbf{B}_0 + g_{a\mathrm{AB}}\left(\mathbf{E}_0^2 - \mathbf{B}_0^2\right), \tag{5.26}$$

where $\mathbf{E}_a$ and $\mathbf{B}_a$ are axion-induced electric and magnetic fields, while $\mathbf{E}_0$ and $\mathbf{B}_0$ are background electric and magnetic fields created in the detector. Note that to study the propagation of light with Eqs. (5.22)–(5.26), it is convenient not to perform the expansion of the electromagnetic fields $\mathbf{E}_\gamma$ and $\mathbf{B}_\gamma$, in which case the equations are the same, but with the substitutions $\mathbf{E}_a$, $\mathbf{E}_0 \to \mathbf{E}_\gamma$ and $\mathbf{B}_a$, $\mathbf{B}_0 \to \mathbf{B}_\gamma$.

6 Targets for axion search experiments

6.1 General lessons

Let us discuss the phenomenology of the new electromagnetic couplings of axions and ALPs found in the previous sections. In our discussion, we will always consider first the general case of ALPs, i.e. Nambu-Goldstone bosons of an arbitrary spontaneously broken global $U(1)$ symmetry, which by definition include QCD axions as a special case, and only then make quantitative predictions in particular axion models.

Due to the scaling of the new $g_{a\mathrm{BB}}$ and $g_{a\mathrm{AB}}$ couplings with the elementary electric and magnetic charge units found in sec. 5.1, in any model where $g_{a\mathrm{AB}} \neq 0$, one expects the ratio $g_{a\mathrm{BB}}/|g_{a\mathrm{AB}}|$ to be proportional to a large number $g_0/e \gg 1$. This means that the possible effects associated to the $g_{a\mathrm{AB}}$ coupling play the dominant role only for those observables, which do not get any sizable contribution from the $g_{a\mathrm{BB}}$ coupling. As we will discuss in the next sections, such observables do exist and can be probed in various experiments by studying the interactions of ALPs with polarized light, searching for EDMs of charged particles and a fifth force or by looking for light ALP dark matter in an external magnetic field with haloscopes.

Still, for most of the processes involving ALP-photon interactions, the dominant effect is associated to the $g_{a\mathrm{BB}}$ coupling. Symmetry of Eq. (5.26) with respect to the interchange of the $g_{a\mathrm{AA}}$ and $g_{a\mathrm{BB}}$ couplings suggests that in any process of creation or disappearance of a certain number of ALPs, the effect of the $g_{a\mathrm{BB}}$ coupling is analogous to the effect of the conventional $g_{a\gamma\gamma}$ coupling [12]. This means that the rates of such processes as ALP-photon conversion in external electromagnetic fields [74], ALP decay, ALP emission through

[12] This statement need not hold for loop effects, as Eq. (5.26) is classical.

Primakoff effect [75] or photon coalescence, are all given by the conventional expressions, but with the $g_{a\gamma\gamma}$ coupling substituted by the $g_{a\mathrm{BB}}$ one.

The same simple rule of replacing the $g_{a\gamma\gamma}$ coupling with the $g_{a\mathrm{BB}}$ coupling in conventional expressions applies to the dispersion relation of light in an ALP background. Indeed, after we omit the subdominant $|g_{a\mathrm{AB}}| \ll g_{a\mathrm{BB}}$ coupling and put $\mathbf{E}_a$, $\mathbf{E}_0 \rightarrow \mathbf{E}_\gamma$ and $\mathbf{B}_a$, $\mathbf{B}_0 \rightarrow \mathbf{B}_\gamma$, the axion Maxwell equations (5.22)–(5.25) become invariant under the interchange of the couplings $g_{a\mathrm{AA}}$ and $g_{a\mathrm{BB}}$ supplemented by the electric-magnetic duality transformation $\mathbf{E}_\gamma \rightarrow \mathbf{B}_\gamma$, $\mathbf{B}_\gamma \rightarrow -\mathbf{E}_\gamma$. Since the propagation of light is electric-magnetic duality invariant, the $g_{a\mathrm{BB}}$ coupling enters the dispersion relation in the same way as the conventional $g_{a\mathrm{AA}}$ coupling. It can also be explicitly checked that the form of the second-order differential equations for $\mathbf{E}_\gamma$ and $\mathbf{B}_\gamma$ does not change.

Let us consider the existing constraints on the ALP-photon $g_{a\gamma\gamma}$ coupling which take advantage of the effects discussed in the previous two paragraphs. It is now clear that the same constraints hold also for the new $g_{a\mathrm{BB}}$ coupling and the corresponding search strategies need not be updated. In particular, this is the case of astrophysical and cosmological constraints [76], helioscope searches [77–79], light-shining-through-wall (LSW) [80–86] and axion interferometry [87–91] experiments as well as the ones of those haloscope searches, where the ALP Compton wavelength fits into the experimental apparatus and thus where the interaction between an ALP and a magnetic field can be described as an ALP-photon conversion in an external field. We present the corresponding constraints on the $g_{a\mathrm{BB}}$ coupling in Fig. 1. Note that as $|g_{a\mathrm{AB}}| \ll g_{a\mathrm{BB}}$, the same constraints obviously hold for $|g_{a\mathrm{AB}}|$ and $\sqrt{|g_{a\mathrm{AB}}|\, g_{a\mathrm{BB}}}$.

The qualitative distinction between the new $g_{a\mathrm{BB}}$ coupling and the conventional $g_{a\gamma\gamma}$ coupling arises whenever a given process cannot be described by Eq. (5.26) and involves observables which are not invariant under the electric-magnetic duality symmetry. In this case, the values of these observables derived from the axion Maxwell equations (5.22)–(5.25) are not symmetric with respect to the interchange $g_{a\mathrm{AA}} \leftrightarrow g_{a\mathrm{BB}}$ of the two couplings, so that there is a qualitative difference in the effects of these couplings. We give a particular example where the latter difference plays a crucial role in sec. 6.3. In particular, in the latter section, we discuss haloscope experiments searching for light ALP dark matter ($m_a \ll \mu$eV). We find that in this case, the constraints obtained for the $g_{a\gamma\gamma}$ coupling need not hold for the $g_{a\mathrm{BB}}$ coupling. This means that to probe the latter coupling, these experiments should exploit a search strategy which is different from the one normally used.

