# Third order post-Newtonian gravitational radiation from two-body scattering: Instantaneous energy and angular momentum radiation

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We compute the third post-Newtonian (3PN) accurate instantaneous contributions to the radiated gravitational wave energy and angular momentum arising from the hyperbolic passages of nonspinning compact objects. The present computations employ 3PN-accurate instantaneous contributions to the far-zone energy and angular momentum fluxes and the 3PN-accurate Keplerian type parametric solution for compact binaries in hyperbolic orbits.

## I. INTRODUCTION

The routine detection of transient gravitational wave (GW) events that arise from merging black hole (BH) binaries in bound orbits has inaugurated the era of GW astronomy [1–3]. Further, observations of a neutron star binary coalescence in GWs, and many electromagnetic frequency windows, have provided a peek into the benefits of the multimessenger GW astronomy [4–6]. In contrast, compact binaries in unbound orbits can provide transient GW burst events in the LIGO, LISA, and IPTA GW frequency windows [7–10]. Interestingly, GW burst events due to hyperbolic encounters of neutron stars may even be accompanied by electromagnetic flares [11]. Therefore, there are ongoing post-Newtonian (PN) efforts to characterize both the dynamics and associated GW emission aspects of compact binaries in hyperbolic orbits in general relativity [12–17].

The present effort extends the classic computations of the radiated energy($\Delta\mathcal{E}$) and angular momentum ($\Delta\mathcal{J}$) during hyperbolic encounters of nonspinning compact objects to 1PN order [18–20]. Recall that PN approximation allows us to write, for example, the orbital dynamics of nonspinning compact binaries as corrections to Newtonian equations of motion in powers of $(v/c)^2 \sim GM/(c^2 r)$, where $v$, $M$, and $r$ are the velocity, total mass, and relative separation of the binary [21,22] with the gravitation constant $G$ and speed of light $c$. Note that the expressions, available in Refs. [19,20], provided the next-to-leading order (1PN) contributions to $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$, influenced by Refs. [23,24]. The present computation provides 3PN-accurate "instantaneous" contributions to $\Delta\mathcal{E}$ and

$\Delta\mathcal{J}$ with the help of Refs. [12,25,26] in the modified Harmonic gauge. It turned out that at PN orders beyond the 1PN, the radiative moments and the resulting far-zone fluxes have two distinct contributions [21]. One part of the radiative moments and their fluxes depends only on the usual retarded time and it is customary to refer to these terms as the instantaneous contributions. In contrast, the second part depends on the dynamics of the compact binary in its entire past and therefore these contributions are usually termed as the hereditary contributions [27,28]. In this paper, we focus our efforts on the instantaneous contributions.

The present computations are not straightforward extensions to 3PN order of what were done in Refs. [19,20]. This is mainly because of logarithmic terms that appear at the 3PN corrections to far-zone energy and angular momentum fluxes associated with compact binaries in noncircular orbits [25,26]. We provide a prescription to compute these logarithmic integrals in terms of Clausen function of order two [29].

The manuscript is structured in the following way. The way of our PN-accurate computations and underlying formalism to obtain $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ and the results are presented in Sec. II. We also briefly present parabolic limit and the implications of the bremsstrahlung limit. The details of our computations are presented in Appendix A, and our PN-accurate expressions in terms of energy and angular momentum are given in Appendix B while Sec. III provides a brief summary and ongoing investigations.

## II. 3PN ACCURATE INSTANTANEOUS CONTRIBUTIONS TO $\Delta\mathcal{E}$ AND $\Delta\mathcal{J}$

By the matching between multipolar post-Minkowskian (MPM) expansion and PN expansion [30,31], gravitational

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fluxes $\mathcal{F}$ (either energy or angular momentum) can be expressed in terms of mechanical variables describing binaries such as mass, radial distance, and velocities, of which explicit expression can be found in Eq. (5.2) in [25] in standard/modified harmonic gauge. Additionally, once the 3PN accurate quasi-Keplerian solution, i.e., the mechanical variables in time (implicitly), obtained as in [12], the fluxes can be fully written as a function of time $t$. Here, we briefly show the PN structure of the fluxes up to 3PN($1/c^6$) order,

$$\mathcal{F}(t) = \mathcal{F}_{\text{inst}}(t) + \mathcal{F}_{\text{hered}}(t), \tag{1a}$$

where

$$\mathcal{F}_{\text{inst}} = \mathcal{F}_{\text{N}} + \frac{1}{c^2}\mathcal{F}_{\text{1PN}} + \frac{1}{c^4}\mathcal{F}_{\text{1PN}} + \frac{1}{c^6}\mathcal{F}_{\text{1PN}} + \mathcal{O}(1/c^7), \tag{1b}$$

$$\mathcal{F}_{\text{hered}} = \frac{1}{c^3}\mathcal{F}_{\text{tail}} + \frac{1}{c^6}\mathcal{F}_{\text{tail(tail)}} + \frac{1}{c^6}\mathcal{F}_{\text{tail}^2} + \mathcal{O}(1/c^7). \tag{1c}$$

By $\mathcal{F}_{\text{hered}}$ (hereditary contribution), we mean all contribution that is dependent on the past history of binaries [27,28]. Otherwise, it is called instantaneous contribution denoted as $\mathcal{F}_{\text{inst}}$. Thus, total radiations $\int_{-\infty}^{+\infty} dt\mathcal{F}$ also have both instantaneous and hereditary contributions and the same PN structure. Note that total radiations are observables, hence gauge invariant but each instantaneous/hereditary contribution is not gauge invariant because of an ambiguous separation of the long and short scales leaving coordinate dependence via $r_0$ (which will be seen shortly). The full result should not be dependent on the scale $r_0$, and hence recover gauge invariance. The leading (Newtonian) order of energy and angular momentum radiations $\Delta\mathcal{E}$, $\Delta\mathcal{J}$ (hence leading order of instantaneous part because hereditary part starts at 1.5PN order) were computed in Ref. [18] for the first time. Its extension up to 1PN, (hence still instantaneous) was made in [19,20]. The higher PN orders including hereditary contribution have never been treated so far. As one of serial works in the line of completing 3PN accurate radiations, we compute and complete the instantaneous contribution first as what follows.

In Sec. II A, we will explain how the computation goes by an example at leading order influenced by Ref. [18,19]. All computation is similar to the leading order one except the logarithmic terms that appear at the 3PN order. The detailed way of tackling these logarithmic terms is provided in Appendix A. These computations are repeated to obtain angular momentum radiation in Sec. II C. Thereafter, we explain why our instantaneous contributions to the radiated energy and angular momentum during hyperbolic encounters are exact up to 3PN order and explore their limiting cases.

## A. Newtonian order $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ computations

We begin by explaining the procedure of Refs. [18,19] for computing the radiated energy and angular momentum in GWs during hyperbolic encounters at the Newtonian order. The natural starting point is the familiar Newtonian (leading) order far-zone GW energy flux $\mathcal{F}_{\text{N}}$ for compact binaries in generic orbits [19]

$$\mathcal{F}_{\text{N}} = \frac{8}{15}\frac{G^3M^2\mu^2}{c^5r^4}(12v^2 - 11\dot{r}^2). \tag{2}$$

The approach of Ref. [19] requires us to employ the standard Keplerian parametric solution for compact binaries in Newtonian hyperbolic orbits, available in Refs. [32,33]. This is for expressing the total orbital velocity $v$, the radial velocity $\dot{r}$, and the radial separation in terms of various elements of the Keplerian parametric solution for hyperbolic orbits. The underlying parametric solution for hyperbolic orbits reads

$$r = a(e\cosh u - 1), \tag{3a}$$

$$\phi - \phi_0 = 2\tan^{-1}\left(\sqrt{\frac{e+1}{e-1}}\tanh\frac{u}{2}\right), \tag{3b}$$

$$n(t - t_0) = e\sinh u - u, \tag{3c}$$

where $r$, $\phi$, and $t$ are radial orbital separation, the angular variable of the reduced mass $\mu$ around the total mass $M$, and coordinate time, respectively. Further, the eccentric anomaly parameter $u$ has the range $-\infty < u < \infty$ while $\phi_0$ and $t_0$ denote some initial value of $\phi$ and $t$. The familiar Newtonian semimajor axis $a$, orbital eccentricity $e$, and the mean motion $n$ are given in terms of the conserved reduced energy $E = \frac{\mathcal{E}}{\mu}$ and reduced angular momentum $j = \frac{\mathcal{J}}{\mu GM}$,

$$a = \frac{GM}{2E},$$
$$e^2 = 1 + 2Ej^2,$$
$$n = \frac{(2E)^{3/2}}{GM}.$$

It is fairly straightforward to express $\dot{r}^2$, $v^2$ in terms of $E, j^2, (e\cosh u - 1)$ with the help of

$$\dot{r}^2 = \left(\frac{dr}{du}\Big/\frac{dt}{du}\right)^2, \tag{4a}$$

$$\dot{\phi}^2 = \left(\frac{d\phi}{du}\Big/\frac{dt}{du}\right)^2, \tag{4b}$$

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2. \tag{4c}$$

This leads to

$$\dot{r}^2 = \left(1 + \frac{2}{(e\cosh u - 1)} - \frac{(2Eh^2)}{(e\cosh u - 1)^2}\right)(2E), \quad (5a)$$

$$v^2 = \left(1 + \frac{2}{(e\cosh u - 1)}\right)(2E). \quad (5b)$$

Using Eqs. (5), we can express the instantaneous energy flux as a polynomial in $(1 - e\cosh u)^{-1}$ and the final expression reads

$$\mathcal{F}_N = \frac{1}{n}\frac{du}{dt}\sum_{N=3}^{5}\frac{\bar{\alpha}_N(E, j)}{(e\cosh u - 1)^N}, \quad (6)$$

$$\bar{\alpha}_N(e_t) = \frac{v^2}{G}\left(\frac{E}{c}\right)^5 \alpha_N(E, j). \quad (7)$$

In the above equation $\nu$ is defined as dimensionless mass parameter of the binary, namely $\nu = m_1 m_2/(m_1 + m_2)^2$, where $m_1$ and $m_2$ are the masses of the binary configuration. The coefficients $\alpha_N(E, j)$ at the Newtonian order are given by

$$\alpha_3 = \frac{256}{15},$$

$$\alpha_4 = \frac{512}{15},$$

$$\alpha_5 = \frac{5632}{15}Ej^2. \quad (8)$$

The fact that we have parametrized the far-zone Newtonian energy flux in terms of Newtonian hyperbolic orbital description allows us to write the total radiated energy in GWs at the Newtonian order as

$$\Delta\mathcal{E}_N = \int_{-\infty}^{\infty}\mathcal{F}_N dt,$$

$$= \frac{1}{n}\sum_{N=3}^{5}\bar{\alpha}_N(E, j)\int_{-\infty}^{\infty}\frac{du}{(e_t\cosh u - 1)^N}. \quad (9)$$

Clearly, we can easily obtain the desired expression for the quadrupolar order $\Delta\mathcal{E}$ during hyperbolic encounters if we can compute the three integrals that appear on the right-hand side of Eq. (9). With the help of Refs. [18,19], we find

$$\int_{-\infty}^{\infty}\frac{du}{(e\cosh u - x)^N}$$

$$= \frac{2}{(N-1)!}\left[\left(\frac{d}{dx}\right)^{N-1}\left(\frac{1}{\sqrt{e^2 - x^2}}\arccos\left(-\frac{x}{e}\right)\right)\right]. \quad (10)$$

This leads to

$$\Delta\mathcal{E}_N = \frac{2M\nu^2}{15c^5 j^7}\left[\sqrt{e^2 - 1}\left(\frac{602}{3} + \frac{673e^2}{3}\right)\right.$$

$$\left. + (96 + 292e^2 + 37e^4)\arccos\left(-\frac{1}{e}\right)\right], \quad (11a)$$

where we have used the following Newtonian accurate relation that connects $E$ to $j$ and $e$, namely $E = \frac{e^2-1}{2j^2}$, to obtain the above result. Indeed, our expression is fully consistent with Refs. [18–20].

