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Resummation of small-x double logarithms in QCD: inclusive deep-inelastic scattering

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Abstract: We present a comprehensive study of high-energy double logarithms in inclusive DIS. They appear parametrically as $\alpha_s^n \ln^{2n-k} x$ at the n-th order in perturbation theory in the splitting functions for the parton evolution and the coefficient functions for the hard scattering process, and represent the leading corrections at small x in the flavour non-singlet case. We perform their resummation, in terms of modified Bessel functions, to all orders in full QCD up to NNLL accuracy, and partly to N^{3}LL and beyond in the large-n_c limit, and provide fixed-order expansions up to five loops. In the flavour-singlet sector, where these double logarithms are sub-dominant at small x compared to single-logarithmic $\alpha_s^n\, x^{-1} \ln^{n-k} x$ BFKL contributions, we construct fixed-order expansions up to five loops at NNLL accuracy in full QCD. The results elucidate the analytic small-x structure underlying inclusive DIS results in fixed-order perturbation theory and provide important information for present and future numerical and analytic calculations of these quantities.

Keywords: Deep Inelastic Scattering or Small-x Physics, Factorization, Renormalization Group, Higher-Order Perturbative Calculations, Resummation

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1 Introduction

Inclusive deep-inelastic lepton-hadron scattering (DIS) is an experimental and theoretical reference process for Quantum Chromodynamics (QCD), the theory of the strong interaction. Important information on the parton (quark and gluon) distribution functions (PDFs) of the proton, in particular, is provided by the dependence of the corresponding cross sections or the structure functions $F_a(x, Q^2)$ on the Bjorken variable x and on the scale $Q^2 = -q^2$ set by the momentum q of the exchanged (gauge) boson. Moreover the scaling violations, i.e., the Q^2-dependence of the structure functions F_2 and F_3, facilitate high-precision determinations of the strong coupling constant α_s.

Due to their relation to propagator-type Feynman integrals via forward Compton amplitudes and the light-cone operator-product expansion (OPE), see, e.g., refs. [1–4] and references therein, structure functions in DIS are particularly well suited for analytical high-order computations in massless perturbative QCD. Indeed, the complete third-order contributions to the (initial-state) splitting functions governing the evolution of the PDFs were obtained more than fifteen years ago [5, 6] in computations that also provided the third-order cross-section projections (coefficient functions) for the most important structure functions in spin-averaged DIS [7–9]. During the past five years those computations have been extended, if only for a limited number of Mellin moments, to the fourth order in α_s [10–13]; for the lowest moments see also refs. [14–18].

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The perturbation series for the splitting functions and coefficient functions appear to be very well behaved (except for the longitudinal structure function F_L [7, 10]) outside the threshold region $1-x \ll 1$ and the high-energy limit $x \to 0$. With the exception of the diagonal (quark-quark and gluon-gluon) splitting functions in the standard $\overline{\mathrm{MS}}$ scheme [19–21], the splitting and coefficient functions include threshold double logarithms at all powers of $1-x$. The dominant $(1-x)^{-1} \ln^{\ell}(1-x)$ contributions to the coefficient functions in DIS have been resummed to a high accuracy [22, 23], see also ref. [24], in the framework of the soft-gluon exponentiation [25–30]. The resummation of the double logarithms has been extended to non-negative powers of $1-x$ by analyzing the physical evolution kernels of the structure functions [31–33], see also ref. [34], and the structure of the 'raw' (unfactorized) expressions in dimensional regularization [35–37].

The latter approach can be applied also to high-energy double logarithms $\sim \alpha_s^n \, x^p \ln^{2n-n_0-k} x$ in splitting functions and coefficient functions, albeit, at least in its present form, not at all powers p of x. In particular, the resummation of the dominant $p = -1$ contributions to the splitting functions for the final-state parton fragmentation functions and to the coefficient functions for semi-inclusive electron-positron annihilation (SIA) [38, 39] have been extended to the $k = 2$ next-to-next-to-leading logarithmic (N^{2}LL) accuracy in refs. [40, 41], see also refs. [42, 43]. Corresponding $p = 0$ results for inclusive DIS were obtained at about the same time. While being formally analogous to their SIA counterparts, these results were not of direct phenomenological relevance, and only one example expression was presented at the time [44].

Such results become relevant, however, for approximate or exact reconstructions of higher-order splitting functions and coefficient functions, if they can be combined with a sufficient amount of other information, such as a sufficiently large number of Mellin moments. Due to the development of the Forcer program [45] for four-loop propagator-type Feynman integrals, this point has now been reached for fourth-order corrections; see refs. [11, 46, 47] for published results on the splitting functions. In fact, in the latter two articles fourth-order predictions of the resummations discussed above and of a complementary proposal of ref. [48] have already been employed and, where feasible, confirmed. Hence it is now timely to present, in sufficient detail to assist future research, the status of the resummation of small-x double logarithms for inclusive DIS.

The remainder of this article is organized as follows: in section 2 we specify our notation and discuss the available formalisms, and their limitations, to the resummation of small-x (double) logarithms. We also briefly indicate how the calculations have been performed. The results for the splitting function for the evolution of flavour differences of sums of quark and antiquark PDFs are presented in section 3. This is the case for which the two complementary approaches overlap. The results include another striking illustration of the phenomenological inadequacy of representing the splitting functions, at any relevant x, solely by a N$^{\ell}$LL small-x approximation at some fixed ℓ.

In sections 4 and 5 we present the N^{2}LL predictions for the corresponding non-singlet coefficient functions and for the flavour-singlet splitting and coefficient functions. In view of the findings in section 3, we focus in these sections on fourth- and fifth-order predictions and the all-order structure of the $x^0 \ln^k x$ contributions. We expect that the former results will become useful in combination with large-x information on these functions, while the leading-logarithmic all-order expressions may provide useful 'data' for future research into the small-x structure of splitting functions and coefficient functions in DIS. We briefly summarize our findings in section 6. Some additional material that may be useful to future research can be found in the appendix.

2 Notation, formalism and calculations

Disregarding $1/Q^2$ power corrections, the structure functions in DIS can be generically written as

$$F_a(x, Q^2) = \Big[C_{a,\mathrm{i}}(a_\mathrm{s}) \, \otimes \, f_\mathrm{i}(Q^2) \Big](x) \tag{2.1}$$

in terms of the coefficient functions $C_{a,\mathrm{i}}(x, a_\mathrm{s})$ and the corresponding (combinations of) parton distributions $f_\mathrm{i}(x, Q^2)$. Here and below we identify the renormalization and mass-factorization scales μ_R^2 and μ_F^2 with the physical scale Q^2; the dependence on μ_R^2 and μ_F^2 can be readily reconstructed a posteriori, see, e.g., sections 2 of refs. [49, 50]. $\otimes$ represents the Mellin convolution, given by

$$[\, a \otimes b\,](x) = \int_x^1 \frac{dz}{z} \, a(z) \, b\left(\frac{x}{z}\right) \tag{2.2}$$

and its generalization for plus-distributions, which corresponds to a simple product in Mellin space. The scale dependence of the PDFs f_i is given by the renormalization-group evolution equations

$$\frac{d}{d \ln Q^2} \, f_\mathrm{i}(x, Q^2) = \Big[P_\mathrm{ik}(a_\mathrm{s}) \, \otimes \, f_\mathrm{k}(Q^2) \Big](x) \, . \tag{2.3}$$

The coefficient functions C_a in eq. (2.1) and the splitting functions P_ik in eq. (2.3) can be expanded in powers of the strong coupling constant, which we normalize as $a_\mathrm{s} = \alpha_\mathrm{s}(Q^2)/(4\pi)$,

$$P\,(x, a_\mathrm{s}) = a_\mathrm{s} \, P^{\,(0)}(x) + a_\mathrm{s}^2 \, P^{\,(1)}(x) + a_\mathrm{s}^3 \, P^{\,(2)}(x) + a_\mathrm{s}^4 \, P^{\,(3)}(x) + \ldots \, , \tag{2.4}$$

$$C_a\,(x, a_\mathrm{s}) = c_a^{(0)}(x) + a_\mathrm{s} \, c_a^{(1)}(x) + a_\mathrm{s}^2 \, c_a^{(2)}(x) + a_\mathrm{s}^3 \, c_a^{(3)}(x) + \ldots \tag{2.5}$$

with

$$c_{a,\mathrm{i}}^{(0)}(x) = \delta_{\,\mathrm{iq}} \, \delta(1-x) \quad \text{for} \quad a = 2, 3\,, \quad c_{L,\mathrm{i}}^{(0)}(x) = 0\,, \tag{2.6}$$

where $\mathrm{i} = \mathrm{q}, \mathrm{g}$. Consequently, the terms up to $c^{(n)}$ and $P^{(n)}$ form the (next-to)^n-leading order (N^{n}LO) approximation of perturbative QCD for F_2 and F_3, while $c_L^{(n+1)}$ and $P^{(n)}$ are required for this accuracy for the longitudinal structure function F_L.

The coefficients in eqs. (2.4) and (2.5) include high-energy double logarithms, with contributions up to $x^p \ln^{2n} x$ for $P^{(n)}$ and $c_L^{(n+1)}$, and terms up to $x^p \ln^{2n+1} x$ for $c_2^{(n+1)}$ and $c_3^{(n+1)}$ at $p \geq 0$.[1] Our main approach to the resummation of these logarithms, i.e., to the determination of the coefficients of $a_\mathrm{s}^n \, x^p \ln^{2n-n_0-k} x$ contributions to eqs. (2.4)

[1]These quantities do not include small-x double logarithms at $p = -1$, see refs. [51–53] and references therein.

and (2.5), is analogous to that presented in ref. [40] for the case of $p = -1$ in semi-inclusive e^+e^- annihilation.

The primary objects of this resummation are the unfactorized partonic structure functions

$$\widehat{F}(x, a_{\rm s}, \epsilon) = [\,\widetilde{C}(a_{\rm s}, \epsilon) \,\otimes\, Z(a_{\rm s}, \epsilon)\,](x) \tag{2.7}$$

in dimensional regularization (we use $D = 4 - 2\,\epsilon$) where, for simplicity, the indices labelling different structure functions and parton distributions have been suppressed. The D-dimensional coefficient functions $\widetilde{C}$ are given by Taylor series in the renormalized coupling $a_{\rm s}$ and ϵ,

$$\widetilde{C}_a(a_{\rm s}, \epsilon) = \sum_{n=0} \sum_{m=0} a_{\rm s}^{\,n}\, \epsilon^{\,m}\, c_a^{(n,m)}\,, \tag{2.8}$$

where the ϵ^m terms include m more powers in $\ln x$ than the 4-dimensional coefficient functions $c_a^{(n)} \equiv c_a^{(n,0)}$ in eq. (2.5). The transition functions Z collect the negative powers of ϵ arising from initial-state collinear singularities; these functions can be written in terms of the splitting functions (2.4) and the expansion coefficients β_n of the D-dimensional beta function,

$$\beta_D(a_{\rm s}) = -\epsilon\, a_{\rm s} - \beta_0\, a_{\rm s}^2 - \beta_1\, a_{\rm s}^3 - \beta_2\, a_{\rm s}^4 - \ldots\,, \tag{2.9}$$

with [54–57]

$$\beta_0 = \frac{11}{3}\, C_A - \frac{2}{3}\, n_f\,, \quad \beta_1 = \frac{34}{3}\, C_A^2 - \frac{10}{3}\, C_A n_f - 2\, C_F n_f \tag{2.10}$$

etc where $C_A = n_c = 3$ and $C_F = (n_c^2 - 1)/(2n_c) = 4/3$ in QCD. Here and below, n_f denotes the number of light flavours.

In Mellin N-space, the transition functions are related to the splitting functions by

$$P = \frac{d\,Z}{d\,\ln Q^2}\, Z^{\,-1} = \beta_D(a_{\rm s})\, \frac{d\,Z}{da_{\rm s}}\, Z^{\,-1}\,. \tag{2.11}$$

Using this relation, $Z(N, a_{\rm s})$ can be expressed order-by-order in terms of the anomalous dimensions $\gamma_n(N) = -P^{\,(n)}(N)$ and β_n. The first four orders in $a_{\rm s}$, allowing for Z and γ_n to be matrices, read

$$\begin{aligned}
Z = {} & 1 + a_{\rm s}\, \frac{1}{\epsilon}\, \gamma_0 + a_{\rm s}^2 \left\{ \frac{1}{2\epsilon^2}\, (\gamma_0 - \beta_0)\gamma_0 + \frac{1}{2\epsilon}\, \gamma_1 \right\} \\
& + a_{\rm s}^3 \left\{ \frac{1}{6\epsilon^3}\, (\gamma_0 - \beta_0)(\gamma_0 - 2\beta_0)\gamma_0 + \frac{1}{6\epsilon^2} \Big[(\gamma_0 - 2\beta_0)\gamma_1 + 2(\gamma_1 - \beta_1)\gamma_0 \Big] + \frac{1}{3\epsilon}\, \gamma_2 \right\} \\
& + a_{\rm s}^4 \Bigg\{ \frac{1}{24\,\epsilon^4}\, (\gamma_0 - \beta_0)(\gamma_0 - 2\beta_0)(\gamma_0 - 3\beta_0)\gamma_0 + \frac{1}{24\,\epsilon^3}\, \Big[(\gamma_0 - 2\beta_0)(\gamma_0 - 3\beta_0)\gamma_1 \\
& \qquad + 2(\gamma_0 - 3\beta_0)(\gamma_1 - \beta_1)\gamma_0 + 3(\gamma_1 - 2\beta_1)(\gamma_0 - \beta_0)\gamma_0 \Big] \\
& \qquad + \frac{1}{24\,\epsilon^2}\, \Big[2(\gamma_0 - 3\beta_0)\gamma_2 + 3(\gamma_1 - 2\beta_1)\gamma_1 + 6(\gamma_2 - \beta_2)\gamma_0 \Big] + \frac{1}{4\epsilon}\, \gamma_3 \Bigg\} + \ldots\,.
\end{aligned} \tag{2.12}$$

The higher-order contributions have been generated using FORM and TFORM [58–60] to a sufficiently high order for the computations of this paper. At order $a_s^{\,n}$, the dependence of Z on β_m and γ_m can be summarized as

$$\epsilon^{\,-n} :\ \gamma_0,\ \beta_0\,, \quad \epsilon^{\,-n+1} :\ \ldots\,,\ \gamma_1,\ \beta_1\,, \quad \epsilon^{\,-n+2} :\ \ldots\,,\ \gamma_2,\ \beta_2\,, \quad \ldots\,, \quad \epsilon^{\,-1} :\ \gamma_{n-1}\,. \tag{2.13}$$

Hence fixed-order knowledge at N^{m}LO (i.e., of the splitting functions to $P^{\,(m)}$, beta function to β_m and the corresponding coefficient functions) fixes the first $m{+}1$ coefficients in the ϵ-expansion of $\widehat{F}$ at all orders in $a_{\rm s}$. Furthermore, the property $P^{\,(n)} \sim \ln^{2n}\!x \ \Leftrightarrow\ \gamma_n \sim N^{\,-2n-1}$ means that β_0 and β_0^2 enter eq. (2.12) at the next-to-leading logarithmic (NLL) and N^{2}LL level, while β_1 contributes only from the fourth (N^{3}LL) logarithms. Similarly, β_2 enters only at N^{5}LL accuracy and beyond.

If the above N^{m}LO knowledge can be extended to all powers in ϵ at a given logarithmic accuracy, e.g., to all coefficients of $a_s^{\,n}\,\epsilon^{-n+k}\,N^{\,-n-k+\ell}$ or $a_s^{\,n}\,\epsilon^{-n+k}\,\ln^{\,n+k-1-\ell}x$ for $\widehat{F}_2$ in eqs. (2.7) and (2.12) for a fixed ℓ, then we arrive at an all-order resummation of these terms, e.g., of the N$^\ell$LL $x^{\,0}$ contributions to the splitting functions and coefficient functions contributing to F_2.