To be more specific, when we make quantitative predictions in the next sections, we will consider a particular kind of ALP: the QCD axion. In this case, we will take advantage of Eqs. (5.3), (5.4) and (5.6) for the axion-photon couplings. As we discussed in sec. 5.1, there are two families of axion models where the new electromagnetic couplings $g_{a\mathrm{AB}}$ and $g_{a\mathrm{BB}}$ can arise: those with Abelian ($\zeta = 3$) and those with non-Abelian ($\zeta = 1$) magnetic monopoles, cf. Eq. (5.7). In the former case, the axion decay constant f_a is obviously related to the QCD anomaly coefficient N and PQ scale v_a in the standard way: $f_a = v_a/2N$, while in the latter case, the relation is non-standard [18]: $f_a = 2\alpha_s^2 v_a/N$, where $\alpha_s = g_s^2/4\pi$. Using these

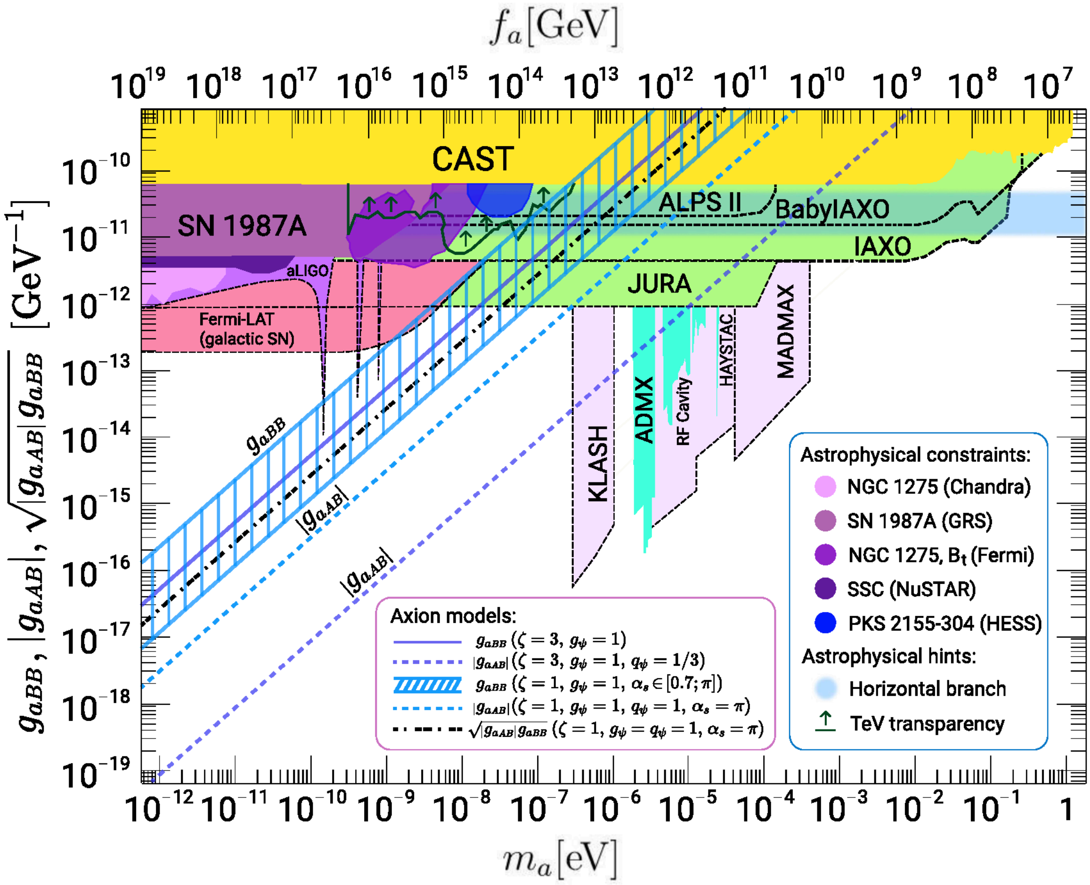

Figure 1. Existing and projected (dashed lines) constraints on the parameter space of ALP-photon $g_{a\mathrm{BB}}$ and $g_{a\mathrm{AB}}$ couplings versus ALP mass and decay constant together with the lines corresponding to $g_{a\mathrm{BB}}$ (solid), $|g_{a\mathrm{AB}}|$ (dashed) and $\sqrt{|g_{a\mathrm{AB}}|\,g_{a\mathrm{BB}}}$ (dash-dotted) in different hadronic axion models with one heavy PQ-charged fermion ψ with the parameters given in a box and $N_{\mathrm{DW}} = 1$. Astrophysical hints are also shown. For further discussion, see main text.

relations, we plot the lines corresponding to $g_{a\mathrm{BB}}$, $|g_{a\mathrm{AB}}|$ and $\sqrt{|g_{a\mathrm{AB}}|\,g_{a\mathrm{BB}}}$ [13] in different hadronic axion models. For simplicity, we choose the models having only one heavy vector-like PQ-charged fermion ψ, which transforms trivially under the $SU(2)_L$ gauge group of the weak interactions and in the fundamental representation under the color $SU(3)_c$ gauge group (electric $SU(3)_{\mathrm{E}}$ in the Abelian monopole case or magnetic $SU(3)_{\mathrm{M}}$ in the non-Abelian monopole case), and has charges q_ψ and g_ψ, see Fig. 1 and the legend therein. In these models, the QCD anomaly coefficient is $N = 1/2$, so that $N_{\mathrm{DW}} = 1$. In the non-Abelian monopole case $\zeta = 1$, there is an uncertainty associated to our ignorance of the exact value of α_s at low energies [92]. Note that the axion models populate a region of the parameter space, which in the analogous plot for the $g_{a\mathrm{AA}}$ coupling would be extensively probed by existing and projected haloscope experiments searching for light ALP dark matter ($m_a < \mu$eV). However, as we discussed briefly in the previous paragraph and will elaborate