We now move on to explain briefly how Ref. [20] computed the quadrupolar order contributions to the radiated angular momentum during hyperbolic encounters of nonspinning compact objects. We begin by the Newtonian order angular momentum flux $\mathcal{G}_N$ for compact binaries in generic orbits [19]:

$$\mathcal{G}_N = \frac{8}{5}\tilde{\mathbf{L}}_N\frac{G^2 M\mu^2}{c^5 r^3}\left\{2v^2 - 3\dot{r}^2 + 2\frac{GM}{r}\right\} \quad (12)$$

where $\tilde{\mathbf{L}}_N = \mathbf{r}\times\mathbf{v}$ stands for the scaled Newtonian angular momentum vector. The fact that the orbital angular momentum vector remains a constant as we consider only nonspinning compact binaries allows us to compute an expression for the angular momentum flux $d\mathcal{J}/dt$ from the above equation. Thereafter, we pursue the steps involved in the $\Delta\mathcal{E}$ computations and this leads to

$$\mathcal{G}_N = \frac{1}{n}\frac{du}{dt}\sum_{N=2}^{4}\frac{\bar{A}_N(E, j)}{(e\cosh u - 1)^N}, \quad (13)$$

$$\bar{A}_N(e_t) = \frac{MjE^4\nu^2}{c^5}A_N(E, j). \quad (14)$$

The three constant coefficients are given by

$$A_2 = -\frac{128}{5},$$

$$A_3 = 0,$$

$$A_4 = \frac{768}{5}Ej^2. \quad (15)$$

The radiated angular momentum $\Delta\mathcal{J}_N$ at the Newtonian order becomes

$$\Delta\mathcal{J}_N = \int_{-\infty}^{\infty}\mathcal{G}_N dt,$$

$$= \frac{1}{n}\sum_{N=2}^{4}\bar{A}_N(E, j)\int_{-\infty}^{\infty}\frac{du}{(e_t\cosh u - 1)^N}. \quad (16)$$

Clearly, these integrals are similar to those we tackled earlier and this eventually leads to

$$\Delta \mathcal{J}_{\rm N} = \frac{8GM^2\nu^2}{5c^5 j^4} \left\{ (13+2e^2)\sqrt{e^2-1} \right.$$

$$\left. + (8+7e^2)\arccos\left(-\frac{1}{e}\right) \right\}. \tag{17}$$

We have verified that the above expression is fully consistent with Refs. [18–20]. We now move on to extend these calculations to 3PN order while focusing on the instantaneous contributions.

## B. 3PN-accurate instantaneous contributions to the radiated energy

It should be obvious that we require two crucial ingredients for our 3PN-accurate $\Delta\mathcal{E}$ computation. The first ingredient is the 3PN accurate "instantaneous" contributions to the far-zone GW energy flux from nonspinning compact binaries in noncircular orbits [21]. These instantaneous contributions depend only on the state of the binary at the usual retarded time and they appear usually at the Newtonian, 1PN, 2PN, 2.5PN, and 3PN orders. In contrast, the hereditary contributions, as the name suggests, are sensitive to the binary dynamics at all epochs prior to the usual retarded time and appear at 1.5PN (relative) order for the first time [21]. In this paper, we focus our efforts on the 3PN-accurate instantaneous contributions to the far-zone fluxes, as given by Eqs. (5.2) in Ref. [25] in the modified harmonic (MH) coordinates. A close inspection of these contributions reveals that $\Delta\mathcal{E}$ computation at 3PN order will be demanding due to the presence of certain "logarithmic" terms, as is evident from Eq. (5.2e) of Ref. [25]. The second ingredient for the present computation is the 3PN-accurate generalized quasi-Keplerian parametric solution for compact binaries in hyperbolic orbits, derived in Ref. [12].

Note that Ref. [12] provided a parametric way to track 3PN-accurate conservative trajectory of compact binaries in hyperbolic orbits. This effort extended the 1PN-accurate derivation of Keplerian type parametric solution for hyperbolic motion that employed the arguments of analytic continuation [32]. At the 3PN order, the radial motion $r(t)$ is conveniently parametrized as

$$r = a_r (e_r \cosh u - 1), \tag{18a}$$

$$n(t - t_0) = e_t \sinh u - u + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right) V$$

$$+ \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)\sin V + \frac{h_{6t}}{c^6}\sin 2V + \frac{i_{6t}}{c^6}\sin 3V, \tag{18b}$$

where $u$ is the eccentric anomaly while $a_r$, $e_r$, $e_t$, $n$, and $t_0$ are certain PN-accurate semimajor axis, radial eccentricity, time eccentricity, mean motion, and initial epoch,

respectively. In addition, we have several orbital functions like $g_{4t}$, $g_{6t}$, $f_{4t}$, $f_{6t}$, $i_{6t}$, and $h_{6t}$ that appear at 2PN and 3PN orders. Further, the angular motion is described by

$$\phi - \phi_0 = (1+k)\left\{ V + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right)\sin 2V \right.$$

$$\left. + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right)\sin 3V + \frac{h_{6\phi}}{c^6}\sin 4V + \frac{i_{6t}}{c^6}\sin 5V \right\} \tag{19a}$$

where

$$V = 2\tan^{-1}\left(\sqrt{\frac{e_\phi + 1}{e_\phi - 1}}\tanh\frac{u}{2}\right). \tag{19b}$$

In the above expressions, $k$ and $e_\phi$ denote PN-accurate rate of periastron advance, and certain angular eccentricity, respectively. Additionally, we have several orbital functions like $f_{4\phi}$, $f_{6\phi}$, $g_{4\phi}$, $g_{6\phi}$, $i_{6\phi}$, and $h_{6\phi}$ that appear at 2PN and 3PN orders. We note that Ref. [12] provided 3PN-accurate expressions for various orbital elements and functions in terms of $E$ and $j$.

It is straightforward but tedious to compute 3PN-accurate expressions for $\dot{r}^2$, $v^2$ in terms of $E$, $j^2$ ($e_t \cosh u - 1$). These dynamical variables that appear in the far-zone energy flux are computed by employing the following relations:

$$\frac{dt}{du} = \frac{\partial t}{\partial u} + \frac{\partial t}{\partial v}\frac{dv}{du}, \tag{20a}$$

$$\dot{r}^2 = \left(\frac{dr}{du} \middle/ \frac{dt}{du}\right)^2, \tag{20b}$$

$$\dot{\phi}^2 = \left(\frac{d\phi}{dv}\frac{dv}{du} \middle/ \frac{dt}{du}\right)^2, \tag{20c}$$

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2. \tag{20d}$$

In what follows, we display 1PN-accurate parametric expressions for $\dot{r}^2$, $v^2$, and $GM/r$ for introducing the reader to the structure of these expressions

$$\dot{r}^2_{\rm MH} = \left\{ 1 + \frac{2}{(e_t \cosh u - 1)} - \frac{(2Ej^2)}{(e_t \cosh u - 1)^2} \right\}(2E)$$

$$+ \frac{(2E)^2}{c^2}\left\{ -\frac{3}{4} + \frac{9\nu}{4} + \frac{1}{(e_t \cosh u - 1)^2}\left[-2 + (2Ej^2)\right.\right.$$

$$\times \left(-\frac{7}{2} - \frac{\nu}{2}\right) + 2\nu \right] + \frac{1}{(e_t \cosh u - 1)}\left[-\frac{3}{2} + \frac{9\nu}{2}\right]\right\}, \tag{21a}$$

$$
v_{\mathrm{MH}}^2 = \left\{ 1 + \frac{2}{(e_t \cosh u - 1)} \right\}(2E)
$$
$$
+ \frac{(2E)^2}{c^2}\left\{ -\frac{3}{4} + \frac{9\nu}{4} + \frac{(2Ej^2)\nu}{(e_t \cosh u - 1)^3} \right.
$$
$$
\left. + \frac{2(\nu-1)}{(e_t \cosh u - 1)^2}, + \frac{1}{(e_t \cosh u - 1)}\left[ -\frac{3}{2} + \frac{9\nu}{2} \right] \right\},
$$
$$(21b)$$

$$
\frac{GM}{r} = \frac{2E}{(e_t \cosh u - 1)} + \frac{(2E)^2}{c^2}\left\{ \frac{1}{(1 - e_t \cosh u)}\left[ \frac{9}{4} - \frac{5\nu}{4} \right] \right.
$$
$$
\left. + \frac{1}{(1 - e_t \cosh u)^2}\left[ 4 - \frac{3}{2}\nu \right] \right\}.
$$
$$(21c)$$

The explicit 3PN-accurate expressions for these dynamical variables can be found in [34].