This situation is completely analogous to that of the large-x double logarithms in refs. [35–37]. In that case, the structure that allows the extensions to all ϵ has been inferred from the calculations of inclusive DIS via suitably projected gauge-boson parton cross sections as carried out at two loops in refs. [61–63]. The same strategy can be applied here. The maximal ($2 \to n+1$ particles) phase space for these processes at order $\alpha_s^{\,n}$ can be schematically written as [64, 65]

$$\left(\frac{1-x}{x}\right)^{-n\,\epsilon} \int_0^1 d(3n{-}1 \text{ other variables})\ f(x,\,\ldots)\,. \tag{2.14}$$

If the integrals for the n-th order purely real (tree graph) contributions $\widehat{F}_{a,\rm i}^{(n)\rm R}$ do not lead to any further factors $x^{\,\epsilon}$, their expansion (for $a \neq L$) around $x = 0$ can be written as

$$\widehat{F}_{a,\rm i}^{\,(n)\rm R} = x^{\,n\,\epsilon} \sum_p\, x^p\, \frac{1}{\epsilon^{2n-1}} \left\{ R_{a,{\rm i},p}^{\,(n)\rm LL} + \epsilon\, R_{a,{\rm i},p}^{\,(n)\rm NLL} + \epsilon^2\, R_{a,{\rm i},p}^{\,(n)\rm NNL} + \ldots \right\}. \tag{2.15}$$

If, furthermore, the mixed real-virtual contributions ($2 \to r+1$ particles with $n-r \geq 1$ loops) include no more than $n-r$ additional factors of $x^{\,\epsilon}$ from the loop integrals, then we arrive at

$$\widehat{F}_{a,\rm i}^{\,(n)}(x)\Big|_{x^p} = \frac{1}{\epsilon^{2n-n_a}} \sum_{l=1}^{l_a} x^{\,\epsilon l} \left\{ A_{a,{\rm i},p}^{(n,l)} + \epsilon\, B_{a,{\rm i},p}^{(n,l)} + \epsilon^2\, C_{a,{\rm i},p}^{(n,l)} + \ldots \right\} \tag{2.16}$$

with $n_a = 1$ and $l_a = n$ for all structure functions considered here, with the exception of F_L for which $n_a = 3$ and $l_a = n-1$. By expanding $x^{\,\epsilon l}$ in powers of $\epsilon l \ln x$, it can be seen that A, B and C are the coefficients of the LL, NLL and NNLL ($\equiv$ N^{2}LL) contributions, respectively. For a given value of p, the N-space counterpart of eq. (2.16) reads

$$\widehat{F}_{a,\rm i}^{\,(n)}(N,p) = \frac{1}{\epsilon^{2n-n_a}} \sum_{l=1}^{l_a} \frac{1}{N+p+\epsilon l} \left\{ A_{a,{\rm i},\,p}^{(n,l)} + \epsilon\, B_{a,{\rm i},\,p}^{(n,l)} + \epsilon^2\, C_{a,{\rm i},\,p}^{(n,l)} + \ldots \right\}. \tag{2.17}$$

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Since eq. (2.12) includes only poles up to ϵ^{-n} at order a_s^n, the terms with $\epsilon^{-2n+1}, \ldots, \epsilon^{-n-1}$ have to cancel (for $a \neq L$) in the sums (2.16) and (2.17). Hence there are $n-1$ 'zero' relations between the LL coefficients $A^{(n,l)}$, $n-2$ relations between the NLL coefficients $B^{(n,l)}$ etc. Moreover, as mentioned above, the N^{m}LO results provide the (non-vanishing) coefficients of $\epsilon^{-n}, \ldots, \epsilon^{-n+m}$ at all orders n, and thus $m+1$ additional relations between the coefficients in eqs. (2.16) and (2.17). Consequently the highest $m+1$ double logarithms, i.e. the N^{m}LL approximation, can be determined and, except for the N^{m}LL terms at order a_s^{m+1}, over-constrained order-by-order from the N^{m}LO results. This feature also holds for $a = L$ but (due to $c_{L,\mathrm{i}}^{(0)}(x) = 0$) with only $n-2$ 'zero' relations but also one term fewer in the above sums.

By comparing eqs. (2.16) and (2.17) to the known N^{k}LO contributions to the unfactorized structure functions (2.7), i.e., its coefficients up to ϵ^{-n+k} at any order n in a_s, it is now possible to specify the cases for which the procedure outlined above can be applied. We find that these equations, and hence the resulting resummation of small-x double logarithms, hold at

$$\text{even } p: \tfrac{1}{x}\, F_{2,L} \ \text{ for e.m., } \nu+\bar{\nu} \text{ DIS}, \quad F_3 \ \text{ for } \nu-\bar{\nu} \text{ DIS}, \quad F_\phi\,, \tag{2.18}$$

$$\text{odd } p: \tfrac{1}{x}\, F_{2,L} \ \text{ for } \nu-\bar{\nu} \text{ DIS}, \quad F_3 \ \text{ for } \nu+\bar{\nu} \text{ DIS}, \quad g_1 \ \text{ for e.m. DIS etc}. \tag{2.19}$$

Here 'e.m.' (electromagnetic) denotes photon exchange, and F_ϕ is the structure function for DIS via the exchange of a scalar that, like the Higgs boson in the heavy-top limit, couples directly only to gluons. The third-order coefficient functions for this structure function, which is experimentally irrelevant but theoretically useful, have been presented in ref. [33]; the second-order results have also been obtained in ref. [66]. For the coefficient functions for $\nu - \bar{\nu}$ charged-current DIS see refs. [67–69]. For completeness, we have included the most important structure function g_1 in spin-dependent DIS, for its coefficient functions and splitting functions at NNLO see refs. [70–74].

It is worthwhile to note that all structure functions in eq. (2.18) are accessible only at even N via forward Compton amplitudes or the OPE; conversely all structure functions in eq. (2.19) are odd-N based (for a detailed discussion see, e.g., ref. [67]) — a fact that can hardly be a mere coincidence. Moreover, the differences between the splitting functions and coefficient functions for the flavour non-singlet $\nu + \bar{\nu}$ and $\nu - \bar{\nu}$ structure functions vanish in the limit of a large number of colours n_c, see refs. [5, 46, 47, 68, 75].

The structural difference between these two cases, and its suppression at large n_c, can already be seen from the $p = 0$ LL resummation of the corresponding even-N and odd-N splitting functions P_{ns}^+ and P_{ns}^- for the flavour differences of quark-antiquark sums and differences [76–78]:

$$P^+_{\mathrm{ns,LL}}(N, a_s) = \frac{N}{2}\left\{1 - \left(1 - \frac{8 a_s C_F}{N^2}\right)^{1/2}\right\}, \tag{2.20}$$

$$P^-_{\mathrm{ns,LL}}(N, a_s) = \frac{N}{2}\left\{1 - \left(1 - \frac{8 a_s C_F}{N^2}\left[1 - \frac{8 a_s n_c}{N}\, \frac{d}{dN}\, \ln\left(e^{\,z^2/4}\, D_{-1/[2n_c^2]}(z)\right)\right]\right)^{1/2}\right\} \tag{2.21}$$

where $z = N\,(2\,a_s n_c)^{-1/2}$, and $D_{\mathrm{p}}(z)$ denotes a parabolic cylinder function [79]. Note that the expansion of eq. (2.21) in powers of a_s is an asymptotic series, in contrast to eq. (2.20).

In ref. [48] a surprisingly simple generalization has been proposed of the equation, first derived in ref. [76], that leads to eq. (2.20). This generalization can be stated as

$$P_{\rm ns}^{+}(N,a_{\rm s})\,\Big(P_{\rm ns}^{+}(N,a_{\rm s}) - N + \beta(a_{\rm s})/a_{\rm s}\Big) = O(1) \tag{2.22}$$

up to terms that are large-n_c suppressed and include even-n values ζ_n of Riemann's ζ-function.[2] Inserting the Laurent expansion

$$P_{\rm ns}^{(n)+}(N) = \sum_{k=0}^{k_{\max}} N^{-2n-1+k}\, p_{n,k}^{+} \tag{2.23}$$

about $N = 0$ into eq. (2.22), one can readily solve this relation to 'any' desired order n for the coefficients $p_{n,k}^{+}$ with $k \leq 2n-1$ that correspond to powers of $\ln x$ in the small-x expansion. Specifically, the coefficients $p_{n,0}^{+}$ to $p_{n,2m+1}^{+}$ can be predicted (with the above restriction) at all $n > m$ from eq. (2.22) with $k_{\max} = 2m+1$ if $P_{\rm ns}^{+}(N)$ is completely known to N^{m}LO. So far these predictions have been verified for the n_f^2 and n_f^3 terms and the complete large-n_c limit at N^{3}LO [46, 47].

The above two approaches to the resummation of small-x terms overlap for the $x^0 \ln^k x$ double logarithms of $P_{\rm ns}^{+}(x,\alpha_{\rm s})$, but are largely complementary otherwise. The predictions of eq. (2.17) cover far more than just x^0 part of $P_{\rm ns}^{+}(x,\alpha_{\rm s})$, while eq. (2.22) is very powerful in this specific case, in particular in the large-n_c limit.

Both approaches require Laurent expansions of the fixed-order input quantities, including non-negative powers of $N+p$, as written down at $p = 0$ for $P_{\rm ns}^{(n)+}(N)$ in eq. (2.23). These expansions can be obtained, for example, by expanding the exact x-space expressions in terms of harmonic polylogarithms (HPLs) [83], using the Harmpol package for Form [58] together with

$$\mathrm{M}\left[x^p\,\ln^k\left(\frac{1}{x}\right)\right](N) \equiv \int_0^1 dx\, x^{\,N-1+p}\,\ln^k\left(\frac{1}{x}\right) = \frac{k!}{(N+p)^{k+1}}\,. \tag{2.24}$$

An easy extension to the coefficients of non-negative powers of $N+p$ is to transform the functions to N-space harmonic sums [84, 85], multiply by a sufficiently large power s of $1/(N+p)$, transform back to x-space, proceed as above, and finally multiply by $(N+p)^s$. Routines for the Mellin transform of the HPLs and its inverse are also provided by the Harmpol package. For the convenience of the reader, the $p = 0$ coefficients employed in this article are collected in appendix A.

The resummation predictions for the splitting and coefficient functions can then be computed order by order in $\alpha_{\rm s}$. Using Form and TForm [58–60], this has been done up to order $\alpha_{\rm s}^{30}$ and $\alpha_{\rm s}^{60}$, respectively, for the flavour singlet and non-singlet cases. Using the formal similarity to the SIA cases covered in refs. [40, 41], these results can then be employed to infer their generating functions via over-constrained systems of linear equations, thus arriving at all-order expressions.

[2] This form of the limitation is a conservative all-order extension of that given in ref. [48]. See refs. [80–82] and references therein for another context in which the even-n values ζ_n, i.e., powers of π^2, play a special role.

3 Results for the non-singlet splitting functions

Non-singlet quantities are dominated at small x by their $x^p \ln^k x$ contributions with $p = 0$ and $k \geq 0$ which correspond to poles at $N = 0$ in Mellin space. These terms can be resummed via eq. (2.17) for the splitting function $P_{\rm ns}^{(n)+}$ which enters the structure functions in eq. (2.18). The resummed N-space expressions can be expressed in terms of

$$S = (1 - 4\,\xi)^{1/2} \quad \text{with} \quad \xi = \frac{2C_F\,a_{\rm s}}{N^2} \equiv \frac{C_F\,\alpha_{\rm s}}{2\pi\,N^2}\,. \tag{3.1}$$

The N^{2}LL result for $P_{\rm ns}^{(n)+}$, already presented in ref. [44], can be written as

$$\begin{aligned} P_{\rm ns}^{+}(N, a_{\rm s}) = &-\frac{1}{2}\,N(S-1) + \frac{1}{2}\,a_{\rm s}(2\,C_F - \beta_0)(S^{-1} - 1) \\ &+ \frac{1}{96\,C_F}\,a_{\rm s}N \Big\{ ([156 - 960\,\zeta_2]\,C_F^2 - [80 - 1152\,\zeta_2]\,C_A C_F - 360\,\zeta_2\,C_A^2 \\ &- 100\,\beta_0 C_F + 3\beta_0^2)(S-1) + 2\,([12 - 576\,\zeta_2]\,C_F^2 + [40 + 576\,\zeta_2]\,C_A C_F \\ &- 180\,\zeta_2\,C_A^2 + 56\,\beta_0\,C_F - 3\beta_0^2)(S^{-1} - 1) + 3\,(2\,C_F - \beta_0)^2 (S^{-3} - 1)\Big\}\,, \end{aligned} \tag{3.2}$$

where β_0 in eq. (2.9) has been used instead of n_f for a more compact representation. The two terms in the first line of eq. (3.2) provide the LL and NLL parts; the former agrees, of course, with the earlier result in eq. (2.20) above. The remaining three lines represent the N^{2}LL contribution.

The expansion of eq. (3.2) in powers of $a_{\rm s}$ yields the N^{3}LO and N^{4}LO contributions

$$\begin{aligned} P_{\rm ns}^{+(3)}(N) = &\; 80\;C_F^4\,N^{-7} + 80\,C_F^3\,(2\,C_F - \beta_0)\,N^{-6} + 8\,C_F^2\,\Big([16 - 200\,\zeta_2]C_F^2 \\ &+ 10\,C_F\beta_0 + [20 + 192\,\zeta_2]C_F C_A + 3\beta_0^2 - 60\,\zeta_2\,C_A^2\Big)\,N^{-5} + \mathcal{O}(N^{-4}) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} P_{\rm ns}^{+(4)}(N) = &\; 448\,C_F^5\,N^{-9} + 560\,C_F^4\,(2\,C_F - \beta_0)\,N^{-8} + 80\,C_F^3\,\Big(\,[16 - 148\zeta_2]\,C_F^2 \\ &+ \frac{8}{3}\,C_F\beta_0 + \left[\frac{40}{3} + 144\,\zeta_2\right]\,C_F C_A + 3\beta_0^2 - 45\,\zeta_2\,C_A^2\Big)\,N^{-7} \;+\; \mathcal{O}(N^{-6}) \end{aligned} \tag{3.4}$$

to the moments of eq. (2.4). The n_f^2 part of eq. (3.3) and its complete large-n_c limit have been employed in refs. [46, 47], respectively, as constraints in the determination of the all-N expressions from a limited number of moments and endpoint constraints. Conversely, the verification of those all-N expressions — via results at higher N and independent form-factor calculations [86–90] that include the large-N limit of $P_{\rm ns}^{+(3)}$, the (light-like) four-loop cusp anomalous dimension [19, 20] — provides a stringent check of eq. (3.3).

Using the N^{3}LO results [46, 47] together with the corresponding coefficient function for F_2 in ref. [8], eqs. (3.2)–(3.4) can be extended to N^{3}LL small-x accuracy for the next-to-leading large-n_f terms and in the large-n_c limit. The latter results will be presented below, together with the predictions of eq. (2.22) at this and higher orders in the small-x expansion.

The structure of the closed-form N-space expression (3.2) is similar to the 'non-singlet' $p = -1$ part of the 'time-like' splitting function $P^{T}_{gg}(N)$ for final-state fragmentation functions. The crucial difference is the sign of ξ in eq. (3.1) which leads to qualitative differences. In x-space, the latter splitting function can be expressed [41] in terms of Bessel functions which exhibit an oscillatory behaviour in the small-x limit. In fact, the resummation is found to completely remove the huge small-x spikes present in the fixed-order results for the time-like splitting functions [91, 92].

In the present 'space-like' (initial-state) case, on the other hand, the N^{2}LL x-space expression is given by

$$
\begin{aligned}
P^{+}_{\rm ns}(x,\alpha_{\rm s}) = {} & 2\, a_{\rm s}\, C_F \left\{ 1 \,+\, (2\, C_F - \beta_0)\, a_{\rm s} \ln\frac{1}{x} \,+\, \frac{1}{2}\, (2\, C_F - \beta_0)^2 a_{\rm s}^2 \ln^2\frac{1}{x} \right\} \widetilde{I}_1(z) \\
& + 2\, a_{\rm s}\, C_F \Big\{ \frac{1}{3}\, (11\beta_0 + 10\, C_A - 6\, C_F) - 4\, C_F \zeta_2 \Big\}\, a_{\rm s}\, \widetilde{I}_0(z) \\
& + 2\, a_{\rm s}\, C_F \Big\{ 8\, C_F^2 - 2\, \zeta_2 (\, 15\, C_A^2 - 48\, C_F C_A + 44\, C_F^2\,) \Big\}\, a_{\rm s}^2 \ln^2\frac{1}{x}\; \widetilde{I}_2(z)
\end{aligned}
\tag{3.5}
$$

with

$$
z \;=\; (8\, C_F\, a_{\rm s})^{1/2}\, \ln\frac{1}{x} \tag{3.6}
$$

and

$$
\widetilde{I}_n(z) \;\equiv\; \left(\frac{2}{z}\right)^{n} I_n(z) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!\,(n+k)!} \left(\frac{z}{2}\right)^{2k} \tag{3.7}
$$

in terms of modified Bessel functions $I_n(z)$, see section 9.6 of ref. [93]. The first two terms in the curly bracket in the first line are the LL and NLL results, respectively; the remaining terms provide the N^{2}LL contribution. The latter can be written in different ways due to the recurrence relation expressing I_{n+1} in terms of I_n and I_{n-1}. The form chosen above yields the most compact coefficients and is in line with our basis choice for the higher-accuracy large-n_c expressions below.

The functions $I_n(z)$ have the exponential form $(2\pi\, z)^{-1/2}\, e^{\,z}\; (1 + \mathcal{O}(z^{\,-1}))$ in the large-z limit, and thus dwarf the fixed-order small-x behaviour in a very unstable manner: the LL result is positive, the NLL contribution is negative (for the physically relevant case $\beta_0 > 2C_F$), the N^{2}LL 'correction' is positive etc. It is interesting to note that the coefficients of the (for $x \to 0$ at fixed $a_{\rm s}$) asymptotically dominant $a_{\rm s}(a_{\rm s} \ln x)^{\ell}\, \widetilde{I}_1(z)$ terms in the first line of eq. (3.5), which correspond to the $S^{-2\ell+1}$ parts of the N$^{\ell}$LL contributions in eq. (3.2), seem to point towards a 'second resummation', a feature already observed in ref. [41] for the time-like splitting functions. We will return to this issue with more 'data' below, see eqs. (3.16) and (3.17).