[13] The $\sqrt{|g_{a\mathrm{AB}}|\,g_{a\mathrm{BB}}}$ line is relevant for LSW searches, see Eq. (6.2) and discussion in sec. 6.2.

later in sec. 6.3, the constraints from such haloscopes cannot be translated to Fig. 1. Also, note that the conventional KSVZ and DFSZ axion lines are obviously missing from the plot in Fig. 1, since this plot depicts the $g_{a\mathrm{BB}}$ and $|g_{a\mathrm{AB}}|$ couplings, but not the $g_{a\mathrm{AA}}$ coupling.

6.2 Purely laboratory-based experiments

A particularly clean way to measure the $g_{a\mathrm{AB}}$ coupling is provided by LSW experiments [93]. As one can see from Eq. (5.26), contrary to the CP-conserving couplings, the CP-violating $g_{a\mathrm{AB}}$ coupling allows interaction between ALPs and light polarized in a plane perpendicular to the external magnetic field. As one can control the polarization of the incoming light in a LSW experiment, it is straightforward to artificially turn off the CP-conserving ALP-photon interaction on the photon to ALP conversion side before the wall. Using $g_{a\mathrm{BB}} \gg |g_{a\mathrm{AB}}| \gg g_{a\mathrm{AA}}$, we find the following LSW probabilities corresponding to different linear polarizations of incoming light:

$$P(\gamma_{\parallel} \to a \to \gamma) \simeq 16\, \frac{(g_{a\mathrm{BB}}\omega B_0)^4}{m_a^8} \sin^4\left(\frac{m_a^2 L_{B_0}}{4\omega}\right), \tag{6.1}$$

$$P(\gamma_{\perp} \to a \to \gamma) \simeq 16\, \frac{(g_{a\mathrm{AB}}\omega B_0)^2 (g_{a\mathrm{BB}}\omega B_0)^2}{m_a^8} \sin^4\left(\frac{m_a^2 L_{B_0}}{4\omega}\right), \tag{6.2}$$

where $\gamma_{\parallel}$ ($\gamma_{\perp}$) denotes the incoming light with frequency $f = \omega/(2\pi)$ and with polarisation parallel (perpendicular) to the magnetic field B_0, which is supposed to be transverse to the direction of the light beam and which is sustained in a cavity of length L_{B_0}, both before and behind the wall. Clearly, from a detection of LSW with some probability $P(\gamma_{\parallel} \to a \to \gamma)$, one can determine the $g_{a\mathrm{BB}}$ coupling in a first measurement. In a second measurement, one can also search for LSW via $\gamma_{\perp} \to a \to \gamma$. Then, for the case of axions, using Eqs. (5.4), (5.6) and (5.7), we see that the coupling $g_{a\mathrm{AB}}$ can be determined from the following ratio:

$$\frac{P(\gamma_{\perp} \to a \to \gamma)}{P(\gamma_{\parallel} \to a \to \gamma)} \simeq \frac{g_{a\mathrm{AB}}^2}{g_{a\mathrm{BB}}^2} = \left(\frac{D}{M}\frac{e}{g_0}\right)^2 = 4\left(\frac{D}{\zeta M}\right)^2 \alpha^2 \simeq 2.13 \times 10^{-4} \left(\frac{D}{\zeta M}\right)^2. \tag{6.3}$$

For example, the experiment ALPS II ($B_0 = 5.3\,\mathrm{T}$, $L_{B_0} = 105.6\,\mathrm{m}$, $\omega = 1.17\,\mathrm{eV}$) has the capability to search for LSW using incoming light with both polarisations, $\gamma_{\parallel}$ and $\gamma_{\perp}$ [94]. For both of them, ALPS II has the projected sensitivity $P_{\mathrm{sens}} \approx 10^{-29}$ to the corresponding LSW probabilities. This would allow for the detection of a light, $m_a \lesssim 10^{-4}\,\mathrm{eV}$, axion featuring a CP-conserving coupling $(g_{a\mathrm{AA}} + g_{a\mathrm{BB}}) \simeq g_{a\mathrm{BB}} \gtrsim 2 \times 10^{-11}\,\mathrm{GeV}^{-1}$ via $\gamma_{\parallel} \to a \to \gamma$, as can be inferred from Eq. (6.1). If this newly discovered axion features also a CP-violating coupling, then the latter has to be in the range $|g_{a\mathrm{AB}}| = 2\alpha(|D|/\zeta M) g_{a\mathrm{BB}} \gtrsim 3 \times 10^{-13}\,\mathrm{GeV}^{-1}(|D|/\zeta M)$. If $|D| \simeq M$, to detect such a coupling via $\gamma_{\perp} \to a \to \gamma$ requires a sensitivity improvement by four order of magnitudes, to $P_{\mathrm{sens}} \sim 10^{-33}$, as can be seen from Eqs. (6.1) (6.2), and (6.3). Intriguingly, such a sensitivity has been argued to be achievable by the next generation LSW experiment JURA (also known as ALPS III [95, 96]). Indeed, see Fig. 1, where we showed the $g_{a\mathrm{BB}}$ ($\sqrt{|g_{a\mathrm{AB}}|\, g_{a\mathrm{BB}}}$) parameter space probed by JURA according to Eq. (6.1) (Eq. (6.2)) together with the lines corresponding to $g_{a\mathrm{BB}}$ and $\sqrt{|g_{a\mathrm{AB}}|\, g_{a\mathrm{BB}}}$ in the axion model with $|D| = M = \zeta = 1$. Thus, our considerations show that an eventual detection of an ALP by ALPS II would strongly motivate the construction of JURA. After all, an experimental verification of Eq. (6.3) would allow a deep view into the UV, provide strong evidence for the existence of heavy dyons and even an insight into their spectrum via the ratio $|D|/\zeta M$.