We are now in a position to replace dynamical variables $r$, $\dot{r}$, and $\dot{\phi}$ that appear in the 3PN-accurate instantaneous far-zone energy flux expression, given by Eqs. (5.2) of Ref. [25], with the 3PN-accurate version of the above equations. The associated 3PN-accurate expression for the radiated energy during hyperbolic encounters reads

$$
\Delta\mathcal{E} = \int_{-\infty}^{+\infty} dt \mathcal{F} = \int_{-\infty}^{+\infty} du \left( \frac{dt}{du} \right)\mathcal{F}. \qquad (22)
$$

The use of 3PN-accurate expressions for $r$, $\dot{r}$, and $\dot{\phi}$ in terms of $E$, $h$, and $(1 - e_t \cosh u)$, as evident from our Eqs. (21), in the Eqs. (5.2) of Ref. [25], leads to

$$
\mathcal{F} = \frac{1}{n}\frac{du}{dt}\left\{ \sum_{N=3}^{11} \frac{\bar{\alpha}_N(E,j)}{(e_t \cosh u - 1)^N} \right.
$$
$$
+ \sum_{N=5}^{9} \bar{\beta}_N(E,j)\frac{\sinh u}{(e_t \cosh u - 1)^N}
$$
$$
\left. + \sum_{N=5}^{9} \bar{\gamma}_N(E,j)\frac{\ln(e_t \cosh u - 1)}{(e_t \cosh u - 1)^N} \right\}, \qquad (23)
$$

where we may write in a compact manner the above constant coefficients as

$$
\bar{W}_N(E,j) = \frac{\nu^2}{G}\left( \frac{E}{c} \right)^5 W_N(E,j),
$$

where $W_N$ stands for $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. We do not list all these lengthy coefficients in the manuscript and they are provided in an ancillary *Mathematica* file at [34]. However, we list below one of them to show the typical structure of these coefficients

$$
\alpha_3(E,j) = \frac{256}{15} + \frac{E}{c^2}\left( \frac{13184}{105} - \frac{128\nu}{21} \right) + \frac{E^2}{c^4}\left( \frac{51328}{63} \right.
$$
$$
+ \frac{5504\nu}{315} + \frac{640\nu^2}{63} \right) + \frac{E^3}{c^6}\left( \frac{5451968}{1155} + \frac{1803328\nu}{3465} \right.
$$
$$
\left. + \frac{181792\nu^2}{3465} - \frac{8000\nu^3}{693} \right). \qquad (24)
$$

We note that all $\bar{\beta}_N$ terms appear at the 2.5PN order while $\bar{\gamma}_N$ terms are accompanied by the logarithmic terms at the 3PN order. Additionally, we have incorporated the dependence on the constant $\log r_0$ into the coefficients $\bar{\alpha}_N$. Recall that $r_0$ is the gauge dependent length scale appearing in the definition of source multiple moments [35] as discussed in Ref. [25]. We have indeed verified that the 1PN-accurate version of these coefficients match with Ref. [19].

We now move to tackle how these coefficients contribute to the instantaneous 3PN-accurate expression for the radiated energy during hyperbolic encounters. It is fairly straightforward to infer that all the 2.5PN terms, namely $\bar{\beta}_N$ terms, do not contribute to $\Delta\mathcal{E}$ expression and this is because

$$
\int_{-\infty}^{\infty} \frac{\sinh u}{(e_t \cosh u - 1)^N} du = 0. \qquad (25)
$$

This is obviously due to the fact that these integrands are an odd function in $u$. The integration of $\bar{\alpha}_N$ terms should be straightforward as they are essentially similar to the terms that we confronted at the Newtonian order. Therefore, we employ Eq. (10) to compute $\Delta\mathcal{E}$ contributions that arise from nine $\bar{\alpha}_N$ terms in Eq. (23).

Clearly, it is very tricky to tackle the logarithmic terms and we pursued an entirely new line of investigations compared to what was done in Ref. [25] for the eccentric orbits. After some detailed efforts, we were able to obtain analytic expressions for these integrals though the final expressions were too lengthy to list here. However, we were eventually able to obtain the most simplified form of these integrals with the help of Clausen identity as detailed in Appendix A. This allowed us to tackle the $\bar{\gamma}_N$ contributions to the 3PN-accurate $\Delta\mathcal{E}$ expression as

$$
\int_{-\infty}^{\infty} \frac{\ln(e\cosh u - x)}{e\cosh u - x} du = \frac{2}{\sqrt{e^2 - x^2}}\left[ \mathrm{Cl}_2\left( 2\arccos\left( -\frac{x}{e} \right) \right) \right.
$$
$$
\left. + \arccos\left( -\frac{x}{e} \right)\ln\left( \frac{2(e^2 - x^2)}{e} \right) \right], \qquad (26)
$$

where $\mathrm{Cl}_2(x)$ is the Clausen function of order 2 given by the integral

$$
\mathrm{Cl}_2(x) = -\int_0^x dy \ln\left| 2\sin\frac{y}{2} \right|. \qquad (27)
$$

Interestingly, the Clausen function admits the following Fourier series representation:

$$Cl_2(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^2}. \tag{28}$$

The crucial integrals that are associated with the $\bar{\gamma}_N$ coefficients in Eq. (23) can now be tackled by noting that the integrals

$$\int_{-\infty}^{\infty} \frac{\ln(e \cosh u - 1)}{(e \cosh u - 1)^N} du$$

can be computed after taking successive derivatives of Eq. (26) with respect to $x$ and then finally taking $x \to 1$.

With these inputs and the following 3PN accurate expression of $E$ in terms of $e_t$ and $j$ [12],

$$E = \frac{e_t^2 - 1}{2j^2} + \frac{1}{c^2 j^4}\left[-\frac{9}{8} + e_t^2\left(\frac{13}{4} - \frac{3\nu}{4}\right) + e_t^4\left(-\frac{17}{8} + \frac{7\nu}{8}\right) - \frac{\nu}{8}\right] + \frac{1}{c^4 j^6}\left[-\frac{81}{16} + \frac{7\nu}{16} - \frac{\nu^2}{16} + e_t^4\left(-\frac{339}{16} + \frac{213\nu}{16} - \frac{35\nu^2}{16}\right)\right.$$
$$+ e_t^2\left(\frac{243}{16} - \frac{29\nu}{16} + \frac{3\nu^2}{16}\right) + e_t^6\left(\frac{177}{16} - \frac{191\nu}{16} + \frac{33\nu^2}{16}\right)\right] + \frac{1}{c^6 j^8}\left[-\frac{3861}{128} + \left(\frac{8833}{384} - \frac{41\pi^2}{64}\right)\nu + \frac{5\nu^2}{64} - \frac{5\nu^3}{128}\right.$$
$$+ e_t^6\left(\frac{4641}{32} - \frac{19037\nu}{96} + \frac{1119\nu^2}{16} - \frac{231\nu^3}{32}\right) + e_t^2\left(\frac{2105}{32} + \left(\frac{157553}{3360} - \frac{41\pi^2}{32}\right)\nu - \frac{105\nu^2}{16} + \frac{\nu^3}{32}\right)$$
$$+ e_t^4\left(-\frac{7479}{64} + \left(\frac{16123}{2240} + \frac{123\pi^2}{64}\right)\nu - \frac{313\nu^2}{32} + \frac{105\nu^3}{64}\right) + e_t^8\left(-\frac{8165}{128} + \frac{15515\nu}{128} - \frac{3435\nu^2}{64} + \frac{715\nu^3}{128}\right)\right], \tag{29}$$

we finally obtain the 3PN-accurate instantaneous contributions to $\Delta\mathcal{E}$ in terms of $e_t$ and $j$,

$$\Delta\mathcal{E} = \frac{2}{15}\frac{M\nu^2}{j^7 c^5}\left(\mathcal{I}_N^{MH} + \frac{1}{c^2 j^2}\mathcal{I}_{1PN}^{MH} + \frac{1}{c^4 j^4}\mathcal{I}_{2PN}^{MH} + \frac{1}{c^6 j^6}\mathcal{I}_{3PN}^{MH}\right), \tag{30}$$

where contributions to $\Delta\mathcal{E}$ that appear at the Newtonian, 1PN, 2PN, and 3PN orders read

$$\mathcal{I}_N^{MH} = \sqrt{e_t^2 - 1}\left[\frac{602}{3} + \frac{673e_t^2}{3}\right] + \arccos\left(-\frac{1}{e_t}\right)[96 + 292e_t^2 + 37e_t^4], \tag{31a}$$

$$\mathcal{I}_{1PN}^{MH} = \sqrt{e_t^2 - 1}\left[\frac{153263}{70} - \frac{1547\nu}{3} + e_t^2\left(\frac{271849}{70} - \frac{13799\nu}{6}\right) + e_t^4\left(-\frac{288513}{280} - 2\nu\right)\right]$$
$$+ \arccos\left(-\frac{1}{e_t}\right)\left[\frac{6578}{7} - 168\nu + e_t^2\left(\frac{31013}{7} - 1982\nu\right) + e_t^4\left(-\frac{223}{4} - \frac{1483\nu}{2}\right) + e_t^6\left(-\frac{15219}{56} + 74\nu\right)\right], \tag{31b}$$

$$\mathcal{I}_{2PN}^{MH} = \sqrt{e_t^2 - 1}\left[\frac{405300022}{19845} - \frac{2947852\nu}{315} + \frac{1173\nu^2}{4} + e_t^2\left(\frac{6673495637}{158760} - \frac{114248429\nu}{2520} + \frac{66217\nu^2}{8}\right)\right.$$
$$+ e_t^4\left(-\frac{3823800817}{158760} + \frac{41499527\nu}{5040} + \frac{3619\nu^2}{2}\right) + e_t^6\left(\frac{39802111}{7840} - \frac{67328\nu}{35} - 103\nu^2\right)\right] + \arccos\left(-\frac{1}{e_t}\right)$$
$$\times\left[\frac{1636769}{189} - \frac{74435\nu}{21} + 48\nu^2 + e_t^2\left(\frac{8638156}{189} - \frac{762901\nu}{21} + \frac{9851\nu^2}{2}\right) + e_t^4\left(-\frac{554104}{63} - \frac{350943\nu}{28} + \frac{48063\nu^2}{8}\right)\right.$$
$$+ e_t^6\left(-\frac{1324649}{336} + \frac{611613\nu}{112} - \frac{1779\nu^2}{2}\right) + e_t^8\left(\frac{1224929}{672} - \frac{10070\nu}{7} + 185\nu^2\right)\right], \tag{31c}$$

$$
\mathcal{I}_{3PN}^{MH} = \sqrt{e_t^2-1}\left[\frac{6713608}{1575} + \frac{17868572e_t^2}{525} + \frac{19300553e_t^4}{525} + \frac{17525209e_t^6}{3150}\right]\log\left[\frac{2r_0(e_t^2-1)}{GMj^2e_t}\right] - \left[\frac{54784}{35} + \frac{465664e_t^2}{21}\right]
$$

$$
+ \frac{4426376e_t^4}{105} + \frac{1498856e_t^6}{105} + \frac{31779e_t^8}{70}\right]\left\{\log\left[\frac{2GMj^2}{e_t r_0}\right]\arccos\left(-\frac{1}{e_t}\right) + \mathrm{Cl}_2\left[2\arccos\left(-\frac{1}{e_t}\right)\right]\right\}
$$

$$
+ \sqrt{e_t^2-1}\left[\frac{1959816183329}{8731800} + \left(-\frac{69960810223}{317520} + \frac{11632643\pi^2}{3360}\right)\nu + \frac{9109459\nu^2}{560} + \frac{689\nu^3}{24}\right.
$$

$$
+ e_t^2\left(\frac{11238026145523}{17463600} + \left(-\frac{264708911281}{317520} + \frac{998227\pi^2}{140}\right)\nu + \frac{518878433\nu^2}{2520} - \frac{641483\nu^3}{48}\right)
$$

$$
+ e_t^4\left(-\frac{10459843311391}{139708800} + \left(\frac{28976695225}{254016} + \frac{99671\pi^2}{896}\right)\nu + \frac{143029027\nu^2}{2880} - 16908\nu^3\right) + e_t^6\left(\frac{135148514527}{739200}\right.
$$