As discussed in section 2, the contributions up to N^{5}LL — with the exception of large-n_c suppressed terms containing ζ_2 — can be obtained from eq. (2.22) [48] and $P^{+}_{\rm ns}$ up to three loops [5]. The resulting coefficients at N^{3}LO (four loops), defined in eq. (2.23), are given by

$$
\begin{aligned}
p^{+}_{3,3} = {} & C_F^4\, [\,212 + 640\, \zeta_3] - C_A\, C_F^3\, [\,20 + 288\, \zeta_3] - \frac{13060}{9}\, C_A^2\, C_F^2 - \frac{2662}{27}\, C_A^3\, C_F \\
& + 192\, \zeta_2\, n_c^3\, C_F + 32\, n_f\, C_F^3 + \frac{4184}{9}\, n_f\, C_A\, C_F^2 + \frac{484}{9}\, n_f\, C_A^2\, C_F - 48\, \zeta_2\, n_f\, n_c^2\, C_F \\
& - \frac{304}{9}\, n_f^2\, C_F^2 - \frac{88}{9}\, n_f^2\, C_A\, C_F + \frac{16}{27}\, n_f^3\, C_F\,,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
p^{+}_{3,4} =\; & -C_F^4\,[224+512\,\zeta_3] + C_A\,C_F^3\left[196+\frac{832}{3}\,\zeta_3\right] + C_A^2\,C_F^2\left[\frac{229480}{81}+64\,\zeta_3\right] \\
& + \frac{50006}{81}\,C_A^3\,C_F - n_c^3\,C_F\left[\frac{7682}{9}\,\zeta_2 - 236\,\zeta_4\right] - n_f\,C_F^3\left[\frac{340}{3}-\frac{512}{3}\,\zeta_3\right] \\
& - n_f\,C_A\,C_F^2\left[\frac{65936}{81}+128\,\zeta_3\right] - \frac{2780}{9}\,n_f\,C_A^2\,C_F + \frac{1552}{9}\,\zeta_2\,n_f\,n_c^2\,C_F \\
& + \frac{4288}{81}\,n_f^2\,C_F^2 + \frac{1288}{27}\,n_f^2\,C_A\,C_F - \frac{80}{9}\,\zeta_2\,n_f^2\,n_c\,C_F - \frac{176}{81}\,n_f^3\,C_F\,,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
p^{+}_{3,5} =\; & C_F^4\,[130+944\,\zeta_3-1920\,\zeta_5] + C_A\,C_F^3\left[\frac{2761}{3}+\frac{12448}{3}\,\zeta_3+960\,\zeta_5\right] \\
& - C_A^2\,C_F^2\left[\frac{254225}{81}+\frac{8984}{3}\,\zeta_3-240\,\zeta_5\right] - C_A^3\,C_F\left[\frac{146482}{81}-264\,\zeta_3\right] \\
& + n_c^3\,C_F\left[\frac{12221}{9}\,\zeta_2-\frac{1312}{3}\,\zeta_4-48\,\zeta_2\,\zeta_3\right] - n_f\,C_F^3\left[\frac{500}{3}+\frac{2080}{3}\,\zeta_3\right] \\
& + n_f\,C_A\,C_F^2\left[\frac{90538}{81}+\frac{1328}{3}\,\zeta_3\right] + n_f\,C_A^2\,C_F\left[\frac{64481}{81}+\frac{32}{3}\,\zeta_3\right] \\
& - n_f\,n_c^2\,C_F\left[\frac{4006}{9}\,\zeta_2-\frac{328}{3}\,\zeta_4\right] - \frac{7736}{81}\,n_f^2\,C_F^2 - n_f^2\,C_A\,C_F\left[\frac{7561}{81}+\frac{32}{3}\,\zeta_3\right] \\
& + \frac{272}{9}\,\zeta_2\,n_f^2\,n_c\,C_F + \frac{64}{27}\,n_f^3\,C_F
\end{aligned}
\tag{3.10}
$$

up to large-n_c suppressed contributions with (powers of) ζ_2 which may include quartic group invariants. The large-n_c limits of eqs. (3.8)–(3.10) have been employed (and verified, recall the discussion below eq. (3.4)) together with that of eq. (3.3) in ref. [47]. The n_f^2 terms agree with eq. (4.14) of ref. [46] which, unlike eqs. (3.8)–(3.10), includes all ζ_2 contributions. The n_f^3 terms in eqs. (3.8)–(3.10), which are part of the leading large-n_f limit derived to all orders in refs. [94, 95], agree with eq. (4.17) of ref. [46].

The corresponding N^{4}LO (five-loop) coefficients read

$$
\begin{aligned}
p^{+}_{4,3} =\; & C_F^5\,[1840+4480\,\zeta_3] + C_A\,C_F^4\left[\frac{7120}{9}-1920\,\zeta_3\right] - \frac{112000}{9}\,C_A^2\,C_F^3 \\
& - \frac{53240}{27}\,C_A^3\,C_F^2 + 960\,\zeta_2\,n_c^4\,C_F + \frac{35360}{9}\,n_f\,C_A\,C_F^3 + \frac{9680}{9}\,n_f\,C_A^2\,C_F^2 \\
& + \frac{1280}{9}\,n_f\,C_F^4 - 240\,\zeta_2\,n_f\,n_c^3\,C_F - \frac{2560}{9}\,n_f^2\,C_F^3 - \frac{1760}{9}\,n_f^2\,C_A\,C_F^2 + \frac{320}{27}\,n_f^3\,C_F^2\,,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
p^{+}_{4,4} =\; & -C_F^5\,[656+512\,\zeta_3] + C_A\,C_F^4\left[\frac{9008}{9}-\frac{13376}{3}\,\zeta_3\right] + \frac{324896}{27}\,C_A^3\,C_F^2 \\
& + C_A^2\,C_F^3\left[\frac{559624}{27}+2496\,\zeta_3\right] + \frac{29282}{81}\,C_A^4\,C_F - n_c^4\,C_F\left[\frac{43478}{9}\,\zeta_2-1240\,\zeta_4\right] \\
& - n_f\,C_F^4\left[\frac{4376}{9}-\frac{5888}{3}\,\zeta_3\right] - n_f\,C_A\,C_F^3\left[\frac{164816}{27}+1152\,\zeta_3\right] - \frac{54304}{9}\,n_f\,C_A^2\,C_F^2 \\
& - \frac{21296}{81}\,n_f\,C_A^3\,C_F + \frac{10544}{9}\,\zeta_2\,n_f\,n_c^3\,C_F + \frac{11776}{27}\,n_f^2\,C_F^3 + 960\,n_f^2\,C_A\,C_F^2 \\
& + \frac{1936}{27}\,n_f^2\,C_A^2\,C_F - \frac{256}{3}\,\zeta_2\,n_f^2\,n_c^2\,C_F - \frac{1280}{27}\,n_f^3\,C_F^2 - \frac{704}{81}\,n_f^3\,C_A\,C_F + \frac{32}{81}\,n_f^4\,C_F\,,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
p^{+}_{4,5} = {} & C_F^5\left[500+5280\,\zeta_3-12800\,\zeta_5\right]+C_A\,C_F^4\left[\frac{85946}{9}+\frac{379168}{9}\,\zeta_3+6400\,\zeta_5\right] \\
& -C_A^2\,C_F^3\left[\frac{204556}{9}+\frac{73984}{3}\,\zeta_3-1440\,\zeta_5\right]-C_A^3\,C_F^2\left[\frac{2795072}{81}-1232\,\zeta_3\right] \\
& -\frac{624118}{243}\,C_A^4\,C_F+n_c^4\,C_F\left[\frac{321290}{27}\,\zeta_2-256\,\zeta_3\,\zeta_2-3948\,\zeta_4\right] \\
& -n_f\,C_F^4\left[\frac{16676}{9}+\frac{56128}{9}\,\zeta_3\right]+n_f\,C_A\,C_F^3\left[\frac{79970}{9}+\frac{7232}{3}\,\zeta_3\right]+\frac{423940}{243}\,n_f\,C_A^3\,C_F \\
& +n_f\,C_A^2\,C_F^2\left[\frac{142534}{9}+832\,\zeta_3\right]-n_f\,n_c^3\,C_F\left[\frac{39236}{9}\,\zeta_2-984\,\zeta_4\right] \\
& -n_f^2\,C_F^3\left[\frac{7516}{9}-\frac{512}{3}\,\zeta_3\right]-n_f^2\,C_A\,C_F^2\left[\frac{59554}{27}+192\,\zeta_3\right]+\frac{3584}{9}\,\zeta_2\,n_f^2\,n_c^2\,C_F \\
& -\frac{34304}{81}\,n_f^2\,C_A^2\,C_F+\frac{7424}{81}\,n_f^3\,C_F^2+\frac{10384}{243}\,n_f^3\,C_A\,C_F-\frac{224}{27}\,\zeta_2\,n_f^3\,n_c\,C_F-\frac{352}{243}\,n_f^4\,C_F\,.
\end{aligned}
\tag{3.13}
$$

The large-n_c limit of eq. (3.11) can also been also obtained via eqs. (2.7) to (2.17) using the $\mathrm{N^3LO}$ result $P_{\rm ns}^{+(3)}$ of ref. [47].

Using the all-N large-n_c expression for $P_{\rm ns}^{+(3)}$, it is possible to predict also the N^{-3} and N^{-2} ($\mathrm{N^6LL}$ and $\mathrm{N^7LL}$) contributions of $P_{\rm ns}^{+(4)}$ in this limit. The corresponding coefficients read

$$
\begin{aligned}
p^{+}_{4,6} = {} & C_F\,n_c^4\left[\frac{83997239}{1944}-\frac{2253859}{81}\,\zeta_2+\frac{13220}{9}\,\zeta_3+\frac{48070}{3}\,\zeta_4+64\,\zeta_3\,\zeta_2\right. \\
& \left.-\frac{4312}{3}\,\zeta_5+176\,\zeta_3^2-1444\,\zeta_6\right]+n_f\,C_F\,n_c^3\left[-\frac{5138330}{243}+\frac{267860}{27}\,\zeta_2\right. \\
& \left.-\frac{4336}{3}\,\zeta_3-\frac{28432}{9}\,\zeta_4+128\,\zeta_3\,\zeta_2+\frac{1504}{3}\,\zeta_5-64\,\zeta_3^2-248\,\zeta_6\right] \\
& +n_f^2\,C_F\,n_c^2\left[\frac{760669}{243}-\frac{9076}{9}\,\zeta_2+\frac{2000}{9}\,\zeta_3+\frac{1408}{9}\,\zeta_4\right] \\
& +n_f^3\,C_F\,n_c\left[-\frac{12826}{81}+\frac{2656}{81}\,\zeta_2-\frac{64}{9}\,\zeta_3\right]+\frac{128}{81}\,n_f^4\,C_F\,,
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
p^{+}_{4,7} = {} & C_F\,n_c^4\left[-\frac{141282997}{2592}+\frac{12219019}{324}\,\zeta_2-\frac{125756}{81}\,\zeta_3-\frac{655423}{27}\,\zeta_4-\frac{5488}{3}\,\zeta_2\,\zeta_3\right. \\
& \left.+\frac{64174}{9}\,\zeta_5-\frac{2056}{3}\,\zeta_3^2+\frac{1874}{3}\,\zeta_6-576\,\zeta_2\,\zeta_5+1104\,\zeta_3\,\zeta_4-1288\,\zeta_7\right] \\
& +n_f\,C_F\,n_c^3\left[\frac{2035745}{72}-\frac{141241}{9}\,\zeta_2+\frac{197588}{81}\,\zeta_3+\frac{71642}{9}\,\zeta_4-\frac{512}{3}\,\zeta_2\,\zeta_3\right. \\
& \left.-\frac{13784}{9}\,\zeta_5-64\,\zeta_2\,\zeta_5+32\,\zeta_3\,\zeta_4+\frac{944}{3}\,\zeta_3^2+368\,\zeta_6-112\,\zeta_7\right] \\
& +n_f^2\,C_F\,n_c^2\left[-\frac{1032713}{243}+\frac{47984}{27}\,\zeta_2-\frac{18992}{27}\,\zeta_3-\frac{4744}{9}\,\zeta_4+96\,\zeta_2\,\zeta_3+\frac{1072}{9}\,\zeta_5\right. \\
& \left.-\frac{64}{3}\,\zeta_3^2-\frac{248}{3}\,\zeta_6\right]+n_f^3\,C_F\,n_c\left[\frac{41497}{243}-\frac{3376}{81}\,\zeta_2+\frac{3008}{81}\,\zeta_3-\frac{176}{27}\,\zeta_4\right] \\
& -n_f^4\,C_F\left[\frac{128}{243}-\frac{64}{81}\,\zeta_3\right]\,.
\end{aligned}
\tag{3.15}
$$

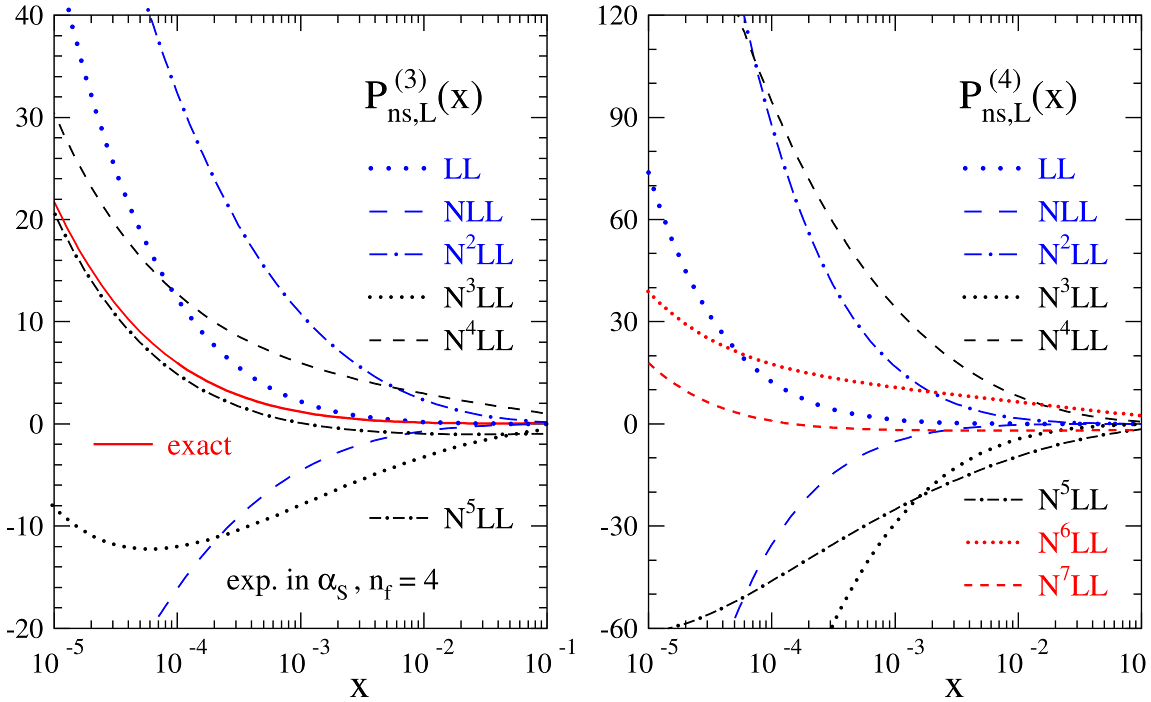

Figure 1. Successive small-x approximations to the N^{n}LO four-flavour splitting functions $P^{(n)}_{\mathrm{ns,\,L}}(x)$ in the large-n_c limit for $n = 3$ (left), compared to the exact result, and $n = 4$ (right). The respective N^{5}LL and N^{7}LL curves include all terms with $x^0 \ln^k x$ at $k > 0$ as specified by eqs. (3.3), (3.4) and (3.8)–(3.15).

The $C_F n_f^4$ terms in eqs. (3.11)–(3.15) agree with refs. [94, 95]; all other contributions have not been presented before.

After transformation to x-space using eq. (2.24), the above results provide all small-x enhanced contributions to the non-singlet splitting functions $P^{\pm}_{\mathrm{ns,\,L}}(x, \alpha_{\mathrm{s}})$ in the large-n_c limit at four and five loops. These results are shown in figure 1 for an expansion in powers of α_{s}, not $a_{\mathrm{s}} = \alpha_{\mathrm{s}}/(4\pi)$ as used in all formulae, at $n_f = 4$ flavours. As seen before for the 3-loop splitting functions, see figures 2 and 3 of ref. [5], and coefficient functions, see figures 7 and 9 of ref. [8] — see also ref. [97] — all logarithms are needed for a meaningful approximation at any physically relevant small values of x. The N^{7}LL small-x limit of $P^{(4)}_{\mathrm{ns,\,L}}(x)$ has already been employed in the first estimate of the 5-loop quark cusp anomalous dimension in the large-n_c limit in ref. [96].