Although LSW experiments can probe the $g_{a\mathrm{AB}}$ coupling, we saw that the effects of the $g_{a\mathrm{BB}}$ coupling are dominant and are expected to be discovered first. To the contrary, there exist purely-laboratory experiments which are primarily sensitive to the $g_{a\mathrm{AB}}$ coupling. These are the experiments which probe CP-violating observables, since only the $g_{a\mathrm{AB}}$ coupling violates CP. One can probe the corresponding CP-violating effects by searching for electric dipole moments of charged particles, such as electrons, protons and muons [97]. Moreover, the CP-violating axion-photon coupling can be probed in various experiments searching for fifth force or monopole-dipole axion-induced interactions [98], since one expects the $g_{a\mathrm{AB}}$ coupling to radiatively induce CP-violating interactions between axions and charged fermions f of the form $g_f a\bar{f}f$. Note however, that although all the couplings of the ALP EFT (5.10) are small, the theory is still essentially non-perturbative, so predicting the exact values for the discussed CP-violating observables is not straightforward. We leave this task for future work.

6.3 Haloscope experiments

As we mentioned in sec. 6.1, the effect of the dominant $g_{a\mathrm{BB}}$ coupling on the ALP-photon conversion in an external electromagnetic field is the same as the effect of the conventional $g_{a\gamma\gamma}$ coupling. This means that in the case of the haloscopes that search for cosmic ALPs with masses $m_a \gtrsim (0.1-1)\,\mu\mathrm{eV}$, such as ADMX [99], CAPP [100], HAYSTAC [101], KLASH [102], MADMAX [103], and others, all the constraints obtained for the $g_{a\gamma\gamma}$ coupling are valid also for the $g_{a\mathrm{BB}}$ coupling. Indeed, in this case, the Compton wavelength of an ALP $\lambda_a = 2\pi/m_a$ is comparable to the physical size of the utilized detectors, or even smaller, and thus one can use a particle-like description of the process.

The same cannot be stated in the case where ALPs have large Compton wavelengths λ_a compared to the length scale L of the experiment. In particular, this is the case of light cosmic ALPs with masses $m_a \ll \mu\mathrm{eV}$, which one aims to detect with such haloscope experiments as ABRACADABRA [104, 105], ADMX SLIC [106], DM Radio [107], SHAFT [108], and others. In these experiments, one maintains a constant magnetic field $\mathbf{B}_0$ in a laboratory and searches for an ALP-induced oscillating magnetic field $\mathbf{B}_a$. Note that due to the condition $\lambda_a \gg L$, interactions of ALPs with the field $\mathbf{B}_0$ cannot be described as a conventional ALP-photon conversion phenomenon. To determine the expected magnitude of the induced fields in this case, one has to use the axion Maxwell equations (5.22)–(5.25). The latter equations can be significantly simplified since most of the terms on the right-hand side are normally suppressed. Indeed, considering the case of axions and assuming $E \simeq M \simeq |D|$, the axion-photon couplings satisfy $g_{a\mathrm{AA}}/g_{a\mathrm{BB}} \simeq (e/g_0)^2 \lesssim 2\cdot 10^{-4}$ and $|g_{a\mathrm{AB}}|/g_{a\mathrm{BB}} \simeq e/g_0 \lesssim 10^{-2}$. Moreover, the cosmic axions that form dark matter have typical velocities $v_a \sim 10^{-3}$, so that the gradient of the oscillating axion field is suppressed with respect to its time derivative: $|\boldsymbol{\nabla} a| \sim 10^{-3}\,\dot{a}$.

Leaving only the first three dominant terms, we obtain the following simplified axion Maxwell equations:

$$\boldsymbol{\nabla}\times\mathbf{B}_a - \dot{\mathbf{E}}_a = 0\,, \tag{6.4}$$

$$\boldsymbol{\nabla}\times\mathbf{E}_a + \dot{\mathbf{B}}_a = -g_{a\mathrm{BB}}\left(\mathbf{B}_0\times\boldsymbol{\nabla}a + \dot{a}\mathbf{E}_0\right) + g_{a\mathrm{AB}}\dot{a}\mathbf{B}_0\,, \tag{6.5}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}_a = 0\,, \tag{6.6}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}_a = 0\,, \tag{6.7}$$

where we included the dominant effect arising from an external electric field $\mathbf{E}_0$. Note that all the existing haloscopes use an external magnetic field $\mathbf{B}_0$ instead, partly because the dominant effect for the usually considered $g_{a\mathrm{AA}}$ coupling is due to the term with an external magnetic, but not electric, field and partly because it is technologically challenging to sustain a large enough electric field in a big enough volume. It is clear from Eq. (6.5) that if the latter technological problem is solved, so that $E_0 \gtrsim 10^{-2}\,(|D|/\zeta M)\,B_0$ in the CP-violating case or $E_0 \gtrsim 10^{-3}\,B_0$ in the CP-conserving case, the $g_{a\mathrm{BB}}\dot{a}\mathbf{E}_0$ term will allow one to search for dark matter axions in an external electric field with the sensitivity which is not worse than the one of the conventional searches conducted in an external magnetic field.