$$
+ \left(-\frac{15436846447}{211680} - \frac{12303239\pi^2}{13440}\right)\nu + \frac{12729151\nu^2}{5040} + \frac{8501\nu^3}{3}\right) + e_t^8\left(-\frac{112472361473}{4139520}\right.
$$

$$
+ \frac{244676087\nu}{8820} - \frac{432849\nu^2}{140} - \frac{1147\nu^3}{3}\right)\right] + \arccos\left(-\frac{1}{e_t}\right)\times\left[\frac{20510192533}{207900} + \left(-\frac{69631105}{756} + 1599\pi^2\right)\nu\right.
$$

$$
+ \frac{145195\nu^2}{28} + 9\nu^3 + e_t^2\left(\frac{269134761733}{415800} + \left(-\frac{736708039}{1080} + \frac{55637\pi^2}{8}\right)\nu + \frac{21177007\nu^2}{168} - \frac{22283\nu^3}{4}\right)
$$

$$
+ e_t^4\left(\frac{125593677691}{554400} + \left(-\frac{2952969469}{10080} + \frac{172405\pi^2}{64}\right)\nu + \frac{112045205\nu^2}{672} - \frac{381255\nu^3}{16}\right) + e_t^6\left(\frac{115347955537}{1108800}\right.
$$

$$
+ \left(\frac{7251474763}{60480} - \frac{174619\pi^2}{128}\right)\nu - \frac{2843351\nu^2}{64} + 3155\nu^3\right) + e_t^8\left(\frac{22565112667}{492800} + \left(-\frac{37470739}{672} - \frac{12177\pi^2}{128}\right)\nu\right.
$$

$$
+ \frac{2686573\nu^2}{112} - 2075\nu^3\right) + e_t^{10}\left(-\frac{484326439}{39424} + \frac{502175\nu}{28} - \frac{25061\nu^2}{4} + 518\nu^3\right)\right].
\tag{31d}
$$

We have found the full agreement with Eq. (1.4) of Ref. [36] up to 3PN($v^7$) order in bremsstrahlung limit. Note that since the choice of the relation between $e_t$, $E$, and $j$ [Eq. (29)] was made in favor of convenience of computation under MH gauge condition [12], hence the apparent expression of Eq. (31) [and also Eq. (35)] is not gauge invariant. We have verified that the above expression is fully consistent with Eqs. (C9) of Ref. [14] up to 2PN order which required us to express $e_t$ and $j$ in terms of $e_r$ and $a_r$ to 2PN order in the MH gauge.

We note that it is customary to characterize hyperbolic encounters with the help of an impact parameter $b$ and an eccentricity parameter as noted in Ref. [20]. Therefore, we provide a 3PN accurate relation that connects $j$ to $b$ and $e_t$ with the help of Ref. [12],

$$
j^2 = \frac{b}{GM}\sqrt{e_t^2-1}\left\{1 + \left[1 + \frac{\nu}{6} + e_t^2\left(-4 + \frac{17\nu}{6}\right)\right]\rho + \left[-\frac{17}{4} - \frac{61\nu}{24} - \frac{\nu^2}{8} + e_t^4\left(-\frac{49}{4} + \frac{307\nu}{24} - \frac{95\nu^2}{24}\right)\right.\right.
$$

$$
+ e_t^2\left(8 + \frac{23\nu}{4} - \frac{5\nu^2}{12}\right)\right]\rho^2 + \left[-\frac{23}{4} + \left(\frac{1375}{48} - \frac{41\pi^2}{64}\right)\nu - \frac{2213\nu^2}{72} + \frac{\nu^3}{12} + e_t^2\left(-\frac{125}{2} + \left(\frac{141199}{1680} - \frac{123\pi^2}{64}\right)\nu\right.\right.
$$

$$
+ \frac{607\nu^2}{24} + \frac{5\nu^3}{4}\right) + e_t^4\left(\frac{333}{4} - \frac{557\nu}{16} - \frac{595\nu^2}{24} + \frac{95\nu^3}{12}\right) + e_t^6\left(-\frac{271}{3} + \frac{7339\nu}{48} - \frac{6391\nu^2}{72} + \frac{217\nu^3}{12}\right)\right]\rho^3\right\},
\tag{32}
$$

where $\rho = \frac{1}{\sqrt{e_t^2-1}}\frac{GM}{bc^2}$. We now move on to briefly list our approach to compute PN accurate the radiated angular momentum during hyperbolic encounters that extends the 1PN-accurate effort of Refs. [14,20].

### C. 3PN-accurate instantaneous contributions to the radiated angular momentum

The crucial input that is required for our $\Delta\mathcal{J}$ computation is the 3PN-accurate instantaneous contributions to the far-zone GW angular momentum flux from compact binaries in noncircular orbits, given by Eqs. (3.4) in Ref. [26] and therefore in the MH gauge. The dynamical variables that appear in these PN contributions that includes $|\hat{\mathbf{L}}_{\mathbf{N}}|$ are expressed in terms of $E$, $j$, and $(e_t\cosh u - 1)$ to 3PN order with the help of Ref. [12]. The resulting 3PN extension of Eq. (13) that provides 3PN-accurate instantaneous contributions to the scalar far-zone GW angular momentum flux may be written as

$$\Delta \mathcal{J} = \frac{1}{n}\frac{du}{dt}\left\{\sum_{N=2}^{10}\frac{\bar{A}_N(E,j)}{(e_t\cosh u - 1)^N}\right.$$

$$+ \sum_{N=4}^{8}\bar{B}_N(E,j)\frac{\sinh u}{(e_t\cosh u - 1)^N}$$

$$\left. + \sum_{N=4}^{8}\bar{\Gamma}_N(E,j)\frac{\ln(e_t\cosh u - 1)}{(e_t\cosh u - 1)^N}\right\}, \qquad (33)$$

where

$$\bar{Y}_N(E,j) = \frac{\nu^2 E^4 M j}{c^5}Y_N(E,j),$$

where $Y$ stands for $A$, $B$, $\Gamma$. It is obvious that we can pursue the similar arguments, detailed in Sec. II B, for computing $\int_{-\infty}^{+\infty}dt\mathcal{G}(j,e_t,u)$. This leads to the following 3PN-accurate instantaneous contributions to the radiated angular momentum during hyperbolic encounters of nonspinning compact objects

$$\Delta \mathcal{J} = \frac{8}{5c^5}\frac{GM^2\nu^2}{j^4}\left(\mathcal{H}_N^{MH} + \frac{1}{c^2 j^2}\mathcal{H}_{1PN}^{MH} + \frac{1}{c^4 j^4}\mathcal{H}_{2PN}^{MH} + \frac{1}{c^6 j^6}\mathcal{H}_{3PN}^{MH}\right), \qquad (34)$$

where the individual contributions that appear at Newtonian, 1PN, 2PN, and 3PN orders are given by

$$\mathcal{H}_N^{MH} = \sqrt{e_t^2 - 1}(13 + 2e_t^2) + \arccos\left(-\frac{1}{e_t}\right)(8 + 7e_t^2), \qquad (35a)$$

$$\mathcal{H}_{1PN}^{MH} = \sqrt{e_t^2 - 1}\left[\frac{14759}{168} + e_t^2\left(\frac{11153}{336} - \frac{1975\nu}{36}\right) - \frac{847\nu}{18} + e_t^4\left(-\frac{62}{7} + 4\nu\right)\right] + \arccos\left(-\frac{1}{e_t}\right)\left[\frac{1777}{42} + e_t^2\left(\frac{3649}{42} - \frac{241\nu}{3}\right)\right.$$

$$\left. + e_t^4\left(-\frac{5713}{336} + \frac{5\nu}{12}\right) - 18\nu\right], \qquad (35b)$$

$$\mathcal{H}_{2PN}^{MH} = \sqrt{e_t^2 - 1}\left[\frac{38409857}{68040} - \frac{3668237\nu}{7560} + \frac{785\nu^2}{18} + e_t^2\left(\frac{2630029}{17010} - \frac{13151567\nu}{15120} + \frac{4423\nu^2}{12}\right) + e_t^4\left(-\frac{853417}{3360}\right.\right.$$

$$\left. + \frac{1691057\nu}{5040} - \frac{617\nu^2}{18}\right) + e_t^6\left(\frac{2480}{63} - \frac{755\nu}{14} + 10\nu^2\right)+\Bigg] + \arccos\left(-\frac{1}{e_t}\right)\left[\frac{326917}{1134} - \frac{27031\nu}{126} + 10\nu^2\right.$$

$$\left.\left. + e_t^2\left(\frac{414821}{756} - \frac{29055\nu}{28} + 298\nu^2\right) + e_t^4\left(-\frac{1154087}{3024} + \frac{72917\nu}{336} + \frac{941\nu^2}{12}\right) + e_t^6\left(\frac{99103}{2016} - \frac{12847\nu}{336} + \frac{3\nu^2}{2}\right)\right]\right],$$

$$\mathcal{H}_{3PN}^{MH} = \sqrt{e_t^2 - 1}\left[\frac{99724}{315} + \frac{351067e_t^2}{315} + \frac{210683e_t^4}{630}\right]\log\left[\frac{2r_0(e_t^2 - 1)}{GMj^2e_t}\right] - \left[\frac{13696}{105} + \frac{98012e_t^2}{105} + \frac{23326e_t^4}{35} + \frac{2461e_t^6}{70}\right]$$

$$\times\left\{\log\left[\frac{2GMj^2}{r_0e_t}\right]\arccos\left(-\frac{1}{e_t}\right) + \mathrm{Cl}_2\left[2\arccos\left(-\frac{1}{e_t}\right)\right]\right\} + \sqrt{e_t^2 - 1}\left[\frac{55475721271}{8382528} + \left(-\frac{17854035221}{1905120}\right.\right.$$

$$\left. + \frac{313363\pi^2}{1920}\right)\nu + \frac{636197\nu^2}{540} - \frac{103\nu^3}{18} + e_t^2\left(\frac{550589812147}{83825280} + \left(-\frac{49917375859}{3810240} + \frac{541733\pi^2}{3840}\right)\nu\right.$$

$$\left. + \frac{12681271\nu^2}{1890} - \frac{33883\nu^3}{36}\right) + e_t^4\left(-\frac{276385167053}{335301120} + \left(\frac{11543781001}{1905120} - \frac{22427\pi^2}{1920}\right)\nu - \frac{116779321\nu^2}{60480}\right.$$

$$\left. - \frac{21775\nu^3}{144}\right) + e_t^6\left(\frac{12794620753}{8279040} - \frac{197812189\nu}{70560} + \frac{9014755\nu^2}{8064} - \frac{23497\nu^3}{288}\right) + e_t^8\left(-\frac{13784}{77} + \frac{2999\nu}{6}\right.$$

$$\left.\left. - 254\nu^2 + 28\nu^3\right)\right] + \arccos\left(-\frac{1}{e_t}\right)\left[\frac{4577461991}{1247400} + \left(-\frac{21428779}{4536} + \frac{369\pi^2}{4}\right)\nu + \frac{7853\nu^2}{18}\right.$$

$$+ e_t^2\left(\frac{13811878057}{1247400} + \left(-\frac{86352541}{5670} + \frac{5781\pi^2}{32}\right)\nu + \frac{646651\nu^2}{126} - 488\nu^3\right) + e_t^4\left(\frac{602403517}{831600} + \left(\frac{150579449}{60480}\right.\right.$$

$$\left. + \frac{6273\pi^2}{256}\right)\nu + \frac{21055\nu^2}{16} - \frac{4289\nu^3}{6}\right) + e_t^6\left(\frac{2135052803}{1108800} + \left(-\frac{1094353}{672} - \frac{615\pi^2}{128}\right)\nu + \frac{8513\nu^2}{168} + \frac{565\nu^3}{12}\right)$$

$$\left.\left. + e_t^8\left(-\frac{94124017}{709632} + \frac{757831\nu}{2016} - \frac{306977\nu^2}{2688} + \frac{129\nu^3}{32}\right)\right]. \qquad (35c)\right.$$

Note that the first line contributions in $\mathcal{H}_{3\text{PN}}^{\text{MH}}$ are due to the log terms in the far-zone angular momentum flux. We have verified that our expressions are consistent with Eqs. (E6) of Ref. [14] at the 2PN order. We now explain why our instantaneous results can be treated to be exact up to 3PN order.