In this context, it is instructive to consider the generalization of the NNLL small-x resummation (3.5), which can be extended to N^{7}LL accuracy in the large-n_c (L) limit:

$$
\begin{aligned}
&P^{\pm}_{\mathrm{ns,L}}(x,\alpha_{\mathrm{s}})/(2\,a_{\mathrm{s}}\,C_F) = \Big\{1-(\beta_0-n_c)\,a_{\mathrm{s}}\ln\frac{1}{x}\Big\}\,\widetilde{I}_1(z)\\
&\quad+\frac{1}{2}\,(\beta_0-n_c)^2\,a_{\mathrm{s}}^2\ln^2\frac{1}{x}\;\widetilde{I}_1(z)+\frac{1}{3}(11\,\beta_0+13\,n_c-18\,\zeta_2\,n_c)\,a_{\mathrm{s}}\,\widetilde{I}_0(z)+2\,(2\,\zeta_2-1)\,n_c\,a_{\mathrm{s}}\;\widetilde{I}_1(z)\\
&\quad-\frac{1}{6}\,(\beta_0-n_c)^3\,a_{\mathrm{s}}^3\ln^3\frac{1}{x}\;\widetilde{I}_1(z)-\frac{1}{3}(11\,\beta_0+13\,n_c-18\,\zeta_2\,n_c)(\beta_0-n_c)\,a_{\mathrm{s}}^2\ln\frac{1}{x}\;\widetilde{I}_0(z)\\
&\quad-\frac{1}{12}(136\,\beta_0-115\,n_c)\,n_c a_{\mathrm{s}}^2\ln\frac{1}{x}\;\widetilde{I}_1(z)+2\,(2\,\zeta_3-1)\,n_c^3\,a_{\mathrm{s}}^3\ln^3\frac{1}{x}\;\widetilde{I}_3(z)\\
&\quad+\frac{1}{24}\,(\beta_0-n_c)^4\,a_{\mathrm{s}}^4\,\ln^4\frac{1}{x}\;\widetilde{I}_1(z)+\frac{1}{6}\,(11\,\beta_0+13\,n_c-18\,\zeta_2\,n_c)(\beta_0-n_c)^2\,a_{\mathrm{s}}^3\ln^2\frac{1}{x}\;\widetilde{I}_0(z)\\
&\quad+\frac{1}{36}\,\Big(\beta_0^2\,(686-72\,\zeta_2)-n_c\,\beta_0\,(181+648\,\zeta_2+144\,\zeta_3)\\
&\qquad+n_c^2\,(647-1008\,\zeta_2+144\,\zeta_3+1620\,\zeta_4)\Big)\,n_c\,a_{\mathrm{s}}^3\,\ln^2\frac{1}{x}\;\widetilde{I}_1(z)\\
&\quad+\frac{1}{72}\,\Big(288\,\beta_0^2+n_c\,\beta_0\,(3811-912\,\zeta_2+576\,\zeta_3)\\
&\qquad-n_c^2\,(4357+852\,\zeta_2+1584\,\zeta_3-1008\,\zeta_4)\Big)\,a_{\mathrm{s}}^2\;\widetilde{I}_0(z)\\
&\quad+\frac{2}{3}\,\Big(\beta_0\,(5-10\,\zeta_2+12\,\zeta_3)+n_c\,(4-2\,\zeta_2-6\,\zeta_3)\Big)\,n_c^2\,a_{\mathrm{s}}^3\,\ln^2\frac{1}{x}\;\widetilde{I}_2(z)\\
&\quad-2\,(2\,\zeta_4-1)\,n_c^4\,a_{\mathrm{s}}^4\,\ln^4\frac{1}{x}\;\widetilde{I}_4(z)\\
&\quad-\frac{1}{120}\,(\beta_0-n_c)^5\,a_{\mathrm{s}}^5\ln^5\frac{1}{x}\;\widetilde{I}_1(z)-\frac{1}{18}\,(11\,\beta_0+13\,n_c-18\,\zeta_2\,n_c)(\beta_0-n_c)^3\,a_{\mathrm{s}}^4\ln^3\frac{1}{x}\;\widetilde{I}_0(z)\\
&\quad-\frac{1}{72}\,\Big(\beta_0^2\,(940-96\,\zeta_2)+n_c\,\beta_0\,(367-1392\,\zeta_2-144\,\zeta_3)\\
&\qquad+n_c^2\,(997-1968\,\zeta_2+144\,\zeta_3+3240\,\zeta_4)\Big)\,(\beta_0-n_c)\,n_c\,a_{\mathrm{s}}^4\,\ln^3\frac{1}{x}\;\widetilde{I}_1(z)\\
&\quad-\frac{1}{72}\,\Big(288\,\beta_0^3+n_c\,\beta_0^2\,(7427-2448\,\zeta_2+864\,\zeta_3)-n_c^2\,\beta_0\,(6730+5508\,\zeta_2+3504\,\zeta_3\\
&\qquad-5040\,\zeta_4)+n_c^3\,(1175+6336\,\zeta_2+336\,\zeta_3-5040\,\zeta_4+1728\,\zeta_2\,\zeta_3)\Big)\,a_{\mathrm{s}}^3\,\ln\frac{1}{x}\;\widetilde{I}_0(z)\\
&\quad-\frac{1}{36}\,\Big(417\,\beta_0^2+n_c\,\beta_0\,(2725-288\,\zeta_2+168\,\zeta_3+432\,\zeta_4)\\
&\qquad-n_c^2\,(4507-1236\,\zeta_3+432\,\zeta_4+2160\,\zeta_5)\Big)\,n_c\,a_{\mathrm{s}}^3\,\ln\frac{1}{x}\;\widetilde{I}_1(z)\\
&\quad-\frac{2}{3}\,\Big(\beta_0\,(2-10\,\zeta_3+18\,\zeta_4)-n_c\,(2-12\,\zeta_2-10\,\zeta_3+9\,\zeta_4+24\,\zeta_2\,\zeta_3)\Big)\,n_c^3\,a_{\mathrm{s}}^4\,\ln^3\frac{1}{x}\;\widetilde{I}_3(z)\\
&\quad+2\,(2\,\zeta_5-1)\,n_c^5\,a_{\mathrm{s}}^5\,\ln^5\frac{1}{x}\;\widetilde{I}_5(z)\\
&\quad+\frac{1}{720}\,(\beta_0-n_c)^6\,a_{\mathrm{s}}^6\ln^6\frac{1}{x}\;\widetilde{I}_1(z)+\frac{1}{72}\,(11\,\beta_0+13\,n_c-18\,\zeta_2\,n_c)(\beta_0-n_c)^4\,a_{\mathrm{s}}^5\,\ln^4\frac{1}{x}\;\widetilde{I}_0(z)\\
&\quad+\,\ldots\,-2\,(2\,\zeta_6-1)\,n_c^6\,a_{\mathrm{s}}^6\,\ln^6\frac{1}{x}\;\widetilde{I}_6(z)\\
&\quad-\frac{1}{5040}\,(\beta_0-n_c)^7\,a_{\mathrm{s}}^7\,\ln^7\frac{1}{x}\;\widetilde{I}_1(z)-\frac{1}{360}\,(11\,\beta_0+13\,n_c-18\,\zeta_2\,n_c)(\beta_0-n_c)^5\,a_{\mathrm{s}}^6\ln^5\frac{1}{x}\;\widetilde{I}_0(z)\\
&\quad+\,\ldots\,+2\,(2\,\zeta_7-1)\,n_c^7\,a_{\mathrm{s}}^7\,\ln^7\frac{1}{x}\;\widetilde{I}_7(z)
\end{aligned}
\tag{3.16}
$$

where we have omitted (with '...') the lengthy and (at least in the present notation) 'irregular' parts of the N^{6}LL and N^{7}LL coefficients for brevity; the ancillary file of this article provides the complete expression. The first two terms at each logarithmic order, which dominate at asymptotically small x for a fixed α_s, can be seen to build up an exponential function, viz

$$
\begin{aligned}
& P^{\pm}_{\rm ns,L}(x,\alpha_s)/(2\,a_s\,C_F) = \\
& \quad \exp\left(-(\beta_0 - n_c)\,a_s \ln\frac{1}{x}\right)\left(\widetilde{I}_1(z) + \frac{1}{3}\,(11\,\beta_0 + 13\,n_c - 18\,\zeta_2\,n_c)\,a_s\,\widetilde{I}_0(z)\right) + \;\ldots\;. \qquad (3.17)
\end{aligned}
$$

This exponential prefactor dampens, and from some unphysically large value of α_s overwhelms, the small-x rise of the modified Bessel functions (3.7).

So far we have presented the $1/N^{n+1} \Leftrightarrow x^0 \ln^n x$ results arising from eqs. (2.17) and (2.22). We now turn to the double logarithmic $1/(N+p)^{n+1} \Leftrightarrow x^p \ln^n x$ contributions which, as discussed in section 2, can be analyzed in a completely analogous manner at even p for $P^+_{\rm ns}$ and odd p for $P^-_{\rm ns}$, and hence at all integer p for their common large-n_c limit $P^{\pm}_{\rm ns,L}$.

Thus the complete Taylor expansions of the coefficients of $\ln^k x$ can be determined at $k = 6, 5, 4$ for $P^{(3)}_{\rm ns,L}$ and at $k = 2n, \ldots, 2n-3$ for $P^{(n)}_{\rm ns,L}$ at $n \geq 4$. In practice, we have performed the necessary computations to $p = 70$, which was more than sufficient to overconstrain the analytic expressions in terms of harmonic polylogarithms [83] for which an ansatz with up to 54 coefficients was used. This partial reconstruction of the analytic form of $P^{\pm}_{\rm ns,L}$ can be carried out to any order in α_s; here we confine ourselves to the N^{3}LO and N^{4}LO expressions. The former is given by

$$
\begin{aligned}
P^{(3)}_{\rm ns,L}(x) \;=& \\
&+ \ln^6 x \left[n_c^3\, C_F \left\{ \frac{5}{24}\left(1 - \frac{16}{15}\,(1-x)^{-1} + x\right)\right\}\right] \\
&+ \ln^5 x \left[n_c^3\, C_F \left\{ -\frac{4}{3}\, p_{qq}\, {\rm H}_1 + \frac{22}{9}\left(1 - \frac{13}{11}\,(1-x)^{-1} + \frac{17}{11}\,x\right)\right\} \right. \\
&\qquad\quad \left. + n_c^2\, C_F\, n_f \left\{ -\frac{7}{9}\left(1 - \frac{8}{7}\,(1-x)^{-1} + x\right)\right\}\right] \\
&+ \ln^4 x \left[n_c^3\, C_F \left\{ -\frac{8}{3}\, p_{qq}\, {\rm H}_{1,1} - \frac{2}{3}\left(1 - 4\,(1-x)^{-1} + x\right) {\rm H}_{0,1} + \frac{58}{9}\left(1 - \frac{70}{29}\,(1-x)^{-1} \right.\right.\right. \\
&\qquad\quad \left. \left. + \frac{41}{29}\,x\right) {\rm H}_1 - \frac{1}{6}\left([227 - 112\,\zeta_2]\,(1-x)^{-1} - [251 - 98\,\zeta_2]\,x - \frac{1}{3}\,[463 - 294\,\zeta_2]\right)\right\} \\
&\qquad\quad + n_c^2\, C_F\, n_f \left\{ \frac{16}{9}\, p_{qq}\, {\rm H}_1 - \frac{65}{9}\left(1 - \frac{92}{65}\,(1-x)^{-1} + \frac{93}{65}\,x\right)\right\} \\
&\qquad\quad \left. + n_c\, C_F\, n_f^2 \left\{ \frac{2}{3}\left(1 - \frac{4}{3}\,(1-x)^{-1} + x\right)\right\}\right]
\end{aligned}
$$

$$+\ln^3 x\Big[n_c^3\, C_F \Big\{\frac{56}{3}\, p_{qq}\, \mathrm{H}_{1,0,1} + \frac{160}{3}\, p_{qq}\, \mathrm{H}_{1,1,1} + \frac{4}{3}\Big(1-4\,(1-x)^{-1}+x\Big)\, \mathrm{H}_{0,0,1}$$

$$-\frac{50}{9}\Big(1-\frac{94}{25}\,(1-x)^{-1}+\frac{109}{25}\,x\Big)\, \mathrm{H}_{0,1} - \frac{88}{3}\Big(1-\frac{14}{11}\,(1-x)^{-1}+x\Big)\, \mathrm{H}_{0,1,1}$$

$$+\frac{160}{9}\Big(1+(1-x)^{-1}-2\,x\Big)\, \mathrm{H}_{1,1} + \frac{1}{81}\Big([2987-4212\,\zeta_2-756\,\zeta_3]+\frac{1}{2}\,[38641$$

$$-18360\,\zeta_2-1512\,\zeta_3]\,x-\frac{1}{4}\,[55291-27216\,\zeta_2-3456\,\zeta_3]\,(1-x)^{-1}\Big)$$

$$+\frac{1}{54}\Big([3977-3312\,\zeta_2]-2\,[4745-3312\,\zeta_2]\,(1-x)^{-1}+[5513-3312\,\zeta_2]\,x\Big)\, \mathrm{H}_1\Big\}$$

$$+n_c^2\, C_F\, n_f \Big\{-\frac{80}{9}\, p_{qq}\, \mathrm{H}_{1,1} - \frac{338}{27}\Big(1-\frac{362}{169}\,(1-x)^{-1}+\frac{193}{169}\,x\Big)\, \mathrm{H}_1$$

$$+\frac{56}{9}\Big(1-\frac{10}{7}\,(1-x)^{-1}+x\Big)\, \mathrm{H}_{0,1} + \frac{1}{9}\Big([725-288\,\zeta_2]\,(1-x)^{-1}$$

$$-\frac{1}{3}\,[851-648\,\zeta_2]-\frac{1}{3}\,[2257-648\,\zeta_2]\,x\Big)\Big\} + C_F\, n_f^3 \Big\{\frac{8}{81}\, p_{qq}\Big\}$$

$$+n_c\, C_F\, n_f^2 \Big\{\frac{8}{27}\, p_{qq}\, \mathrm{H}_1 + \frac{92}{27}\Big(1-\frac{53}{23}\,(1-x)^{-1}+\frac{45}{23}\,x\Big)\Big\}\Big] \;+\; \ldots \tag{3.18}$$

with $p_{qq} = 2\,(1-x)^{-1}-1-x$. We have suppressed the argument x of the HPLs for brevity. The above LL, NLL and NNLL predictions of the resummation have been used in and verified by ref. [47], of which the $\ln^3 x$ part is a result. The corresponding N^4LO (five-loop) predictions read

$$P^{(4)}_{\mathrm{ns,L}}(x) \;=$$

$$+\ln^8 x\Big[n_c^4\, C_F \Big\{\frac{31}{1440}\Big(1-\frac{32}{31}\,(1-x)^{-1}+x\Big)\Big\}\Big]$$

$$+\ln^7 x\Big[n_c^4\, C_F \Big\{-\frac{2}{9}\, p_{qq}\, \mathrm{H}_1 + \frac{4}{9}\Big(1-\frac{13}{12}\,(1-x)^{-1}+\frac{3}{2}\,x\Big)\Big\}$$

$$+n_c^3\, C_F\, n_f \Big\{-\frac{5}{36}\Big(1-\frac{16}{15}\,(1-x)^{-1}+x\Big)\Big\}\Big]$$

$$+\ln^6 x\Big[n_c^4\, C_F \Big\{-\frac{4}{3}\, p_{qq}\, \mathrm{H}_{1,1} + 2\Big(1-\frac{8}{3}\,(1-x)^{-1}+\frac{5}{3}\,x\Big)\, \mathrm{H}_1 + \frac{1}{3}\Big(1+x\Big)\, \mathrm{H}_{0,1}$$

$$+\frac{1}{36}\Big([283-150\,\zeta_2]-\frac{10}{9}\,[305-144\,\zeta_2]\,(1-x)^{-1}+\frac{2}{3}\,[641-225\,\zeta_2]\,x\Big)\Big\}$$

$$+n_c^3\, C_F\, n_f \Big\{\frac{2}{3}\, p_{qq}\, \mathrm{H}_1 - \frac{67}{27}\Big(1-\frac{236}{201}\,(1-x)^{-1}+\frac{89}{67}\,x\Big)\Big\}$$

$$+n_c^2\, C_F\, n_f^2 \Big\{\frac{7}{27}\Big(1-\frac{8}{7}\,(1-x)^{-1}+x\Big)\Big\}\Big]$$

$$
\begin{aligned}
&+\ln^5 x\,\Bigg[n_c^4\,C_F\,\bigg\{\frac{40}{3}\,p_{qq}\,\mathrm{H}_{1,0,1}+\frac{80}{3}\,p_{qq}\,\mathrm{H}_{1,1,1}-\frac{56}{3}\,\Big(1-\frac{10}{7}\,(1-x)^{-1}+x\Big)\,\mathrm{H}_{0,1,1}\\
&\quad-\frac{16}{3}\,\Big(1-(1-x)^{-1}+x\Big)\,\mathrm{H}_{0,0,1}+\frac{124}{9}\,\Big(1-\frac{2}{31}\,(1-x)^{-1}-\frac{29}{31}\,x\Big)\,\mathrm{H}_{1,1}\\
&\quad+\frac{2}{3}\,\Big(1+14\,(1-x)^{-1}-19\,x\Big)\,\mathrm{H}_{0,1}+\frac{11}{27}\,\Big([115-72\,\zeta_2]\,x+\frac{1}{11}\,[1049-792\,\zeta_2]\\
&\quad-\frac{2}{11}\,[1157-792\,\zeta_2]\,(1-x)^{-1}\Big)\,\mathrm{H}_1+\frac{1}{72}\,\Big([8129-4704\,\zeta_2-360\,\zeta_3]\,x\\
&\quad+\frac{1}{3}\,[10643-7776\,\zeta_2-1080\,\zeta_3]-\frac{2}{9}\,[26449-14256\,\zeta_2-1728\,\zeta_3]\,(1-x)^{-1}\Big)\bigg\}\\
&\quad+n_c^3\,C_F\,n_f\,\bigg\{\frac{74}{9}\,p_{qq}\,\mathrm{H}_1-\frac{32}{9}\,p_{qq}\,\mathrm{H}_{1,1}+\frac{10}{3}\,\Big(1-\frac{8}{5}\,(1-x)^{-1}+x\Big)\,\mathrm{H}_{0,1}\\
&\quad-\frac{7}{27}\,\Big([169-54\,\zeta_2]\,x+\frac{1}{7}\,[745-378\,\zeta_2]-\frac{1}{7}\,[1091-432\,\zeta_2]\,(1-x)^{-1}\Big)\bigg\}\\
&\quad+n_c^2\,C_F\,n_f^2\,\bigg\{-\frac{4}{27}\,p_{qq}\,\mathrm{H}_1+\frac{94}{27}\,\Big(1-\frac{66}{47}\,(1-x)^{-1}+\frac{61}{47}\,x\Big)\bigg\}\\
&\quad+n_c\,C_F\,n_f^3\,\bigg\{-\frac{4}{27}\,\Big(1-\frac{4}{3}\,(1-x)^{-1}+x\Big)\bigg\}\Bigg]\;+\;\ldots\,.
\end{aligned}
\tag{3.19}
$$