Returning to the case of the existing haloscopes where $\mathbf{E}_0 = 0$, $\mathbf{B}_0 \neq 0$, we see that the axion Maxwell equations (6.4)–(6.7) have significantly different structure compared to the conventional axion Maxwell equations which take into account solely the $g_{a\mathrm{AA}}$ coupling. While in the latter case axions generate an effective electric current $\mathbf{j}^e_{\mathrm{eff}} = -g_{a\mathrm{AA}}\dot{a}\mathbf{B}_0$, in the former case an effective magnetic current is generated:

$$\mathbf{j}^m_{\mathrm{eff}} = g_{a\mathrm{BB}}\mathbf{B}_0\times\boldsymbol{\nabla}a - g_{a\mathrm{AB}}\dot{a}\mathbf{B}_0\,. \tag{6.8}$$

Note that in the case $M \simeq |D|$, the term proportional to the CP-violating $g_{a\mathrm{AB}}$ coupling is dominant. To the contrary, if the underlying UV theory is CP-conserving, then $D = 0$ and $g_{a\mathrm{AB}} = 0$, so that only the term proportional to the $g_{a\mathrm{BB}}$ coupling contributes.

Let us now find experimental implications of the magnetic current (6.8). Applying the curl differential operator to the Eqs. (6.4) and (6.5), and using the other equations from the system (6.4)–(6.7), we obtain:

$$\Delta\mathbf{E}_a - \ddot{\mathbf{E}}_a = \boldsymbol{\nabla}\times\mathbf{j}^m_{\mathrm{eff}}\,, \tag{6.9}$$

$$\Delta\mathbf{B}_a - \ddot{\mathbf{B}}_a = \partial\mathbf{j}^m_{\mathrm{eff}}/\partial t\,. \tag{6.10}$$

The leading terms contributing to the right-hand side are:

$$\boldsymbol{\nabla}\times\mathbf{j}^m_{\mathrm{eff}} = g_{a\mathrm{BB}}\left(\boldsymbol{\nabla}a\cdot\boldsymbol{\nabla}\right)\mathbf{B}_0 - g_{a\mathrm{AB}}\dot{a}\,\boldsymbol{\nabla}\times\mathbf{B}_0\,, \tag{6.11}$$

$$\partial\mathbf{j}^m_{\mathrm{eff}}/\partial t = g_{a\mathrm{BB}}\mathbf{B}_0\times\boldsymbol{\nabla}\dot{a} - g_{a\mathrm{AB}}\ddot{a}\mathbf{B}_0\,. \tag{6.12}$$

The axion dark matter field is given by the following expression: $a(t,\mathbf{r}) = a_0\cos(\omega_a t - \mathbf{k}_a\cdot\mathbf{r})$, where $\omega_a = m_a$ and $\mathbf{k}_a = m_a\mathbf{v}_a$. The leading CP-conserving effect then depends on the direction of the axion wind and thus experiences modulations with the periods of one sidereal day T_d and one sidereal year T_y due to the rotation of the Earth around its axis and around the Sun, respectively. To find the axion-induced $\mathbf{E}_a$ and $\mathbf{B}_a$ fields, Eqs. (6.9) and (6.10) have to be solved for a particular geometry of a given haloscope experiment.

Let us illustrate the general features of the solution by considering the example of a very long solenoid of radius R with magnetic field $\mathbf{B}_0$ directed along the z-axis. In this case, Eqs. (6.9) and (6.10) become:

$$\Delta \mathbf{E}_a - \ddot{\mathbf{E}}_a = -\left(g_{a\mathrm{BB}}\, \partial_\rho a\, \mathbf{e}_z + g_{a\mathrm{AB}} \dot{a}\, \mathbf{e}_\phi\right) B_0\, \delta(\rho - R)\ , \tag{6.13}$$

$$\Delta \mathbf{B}_a - \ddot{\mathbf{B}}_a = g_{a\mathrm{BB}} \mathbf{B}_0 \times \boldsymbol{\nabla} \dot{a} - g_{a\mathrm{AB}} \ddot{a} \mathbf{B}_0\, , \tag{6.14}$$

where we work in cylindrical coordinates (ρ, ϕ, z) with unit vectors $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$. Let us parameterize the direction of the axion wind $\hat{\boldsymbol{v}}_a = (\sin\theta \cos(\phi - \xi),\, -\sin\theta \sin(\phi - \xi),\, \cos\theta)$ in cylindrical coordinates by two angles θ and ξ. Assuming $T_d \gg 2\pi/\omega_a$, which corresponds to $m_a \gg 5 \cdot 10^{-20}$ eV, we neglect the terms proportional to $\dot{\xi}$ and $\dot{\theta}$ in the Eqs. (6.13) and (6.14). It is then straightforward to obtain the solutions of these equations in terms of Bessel functions. All we need however are these solutions in the limit $\omega_a R \ll 1$, as we are interested in axions with large Compton wavelengths. In the latter limit, solutions to Eqs. (6.13), (6.14) with physical boundary conditions are:

$$\mathbf{E}_a = \begin{cases} \frac{1}{2}\, a_0\, \omega_a \rho\, B_0 \Big(g_{a\mathrm{AB}}\, \mathbf{e}_\phi - g_{a\mathrm{BB}}\, v_a \sin\theta\, \cos(\phi - \xi)\, \mathbf{e}_z\Big) \sin\omega_a t\, , & \rho < R \\ \frac{1}{2}\, a_0\, \omega_a \frac{R^2}{\rho} B_0 \Big(g_{a\mathrm{AB}}\, \mathbf{e}_\phi - g_{a\mathrm{BB}}\, v_a \sin\theta\, \cos(\phi - \xi)\, \mathbf{e}_z\Big) \sin\omega_a t\, , & \rho > R \end{cases}\ , \tag{6.15}$$

$$\mathbf{B}_a = \begin{cases} \frac{1}{2}\, a_0\, (\omega_a R)^2\, B_0 \Big(g_{a\mathrm{AB}}\, \mathbf{e}_z + g_{a\mathrm{BB}}\, v_a \sin\theta \left\{\cos(\phi - \xi)\, \mathbf{e}_\phi + \right. & \\ \qquad\qquad \left. \sin(\phi - \xi)\, \mathbf{e}_\rho \right\}\Big) \left(\ln \omega_a R + \frac{\rho^2}{2R^2}\right) \cos\omega_a t\, , & \rho < R \\ \frac{1}{2}\, a_0\, (\omega_a R)^2\, B_0 \Big(g_{a\mathrm{AB}}\, \mathbf{e}_z + g_{a\mathrm{BB}}\, v_a \sin\theta \left\{\cos(\phi - \xi)\, \mathbf{e}_\phi + \right. & \\ \qquad\qquad \left. \sin(\phi - \xi)\, \mathbf{e}_\rho \right\}\Big) \ln \omega_a \rho\, \cos\omega_a t\, , & \rho > R \end{cases}\ . \tag{6.16}$$