### D. On the exact nature of our 3PN results

We note that two crucial inputs are required to compute 3PN-accurate expressions for the instantaneous contributions to $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$. The first input is the 3PN-accurate generalized quasi-Keplerian parametric solution for compact binaries in hyperbolic orbits [12]. This solution, presented in the modified harmonic gauge, provided analytic expressions for the angular and radial dynamical variables of the 3PN accurate conservative dynamics of compact binaries in hyperbolic orbits. With the help of Ref. [12], we write schematically analytic expressions for these dynamical variables as

$$
\begin{aligned}
r &= \mathcal{R}(j, e_t, u), \\
\phi &= \mathcal{P}(j, e_t, u), \\
\dot{r} &= \mathcal{S}(j, e_t, u), \\
\dot{\phi} &= \mathcal{Q}(j, e_t, u),
\end{aligned}
\tag{36}
$$

where $r$ and $\dot{r}$ stand for the radial orbital separation and its time derivative. Further, $\phi$ denotes the angular variable of the reduced mass $\mu$ around the total mass $M$ while $\dot{\phi}$ is the time derivative of the above orbital phase. For the present discussion, we employed certain time eccentricity $e_t$ and the reduced angular momentum as orbital parameters to characterize the hyperbolic orbit. The temporal evolution arises via the eccentric anomaly $u$ and it is related to coordinate time $t$ via the PN-accurate Kepler equation that we symbolically write as

$$
t = \mathcal{T}(j, e_t, u).
\tag{37}
$$

Let us emphasize that the above parametric solution incorporates only the conservative temporal evolution of orbital variables to the 3PN order.

The second crucial ingredient for our present computation is the 3PN-accurate instantaneous contributions to the energy and angular momentum fluxes, given by Eqs. (5.2) of Ref. [25] and Eqs. (3.4) of Ref. [26], in the MH gauge. For the present discussion, we write these fluxes schematically as

$$
\begin{aligned}
\mathcal{F} &= \mathcal{F}(r, \dot{r}, \dot{\phi}), \\
\mathcal{G} &= \mathcal{G}(r, \dot{r}, \dot{\phi}).
\end{aligned}
\tag{38}
$$

Following Refs. [19,20], we estimate the radiated energy and angular momentum during hyperbolic encounters by integrating the above fluxes from $t = -\infty$ to $t = \infty$. The analytic treatment of these integrals require us to express the dynamical variables that appear in Eqs. (36) and (37) with the help of PN-accurate Keplerian parametric solution of Ref. [12]. This leads to

$$
\Delta\mathcal{E} := \int_{-\infty}^{+\infty} dt \mathcal{F}(j, e_t, u),
\tag{39a}
$$

$$
\Delta\mathcal{J} := \int_{-\infty}^{+\infty} dt \mathcal{G}(j, e_t, u).
\tag{39b}
$$

The above approach is appropriate as it is customary to write these fluxes as $\mathcal{F} = -\frac{d\mathcal{E}}{dt}$ and $\mathcal{G} = -\frac{d\mathcal{J}}{dt}$. However, the far-zone energy and angular momentum fluxes and the time derivatives of orbital energy and angular momentum are related to each other modulo certain total time derivatives that appear at the 2.5PN order [37–39]. This is why the above equalities hold in an orbital averaged sense in the case of bound elliptical orbits [19].

When we pursue the computations of $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ to 3PN order, there are certain subtleties that we need to address. This is related to the fact that both $h$ and $e_t$ vary with time due to the gravitational radiation reaction effects that appear at the 2.5PN order. This implies that the temporal evolution in the above integrands occur not only through $u$ but also through $j$ and $e_t$. However, the perturbative nature of GW emission allows us to write

$$
\begin{aligned}
e_t(t) &= e_{t0} + \frac{\delta e_t(t)}{c^5} + \mathcal{O}(c^{-7}), \\
j(t) &= j_0 + \frac{\delta j(t)}{c^5} + \mathcal{O}(c^{-7}),
\end{aligned}
\tag{40}
$$

where $e_{t0}$ and $j_0$ are the values of time eccentricity and the scaled angular momentum at periastron (defined by $u = 0$) and hence constants. Therefore, we could ignore temporal evolution in $e_t$ and $j$ while computing $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions up to 2PN order. Further, the resulting expressions involve only the constant scaled orbital angular momentum and time eccentricity along with the two mass parameters $m_1$ and $m_2$. However, a close look of Eqs. (40) and their implications for the integrands of Eqs. (39) reveal that the dissipative corrections $\delta e_t$, $\delta j$ are required if we plan to obtain 3PN extension of Refs. [19,20].

This mainly arises due to the structure of the relative acceleration at 2.5PN order which may be written as $\ddot{\mathbf{x}} = \mathbf{a}_{2.5PN}(r, \dot{r}, \dot{\phi}, \mathbf{r}, \mathbf{v})$ and this ensures both $d\mathcal{E}/dt \neq 0$; $d\mathcal{J}/dt \neq 0$ at the 2.5PN order and hence contain terms of $\mathcal{O}(1/c^5)$. Therefore, we may try to parametrize the orbital dynamics at 2.5PN in the following manner:

$$r = \mathcal{R}_0 + \frac{1}{c^5} \partial \mathcal{R}_0 \cdot \delta,$$

$$t = \mathcal{T}_0 + \frac{1}{c^5} \partial \mathcal{T}_0 \cdot \delta + \frac{1}{c^5} C_t(t),$$

$$\phi = \mathcal{P}_0 + \frac{1}{c^5} \partial \mathcal{P}_0 \cdot \delta + \frac{1}{c^5} C_\phi(t),$$

$$\dot{r} = \mathcal{S}_0 + \frac{1}{c^5} \partial \mathcal{S}_0 \cdot \delta,$$

$$\dot{\phi} = \mathcal{Q}_0 + \frac{1}{c^5} \partial \mathcal{Q}_0 \cdot \delta, \tag{41}$$

where we used a few shorthand notations such that

$$\mathcal{R}_0 = \mathcal{R}(j_0, e_{0t}, u),$$

$$\partial \mathcal{R}_0 = \left\{ \frac{\partial}{\partial j} \mathcal{R}(j_0, e_{0t}, u), \frac{\partial}{\partial e_t} \mathcal{R}(j_0, e_{0t}, u) \right\},$$

$$\delta = \{\delta j, \delta e_t\},$$

and similar notational conventions apply for the other dynamical variables. The above expressions are influenced by the improved "method of variation of constants," detailed in Ref. [40] that provided a way to include the effects of quadrupolar GW emission on the 2PN-accurate Keplerian type parametric solution for eccentric compact binaries. Further, the two new variables $C_t$ and $C_\phi$ that appear at the 2.5PN order are influenced by the $c_l$ and $c_m$ variables of Ref. [40]. It is not difficult to argue that the temporal evolution of these new variables should follow

$$\frac{dC_t}{dt} = \delta \cdot \left( \frac{\partial_u \partial \mathcal{R}_0}{\partial_u \mathcal{R}_0} - \frac{\partial_u \partial \mathcal{T}_0}{\partial_u \mathcal{T}_0} - \frac{\partial \mathcal{S}_0 \partial_u \mathcal{T}_0}{\partial_u \mathcal{R}_0} \right)$$
$$+ \frac{d\delta}{dt} \cdot \left( -\partial \mathcal{T}_0 + \frac{\partial \mathcal{R}_0 \partial_u \mathcal{T}_0}{\partial_u \mathcal{R}_0} \right),$$

$$\frac{dC_\phi}{dt} = \frac{d\delta}{dt} \cdot \left( -\partial \mathcal{P}_0 + \frac{\partial \mathcal{R}_0 \partial_u \mathcal{P}_0}{\partial_u \mathcal{R}_0} \right)$$
$$+ \delta \cdot \left( \partial \mathcal{Q}_0 - \frac{\partial \mathcal{S}_0 \partial_u \mathcal{P}_0}{\partial_u \mathcal{R}_0} - \frac{\partial_u \partial \mathcal{P}_0}{\partial_u \mathcal{T}_0} + \frac{\partial_u \mathcal{P}_0 \partial_u \partial \mathcal{R}_0}{\partial_u \mathcal{R}_0 \partial_u \mathcal{T}_0} \right).$$

The fact that $\mathbf{a}_{2.5\mathrm{PN}}$ does not explicitly depend on $\phi$ and $t$ ensures that $C_t$ and $C_\phi$ cannot contribute to the variations in

$\mathcal{E}$ and $\mathcal{J}$ at 2.5PN order. In other words, we may write the total time derivative of the conserved energy at 2.5PN to be

$$\frac{d\mathcal{E}}{dt} = \dot{\mathcal{E}}_0 + \delta \cdot \partial \dot{\mathcal{E}}_0, \tag{42}$$

where $\dot{\mathcal{E}}_0$ stands for 2.5PN accurate energy flux, evaluated at the periastron $e_t = e_{t0}$ and $j = j_0$. It is easily seen that $\delta \cdot \partial \dot{\mathcal{E}}_0$ represents radiation reaction correction to leading order radiation, which is induced by the deflection. Note that $\dot{\mathcal{E}}_0$ is even function in time, because the choice of $z$ axis (perpendicular to the orbital plane) in the opposite way should be formally equivalent to time reverse operation $t \leftrightarrow -t$, which cannot make any difference in the result and in other words, there is no radiation reaction in $\dot{\mathcal{E}}_0$. Further, $\partial \dot{\mathcal{E}}_0$ is also even in that they are just partial derivatives of $\dot{\mathcal{E}}_0$ with respect to the $e_t, j$, so they cannot affect to its time dependency structure. On the other hand, $\delta(t)$ comes from time integration of $\dot{\mathcal{E}}_0$ and therefore it should be odd in time. We may now conclude that the reaction correction $\delta \cdot \partial \dot{\mathcal{E}}_0$ is also odd function. Hence, when it comes to total radiation, which involves the integration from $t = -\infty$ to $t = +\infty$, $\delta \cdot \partial \dot{\mathcal{E}}_0$, does not contribute to $\Delta \mathcal{E}$ (likewise to $\Delta \mathcal{J}$) at all.