4 Results for the non-singlet coefficient functions

The order-by-order resummation of the highest $1/N$ powers of the even-N based ('+') non-singlet coefficient functions C_a^+ for the structure functions in eq. (2.18) is 'automatically' included in the calculations towards the corresponding expressions for the splitting function $P_{\rm ns}^+$ reported in the previous section. Using the short-hand $F \equiv S^{-1/2}$ with S as defined in eq. (3.1), the NNLL results for C_2^+ and C_L^+ analogous to eq. (3.2) for $P_{\rm ns}^+$ are found to be

$$
\begin{aligned}
&C_2^+(N)\;=\\
&F+\frac{1}{192C_F}N\Big\{-3(32C_F+11\beta_0)(F^{-1}-1)+4(18C_F+11\beta_0)(F-1)+6\beta_0(F^3-1)\\
&+12(2C_F-\beta_0)(F^5-1)-5\beta_0(F^7-1)\Big\}+\frac{1}{9216C_F}a_s\Big\{-128([333-1368\,\zeta_2]C_F^2\\
&-[60-1728\,\zeta_2]C_AC_F-540\,\zeta_2C_A^2-87\beta_0C_F-10\beta_0^2)\frac{1}{\xi}(F^{-3}-F^{-1}+2\xi)\\
&-([144000-442368\,\zeta_2]C_F^2-[7680-552960\,\zeta_2]C_AC_F-172800\,\zeta_2C_A^2-16320\beta_0C_F\\
&-5111\beta_0^2)(F-1)+8(576C_F^2-90\beta_0C_F-79\beta_0^2)(F^3-1)+([5184-110592\,\zeta_2]C_F^2\\
&+[7680+110592\,\zeta_2]C_AC_F-34560\,\zeta_2C_A^2+10368\beta_0C_F-2093\beta_0^2)(F^5-1)\\
&+16\beta_0(54C_F-77\beta_0)(F^7-1)+(2880C_F(C_F-\beta_0)+181\beta_0^2)(F^9-1)\\
&-840\beta_0(2C_F-\beta_0)(F^{11}-1)+385\beta_0^2(F^{13}-1)\Big\},
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
&C_L^{+}(N) = \\
&4\,C_F a_s F + \frac{1}{48} a_s N \Big\{ -3(64C_F - 5\beta_0)(F^{-1}-1) - 4(6C_F+\beta_0)(F-1) + 6\beta_0(F^{3}-1) \\
&+12(2C_F-\beta_0)(F^{5}-1) - 5\beta_0(F^{7}-1) \Big\} + \frac{1}{2304} a_s^{\,2} \Big\{ -9216 C_F^2 \frac{1}{\xi}(F^{-3} - 3F^{-1} + 2) \\
&-128([153+72\,\zeta_2]C_F^2 - 60C_AC_F - 69\beta_0C_F + 2\beta_0^{2})\frac{1}{\xi}(F^{-3} - F^{-1} + 2\xi) \\
&-([222336 - 73728\,\zeta_2]C_F^2 - [38400 - 110592\,\zeta_2]C_AC_F - 34560\,\zeta_2 C_A^2 - 46080\beta_0C_F \\
&+1321\beta_0^{2})(F-1) + 8(1152C_F^2 - 690\beta_0C_F + 53\beta_0^{2})(F^{3}-1) + ([576 - 110592\,\zeta_2]C_F^2 \\
&+[7680 + 110592\,\zeta_2]C_AC_F - 34560\,\zeta_2C_A^2 + 9024\beta_0C_F - 269\beta_0^{2})(F^{5}-1) \\
&+16\beta_0(114C_F - 47\beta_0)(F^{7}-1) + (2880C_F(C_F-\beta_0) + 181\beta_0^{2})(F^{9}-1) \\
&-840\beta_0(2C_F-\beta_0)(F^{11}-1) + 385\beta_0^2(F^{13}-1) \Big\}\,. \qquad (4.2)
\end{aligned}
$$

The corresponding result for the $\nu - \bar{\nu}$ charged-current (CC) structure function F_3 reads

$$
\begin{aligned}
&C_3^{+}(N) = \\
&\quad F + \frac{1}{192C_F} N \Big\{ -33\beta_0(F^{-1}-1) - 4[6C_F - 11\beta_0](F-1) + 6\beta_0(F^3-1) \\
&\quad + 12[2C_F-\beta_0](F^5-1) - 5\beta_0(F^7-1) \Big\} + \frac{1}{9216C_F} a_s \Big\{ +128(63\beta_0C_F \\
&\quad + 10\beta_0^2 + 540\zeta_2C_A^2 + 12[5-144\zeta_2]C_AC_F - 9[49-152\zeta_2]C_F^2)\frac{1}{\xi}(F^{-3} - F^{-1} + 2\xi) \\
&\quad + (3456\beta_0C_F + 5111\beta_0^2 + 172800\zeta_2C_A^2 + 7680[1-72\zeta_2]C_AC_F \\
&\quad - 1152[157-384\zeta_2]C_F^2)(F-1) + 8(174\beta_0C_F - 79\beta_0^2)(F^3-1) + (14016\beta_0C_F \\
&\quad - 2093\beta_0^2 - 34560\zeta_2C_A^2 + 576[1-192\zeta_2]C_F^2 + 1536[5+72\zeta_2]C_AC_F)(F^5-1) \\
&\quad + 16(114\beta_0C_F - 77\beta_0^2)(F^7-1) + (2880C_F(C_F-\beta_0) + 181\beta_0^2)(F^{9}-1) \\
&\quad - 840(2\beta_0C_F - \beta_0^2)(F^{11}-1) + 385\beta_0^2(F^{13}-1) \Big\}\,. \qquad (4.3)
\end{aligned}
$$

Like the splitting function, these coefficient functions exhibit a structure similar to that of their time-like counterparts in ref. [41], apart from the all-important fact that the sign of ξ is different there. The corresponding x-space results can be expressed in terms of generalized hypergeometric functions listed in appendix B. These hypergeometric functions related to the function F are, however, not present in the non-singlet physical evolution kernels K_a^+ given by

$$
K_a^{+} = P^{+} + \beta(a_s)\frac{d}{da_s}\ln C_a^{+}\,. \qquad (4.4)
$$

Eqs. (4.1)–(4.3) can be expanded to produce the explicit four- and five-loop results

$$c_2^+(N)\Big|_{a_s^4} = 390\,C_F^4 N^{-8} + C_F^3\left(1052C_F - \frac{1822}{3}\beta_0\right)N^{-7} + C_F^2\Big([336-5872\,\zeta_2]C_F^2 \\ -448C_F\beta_0 + \left[\frac{2180}{3}+4992\,\zeta_2\right]C_FC_A + \frac{1951}{6}\beta_0^2 - 1560\,\zeta_2 C_A^2\Big)N^{-6} + \mathcal{O}(N^{-5}), \tag{4.5}$$

$$c_L^+(N)\Big|_{a_s^4} = 240\,C_F^4 N^{-6} + C_F^3\left(472C_F - \frac{992}{3}\beta_0\right)N^{-5} + C_F^2\Big(-[644+4016\,\zeta_2]C_F^2 \\ +56C_F\beta_0 + [480+3840\,\zeta_2]C_FC_A + \frac{460}{3}\beta_0^2 - 1200\,\zeta_2 C_A^2\Big)N^{-4} + \mathcal{O}(N^{-3}), \tag{4.6}$$

$$c_3^+(N)\Big|_{a_s^4} = 390C_F^4 N^{-8} + C_F^3\left(780C_F - \frac{1822}{3}\beta_0\right)N^{-7} + C_F^2\Big(-[496+5872\zeta_2]C_F^2 \\ -\frac{8}{3}C_F\beta_0 + \left[\frac{2180}{3}+4992\zeta_2\right]C_FC_A + \frac{1951}{6}\beta_0^2 - 1560\zeta_2 C_A^2\Big)N^{-6} + \mathcal{O}(N^{-5}) \tag{4.7}$$

and

$$c_2^+(N)\Big|_{a_s^5} = 2652C_F^5 N^{-10} + C_F^4\left(8418C_F - \frac{17012}{3}\beta_0\right)N^{-9} + C_F^3\Big([6438-56508\zeta_2]C_F^2 \\ -\frac{23546}{3}C_F\beta_0 + [6040+50688\zeta_2]C_FC_A + \frac{14363}{3}\beta_0^2 - 15840\zeta_2C_A^2\Big)N^{-8} \\ +\mathcal{O}(N^{-7}), \tag{4.8}$$

$$c_L^+(N)\Big|_{a_s^5} = 1560C_F^5 N^{-8} + C_F^4\left(3736C_F - \frac{8920}{3}\beta_0\right)N^{-7} + C_F^3\Big(-[2064+35648\zeta_2]\,C_F^2 \\ -1504C_F\beta_0 + \left[\frac{11120}{3}+34560\zeta_2\right]C_FC_A + \frac{6574}{3}\beta_0^2 - 10800\zeta_2C_A^2\Big)N^{-6} \\ +\mathcal{O}(N^{-5}), \tag{4.9}$$

$$c_3^+(N)\Big|_{a_s^5} = 2652C_F^5 N^{-10} + C_F^4\left(6630C_F - \frac{17012}{3}\beta_0\right)N^{-9} + C_F^3\Big([66-56508\zeta_2]\,C_F^2 \\ -\frac{11374}{3}C_F\beta_0 + [6040+50688\zeta_2]\,C_FC_A + \frac{14363}{3}\beta_0^2 - 15840\zeta_2C_A^2\Big)N^{-8} \\ +\mathcal{O}(N^{-7}). \tag{4.10}$$

The LL, NLL and NNLL x-space approximations resulting from eqs. (4.5), (4.6), (4.8) and (4.9) with eq. (2.24) are shown for four flavours in figures 2 and 3. Unsurprisingly, these results alone do not provide relevant information about the behaviour of these coefficient functions at any physically interesting value of x. The results in eqs. (4.5)–(4.10) can become useful, however, once combined with other partial results such as the large-x limit and a sufficient number of moments.

As for the non-singlet splitting functions, the resummation can be performed for any even or odd power of x, and for all powers in the common large-n_c limit $C_{a,\mathrm{L}}$ of the coefficient functions C_a^+ and C_a^-. Since the N^{3}LO splitting functions are known in this limit [47], the all-x coefficients of the four highest small-x logarithms can be predicted at any higher order

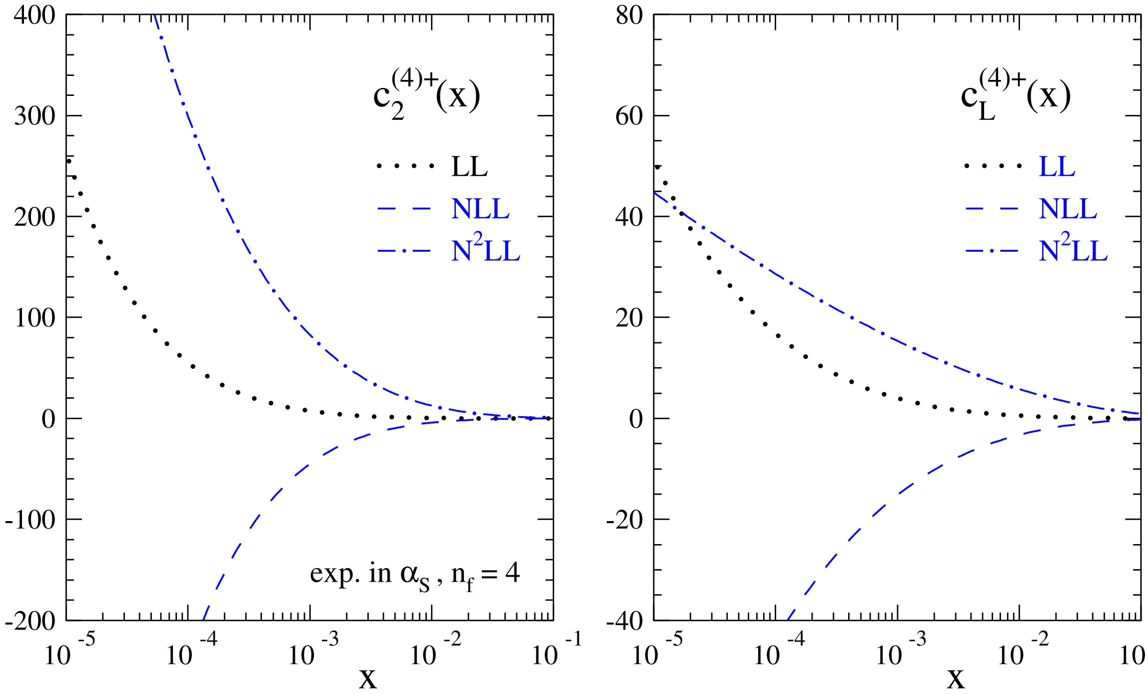

Figure 2. The LL, NLL and NNLL small-x approximations to the fourth-order non-singlet coefficient functions $c_2^{(4)+}(x)$ and $c_L^{(4)+}(x)$ for $n_f = 4$ light flavours. Eqs. (4.5) and (4.6) have been transformed to x-space using eq. (2.24) and converted to an expansion in powers of α_s by a multiplication with $1/(4\,\pi)^4$.

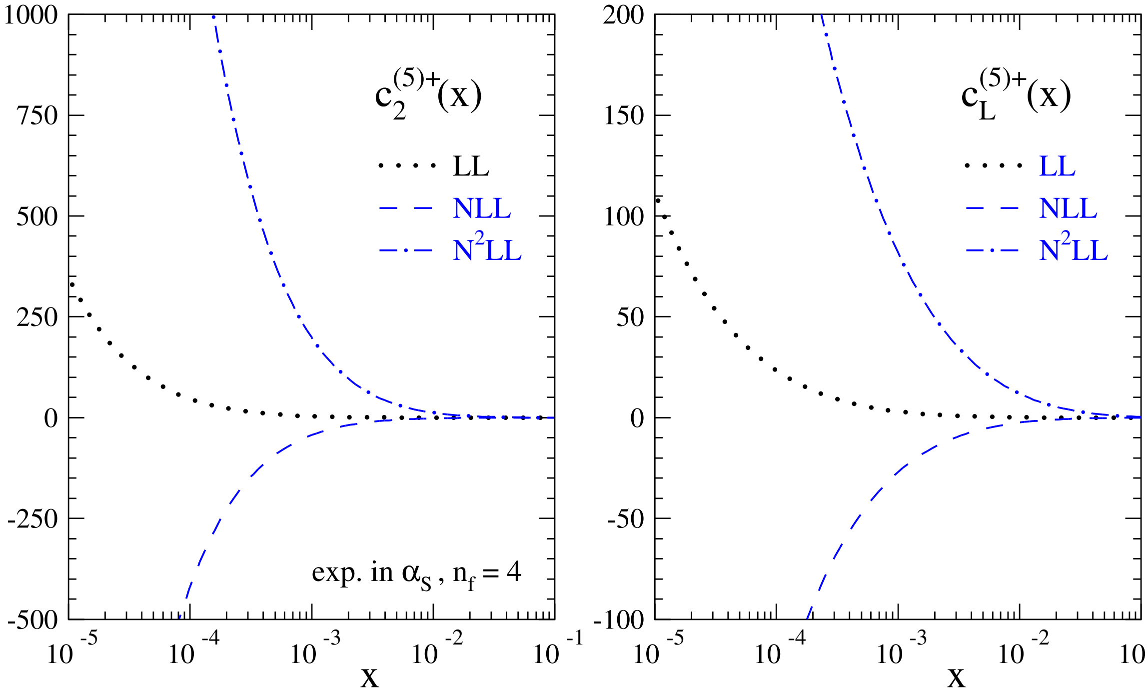

Figure 3. As figure 2, but for fifth-order coefficient functions for the structure functions F_2 and F_L in eq. (2.18).