It is immediately clear that $E_a \gg B_a$, which means that in an experiment with the geometry of our example one has to search for axion-induced electric, but not magnetic, fields. Moreover, it turns out that this feature persists for any other possible geometry as well. Indeed, our Eqs. (6.9) and (6.10) can be rendered equivalent to the equations studied in Ref. [109] by substituting $\mathbf{E}_a \to \mathbf{B}_a$, $\mathbf{B}_a \to -\mathbf{E}_a$ and $\mathbf{j}^m_{\mathrm{eff}} \to \mathbf{j}^e_{\mathrm{eff}}$. In the latter work, it was found that for any haloscope geometry with characteristic length scale L, equation involving the time derivative of the effective current yields solutions which are suppressed by powers of $\omega_a L \ll 1$ with respect to the solutions of the equation involving the curl of this current. In our case, this means that in any haloscope probing $m_a \ll \mu$eV, the axion-induced magnetic field $\mathbf{B}_a$ is suppressed with respect to the axion-induced electric field $\mathbf{E}_a$.

Note however that all the existing as well as many projected haloscopes searching for such light axions – such as ABRACADABRA, ADMX SLIC, DM Radio, SHAFT, ...

– aim to measure only the axion-induced magnetic, but not electric, fields. Thus, the constraints on the conventional $g_{a\mathrm{AA}}$ coupling obtained by these experiments do not hold for the dominant axion-photon couplings $g_{a\mathrm{AB}}$ and $g_{a\mathrm{BB}}$. The latter couplings can be probed by future haloscopes equipped with electric field sensors. One haloscope of such kind has already been proposed, see Ref. [110]. We hope that our work will encourage more experimental effort in this direction, as we provided a sound theoretical motivation for such an endeavor. Since the electromagnetic fields generated in a haloscope by the $g_{a\mathrm{BB}}$ and $g_{a\mathrm{AB}}$ couplings are qualitatively different from the fields generated by the conventional $g_{a\mathrm{AA}}$ coupling, for which $B_a \gg E_a$ [109], the first detection of cosmic axions with electric, but not magnetic, sensor haloscope would not only constitute the discovery of axions and dark matter, but also provide a circumstantial experimental evidence for the existence of heavy magnetically charged particles. Furthermore, as one can see from Eq. (6.15), the direction of the detected electric field could allow one to infer the ratio $g_{a\mathrm{AB}}/g_{a\mathrm{BB}} = 2\alpha(D/\zeta M)$ and thus get information about the spectrum of dyons in the UV.

	Axions	Gravitational waves
$\mathbf{P}_e$	$-g_{a\mathrm{AA}}\, a\mathbf{B}_0 + g_{a\mathrm{AB}}\, a\mathbf{E}_0$	$\mathbf{h}\times\mathbf{B}_0 + H\mathbf{E}_0 - \big(h_{00} + \frac{1}{2}\,h\big)\mathbf{E}_0$
$\mathbf{M}_m$	$g_{a\mathrm{BB}}\, a\mathbf{B}_0 + g_{a\mathrm{AB}}\, a\mathbf{E}_0$	
$\mathbf{M}_e$	$-g_{a\mathrm{AA}}\, a\mathbf{E}_0 - g_{a\mathrm{AB}}\, a\mathbf{B}_0$	$-\mathbf{h}\times\mathbf{E}_0 + H\mathbf{B}_0 - \big(h_{00} + \frac{1}{2}\,h\big)\mathbf{B}_0$
$\mathbf{P}_m$	$g_{a\mathrm{BB}}\, a\mathbf{E}_0 - g_{a\mathrm{AB}}\, a\mathbf{B}_0$	

Table 1. Comparison between axion and gravitational wave electrodynamics in external electric $\mathbf{E}_0$ and magnetic $\mathbf{B}_0$ fields in terms of the effective induced polarization and magnetization vectors, defined by Eqs. (6.17)–(6.20); $\mathbf{h}$, H, h_{00} and h are characteristics of the gravitational wave defined in Ref. [72].

Finally, to compare different extensions of electrodynamics which could be probed by haloscope experiments, it is convenient to reexpress the axion Maxwell equations in terms of the axion-induced polarization and magnetization vectors, which are defined as follows:

$$\boldsymbol{\nabla}\times\mathbf{B}_a - \dot{\mathbf{E}}_a = \frac{\partial \mathbf{P}^e_{\text{eff}}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{M}^e_{\text{eff}}\,, \tag{6.17}$$

$$\boldsymbol{\nabla}\times\mathbf{E}_a + \dot{\mathbf{B}}_a = -\frac{\partial \mathbf{P}^m_{\text{eff}}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{M}^m_{\text{eff}}\,, \tag{6.18}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B}_a = -\boldsymbol{\nabla}\cdot\mathbf{P}^m_{\text{eff}}\,, \tag{6.19}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}_a = -\boldsymbol{\nabla}\cdot\mathbf{P}^e_{\text{eff}}\,. \tag{6.20}$$