These arguments ensure that we can express the various PN contributions to the $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ integrands in terms of variables that are associated with the Keplerian type solution to the PN-accurate conservative dynamics of hyperbolic encounters. In other words, we are fully justified to ignore Newtonian order GW emission induced variations in $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ while characterizing the orbital dynamics during the 3PN order $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ computations. Therefore, we employ $e_t$ and $j$ to characterize PN-accurate hyperbolic orbits and can treat them as constant parameters during computing 3PN-accurate expressions for the radiated energy and angular momentum. This is why we have not considered the radiation reaction while we computed the 3PN accurate instantaneous radiation, and hence this argument fully completes our computation.

### E. Parabolic limit

Here, we list the exact values of the parabolic limit $e_t = e = 1$, or $E = 0$ of the total radiations.

$$\Delta \mathcal{E}(E = 0) = \frac{M\nu^2}{15} \left[ \frac{850\pi}{j^7} + \frac{7\pi(5763 - 3220\nu)}{4j^9} + \frac{\pi(29198255 + 378\nu(-86017 + 18270\nu))}{336j^{11}} \right.$$
$$+ \frac{\pi}{80640j^{13}} \left( 179020439969 + 440\nu(-361091813 + 3587787\pi^2 + 945(105219 - 10780\nu)\nu) \right.$$
$$\left. \left. - 13003119360 \log \left( 2\frac{GM}{r_0} j^2 \right) \right) \right], \tag{43}$$

$$\Delta \mathcal{J}(E=0) = \frac{8GM^2\nu^2}{5}\left[\frac{15\pi}{j^4} - \frac{5\pi(-1077+940\nu)}{48j^6} + \frac{\pi(1307683 - 2782332\nu + 1005480\nu^2)}{2592j^8}\right.$$

$$+ \frac{1}{322560j^{10}}\left(\pi(5567205457 + 20(-301863524 + 4719141\pi^2)\nu + 2200128840\nu^2 - 371498400\nu^3)\right.$$

$$\left.\left.- 569479680\pi\log\left(2\frac{GM}{r_0}j^2\right)\right)\right]. \tag{44}$$

We have verified that these expressions are in full agreement with the parabolic limit of the radiated energy and angular momentum during one radial period of an eccentric binary available in Refs. [25,26].

### F. Implications of post-bremsstrahlung expansion

We now probe the implications of the post-bremsstrahlung limit of our $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions. Recall that the bremsstrahlung limit arises by allowing the eccentricity parameter $\to \infty$ as noted in Ref. [19], and we are exploring the implications of eccentricity corrections to such bremsstrahlung limit of our 3PN order hyperbolic expressions. This effort is also influenced by the fact that it is rather difficult to obtain closed form expressions for $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions even when the leading order hereditary contributions are included [16]. Therefore, it is reasonable that our ongoing effort to obtain fully 3PN-accurate expressions for the radiated energy and angular momentum in GWs during hyperbolic encounters will not be exact in orbital eccentricity. In what follows, we probe the implications of post-bremsstrahlung expansion of our 3PN order hyperbolic $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions with respect to their bound orbit counterparts. These counterparts are essentially 3PN-accurate instantaneous $\delta\mathcal{E}$ and $\delta\mathcal{J}$ expressions that provide the radiated energy and angular momentum during one radial period of an eccentric binary. These expressions can easily be obtained from Refs. [25,26], and are exact in orbital eccentricity. Further, we find it convenient to employ the Newtonian eccentricity parameter $e = \sqrt{1 + 2Ej^2}$ to characterize both the bound and unbound far-zone quantities to ensure that the same eccentricity parameter is used in our comparisons. We display the relevant $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions in Appendix. B. Further, the explicit 3PN-accurate instantaneous post-bremsstrahlung $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions for hyperbolic encounters are available at [34], where $\Delta\mathcal{E}$ is expanded up to $\mathcal{O}(1/e^5)$ while $\Delta\mathcal{J}$ has been expanded up to $\mathcal{O}(1/e^7)$.

In Fig. 1, we plot the fractional differences between Refs. [25,26] based $\delta\mathcal{E}$ and $\delta\mathcal{J}$ 3PN order expression that are exact in $e$ and the post-bremsstrahlung expansion of our $\Delta\mathcal{E}$, $\Delta\mathcal{J}$ expressions while allowing $e \leq 1$.

These plots reveal that the post-bremsstrahlung versions of our $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions provide excellent proxies to compact binaries both in parabolic and high eccentric (bound) orbits. These post-bremsstrahlung approximants

are substantially different from their eccentric counterparts near their circular limits. This is most likely due to the presence of $\frac{1}{e}$ terms and their multiples and similar conclusions are drawn from plots where we change values of $\nu$ and the dimensionless $hc$ as evident from Fig. 2. We infer that our post-bremsstrahlung $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions, which should be accurate to describe large eccentric hyperbolic orbit, are not only convergent at the parabolic limit ($e = 1$) but also highly eccentric cases ($e \lesssim 1$). It will be interesting to explore if numerical relativity simulations display a similar behavior. This natural convergence at parabolic limit is contrary to the attempt to cover the parabolic limit adopted in Sec. IX in Ref. [41], where the physical quantities are divergent at parabolic limit, so the parabolic limit is incorporated by numerical methodologies such as fitting and Pade approximation.
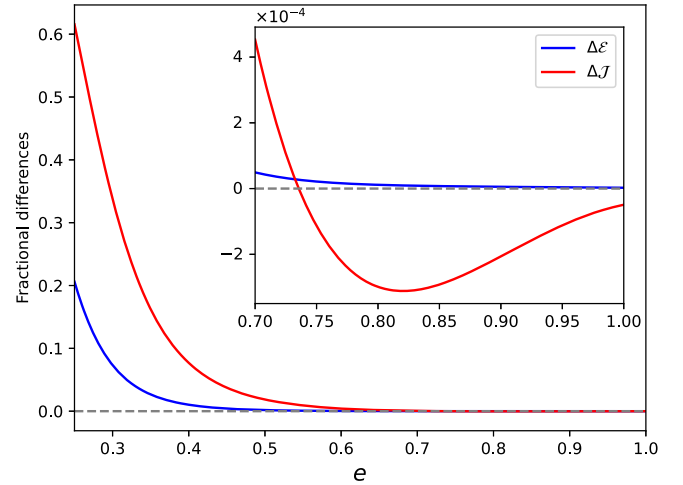


FIG. 1. The fractional difference between the 3PN accurate instantaneous fluxes (i.e., $\frac{\Delta\mathcal{Q}_{\text{hyp}} - \Delta\mathcal{Q}_{\text{ecc}}}{\Delta\mathcal{Q}_{\text{hyp}}}$, where $\mathcal{Q} = \mathcal{E}, \mathcal{J}$) vs Newtonian eccentricity. Hyperbolic fluxes are expanded in the post-bremsstrahlung limit ($e \to \infty$) while the elliptic fluxes have been kept in their original forms, namely in terms of $e, j, \nu$. The blue curve represents the fractional difference of energy fluxes while the red curve represents the fractional difference of angular momentum fluxes. The closeup version focuses on the 0.7 to 1 eccentricity range. The hyperbolic energy flux, as noted in the text, is expanded up to $\mathcal{O}(1/e^5)$ while the angular momentum flux has been expanded up to $\mathcal{O}(1/e^7)$. Additionally, we have chosen $c^2 j^2 = 10$, $\frac{GM}{2r_0}j^2 = 1/5$, and $\nu = 1/4$ for these plots.
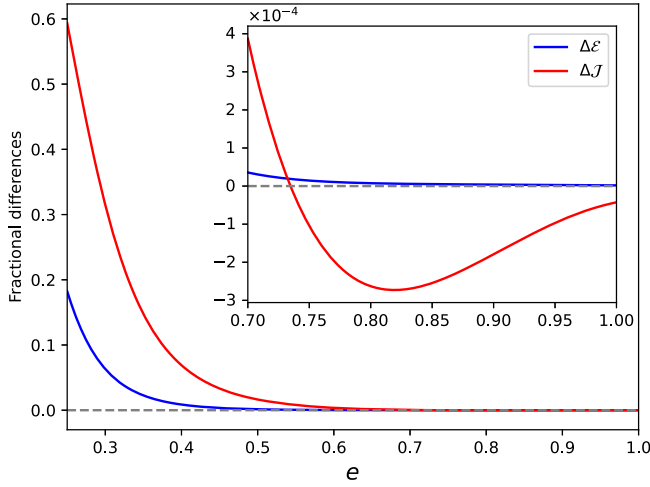
FIG. 2. Plots that are similar to what are displayed in Fig. 1. For these plots, we let $c^2 j^2 = 10$, $\frac{GM}{2r_0} j^2 = 1/5$, and $\nu = 1/20$. Clearly, our conclusions are rather independent of $\nu$ values.

## III. SUMMARY AND ONGOING EFFORTS

We have provided explicit expressions for the 3PN-accurate instantaneous contributions to the radiated energy and angular momentum during hyperbolic encounters. These computations, pursued in the time domain, are not straightforward extensions of the classic 1PN-accurate efforts by Schäfer and his collaborators [19,20]. This is essentially due to the presence of certain logarithmic terms in the 3PN order contributions to the far-zone flux expressions for the general orbits [25]. Additionally, we explored the implications of post-bremsstrahlung expansion of our results from the perspective of eccentric orbits.

There are ongoing efforts to compute hereditary contributions to the $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ expressions that are accurate to 3PN order, influenced by Ref. [15]. We are also pursuing efforts to compare our PN-accurate results with those arising from numerical relativity [13,42]. This will be helpful to explore the validity of PN approximation while exploring GWs from hyperbolic encounters. Further, these efforts may allow us to develop a prescription for describing GW emission aspects of highly eccentric compact binaries with constructs that arise from our present and ongoing PN-accurate hyperbolic computations.