for $a = 2$ and $a = 3$. The four-loop (N^{4}LO) results are, using the same abbreviations as in eqs. (3.18) and (3.19) above,

$$
\begin{aligned}
C^{(4)}_{2,\mathrm{L}} \;=\; & \\
& + \ln^7 x \Bigg[n_c^3\, C_F \bigg\{ \frac{65}{448} \Big(1 - \frac{16}{15}\, (1-x)^{-1} + x \Big) \bigg\} \Bigg] \\
& + \ln^6 x \Bigg[n_c^3\, C_F \bigg\{ -\frac{229}{180}\, p_{qq}\, \mathrm{H}_1 + \frac{6247}{1296} \Big(1 - \frac{35062}{31235}\, (1-x)^{-1} + \frac{38867}{31235}\, x \Big) \bigg\} \\
& \qquad + n_c^2\, C_F\, n_f \bigg\{ -\frac{6377}{6480} \Big(1 - \frac{8}{7}\, (1-x)^{-1} + x \Big) \bigg\} \Bigg] \\
& + \ln^5 x \Bigg[n_c^3\, C_F \bigg\{ -\frac{19}{2}\, p_{qq}\, \mathrm{H}_{1,1} + \frac{29137}{1080} \Big(1 - \frac{61226}{29137}\, (1-x)^{-1} + \frac{35869}{29137}\, x \Big) \mathrm{H}_1 \\
& \qquad + \frac{329}{120} \Big(1 - \frac{148}{329}\, (1-x)^{-1} + x \Big) \mathrm{H}_{0,1} + \frac{1}{540} \Big([54733 - 9603\, \zeta_2]\, x \\
& \qquad + \frac{1}{4}\, [152641 - 38412\, \zeta_2] - \frac{1}{24}\, [1131721 - 259200\, \zeta_2]\, (1-x)^{-1} \Big) \bigg\} \\
& \qquad + n_c^2\, C_F\, n_f \bigg\{ \frac{662}{135}\, p_{qq}\, \mathrm{H}_1 - \frac{8543}{360} \Big(1 - \frac{96332}{76887}\, (1-x)^{-1} + \frac{10299}{8543}\, x \Big) \bigg\} \\
& \qquad + n_c\, C_F\, n_f^2 \bigg\{ \frac{1951}{1080} \Big(1 - \frac{4}{3}\, (1-x)^{-1} + x \Big) \bigg\} \Bigg] \\
& + \ln^4 x \Bigg[n_c^3\, C_F \bigg\{ \frac{53}{6}\, p_{qq}\, \mathrm{H}_{1,0,1} - \frac{142}{9}\, p_{qq}\, \mathrm{H}_{1,1,1} + \frac{8323}{54} \Big(1 + \frac{1584}{41615}\, x^{-2} - \frac{14864}{8323}\, (1-x)^{-1} \\
& \qquad + \frac{8833}{8323}\, x - \frac{1728}{8323}\, x^2 + \frac{14256}{41615}\, x^3 \Big) \mathrm{H}_{1,1} - \frac{85}{18} \Big(1 - \frac{416}{85}\, (1-x)^{-1} + x \Big) \mathrm{H}_{0,1,1} \\
& \qquad + \frac{69}{2} \Big(1 + \frac{1573}{3726}\, (1-x)^{-1} + \frac{85}{69}\, x - \frac{64}{69}\, x^2 + \frac{176}{115}\, x^3 \Big) \mathrm{H}_{0,1} - \frac{53}{18} \Big(1 + \frac{48}{53}\, (1-x)^{-1} \\
& \qquad + x \Big) \mathrm{H}_{0,0,1} - \frac{88}{15} \Big(x^{-1} + 9\, x^2 + \frac{5}{9504}\, [186223 - 66672\, \zeta_2]\, (1-x)^{-1} \\
& \qquad - \frac{1}{19008}\, [808271 - 333360\, \zeta_2] - \frac{1}{19008}\, [1282259 - 333360\, \zeta_2]\, x \Big) \mathrm{H}_1 \\
& \qquad + \frac{8}{5} \Big([33 - 20\, \zeta_2]\, x^2 - \frac{25}{20736}\, [406651 - 190080\, \zeta_2 - 30240\, \zeta_3]\, (1-x)^{-1} \\
& \qquad + \frac{1}{124416}\, [42910871 - 23142960\, \zeta_2 - 4034880\, \zeta_3] \\
& \qquad + \frac{1}{124416}\, [74167091 - 29523600\, \zeta_2 - 4034880\, \zeta_3]\, x + 33\, \zeta_2\, x^3 \Big) \bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ n_c^2\, C_F\, n_f \Big\{ -\tfrac{422}{27} \Big(1 + \tfrac{24}{211}\, x^{-2} - 2\,(1-x)^{-1} + \tfrac{259}{211}\, x - \tfrac{144}{211}\, x^2 + \tfrac{216}{211}\, x^3\Big)\, \mathrm{H}_{1,1} \\
&- \tfrac{8177}{108} \Big(1 - \tfrac{192}{8177}\, x^{-1} - \tfrac{16782}{8177}\,(1-x)^{-1} + \tfrac{10341}{8177}\, x - \tfrac{1728}{8177}\, x^2\Big)\, \mathrm{H}_1 \\
&+ \tfrac{10}{9} \Big(1 - \tfrac{22}{3}\,(1-x)^{-1} - \tfrac{11}{5}\, x + \tfrac{48}{5}\, x^2 - \tfrac{72}{5}\, x^3\Big)\, \mathrm{H}_{0,1} - \tfrac{16}{3} \Big([3 - 2\,\zeta_2]\, x^2 \\
&+ \tfrac{5}{10368}\, [92215 - 23364\,\zeta_2] + \tfrac{1}{5184}\, [338959 - 56250\,\zeta_2]\, x \\
&- \tfrac{1}{20736}\, [1323835 - 295488\,\zeta_2]\,(1-x)^{-1} + 3\,\zeta_2\, x^3\Big)\Big\} + C_F\, n_f^3 \Big\{\tfrac{119}{162}\, p_{qq}\Big\} \\
&+ n_c\, C_F\, n_f^2 \Big\{ -\tfrac{671}{162}\, p_{qq}\, \mathrm{H}_1 + \tfrac{111}{4} \Big(1 - \tfrac{41134}{26973}\,(1-x)^{-1} + \tfrac{11249}{8991}\, x\Big)\Big\}\Big]\,, \qquad (4.11)
\end{aligned}
$$

$$
\begin{aligned}
C_{3,\mathrm{L}}^{(4)} \;=\; & \\
&+ \ln^7 x \Big[n_c^3\, C_F \Big\{ \tfrac{65}{448} \Big(1 - \tfrac{16}{15}\,(1-x)^{-1} + x\Big)\Big\}\Big] \\
&+ \ln^6 x \Big[n_c^3\, C_F \Big\{ -\tfrac{229}{180}\, p_{qq}\, \mathrm{H}_1 + \tfrac{30929}{6480} \Big(1 - \tfrac{35062}{30929}\,(1-x)^{-1} + \tfrac{38561}{30929}\, x\Big)\Big\} \\
&\quad + n_c^2\, C_F\, n_f \Big\{ -\tfrac{6377}{6480} \Big(1 - \tfrac{8}{7}\,(1-x)^{-1} + x\Big)\Big\}\Big] \\
&+ \ln^5 x \Big[n_c^3\, C_F \Big\{ -\tfrac{19}{2}\, p_{qq}\, \mathrm{H}_{1,1} + \tfrac{28057}{1080} \Big(1 - \tfrac{61226}{28057}\,(1-x)^{-1} + \tfrac{34789}{28057}\, x\Big)\, \mathrm{H}_1 \\
&\quad + \tfrac{329}{120} \Big(1 - \tfrac{148}{329}\,(1-x)^{-1} + x\Big)\, \mathrm{H}_{0,1} + \tfrac{1}{360} \Big([35675 - 6402\,\zeta_2]\, x \\
&\quad + \tfrac{1}{2}\, [49055 - 12804\,\zeta_2] - \tfrac{1}{36}\, [1131721 - 259200\,\zeta_2]\,(1-x)^{-1}\Big)\Big\} \\
&\quad + n_c^2\, C_F\, n_f \Big\{ \tfrac{662}{135}\, p_{qq}\, \mathrm{H}_1 - \tfrac{24961}{1080} \Big(1 - \tfrac{96332}{74883}\,(1-x)^{-1} + \tfrac{30229}{24961}\, x\Big)\Big\} \\
&\quad + n_c\, C_F\, n_f^2 \Big\{ \tfrac{1951}{1080} \Big(1 - \tfrac{4}{3}\,(1-x)^{-1} + x\Big)\Big\}\Big] \\
&+ \ln^4 x \Big[n_c^3\, C_F \Big\{ \tfrac{53}{6}\, p_{qq}\, \mathrm{H}_{1,0,1} - \tfrac{142}{9}\, p_{qq}\, \mathrm{H}_{1,1,1} + \tfrac{7963}{54} \Big(1 - \tfrac{528}{7963}\, x^{-1} - \tfrac{14864}{7963}\,(1-x)^{-1} \\
&\quad + \tfrac{7609}{7963}\, x + \tfrac{528}{7963}\, x^2\Big)\, \mathrm{H}_{1,1} - \tfrac{85}{18} \Big(1 - \tfrac{416}{85}\,(1-x)^{-1} + x\Big)\, \mathrm{H}_{0,1,1} + \tfrac{65}{2} \Big(1 \\
&\quad + \tfrac{121}{270}\,(1-x)^{-1} + \tfrac{49}{65}\, x + \tfrac{176}{585}\, x^2\Big)\, \mathrm{H}_{0,1} - \tfrac{53}{18} \Big(1 + \tfrac{48}{53}\,(1-x)^{-1} + x\Big)\, \mathrm{H}_{0,0,1}
\end{aligned}
$$

$$+\tfrac{1}{648}\Big([159931-66672\,\zeta_2]-2\,[186223-66672\,\zeta_2]\,(1-x)^{-1}$$

$$+[235015-66672\,\zeta_2]\,x\Big)\,\mathrm{H}_1-\tfrac{5}{2592}\Big([406651-190080\,\zeta_2$$

$$-30240\,\zeta_3]\,(1-x)^{-1}-\tfrac{1}{30}\,[7672909-4526856\,\zeta_2-806976\,\zeta_3]$$

$$-\tfrac{1}{30}\,[14040109-5927400\,\zeta_2-806976\,\zeta_3]\,x-\tfrac{25344}{5}\,\zeta_2\,x^2\Big)\Big\}$$

$$+n_c^2\,C_F\,n_f\Big\{-\tfrac{422}{27}\Big(1-\tfrac{24}{211}\,x^{-1}-2\,(1-x)^{-1}+x+\tfrac{24}{211}\,x^2\Big)\,\mathrm{H}_{1,1}$$

$$+\tfrac{10}{9}\Big(1-\tfrac{22}{3}\,(1-x)^{-1}+x-\tfrac{8}{5}\,x^2\Big)\,\mathrm{H}_{0,1}-\tfrac{8065}{108}\Big(1-\tfrac{16782}{8065}\,(1-x)^{-1}$$

$$+\tfrac{9461}{8065}\,x\Big)\,\mathrm{H}_1-\tfrac{1}{972}\Big([208597-58410\,\zeta_2]+\tfrac{1}{2}\,[633641-115956\,\zeta_2]\,x$$

$$-\tfrac{1}{4}\,[1323835-295488\,\zeta_2]\,(1-x)^{-1}+1728\,\zeta_2\,x^2\Big)\Big\}+C_F\,n_f^3\Big\{\tfrac{119}{162}\,p_{qq}\Big\}$$

$$+n_c\,C_F\,n_f^2\Big\{-\tfrac{671}{162}\,p_{qq}\,\mathrm{H}_1+\tfrac{8239}{324}\Big(1-\tfrac{41134}{24717}\,(1-x)^{-1}+\tfrac{10497}{8239}\,x\Big)\Big\}\Big]\,. \qquad (4.12)$$

We have also determined the corresponding five-loop results. Since they would become useful mainly in the context of research towards N^5LO all-x expressions, which we do not expect in the foreseeable future, we skip these here for brevity but provide them in the ancillary file of this paper.

In the case of C_L, the fixed order results are restricted to NNLO even in the large-n_c limit, since the four-loop coefficient function would be required for N^3LO accuracy, recall the discussion below eq. (2.6). Hence only the highest three small-x logarithms can be resummed using eq. (2.17). The resulting N^3LO and N^4LO predictions read

$$C_{L,\mathrm{L}}^{(4)}\;=$$

$$+\ln^6 x\Big[n_c^3\,C_F\Big\{\frac{17}{180}\,x\Big\}\Big]$$

$$+\ln^5 x\Big[n_c^3\,C_F\Big\{-\frac{1}{4}\Big(1-\frac{3386}{135}\,x\Big)+2\,x\,\mathrm{H}_1\Big\}+n_c^2\,C_F\,n_f\Big\{-\frac{167}{135}\,x\Big\}\Big]$$

$$+\ln^4 x\Big[n_c^3\,C_F\Big\{-\frac{116}{15}\Big(1+\frac{88}{29}\,x^{-1}-\frac{5869}{1044}\,x+\frac{132}{29}\,x^2\Big)\,\mathrm{H}_1$$

$$+\frac{1687}{1080}\Big(1+\frac{1152}{1687}\,[33-20\,\zeta_2]\,x^2+\frac{1}{10122}\,[867587-32940\,\zeta_2]\,x+\frac{38016}{1687}\,\zeta_2\,x^3\Big)$$

$$+\frac{352}{15}\Big(x^{-2}-\frac{5}{11}\,x^{-1}+\frac{5}{4}\,x-\frac{10}{11}\,x^2+\frac{3}{2}\,x^3\Big)\,\mathrm{H}_{1,1}+20\,\Big(x-\frac{16}{15}\,x^2+\frac{44}{25}\,x^3\Big)\,\mathrm{H}_{0,1}\Big\}$$

$$+\,n_c^2\,C_F\,n_f\biggl\{-\frac{34}{27}\Bigl(1+\frac{48}{17}\,[3-2\,\zeta_2]\,x^2+\frac{1}{408}\,[16361+576\,\zeta_2]\,x+\frac{144}{17}\,\zeta_2\,x^3\Bigr)$$
$$-\frac{64}{9}\Bigl(x^{-2}-\frac{1}{2}\,x^{-1}+\frac{1}{2}\,x-x^2+\frac{3}{2}\,x^3\Bigr)\,\mathrm{H}_{1,1}+\frac{64}{9}\Bigl(x^{-1}-\frac{31}{24}\,x+\frac{3}{2}\,x^2\Bigr)\,\mathrm{H}_1$$
$$-\frac{32}{9}\Bigl(x-2\,x^2+3\,x^3\Bigr)\,\mathrm{H}_{0,1}\biggr\}+n_c\,C_F\,n_f^2\biggl\{\frac{376}{81}\,x\biggr\}\biggr] \tag{4.13}$$

and

$$C_{L,\mathrm{L}}^{(5)} =$$
$$+\ln^8\!x\biggl[n_c^4\,C_F\biggl\{\frac{149}{26880}\,x\biggr\}\biggr]$$
$$+\ln^7\!x\biggl[n_c^4\,C_F\biggl\{-\frac{13}{672}\Bigl(1-\frac{61168}{1755}\,x\Bigr)+\frac{13}{42}\,x\,\mathrm{H}_1\biggr\}+n_c^3\,C_F\,n_f\biggl\{-\frac{3043}{22680}\,x\biggr\}\biggr]$$
$$+\ln^6\!x\biggl[n_c^4\,C_F\biggl\{-\frac{971}{450}\Bigl(1+\frac{2552}{971}\,x^{-1}-\frac{106633}{17478}\,x+\frac{3828}{971}\,x^2\Bigr)\,\mathrm{H}_1$$
$$+\frac{82109}{64800}\Bigl(1+\frac{16704}{82109}\,[33-20\,\zeta_2]\,x^2+\frac{1}{246327}\,[6243947-181980\,\zeta_2]\,x$$
$$+\frac{551232}{82109}\,\zeta_2\,x^3\Bigr)+\frac{1276}{225}\Bigl(x^{-2}-\frac{5}{11}\,x^{-1}+\frac{1725}{1276}\,x-\frac{10}{11}\,x^2+\frac{3}{2}\,x^3\Bigr)\,\mathrm{H}_{1,1}$$
$$+\frac{46}{9}\Bigl(x-\frac{116}{115}\,x^2+\frac{957}{575}\,x^3\Bigr)\,\mathrm{H}_{0,1}\biggr\}+n_c^3\,C_F\,n_f\biggl\{+\frac{232}{135}\Bigl(x^{-1}-\frac{1115}{696}\,x+\frac{3}{2}\,x^2\Bigr)\,\mathrm{H}_1$$
$$-\frac{232}{135}\Bigl(x^{-2}-\frac{1}{2}\,x^{-1}+\frac{1}{2}\,x-x^2+\frac{3}{2}\,x^3\Bigr)\,\mathrm{H}_{1,1}-\frac{116}{135}\Bigl(x-2\,x^2+3\,x^3\Bigr)\,\mathrm{H}_{0,1}$$
$$-\frac{1669}{3240}\Bigl(1+\frac{2784}{1669}\,[3-2\,\zeta_2]\,x^2+\frac{1}{5007}\,[115745+2088\,\zeta_2]\,x+\frac{8352}{1669}\,\zeta_2\,x^3\Bigr)\biggr\}$$
$$+\,n_c^2\,C_F\,n_f^2\biggl\{\frac{10891}{9720}\,x\biggr\}\biggr]\,. \tag{4.14}$$

5 Results for flavour-singlet quantities

We now turn to the singlet case, and first present the results for the splitting functions P_{ik}. The diagonal ($\mathrm{i}=\mathrm{k}$) quantities can be written as sums of 'non-singlet' (ns) and pure-singlet (ps) pieces,

$$P_{\mathrm{qq}}(N) \equiv P_{\mathrm{ns}}^{+}(N)+P_{\mathrm{qq}}^{\mathrm{ps}}(N)\,, \qquad\qquad P_{\mathrm{gg}}(N) \equiv P_{\mathrm{gg}}^{+}(N)+P_{\mathrm{gg}}^{\mathrm{ps}}(N)\,, \tag{5.1}$$

where P_{gg}^{+} is an non-singlet-like quantity, i.e., P_{gg} in the limit $C_F=0$ (cf. ref. [98]). At leading-logarithmic (LL) accuracy,

$$P_{\mathrm{gg}}^{+}(N) \;=\; -\frac{1}{2}\,N(S'-1)\,, \tag{5.2}$$

where

$$S' = (1 - 4\,\xi')^{1/2} \quad \text{with} \quad \xi' = -\frac{4C_A a_{\rm s}}{N^2} \equiv -\frac{C_A \alpha_{\rm s}}{\pi N^2}\,. \tag{5.3}$$

Consequently the LL $P_{gg}^{+}(x)$ is an oscillatory function, in notable contrast to the corresponding quark quantity in eqs. (3.2) and (3.5). The other LL contributions are found to be

$$P_{\rm qq}^{\rm ps\,(n)}(N) = \mathcal{C}_n\,\frac{2^{n+1}}{N^{2n+1}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=0}^{n-1-2i}(-2)^{i+1+k}\,(n_f C_F)^{i+1} C_A^k C_F^\rho \binom{k+i}{k}\binom{\rho+i+1}{\rho}, \tag{5.4}$$

$$P_{\rm gg}^{\rm ps\,(n)}(N) = \mathcal{C}_n\,\frac{2^{n+1}}{N^{2n+1}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=0}^{n-1-2i}(-2)^{i+1+k}\,(n_f C_F)^{i+1} C_A^k C_F^\rho \binom{k+i+1}{k}\binom{\rho+i}{\rho}, \tag{5.5}$$

$$P_{\rm qg}^{(n)}(N) = n_f\mathcal{C}_n\,\frac{2^{n+1}}{N^{2n+1}}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{k=0}^{n-2i}(-2)^{i+k}\,(n_f C_F)^{i} C_A^k C_F^\delta \binom{k+i}{k}\binom{\delta+i}{\delta}, \tag{5.6}$$

$$P_{\rm gq}^{(n)}(N) = -\,\frac{2\,C_F}{n_f}\,P_{qg}^{(n)}(N)\,, \tag{5.7}$$

where the relation between $P_{\rm qg}$ and $P_{\rm gq}$ holds only at LL. In these expressions, $\lfloor\ldots\rfloor$ denotes the Gauß bracket (floor function), $\rho = n-k-2\,i-1$, $\delta = n-k-2\,i$, and $\mathcal{C}_n$ are the Catalan numbers,

$$\mathcal{C}_n = \frac{(2n)!}{n!\,(n+1)!}\,. \tag{5.8}$$

The first (second) binomial factors in eqs. (5.4)–(5.6) may be interpreted as counting the number of ways k (ρ/δ) gluons can be emitted from the gluon (quark) propagators, the number of which is directly related to the number of factors $n_f C_F$. In this interpretation, the quark and gluon emissions contribute to these logarithms equally, with emission strengths proportional to the colour factors.