Note that along with the ordinary effective electric polarization and magnetization vectors $\mathbf{P}^e_{\text{eff}}$ and $\mathbf{M}^e_{\text{eff}}$, we introduced effective magnetic polarization and magnetization vectors $\mathbf{P}^m_{\text{eff}}$ and $\mathbf{M}^m_{\text{eff}}$, which describe electromagnetic properties of an effective medium consisting of magnetically charged particles. In Table 1, we give explicit expressions for the effective polarization and magnetization vectors corresponding to a generic axion in an external electromagnetic field. The fact that it is possible to bring the axion Maxwell equations (5.22)–(5.25) into the form (6.17)–(6.20) suggests that the effects of axions in external electromagnetic fields are analogous to the effects of a certain medium consisting of both electrically and magnetically charged particles. From what has been discussed before, it is clear that for a generic axion, effects of the magnetic polarization (magnetization) vectors $\mathbf{P}^m_{\text{eff}}$ ($\mathbf{M}^m_{\text{eff}}$) in an external magnetic field dominate the effects of the electric polarization (magnetization) vectors $\mathbf{P}^e_{\text{eff}}$ ($\mathbf{M}^e_{\text{eff}}$). Such asymmetry between the effects exhibited by electric and magnetic constituents of the effective medium is to be contrasted with the case of gravitational wave electrodynamics [72], where there exists an electric-magnetic $U(1)$ duality symmetry rendering electric and magnetic variables equivalent. This difference in the symmetry properties of the two theories can be easily understood from the fact that the axion-photon interactions are fundamentally mediated by heavy charged particles which break the duality symmetry, while in General Relativity, the interaction between gravity and electromagnetic field is direct and independent of any charges. For an experimentalist, this distinction signifies a substantial difference in the distribution of the induced electromagnetic fields inside the haloscope. Indeed, as it was discussed before, for a generic light axion ($m_a \ll \mu\text{eV}$, $g_{a\text{BB}} \gg |g_{a\text{AB}}| \gg g_{a\text{AA}}$) in an external magnetic field one expects $\mathbf{j}^m_{\text{eff}} \gg \mathbf{j}^e_{\text{eff}}$ and thus $E_a \gg B_a$, while for a gravitational wave in an external magnetic field, plugging the expressions given in Table 1 into the Eqs. (6.17)–(6.20), one obtains $\mathbf{j}^m_{\text{eff}} \sim \mathbf{j}^e_{\text{eff}}$ and thus $E_a \sim B_a$.

7 Conclusion

As it was asserted by J. Polchinski [31], “the existence of magnetic monopoles seems like one of the safest bets that one can make about physics not yet seen”. Indeed, in the beginning of this article, we reviewed a number of theoretical arguments which together provide an overwhelming theoretical evidence for magnetic monopoles, like there probably exists for no other kind of hypothetical particles. There are nevertheless multiple practical challenges behind the experimental study of monopoles. While some of the previously mentioned arguments for the existence of monopoles do not restrict their masses, another part of these arguments suggests that monopoles are super heavy, with masses well beyond the energy reach of the present-day collider experiments. Thus, although monopoles almost certainly exist from the theoretical point of view, it might be difficult to directly probe them with existing and near-future experiments.

In this work, we proposed experiments which could probe magnetic monopoles indirectly, in particular through the influence virtual monopoles would exhibit on the interactions of axions with an electromagnetic field. To study the effects of virtual monopoles, we developed an EFT approach to QEMD classifying all independent marginal operators preserving the symmetries and degrees of freedom of the theory. Then, we built a generic axion-photon EFT, which holds for the QCD axion as well as for other ALPs. We showed that this EFT features previously unknown couplings which arise from UV models containing magnetic charges. We then found that these new couplings give rise to unique experimental signatures in LSW experiments and in some kinds of axion searches, such as haloscopes searching for low-mass axions ($m_a \ll \mu\text{eV}$). In the latter case, we showed that the best sensitivity to the new electromagnetic couplings of axions could be achieved by measuring an induced oscillating electric field, instead of an induced oscillating magnetic field, contrary to the setup of existing experiments. The case of low mass axions is particularly interesting, since, as it can be seen from Fig. 1, the simplest axion models predict rather large $g_{a\mathrm{BB}}$ and $g_{a\mathrm{AB}}$ couplings, which can be probed by many projected experiments. Thus, we encourage the development of electric sensor haloscopes which would search for axion dark matter in the corresponding parameter region.

Apart from unveiling these intriguing experimental applications, our work reconsidered several questions in axion theory. First, we showed that contrary to the existing statements, the main contribution to the axion-photon coupling need not be quantized in units proportional to e^2. Second, contrary to what has been advocated recently in the literature, we found that magnetic monopoles of an Abelian gauge field need not give mass to axions coupled to this gauge field. We showed that the axions do get mass in theories with magnetic monopoles only if these monopoles carry an additional instanton degree of freedom interacting with the axion a, e.g. in the case of the spontaneously broken symmetry phase of a non-Abelian gauge theory with aGG^d term in the Lagrangian, where G (G^d) is the (dual) field strength tensor of the non-Abelian gauge field. This axion mass is generated via the conventional mechanism through instantons of the non-Abelian theory, which however live on the 't Hooft-Polyakov monopoles in the low energy phase. Finally, we found that the interaction between axions and an electromagnetic field need not preserve CP. In particular, as long as one has heavy dyons in the high energy theory, there exists a natural source of CP-violation which can be transferred to low energy physics through axion-photon interactions. We discussed experiments which can probe this new CP-violating axion coupling.

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A Calculation of the anomaly coefficients

Let us calculate the anomaly coefficients E, M and D, which enter the expressions (5.3), (5.4) and (5.6) for the axion-photon couplings. We start with a high energy QEMD theory and integrate out heavy fermions ψ carrying PQ charges. In the PQ-symmetric phase, the Lagrangian includes the following terms for each of the fermions ψ:

$$\mathcal{L} \supset i\bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi + y\left(\Phi\, \bar{\psi}_L \psi_R + \text{h.c.}\right) , \tag{A.1}$$

where y is a dimensionless Yukawa constant, $\mathcal{D}_\mu = \partial_\mu - e\, q_\psi A_\mu - g_0\, g_\psi B_\mu$ is a covariant derivative including both magnetic and electric four-potentials multiplied by the corresponding electric and magnetic charges, and Φ is the PQ complex scalar field.