## APPENDIX A: HYPERBOLIC LOG INTEGRALS

We provide the details of evaluating certain 3PN order log integrals that are crucial for our results. We begin with the following expression:

$$I(x, y) := \int_{-\infty}^{+\infty} du \, \frac{\ln(e \cosh u - x)}{e \cosh u - y}. \quad (A1)$$

The goal is to first take a differentiation followed by an integration of $I(x, y)$ with respect to $x$. Differentiating the above expression with respect to $x$ yields

$$-\partial_x I(x, y) = \int_{-\infty}^{+\infty} du \, \frac{1}{e \cosh u - x} \frac{1}{e \cosh u - y},$$
$$= J_1(x, y) + J_2(x, y), \quad (A2a)$$

where

$$J_1(x, y) := \frac{2}{\sqrt{e^2 - x^2}(x - y)} \arccos\left(-\frac{x}{e}\right), \quad (A2b)$$

$$J_2(x, y) := \frac{2}{\sqrt{e^2 - y^2}(y - x)} \arccos\left(-\frac{y}{e}\right). \quad (A2c)$$

The antiderivative of $-J_2$ with respect to $x$ is

$$I_2 := -\int dx J_2 = \frac{2\ln(y - x)}{\sqrt{e^2 - y^2}} \arccos\left(-\frac{y}{e}\right). \quad (A3)$$

To integrate $J_1$, we make a change of variable $f := \arccos\left(-\frac{x}{e}\right)$, then

$$I_1 := -\int dx J_1 = \int df \, \frac{2f}{e \cos f + y}$$
$$= \frac{2i}{\sqrt{e^2 - y^2}} [\text{Li}_2(g) - \text{Li}_2(g^*)]$$
$$- \frac{2}{\sqrt{e^2 - y^2}} \left[ \arccos\left(-\frac{y}{e}\right) \right.$$
$$\left. \times \log\left(-\frac{e^2 - xy + \sqrt{e^2 - x^2}\sqrt{e^2 - y^2}}{4e(x - y)}\right) \right] + \alpha(y; e), \quad (A4)$$

where

$$g = \frac{(y - i\sqrt{e^2 - y^2})}{e} \frac{(e+y)\sqrt{e^2 - x^2} - (e+x)\sqrt{e^2 - y^2}}{(e+y)\sqrt{e^2 - x^2} + (e+x)\sqrt{e^2 - y^2}},$$

(A5a)

$$g^* = \frac{(y + i\sqrt{e^2 - y^2})}{e} \frac{(e+y)\sqrt{e^2 - x^2} - (e+x)\sqrt{e^2 - y^2}}{(e+y)\sqrt{e^2 - x^2} + (e+x)\sqrt{e^2 - y^2}}.$$

(A5b)

Therein, $\text{Li}_2$ is the polylogarithm function of order 2. These results are valid up to modulo a constant $\alpha(y; e)$ that is independent of $x$. To determine the expression of $\alpha$, let us consider $x \to -\infty$ behavior such that

$$\lim_{x \to -\infty} I(x, y) = \lim_{x \to -\infty} \frac{2 \log(-x)}{\sqrt{e^2 - y^2}} \arccos\left(-\frac{y}{e}\right).$$

(A6)

An, it can be easily checked that

$$\lim_{x \to -\infty} I(x, y) = \lim_{x \to -\infty} I_2(x, y),$$

(A7)

which implies

$$\lim_{x \to -\infty} I_1(x, y) = 0,$$

(A8)

to keep $I = I_1 + I_2$ held. Making use of this condition at $x \to -\infty$ on Eq. (A4), we reach

$$\alpha(y; e) = \frac{2i}{\sqrt{e^2 - y^2}} \left\{ \frac{\pi^2}{6} - \text{Li}_2\left[ -\frac{(iy + \sqrt{e^2 - y^2})^2}{e^2} \right] - i \arccos\left(-\frac{y}{e}\right) \log\left[ \frac{4(y - i\sqrt{e^2 - y^2})}{e} \right] \right\}.$$

(A9)

In the above we used $\lim_{x \to -\infty} \text{Li}_2(g^*) = \frac{\pi^2}{6}$. Thus, we finally obtain

$$I = \frac{2i}{\sqrt{e^2 - y^2}} \left\{ \frac{\pi^2}{6} + \text{Li}_2(g) - \text{Li}_2(g^*) - \text{Li}_2\left( -\frac{(iy + \sqrt{e^2 - y^2})^2}{e^2} \right) - i \arccos\left(-\frac{y}{e}\right) \right.$$
$$\left. \times \log\left[ \frac{(y - i\sqrt{e^2 - y^2})(e^2 - xy + \sqrt{e^2 - x^2}\sqrt{e^2 - y^2})}{e^2} \right] \right\}.$$

(A10)

Now, we focus on a special condition on Eq. (A10) where $x \to y$, the integral we are interested to compute. Note that when $y \to x$, there are some divergences because of $\sim \log(y - x)$ term in $I_1$, but with proper calculus, these divergences are canceled while taking $y \to x$ limit, and we get

$$\int_{-\infty}^{\infty} du \frac{\ln(e \cosh u - x)}{e \cosh u - x} = \frac{2i}{\sqrt{e^2 - x^2}} \left\{ \frac{\pi^2}{6} - \text{Li}_2\left( -\frac{(ix + \sqrt{e^2 - x^2})^2}{e^2} \right) - i \arccos\left(-\frac{x}{e}\right) \log\left[ \frac{2(e^2 - x^2)(x - i\sqrt{e^2 - x^2})}{e^2} \right] \right\}.$$

(A11)

Equation (A11) can be further simplified using the usual definition of Clausen function $\text{Cl}_2(x)$ [29] of order 2 given by the integral

$$\text{Cl}_2(x) = -\int_0^x dx \log\left| 2 \sin\frac{x}{2} \right|.$$

(A12)

There exists a nice expression of the dilogarithm function :$\text{Li}_2(x)$ in terms of Clausen function, given by

$$\text{Li}_2(e^{i\theta}) = \frac{\pi^2}{6} - \frac{1}{4}|\theta|(2\pi - |\theta|) + i\text{Cl}_2(\theta), \quad |\theta| \le 2\pi.$$

(A13)

By virtue of Eq. (A13) we obtain the most simplified form of the integral $I(x)$. Our final expression is

$$\int_{-\infty}^{\infty} du \frac{\ln(e \cosh u - x)}{e \cosh u - x} = \frac{2}{\sqrt{e^2 - x^2}} \left\{ \text{Cl}_2\left(-\frac{x}{e}\right) + \arccos\left(-\frac{x}{e}\right) \ln\left( 2e - \frac{2x^2}{e} \right) \right\}.$$

(A14)

## APPENDIX B: ENERGY AND ANGULAR MOMENTUM RADIATIONS IN TERMS OF ENERGY AND ANGULAR MOMENTUM

Since the time eccentricity $e_t$ is not gauge invariant, we provide maximally gauge invariant expressions of $\Delta\mathcal{E}$ and $\Delta\mathcal{J}$ in terms of energy $E$ and angular momentum $j$. Note that it could be never possible to provide a fully gauge invariant expression, because the instantaneous part of $\Delta\mathcal{E}$, $\Delta\mathcal{J}$ is not gauge invariant, since they depend on the separation of the scale $r_0$ differentiating short/long scales. Except the issue of $r_0$ which will be removed by hereditary contribution, the following expressions are gauge invariant. Here, we use the Newtonian eccentricity $e$ as a shorthand symbol of $\sqrt{1 + 2Ej^2}$ without implying any geometric meaning.

$$\Delta\mathcal{E} = \frac{2}{15}\frac{M\nu^2}{j^7c^5}\left(\mathcal{I}_N^{\mathrm{MH}} + \frac{1}{j^2c^2}\mathcal{I}_{1\mathrm{PN}}^{\mathrm{MH}} + \frac{1}{j^4c^4}\mathcal{I}_{2\mathrm{PN}}^{\mathrm{MH}} + \frac{1}{j^6c^6}\mathcal{I}_{3\mathrm{PN}}^{\mathrm{MH}}\right),$$

(B1)

where

$$\mathcal{I}_N^{\mathrm{MH}} = \sqrt{e^2 - 1}\left[\frac{602}{3} + \frac{673e^2}{3}\right] + \arccos\left(-\frac{1}{e}\right)[96 + 292e^2 + 37e^4],$$

(B2a)

$$\mathcal{I}_{1\mathrm{PN}}^{\mathrm{MH}} = \sqrt{e^2 - 1}\left[\frac{108 + 12\nu}{e^2} + \left(\frac{90754}{35} - \frac{2185\nu}{6}\right) + e^2\left(\frac{141439}{70} - \frac{11441\nu}{6}\right) + e^4\left(\frac{89907}{280} - \frac{1117\nu}{2}\right)\right]$$
$$+ \arccos\left(-\frac{1}{e}\right)\left[\left(\frac{11177}{7} - 95\nu\right) + e^2\left(\frac{37785}{14} - \frac{3051\nu}{2}\right) + e^4\left(\frac{2817}{4} - \frac{2283\nu}{2}\right) + e^6\left(\frac{2393}{56} - \frac{111\nu}{2}\right)\right],$$

(B2b)

$$\mathcal{I}_{2\mathrm{PN}}^{\mathrm{MH}} = \sqrt{e^2 - 1}\left[\frac{1}{e^4}\left(-\frac{243}{2} - 27\nu - \frac{3\nu^2}{2}\right) + \frac{1}{e^2}\left(\frac{207423}{112} - \frac{2515\nu}{56} - \frac{179\nu^2}{16}\right) + \left(\frac{8409586747}{317520}\right.\right.$$
$$- \frac{9858601\nu}{720} - \frac{1569\nu^2}{4}\right) + e^2\left(\frac{10125905183}{635040} - \frac{38505773\nu}{1440} + 4098\nu^2\right) + e^4\left(-\frac{154205167}{158760} - \frac{36653233\nu}{5040}\right.$$
$$\left.\left. + \frac{11581\nu^2}{2}\right) + e^6\left(\frac{509759}{1960} - \frac{680471\nu}{1120} + \frac{12693\nu^2}{16}\right)\right] + \arccos\left(-\frac{1}{e}\right)\left[\frac{52925837}{3024} - \frac{158383\nu}{24} - \frac{4787\nu^2}{16}\right.$$
$$+ e^2\left(\frac{36730439}{1512} - \frac{4415855\nu}{168} + \frac{3877\nu^2}{2}\right) + e^4\left(\frac{479005}{288} - \frac{3069361\nu}{224} + \frac{25641\nu^2}{4}\right) + e^6\left(-\frac{5885}{84}\right.$$
$$\left.\left. - \frac{3489\nu}{2} + \frac{4339\nu^2}{2}\right) + e^8\left(\frac{745}{12} - \frac{11965\nu}{224} + \frac{925\nu^2}{16}\right)\right],$$