Incidentally, the Catalan numbers (5.8) are the coefficients of the Taylor expansion of S in the LL non-singlet splitting function, suggesting the singlet results could be generalizations of their simpler non-singlet counterparts. However, we have not found closed-form expressions. The NLL and NNLL results are considerably more complicated, and even expressions in the form of the sum representations above are not available. To achieve this could require defining the LL results in terms of new special functions, and suitably generalizing them for the sub-leading results. Potential complications also arise from the observation that the NLL and NNLL results have terms with denominators similar to that of the N-space logarithmic functions in the SIA case [41], which could indicate the presence of logarithmic functions that depend on the special functions yet to be found.

The NNLL double-logarithmic resummation of the $1/N$ pole terms leads to the following results for the four-loop (N^{3}LO) splitting functions ($P_{\rm ns}^{+(3)}(N)$ has been given in eq. (3.4) above)

$$
\begin{aligned}
P^{(3)}_{\rm qq}(N) = {} & P^{+(3)}_{\rm ns}(N) + n_f\, C_F \Big\{ N^{-7} \Big(-640\, C_A^2 + 640\, C_F\, C_A - 480\, C_F^2 + 320\, n_f\, C_F \Big) \\
& + N^{-6} \Big(-\frac{2176}{3} C_A^2 + \frac{3424}{3} C_F\, C_A - \frac{1024}{3} C_F^2 - 256\, n_f\, C_F \Big) \\
& + N^{-5} \Big(-288\, n_f\, C_A + \frac{15232}{9} n_f\, C_F - \frac{32}{9} n_f^2 - \frac{8}{3} (519 - 524\,\zeta_2)\, C_F^2 \\
& + \frac{8}{9} (541 - 1332\,\zeta_2)\, C_F\, C_A - \frac{8}{9} (1709 - 192\,\zeta_2)\, C_A^2 \Big) \Big\} + \mathcal{O}(N^{-4})\,,
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
P^{(3)}_{\rm qg}(N) = {} & n_f \Big\{ N^{-7} \Big(-640\, C_A^3 + 320\, C_F\, C_A^2 - 160\, C_F^2\, C_A + 80\, C_F^3 + 640\, n_f\, C_F\, C_A \\
& - 320\, n_f\, C_F^2 \Big) + N^{-6} \Big(-\frac{416}{3} C_A^3 + 192\, C_F\, C_A^2 - \frac{632}{3} C_F^2\, C_A - \frac{32}{3} C_F^3 \\
& - \frac{320}{3} n_f\, C_A^2 - \frac{1408}{3} n_f\, C_F\, C_A + 432\, n_f\, C_F^2 \Big) + N^{-5} \Big(-\frac{32}{9} n_f^2\, C_A + \frac{2224}{27} n_f^2\, C_F \\
& - \frac{32}{27} (148 + 81\,\zeta_2)\, n_f\, C_A^2 + \frac{2}{3} (557 - 1448\,\zeta_2)\, C_F^3 - \frac{40}{27} (1711 + 108\,\zeta_2)\, C_A^3 \\
& + \frac{8}{9} (2951 + 300\,\zeta_2)\, n_f\, C_F\, C_A + \frac{8}{27} (6427 - 3960\,\zeta_2)\, C_F\, C_A^2 - \frac{2}{27} (6707 \\
& - 19368\,\zeta_2)\, C_F^2\, C_A - \frac{4}{27} (13583 - 3600\,\zeta_2)\, n_f\, C_F^2 \Big) \Big\} + \mathcal{O}(N^{-4})\,,
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
P^{(3)}_{\rm gq}(N) = {} & C_F \Big\{ N^{-7} \Big(1280\, C_A^3 - 640\, C_F\, C_A^2 + 320\, C_F^2\, C_A - 160\, C_F^3 - 1280\, n_f\, C_F\, C_A \\
& + 640\, n_f\, C_F^2 \Big) + N^{-6} \Big(\frac{4160}{3} C_A^3 - 1280\, C_F\, C_A^2 + \frac{2800}{3} C_F^2\, C_A - 320\, C_F^3 \\
& + \frac{640}{3} n_f\, C_A^2 - 640\, n_f\, C_F\, C_A + \frac{800}{3} n_f\, C_F^2 \Big) + N^{-5} \Big(\frac{64}{9} n_f^2\, C_A - \frac{12256}{27} n_f^2\, C_F \\
& - \frac{4}{3} (25 - 1248\,\zeta_2)\, C_F^3 + \frac{64}{27} (542 + 81\,\zeta_2)\, n_f\, C_A^2 - \frac{16}{3} (817 + 164\,\zeta_2)\, n_f\, C_F\, C_A \\
& + \frac{16}{27} (1969 + 936\,\zeta_2)\, C_F\, C_A^2 + \frac{16}{27} (3871 + 2340\,\zeta_2)\, C_A^3 + \frac{8}{27} (7747 \\
& - 2448\,\zeta_2)\, n_f\, C_F^2 - \frac{4}{27} (8633 + 12672\,\zeta_2)\, C_F^2\, C_A \Big) \Big\} + \mathcal{O}(N^{-4})\,,
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
P^{(3)}_{\rm gg}(N) = {} & N^{-7} \Big(1280\, C_A^4 - 1920\, n_f\, C_F\, C_A^2 + 640\, n_f\, C_F^2\, C_A - 160\, n_f\, C_F^3 + 320\, n_f^2\, C_F^2 \Big) \\
& + N^{-6} \Big(\frac{640}{3} C_A^4 + \frac{1280}{3} n_f\, C_A^3 + \frac{1856}{3} n_f\, C_F\, C_A^2 + \frac{256}{3} n_f\, C_F^2\, C_A + \frac{64}{3} n_f\, C_F^3 \\
& - \frac{640}{3} n_f^2\, C_F\, C_A - \frac{1472}{3} n_f^2\, C_F^2 \Big) + N^{-5} \Big(\frac{128}{3} n_f^2\, C_A^2 - \frac{4768}{9} n_f^2\, C_F\, C_A \\
& + \frac{19904}{9} n_f^2\, C_F^2 - \frac{32}{9} n_f^3\, C_F + \frac{128}{3} (20 + 9\,\zeta_2)\, n_f\, C_A^3 + 32\, (137 + 64\,\zeta_2)\, C_A^4 \\
& - \frac{8}{3} (195 - 148\,\zeta_2)\, n_f\, C_F^3 + \frac{8}{9} (1997 - 756\,\zeta_2)\, n_f\, C_F^2\, C_A \\
& - \frac{8}{3} (2751 + 688\,\zeta_2)\, n_f\, C_F\, C_A^2 \Big) + \mathcal{O}(N^{-4})\,,
\end{aligned} \tag{5.12}
$$

which we expect to become relevant in the near future in combination with the fixed-N moments in ref. [11] and other constraints. Their N^{4}LO counterparts read

$$
\begin{aligned}
P_{\mathrm{qq}}^{(4)}(N) = P_{\mathrm{ns}}^{+(4)}(N) + n_f\, C_F \Big\{ & N^{-9} \Big(7168\, C_A^3 - 7168\, C_F\, C_A^2 + 5376\, C_F^2\, C_A - 3584\, C_F^3 \\
& - 7168\, n_f\, C_F\, C_A + 5376\, n_f\, C_F^2 \Big) + N^{-8} \Big(7936\, C_A^3 - \frac{38720}{3} C_F\, C_A^2 + \frac{41984}{3} C_F^2\, C_A \\
& - \frac{12272}{3} C_F^3 + \frac{1792}{3} n_f\, C_A^2 - 1088\, n_f\, C_F\, C_A - \frac{6656}{3} n_f\, C_F^2 + \frac{896}{3} n_f^2\, C_F \Big) \\
& + N^{-7} \Big(\frac{256}{9} n_f^2\, C_A - \frac{20480}{9} n_f^2\, C_F + \frac{32}{3} (442 + 105\, \zeta_2)\, n_f\, C_A^2 + \frac{32}{9} (7054 \\
& + 243\, \zeta_2)\, C_A^3 - \frac{4}{3} (9109 - 19668\, \zeta_2)\, C_F^3 + \frac{4}{9} (9211 - 72108\, \zeta_2)\, C_F^2\, C_A \\
& - \frac{16}{9} (16829 + 1602\, \zeta_2)\, n_f\, C_F\, C_A - \frac{8}{9} (24337 - 22320\, \zeta_2)\, C_F\, C_A^2 \\
& + \frac{8}{9} (33715 - 9216\, \zeta_2)\, n_f\, C_F^2 \Big) \Big\} + \mathcal{O}(N^{-6})\,,
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
P_{\mathrm{qg}}^{(4)}(N) = n_f \Big\{ & N^{-9} \Big(7168\, C_A^4 - 3584\, C_F\, C_A^3 + 1792\, C_F^2\, C_A^2 - 896\, C_F^3\, C_A + 448\, C_F^4 \\
& - 10752\, n_f\, C_F\, C_A^2 + 7168\, n_f\, C_F^2\, C_A - 2688\, n_f\, C_F^3 + 1792\, n_f^2\, C_F^2 \Big) \\
& + N^{-8} \Big(\frac{4096}{3} C_A^4 - 2368\, C_F\, C_A^3 + \frac{7840}{3} C_F^2\, C_A^2 - \frac{6064}{3} C_F^3\, C_A + \frac{584}{3} C_F^4 \\
& + 1792\, n_f\, C_A^3 + 4736\, n_f\, C_F\, C_A^2 - \frac{6272}{3} n_f\, C_F^2\, C_A + \frac{7424}{3} n_f\, C_F^3 - \frac{1792}{3} n_f^2\, C_F\, C_A \\
& - \frac{11648}{3} n_f^2\, C_F^2 \Big) + N^{-7} \Big(128\, n_f^2\, C_A^2 - \frac{52672}{27} n_f^2\, C_F\, C_A + \frac{427424}{27} n_f^2\, C_F^2 \\
& - \frac{128}{9} n_f^3\, C_F + \frac{2}{3} (2915 - 13216\, \zeta_2)\, C_F^4 + \frac{16}{27} (5216 + 3375\, \zeta_2)\, n_f\, C_A^3 \\
& - \frac{2}{27} (29293 - 244260\, \zeta_2)\, C_F^3\, C_A + \frac{16}{27} (59326 + 8199\, \zeta_2)\, C_A^4 \\
& + \frac{4}{27} (73415 - 115992\, \zeta_2)\, C_F^2\, C_A^2 - \frac{8}{27} (81626 - 37539\, \zeta_2)\, C_F\, C_A^3 \\
& - \frac{4}{27} (114685 - 57816\, \zeta_2)\, n_f\, C_F^3 + \frac{8}{27} (118813 - 41067\, \zeta_2)\, n_f\, C_F^2\, C_A \\
& - \frac{8}{27} (181400 + 18351\, \zeta_2)\, n_f\, C_F\, C_A^2 \Big) \Big\} + \mathcal{O}(N^{-6})\,,
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
P^{(4)}_{\rm gq}(N) = C_F \Big\{ & N^{-9} \Big(-14336\, C_A^4 + 7168\, C_F\, C_A^3 - 3584\, C_F^2\, C_A^2 + 1792\, C_F^3\, C_A \\
& - 896\, C_F^4 + 21504\, n_f\, C_F\, C_A^2 - 14336\, n_f\, C_F^2\, C_A + 5376\, n_f\, C_F^3 - 3584\, n_f^2\, C_F^2 \Big) \\
& + N^{-8} \Big(-16128\, C_A^4 + \frac{43904}{3}\, C_F\, C_A^3 - \frac{31808}{3}\, C_F^2\, C_A^2 + 6944\, C_F^3\, C_A - 2240\, C_F^4 \\
& - 3584\, n_f\, C_A^3 + \frac{46592}{3}\, n_f\, C_F\, C_A^2 - \frac{51968}{3}\, n_f\, C_F^2\, C_A + 4928\, n_f\, C_F^3 + \frac{3584}{3}\, n_f^2\, C_F\, C_A \\
& + \frac{7168}{3}\, n_f^2\, C_F^2 \Big) + N^{-7} \Big(-256\, n_f^2\, C_A^2 + \frac{318208}{27}\, n_f^2\, C_F\, C_A - \frac{750464}{27}\, n_f^2\, C_F^2 \\
& + \frac{256}{9}\, n_f^3\, C_F - \frac{112}{3}\, (42 - 437\, \zeta_2)\, C_F^4 - \frac{112}{27}\, (191 + 1017\, \zeta_2)\, C_F\, C_A^3 \\
& + \frac{64}{27}\, (8005 - 5517\, \zeta_2)\, n_f\, C_F^3 - \frac{8}{27}\, (13313 + 104940\, \zeta_2)\, C_F^3\, C_A \\
& - \frac{32}{27}\, (14392 + 3375\, \zeta_2)\, n_f\, C_A^3 + \frac{8}{27}\, (17711 + 77652\, \zeta_2)\, C_F^2\, C_A^2 \\
& - \frac{32}{27}\, (37616 + 17019\, \zeta_2)\, C_A^4 - \frac{16}{27}\, (63557 - 20547\, \zeta_2)\, n_f\, C_F^2\, C_A \\
& + \frac{16}{27}\, (149746 + 32031\, \zeta_2)\, n_f\, C_F\, C_A^2 \Big) \Big\} + \mathcal{O}(N^{-4})\,,
\end{aligned} \tag{5.15}
$$

$$
\begin{aligned}
P^{(4)}_{\rm gg}(N) = & N^{-9} \Big(-14336\, C_A^5 + 28672\, n_f\, C_F\, C_A^3 - 10752\, n_f\, C_F^2\, C_A^2 + 3584\, n_f\, C_F^3\, C_A \\
& - 896\, n_f\, C_F^4 - 10752\, n_f^2\, C_F^2\, C_A + 3584\, n_f^2\, C_F^3 \Big) + N^{-8} \Big(-\frac{8960}{3}\, C_A^5 \\
& - \frac{17920}{3}\, n_f\, C_A^4 - \frac{14848}{3}\, n_f\, C_F\, C_A^3 - \frac{15040}{3}\, n_f\, C_F^2\, C_A^2 + \frac{9536}{3}\, n_f\, C_F^3\, C_A \\
& - \frac{1168}{3}\, n_f\, C_F^4 + 5376\, n_f^2\, C_F\, C_A^2 + 10048\, n_f^2\, C_F^2\, C_A - \frac{11264}{3}\, n_f^2\, C_F^3 - \frac{896}{3}\, n_f^3\, C_F^2 \Big) \\
& + N^{-7} \Big(-\frac{640}{9}\, [907 + 396\, \zeta_2]\, C_A^5 - \frac{640}{9}\, [164 + 81\, \zeta_2]\, n_f\, C_A^4 - \frac{2560}{3}\, n_f^2\, C_A^3 \\
& + \frac{224}{9}\, [5438 + 1431\, \zeta_2]\, n_f\, C_F\, C_A^3 - \frac{4}{3}\, [2171 - 7692\, \zeta_2]\, n_f\, C_F^4 \\
& - \frac{8}{9}\, [43463 - 14940\, \zeta_2]\, n_f\, C_F^2\, C_A^2 + \frac{4}{9}\, [18349 - 39132\, \zeta_2]\, n_f\, C_F^3\, C_A \\
& + \frac{32}{9}\, [3274 + 495\, \zeta_2]\, n_f^2\, C_F\, C_A^2 - \frac{16}{9}\, [38371 + 3978\, \zeta_2]\, n_f^2\, C_F^2\, C_A \\
& + \frac{8}{9}\, [26605 - 5184\, \zeta_2]\, n_f^2\, C_F^3 + 256\, n_f^3\, C_F\, C_A - \frac{14080}{9}\, n_f^3\, C_F^2 \Big) + \mathcal{O}(N^{-6})\,.
\end{aligned} \tag{5.16}
$$

Finally we present the corresponding results for the singlet structure functions F_2 and F_L. The quark coefficient functions can be written as sum of non-singlet and pure-singlet pieces,

$$
C_{a,\rm q}(N) \;=\; C_a^{+}(N) + C_{a,\rm ps}(N)\,. \tag{5.17}
$$

The leading-logarithmic resummations of the pure-singlet and gluon coefficient functions are

$$C^{(n)}_{2,\mathrm{ps}}(N) = \mathcal{D}_n \frac{2^n}{N^{2n}} \sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor} \sum_{k=0}^{n-2-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^{\rho'} \binom{k+i}{k} \binom{\rho'+i+1}{\rho'}, \tag{5.18}$$

$$C^{(n)}_{2,\mathrm{g}}(N) = n_f \mathcal{D}_n \frac{2^n}{N^{2n}} \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+k} (n_f C_F)^{i} C_A^k C_F^{\delta'} \binom{k+i}{k} \binom{\delta'+i}{\delta'}, \tag{5.19}$$

$$C^{(n)}_{L,\mathrm{ps}}(N) = \mathcal{D}_{n-1} \frac{2^{n+1}}{N^{2n-2}} \sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor} \sum_{k=0}^{n-2-2i} (-2)^{i+1+k} (n_f C_F)^{i+1} C_A^k C_F^{\rho'} \binom{k+i}{k} \binom{\rho'+i+1}{\rho'}, \tag{5.20}$$

$$C^{(n)}_{L,\mathrm{g}}(N) = n_f \mathcal{D}_{n-1} \frac{2^{n+1}}{N^{2n-2}} \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \sum_{k=0}^{n-1-2i} (-2)^{i+k} (n_f C_F)^{i} C_A^k C_F^{\delta'} \binom{k+i}{k} \binom{\delta'+i}{\delta'}, \tag{5.21}$$

where $\rho' = n-k-2i-2$, $\delta' = n-k-2i-1$, and the $\mathcal{D}_n$ are defined as

$$\mathcal{D}_n = \frac{1}{n!} \prod_{k=0}^{n-1} (1+4k) \; . \tag{5.22}$$

These are the coefficients of the Taylor expansion of

$$F \;=\; S^{-1/2} \;=\; (1-4\,\xi)^{-1/4}\,, \tag{5.23}$$

i.e., those of the leading-logarithmic contributions to the non-singlet coefficient functions (4.1)–(4.3). Similarly to the splitting functions above, the first (second) binomial factor could be interpreted as the number of ways k (ρ'/δ') gluons can be emitted from the gluon (quark) propagators. Again, analytic NLL and NNLL results, which could require generalization of the LL singlet coefficient functions, are not available at this point. Corresponding results for the scalar-exchange structure function F_ϕ in eq. (2.18), which is of only theoretical relevance, can be found in appendix C.