Below the PQ symmetry breaking scale, one can expand $\Phi = (v_a + \rho)\exp(ia/v_a)/\sqrt{2}$, where ρ is a heavy radial mode and a is an axion. The terms in the resulting Lagrangian which are relevant for the low energy phenomenology are:

$$\mathcal{L} \supset i\bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi + \frac{y v_a}{\sqrt{2}} \left\{ \exp\left(\frac{ia}{v_a}\right) \bar{\psi}_L \psi_R + \text{h.c.} \right\}. \tag{A.2}$$

We perform an axial rotation of the fermion $\psi \to \exp(ia\gamma_5/2v_a)\cdot\psi$, after which there arise an anomalous contribution $\mathcal{L}_\mathrm{F}$ from the transformation of the measure of the path integral and a derivative coupling of a to the axial current of ψ:

$$\mathcal{L} \supset i\bar{\psi}\gamma^\mu \mathcal{D}_\mu \psi + \frac{y v_a}{\sqrt{2}}\, \bar{\psi}\psi - \frac{\partial^\mu a}{2v_a}\, \bar{\psi}\gamma_\mu\gamma_5\psi - \mathcal{L}_\mathrm{F}\,. \tag{A.3}$$

The fermion ψ gets its mass $m = y v_a/\sqrt{2}$, which we assume to be large compared to the energy scales probed in experiments. In the large m limit, the derivative axion coupling from Eq. (A.3) does not contribute to the low energy Lagrangian of axion-photon interactions due to the Sutherland-Veltman theorem [111, 112]. The axion-photon couplings are thus given by the anomalous terms $\mathcal{L}_\mathrm{F}$ which can be calculated using the Fujikawa method [113]. In particular, following Fujikawa, we apply the gauge invariant heat kernel regularization to the path integral measure which yields:

$$\mathcal{L}_\mathrm{F} = -\frac{a}{v_a} \cdot \lim_{\substack{\Lambda\to\infty \\ x\to y}} \mathrm{tr}\left\{\gamma_5 \exp\left(\not{\mathcal{D}}^2/\Lambda^2\right) \delta^4(x-y)\right\}. \tag{A.4}$$

We then notice that the commutators Eqs. (4.1) and (4.2) vanish if the four-potentials are taken at the same space-time point. This means that these commutators do not contribute to the expression for $\not{\mathcal{D}}^2$, which becomes:

$$\not{\mathcal{D}}^2 = \mathcal{D}^2 - i\gamma_\mu\gamma_\nu \left(e q_\psi\, \partial^\mu A^\nu + g_0 g_\psi\, \partial^\mu B^\nu\right). \tag{A.5}$$

It is then convenient to express the delta-function as a superposition of plane waves: $\delta^4(x-y) = \int d^4k\, e^{ik(x-y)}/\left(2\pi\right)^4$, each of which shifts the derivative operator $\mathcal{D}_\mu \to \mathcal{D}_\mu + ik_\mu$. Taking into account Eq. (A.5), we obtain:

$$\mathcal{L}_\mathrm{F} = -\frac{a}{v_a}\cdot \lim_{\Lambda\to\infty} \int \frac{d^4k}{\left(2\pi\right)^4}\, \mathrm{tr}\left\{\gamma_5 \exp\left(-i\gamma_\mu\gamma_\nu\left(e q_\psi\,\partial^\mu A^\nu + g_0 g_\psi\,\partial^\mu B^\nu\right)/\Lambda^2 + \left(\mathcal{D} + ik\right)^2/\Lambda^2\right)\right\}. \tag{A.6}$$

Any terms in the integrand which are $o\big(1/\Lambda^4\big)$ vanish after performing the integration and sending $\Lambda \to \infty$. Taylor expanding the exponent, we are then left with the finite number of terms, which after taking the trace, the integral and the limit simplify into:

$$\mathcal{L}_\mathrm{F} = \frac{a\, d(C_\psi)}{8\pi^2 v_a}\cdot \epsilon_{\mu\nu\lambda\rho}\Big(e q_\psi\,\partial^\mu A^\nu + g_0 g_\psi\,\partial^\mu B^\nu\Big)\Big(e q_\psi\,\partial^\lambda A^\rho + g_0 g_\psi\,\partial^\lambda B^\rho\Big), \tag{A.7}$$

where $d(C_\psi)$ is the dimension of the color representation of ψ. Using the notations of sec. 4, Eq. (A.7) can be rewritten as follows:

$$\mathcal{L}_\mathrm{F} = -\frac{a\, d(C_\psi)}{16\pi^2 v_a}\cdot \Big(q_\psi^2 e^2\, \mathrm{tr}\left\{(\partial\wedge A)\,(\partial\wedge A)^d\right\} + g_\psi^2 g_0^2\, \mathrm{tr}\left\{(\partial\wedge B)\,(\partial\wedge B)^d\right\} + 2\, q_\psi g_\psi\, e g_0\, \mathrm{tr}\left\{(\partial\wedge A)\,(\partial\wedge B)^d\right\}\Big). \tag{A.8}$$

If there are multiple dyons ψ, each of them gives a similar contribution to the resulting axion-photon Lagrangian. Thus, the expressions for the axion-photon couplings $g_{a\mathrm{AA}}, g_{a\mathrm{BB}}, g_{a\mathrm{AB}}$ and the corresponding anomaly coefficients E, M, D are:

$$g_{a\mathrm{AA}} = \frac{Ee^2}{4\pi^2 v_a}\,, \quad E = \sum_\psi q_\psi^2 \cdot d(C_\psi)\ , \tag{A.9}$$

$$g_{a\mathrm{BB}} = \frac{Mg_0^2}{4\pi^2 v_a}\,, \quad M = \sum_\psi g_\psi^2 \cdot d(C_\psi)\ , \tag{A.10}$$

$$g_{a\mathrm{AB}} = \frac{Deg_0}{4\pi^2 v_a}\,, \quad D = \sum_\psi q_\psi g_\psi \cdot d(C_\psi)\ . \tag{A.11}$$

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