(B2c)

$$\mathcal{I}_{3\mathrm{PN}}^{\mathrm{MH}} = \sqrt{e^2 - 1}\left[\frac{6713608}{1575} + \frac{17868572e^2}{525} + \frac{19300553e^4}{525} + \frac{17525209e^6}{3150}\right]\log\left[\frac{2r_0(e^2 - 1)}{GMj^2e}\right] - \left[\frac{54784}{35}\right.$$
$$+ \frac{465664e^2}{21} + \frac{4426376e^4}{105} + \frac{1498856e^6}{105} + \frac{31779e^8}{70}\right]\left\{\log\left[\frac{2GMj^2}{r_0e}\right]\arccos\left(-\frac{1}{e}\right)\right.$$
$$\left. + \mathrm{Cl}_2\left[2\arccos\left(-\frac{1}{e}\right)\right]\right\} + \sqrt{e^2 - 1}\left[\frac{1}{4e^6}(729 + 243\nu + 27\nu^2 + \nu^3) + \frac{1}{e^4}\left(-\frac{1106055}{448}\right.\right.$$
$$- \frac{63027\nu}{448} + \frac{10159\nu^2}{448} + \frac{167\nu^3}{192}\right) + \frac{1}{e^2}\left(\frac{15311843}{672} + \left(-\frac{2862985}{378} + \frac{123\pi^2}{2}\right)\nu - \frac{426791\nu^2}{672} - \frac{79\nu^3}{4}\right)$$
$$+ \frac{40066564486859}{139708800} + \left(-\frac{205531241131}{635040} + \frac{13414913\pi^2}{3360}\right)\nu + \frac{833407139\nu^2}{40320} + \frac{206305\nu^3}{192}$$
$$+ e^2\left(\frac{8653194752669}{23284800} + \left(-\frac{243329285581}{423360} + \frac{2172959\pi^2}{280}\right)\nu + \frac{310440959\nu^2}{2240} - \frac{4703\nu^3}{48}\right)$$
$$+ e^4\left(\frac{10874967898483}{46569600} + \left(-\frac{13311394987}{141120} - \frac{995521\pi^2}{896}\right)\nu + \frac{131367287\nu^2}{1344} - \frac{1086249\nu^3}{64}\right)$$

$$
+ e^6 \left( \frac{723666251779}{19958400} + \left( \frac{19905131933}{1270080} - \frac{12303239\pi^2}{13440} \right) \nu + \frac{284599649\nu^2}{20160} - \frac{261641\nu^3}{24} \right) + e^8 \left( \frac{248312131}{4139520} \right.
$$

$$
\left. - \frac{11427029\nu}{28224} + \frac{3374859\nu^2}{4480} - \frac{56047\nu^3}{64} \right) \bigg] + \arccos\left( -\frac{1}{e} \right) \left[ \frac{642008180503}{3326400} + \left( -\frac{21604997}{126} + \frac{15785\pi^2}{8} \right) \nu \right.
$$

$$
+ \frac{7846337\nu^2}{1344} + \frac{12633\nu^3}{32} + e^2 \left( \frac{2827833857903}{6652800} + \left( -\frac{6912058987}{12096} + \frac{124763\pi^2}{16} \right) \nu + \frac{266844577\nu^2}{2688} \right)
$$

$$
+ \frac{26815\nu^3}{16} \right) + e^4 \left( \frac{1246592422921}{3326400} + \left( -\frac{3120760105}{12096} + \frac{112709\pi^2}{64} \right) \nu + \frac{174794687\nu^2}{1344} - \frac{91945\nu^3}{8} \right)
$$

$$
+ e^6 \left( \frac{31441963853}{277200} + \left( \frac{86715535}{6048} - \frac{211027\pi^2}{128} \right) \nu + \frac{7483281\nu^2}{224} - \frac{245475\nu^3}{16} \right) + e^8 \left( \frac{5518709201}{1478400} + \left( \frac{2020985}{1344} \right. \right.
$$

$$
\left. \left. - \frac{12177\pi^2}{128} \right) \nu + \frac{296053\nu^2}{112} - \frac{95255\nu^3}{32} \right) + e^{10} \left( \frac{2075735}{118272} - \frac{745\nu}{12} + \frac{45467\nu^2}{896} - \frac{407\nu^3}{8} \right) \bigg]. \tag{B2d}
$$

Similarly, the 3PN-accurate instantaneous contributions to the radiated angular momentum terms of $e$ and $j$ become

$$
\Delta \mathcal{J} = \frac{8}{5c^5} \frac{GM^2\nu^2}{j^4} \left( \mathcal{H}_N^{\mathrm{MH}} + \frac{1}{j^2 c^2} \mathcal{H}_{1\mathrm{PN}}^{\mathrm{MH}} + \frac{1}{j^4 c^4} \mathcal{H}_{2\mathrm{PN}}^{\mathrm{MH}} + \frac{1}{j^6 c^6} \mathcal{H}_{3\mathrm{PN}}^{\mathrm{MH}} \right), \tag{B3}
$$

where

$$
\mathcal{H}_N^{\mathrm{MH}} = \sqrt{e^2 - 1} [13 + 2e^2] + \arccos\left( -\frac{1}{e} \right) [8 + 7e^2], \tag{B4a}
$$

$$
\mathcal{H}_{1\mathrm{PN}}^{\mathrm{MH}} = \sqrt{e^2 - 1} \left[ \frac{9 + \nu}{e^2} + \left( \frac{11525}{168} - \frac{1451\nu}{36} \right) + e^2 \left( \frac{10313}{336} - \frac{2065\nu}{36} \right) + e^4 \left( \frac{109}{28} - \frac{5\nu}{4} \right) \right]
$$

$$
+ \arccos\left( -\frac{1}{e} \right) \left[ \frac{4877}{84} - \frac{65\nu}{4} + e^2 \left( \frac{869}{21} - \frac{419\nu}{6} \right) + e^4 \left( \frac{4283}{336} - \frac{71\nu}{6} \right) \right], \tag{B4b}
$$

$$
\mathcal{H}_{2\mathrm{PN}}^{\mathrm{MH}} = \sqrt{e^2 - 1} \left[ -\frac{1}{8e^4} (81 + 18\nu + \nu^2) + \frac{1}{e^2} \left( \frac{41169}{448} - \frac{11839\nu}{672} - \frac{109\nu^2}{64} \right) + \frac{139894943}{272160} - \frac{17944223\nu}{30240} \right.
$$

$$
+ \frac{1915\nu^2}{144} + e^2 \left( -\frac{18473111}{272160} - \frac{201089\nu}{540} + \frac{25595\nu^2}{96} \right) + e^4 \left( -\frac{13557}{560} - \frac{870131\nu}{10080} + \frac{15701\nu^2}{144} \right)
$$

$$
+ e^6 \left( \frac{3245}{4032} - \frac{327\nu}{224} + \frac{51\nu^2}{64} \right) \right] + \arccos\left( -\frac{1}{e} \right) \left[ \frac{2051491}{4536} - \frac{46297\nu}{126} - \frac{79\nu^2}{12} + e^2 \left( \frac{752767}{6048} \right. \right.
$$

$$
\left. \left. - \frac{121675\nu}{224} + \frac{4433\nu^2}{24} \right) + e^4 \left( -\frac{28297}{378} - \frac{24611\nu}{168} + \frac{783\nu^2}{4} \right) + e^6 \left( \frac{1327}{504} - \frac{3653\nu}{224} + \frac{337\nu^2}{24} \right) \right], \tag{B4c}
$$

$$
\mathcal{H}_{3\mathrm{PN}}^{\mathrm{MH}} = \sqrt{e^2 - 1} \left[ \frac{99724}{315} + \frac{351067e^2}{315} + \frac{210683e^4}{630} \right] \log\left[ \frac{r_0(e^2 - 1)}{GMj^2 e} \right] - \left[ \frac{13696}{105} + \frac{98012e^2}{105} + \frac{23326e^4}{35} + \frac{2461e^6}{70} \right]
$$

$$
\times \left\{ \log\left[ \frac{2GMj^2}{r_0 e} \right] \arccos\left( -\frac{1}{e} \right) + \mathrm{Cl}_2\left[ 2\arccos\left( -\frac{1}{e} \right) \right] \right\} + \sqrt{e^2 - 1} \left[ \frac{1}{16e^6} (243 + 81\nu + 9\nu^2 + \nu^3) \right.
$$

$$
+ \frac{1}{e^4} \left( -\frac{257553}{1792} + \frac{16035\nu}{1792} + \frac{22507\nu^2}{5376} + \frac{121\nu^3}{768} \right) + \frac{1}{e^2} \left( \frac{27158603}{32256} + \left( -\frac{165077725}{290304} + \frac{41\pi^2}{8} \right) \nu \right.
$$

$$
\left. - \frac{718787\nu^2}{32256} - \frac{237\nu^3}{512} \right) + \frac{387361378703}{67060224} + \left( -\frac{337799063429}{30481920} + \frac{340423\pi^2}{1920} \right) \nu + \frac{878650477\nu^2}{483840} + \frac{344695\nu^3}{4608}
$$

$$
+ e^2 \left( \frac{1005116464409}{167650560} + \left( -\frac{23645568707}{3048192} + \frac{512213\pi^2}{3840} \right) \nu + \frac{134389529\nu^2}{34560} - \frac{708305\nu^3}{2304} \right) + e^4 \left( \frac{432742948027}{335301120} \right.
$$

$$+ \left( \frac{5045091193}{7620480} - \frac{44567\pi^2}{1920} \right)\nu + \frac{60548279\nu^2}{60480} - \frac{886135\nu^3}{1152} \right) + e^6 \left( -\frac{1172569271}{24837120} + \frac{17091961\nu}{1128960} + \frac{4248733\nu^2}{32256} \right.$$

$$\left. - \frac{687113\nu^3}{4608} \right) + e^8 \left( -\frac{6973}{354816} - \frac{3245\nu}{32256} + \frac{3379\nu^2}{3584} - \frac{249\nu^3}{512} \right) \Bigg] + \arccos\left( -\frac{1}{e} \right) \Bigg[ \frac{26951805241}{5702400}$$

$$+ \left( -\frac{503311993}{72576} + \frac{3239\pi^2}{32} \right)\nu + \frac{12801961\nu^2}{16128} + \frac{2479\nu^3}{64} + e^2 \left( \frac{83623027931}{9979200} + \left( -\frac{408126835}{36288} + \frac{6355\pi^2}{32} \right)\nu \right.$$

$$\left. + \frac{15335105\nu^2}{4032} - \frac{3907\nu^3}{48} \right) + e^4 \left( \frac{6781095059}{1663200} + \left( -\frac{18560153}{24192} - \frac{615\pi^2}{256} \right)\nu + \frac{5148791\nu^2}{2688} - \frac{71389\nu^3}{96} \right)$$

$$+ e^6 \left( \frac{254451209}{3326400} + \left( \frac{1448255}{6048} - \frac{615\pi^2}{128} \right)\nu + \frac{391981\nu^2}{1344} - \frac{16867\nu^3}{48} \right) + e^8 \left( -\frac{227005}{709632} - \frac{14411\nu}{2016} + \frac{29727\nu^2}{1792} \right.$$

$$\left. - \frac{2693\nu^3}{192} \right) \Bigg].$$

(B4d)

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