Hence, at least for the time being, the coefficients for the NLL and NNLL contributions to these coefficient functions are only known at each order separately. Since we do not expect yet higher orders to be become relevant in the foreseeable future, we finally present these results at the fourth and fifth order in $a_\mathrm{s} = \alpha_\mathrm{s}/(4\,\pi)$. The former read

$$\begin{aligned} c^{(4)}_{2,\mathrm{q}}(N) = c_2^+(N)\big|_{a_\mathrm{s}^4} + n_f\, C_F \Big\{ & N^{-8} \Big(-3120\, C_A^2 + 3120\, C_F\, C_A - 2340\, C_F^2 + 1560\, n_f\, C_F \Big) \\ & + N^{-7} \Big(-\frac{60872}{9} C_A^2 + \frac{86228}{9} C_F\, C_A - \frac{7798}{3} C_F^2 + \frac{5216}{9} n_f\, C_A - \frac{16688}{9} n_f\, C_F \Big) \\ & + N^{-6} \Big(\frac{9848}{27} n_f\, C_A - \frac{952}{9} n_f^2 - \frac{1}{3} (16611 - 21752\,\zeta_2)\, C_F^2 \\ & + \frac{8}{27} (24251 - 20439\,\zeta_2)\, C_F\, C_A + \frac{2}{27} (124393 - 14688\,\zeta_2)\, n_f\, C_F \\ & - \frac{2}{27} (242611 - 22752\,\zeta_2)\, C_A^2 \Big) \Big\} + \mathcal{O}(N^{-5})\,, \end{aligned} \tag{5.24}$$

$$
\begin{aligned}
c_{2,\mathrm{g}}^{(4)}(N) = n_f \Big\{ & N^{-8}\Big(-3120\,C_A^3+1560\,C_F\,C_A^2-780\,C_F^2\,C_A+390\,C_F^3+3120\,n_f\,C_F\,C_A \\
& -1560\,n_f\,C_F^2\Big)+N^{-7}\Big(-\frac{35132}{9}\,C_A^3+\frac{30052}{9}\,C_F\,C_A^2-\frac{21101}{9}\,C_F^2\,C_A \\
& +\frac{889}{3}\,C_F^3+\frac{536}{9}\,n_f\,C_A^2-\frac{2056}{3}\,n_f\,C_F\,C_A+\frac{13778}{9}\,n_f\,C_F^2-\frac{2608}{9}\,n_f^2\,C_F\Big) \\
& +N^{-6}\Big(-\frac{248}{27}\,n_f^2\,C_A+\frac{20300}{27}\,n_f^2\,C_F+\frac{52}{3}\,(771-41\,\zeta_2)\,n_f\,C_F\,C_A \\
& +\frac{1}{6}\,(2453-23816\,\zeta_2)\,C_F^3-\frac{4}{27}\,(2882+1647\,\zeta_2)\,n_f\,C_A^2 \\
& +\frac{67}{27}\,(3265-1512\,\zeta_2)\,C_F\,C_A^2-\frac{1}{27}\,(19957-145440\,\zeta_2)\,C_F^2\,C_A \\
& -\frac{8}{27}\,(25579-9972\,\zeta_2)\,n_f\,C_F^2-\frac{10}{27}\,(48911+846\,\zeta_2)\,C_A^3\Big)\Big\}+\mathcal{O}(N^{-5})\,,
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
c_{L,\mathrm{q}}^{(4)}(N) = c_L^{+}(N)\big|_{a_s^4}+n_f\,C_F\Big\{ & N^{-6}\Big(-1920\,C_A^2+1920\,C_F\,C_A-1440\,C_F^2+960\,n_f\,C_F\Big) \\
& +N^{-5}\Big(-\frac{24640}{9}\,C_A^2+\frac{37408}{9}\,C_F\,C_A-\frac{2048}{3}\,C_F^2+\frac{2176}{9}\,n_f\,C_A-\frac{13024}{9}\,n_f\,C_F\Big) \\
& +N^{-4}\Big(-\frac{5696}{27}\,n_f\,C_A-\frac{32}{3}\,(49-361\,\zeta_2)\,C_F^2-\frac{224}{27}\,(698-207\,\zeta_2)\,C_A^2 \\
& -\frac{8}{27}\,(4913+11988\,\zeta_2)\,C_F\,C_A+\frac{16}{27}\,(8461-1188\,\zeta_2)\,n_f\,C_F-\frac{128}{3}\,n_f^2\Big)\Big\} \\
& +\mathcal{O}(N^{-3})
\end{aligned}
\tag{5.26}
$$

and

$$
\begin{aligned}
c_{L,\mathrm{g}}^{(4)}(N) = n_f \Big\{ & N^{-6}\Big(-1920\,C_A^3+960\,C_F\,C_A^2-480\,C_F^2\,C_A+240\,C_F^3+1920\,n_f\,C_F\,C_A \\
& -960\,n_f\,C_F^2\Big)+N^{-5}\Big(-\frac{8800}{9}\,C_A^3+\frac{9248}{9}\,C_F\,C_A^2-\frac{8296}{9}\,C_F^2\,C_A \\
& -\frac{16}{3}\,C_F^3-\frac{704}{9}\,n_f\,C_A^2-\frac{4640}{3}\,n_f\,C_F\,C_A+\frac{13648}{9}\,n_f\,C_F^2-\frac{1088}{9}\,n_f^2\,C_F\Big) \\
& +N^{-4}\Big(-\frac{4}{3}\,(115+1964\,\zeta_2)\,C_F^3-\frac{32}{27}\,(263+162\,\zeta_2)\,n_f\,C_A^2 \\
& +\frac{16}{3}\,(1231-118\,\zeta_2)\,n_f\,C_F\,C_A-\frac{32}{27}\,(6314-459\,\zeta_2)\,C_A^3+\frac{11776}{27}\,n_f^2\,C_F \\
& +\frac{4}{27}\,(6487+24048\,\zeta_2)\,C_F^2\,C_A+\frac{8}{27}\,(8785-7722\,\zeta_2)\,C_F\,C_A^2-\frac{64}{27}\,n_f^2\,C_A \\
& -\frac{8}{27}\,(14249-5256\,\zeta_2)\,n_f\,C_F^2\Big)\Big\}+\mathcal{O}(N^{-3})\,.
\end{aligned}
\tag{5.27}
$$

The non-singlet parts of eqs. (5.24) and (5.26) have been given in eqs. (4.5) and (4.6) above. The highest three $1/N$ poles of the corresponding 5-loop coefficient functions are given by

$$
\begin{aligned}
c_{2,\mathrm{q}}^{(5)}(N) = {} & c_2^{+}(N)\big|_{a_s^5} + n_f\, C_F \Big\{ N^{-10} \Big(42432\, C_A^3 - 42432\, C_F\, C_A^2 \\
& + 31824\, C_F^2\, C_A - 21216\, C_F^3 - 42432\, n_f\, C_F\, C_A + 31824\, n_f\, C_F^2 \Big) \\
& + N^{-9} \Big(\frac{5366608}{45}\, C_A^3 - \frac{7102528}{45}\, C_F\, C_A^2 + \frac{2208812}{15}\, C_F^2\, C_A - \frac{511648}{15}\, C_F^3 \\
& - \frac{81248}{9}\, n_f\, C_A^2 - \frac{1361056}{45}\, n_f\, C_F\, C_A - \frac{243376}{15}\, n_f\, C_F^2 + \frac{72448}{9}\, n_f^2\, C_F \Big) \\
& + N^{-8} \Big(-\frac{4}{5}(74593 - 180392\,\zeta_2)\, C_F^3 + \frac{2}{135}(7465355 - 11586096\,\zeta_2)\, C_F^2\, C_A \\
& - \frac{16}{135}(3126887 - 924570\,\zeta_2)\, C_F\, C_A^2 + \frac{16}{135}(3063709 - 69039\,\zeta_2)\, C_A^3 \\
& + \frac{16}{135}(1390214 - 523683\,\zeta_2)\, n_f\, C_F^2 - \frac{16}{45}(429100 - 30021\,\zeta_2)\, n_f\, C_F\, C_A \\
& - \frac{16}{135}(102961 - 37125\,\zeta_2)\, n_f\, C_A^2 - \frac{2472352}{135}\, n_f^2\, C_F + \frac{17696}{9}\, n_f^2\, C_A \Big) \Big\} \\
& + \mathcal{O}(N^{-7})\,,
\end{aligned}
\tag{5.28}
$$

$$
\begin{aligned}
c_{2,\mathrm{g}}^{(5)}(N) = {} & n_f \Big\{ N^{-10} \Big(42432\, C_A^4 - 21216\, C_F\, C_A^3 + 10608\, C_F^2\, C_A^2 - 5304\, C_F^3\, C_A \\
& + 2652\, C_F^4 - 63648\, n_f\, C_F\, C_A^2 + 42432\, n_f\, C_F^2\, C_A - 15912\, n_f\, C_F^3 + 10608\, n_f^2\, C_F^2 \Big) \\
& + N^{-9} \Big(\frac{3616288}{45}\, C_A^4 - \frac{2668904}{45}\, C_F\, C_A^3 + \frac{1762432}{45}\, C_F^2\, C_A^2 - \frac{1089206}{45}\, C_F^3\, C_A \\
& + \frac{50356}{15}\, C_F^4 - \frac{17600}{9}\, n_f\, C_A^3 - \frac{1868624}{45}\, n_f\, C_F\, C_A^2 + \frac{1566016}{45}\, n_f\, C_F^2\, C_A \\
& + \frac{334184}{45}\, n_f\, C_F^3 + \frac{81248}{9}\, n_f^2\, C_F\, C_A - \frac{1239104}{45}\, n_f^2\, C_F^2 \Big) + N^{-8} \Big(\frac{12464}{27}\, n_f^2\, C_A^2 \\
& - \frac{1105856}{135}\, n_f^2\, C_F\, C_A - \frac{8848}{9}\, n_f^3\, C_F + \frac{16}{135}(39757 + 61020\,\zeta_2)\, n_f\, C_A^3 \\
& + \frac{1}{15}(59357 - 673108\,\zeta_2)\, C_F^4 - \frac{4}{135}(365911 - 3205476\,\zeta_2)\, C_F^3\, C_A \\
& - \frac{4}{15}(648293 - 174498\,\zeta_2)\, C_F\, C_A^3 + \frac{2}{45}(1977587 - 1966200\,\zeta_2)\, C_F^2\, C_A^2 \\
& + \frac{8}{135}(2407760 - 1256427\,\zeta_2)\, n_f\, C_F^2\, C_A + \frac{4}{135}(3500111 - 241380\,\zeta_2)\, n_f^2\, C_F^2 \\
& - \frac{2}{135}(4630465 - 3452868\,\zeta_2)\, n_f\, C_F^3 + \frac{4}{135}(11350279 + 666720\,\zeta_2)\, C_A^4 \\
& - \frac{4}{135}(12031717 + 12510\,\zeta_2)\, n_f\, C_F\, C_A^2 \Big) \Big\} + \mathcal{O}(N^{-7})\,,
\end{aligned}
\tag{5.29}
$$

$$
\begin{aligned}
c^{(5)}_{L,\mathrm{q}}(N) = c^{+}_{L}(N)\big|_{a_s^5} + n_f\, C_F \Big\{ & N^{-8} \Big(24960\, C_A^3 - 24960\, C_F\, C_A^2 \\
& + 18720\, C_F^2\, C_A - 12480\, C_F^3 - 24960\, n_f\, C_F\, C_A + 18720\, n_f\, C_F^2 \Big) \\
& + N^{-7} \Big(+ \frac{436192}{9}\, C_A^3 - \frac{602368}{9}\, C_F\, C_A^2 + \frac{198248}{3}\, C_F^2\, C_A - \frac{33904}{3}\, C_F^3 \\
& - \frac{30400}{9}\, n_f\, C_A^2 - \frac{27328}{9}\, n_f\, C_F\, C_A - \frac{56272}{3}\, n_f\, C_F^2 + \frac{33920}{9}\, n_f^2\, C_F \Big) \\
& + N^{-6} \Big(- \frac{56}{3}\, (359 - 4436\, \zeta_2)\, C_F^3 - \frac{8}{27}\, (52207 + 341604\, \zeta_2)\, C_F^2\, C_A \\
& - \frac{16}{27}\, (180227 - 109944\, \zeta_2)\, C_F\, C_A^2 + \frac{16}{27}\, (218827 - 20610\, \zeta_2)\, C_A^3 \\
& - \frac{32}{9}\, (18590 - 2271\, \zeta_2)\, n_f\, C_F\, C_A + \frac{8}{27}\, (278473 - 110304\, \zeta_2)\, n_f\, C_F^2 \\
& + \frac{160}{27}\, (278 + 513\, \zeta_2)\, n_f\, C_A^2 + \frac{6208}{9}\, n_f^2\, C_A - \frac{305536}{27}\, n_f^2\, C_F \Big) \Big\} + \mathcal{O}(N^{-5})
\end{aligned}
\tag{5.30}
$$

and

$$
\begin{aligned}
c^{(5)}_{L,\mathrm{g}}(N) = n_f \Big\{ & N^{-8} \Big(24960\, C_A^4 - 12480\, C_F\, C_A^3 + 6240\, C_F^2\, C_A^2 - 3120\, C_F^3\, C_A \\
& + 1560\, C_F^4 - 37440\, n_f\, C_F\, C_A^2 + 24960\, n_f\, C_F^2\, C_A - 9360\, n_f\, C_F^3 + 6240\, n_f^2\, C_F^2 \Big) \\
& + N^{-7} \Big(\frac{230272}{9}\, C_A^4 - \frac{178352}{9}\, C_F\, C_A^3 + \frac{139120}{9}\, C_F^2\, C_A^2 - \frac{93860}{9}\, C_F^3\, C_A \\
& + \frac{2140}{3}\, C_F^4 + \frac{7040}{9}\, n_f\, C_A^3 + \frac{22528}{9}\, n_f\, C_F\, C_A^2 - \frac{1856}{9}\, n_f\, C_F^2\, C_A + \frac{96152}{9}\, n_f\, C_F^3 \\
& + \frac{30400}{9}\, n_f^2\, C_F\, C_A - \frac{164144}{9}\, n_f^2\, C_F^2 \Big) + N^{-6} \Big(\frac{3424}{27}\, n_f^2\, C_A^2 - \frac{200672}{27}\, n_f^2\, C_F\, C_A \\
& - \frac{3104}{9}\, n_f^3\, C_F - \frac{2}{3}\, (1261 + 42056\, \zeta_2)\, C_F^4 + \frac{32}{27}\, (4967 + 4212\, \zeta_2)\, n_f\, C_A^3 \\
& + \frac{16}{27}\, (101897 - 7344\, \zeta_2)\, n_f^2\, C_F^2 + \frac{8}{27}\, (191519 - 130122\, \zeta_2)\, n_f\, C_F^2\, C_A \\
& + \frac{8}{27}\, (458999 + 12744\, \zeta_2)\, C_A^4 - \frac{8}{27}\, (542135 - 18342\, \zeta_2)\, n_f\, C_F\, C_A^2 \\
& + 4\, (5905 - 13404\, \zeta_2)\, C_F^2\, C_A^2 + \frac{16}{27}\, (11939 + 102024\, \zeta_2)\, C_F^3\, C_A \\
& - \frac{8}{3}\, (22553 - 9334\, \zeta_2)\, C_F\, C_A^3 - \frac{20}{27}\, (46153 - 35208\, \zeta_2)\, n_f\, C_F^3 \Big) \Big\} + \mathcal{O}(N^{-5}),
\end{aligned}
\tag{5.31}
$$

where the non-singlet contributions can be found in eqs. (4.8) and (4.9).

6 Summary and outlook

We have presented a comprehensive study of high-energy double logarithms appearing at the n-th order in perturbation theory in the splitting functions for the evolution of the parton distribution functions and in the coefficient functions of inclusive deep-inelastic scattering.

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These have the structure $\alpha_s^n\, x^p \ln^{2n-n_0-k} x$, where $p \geq 0$ in gauge-boson-exchange DIS, and the parameter n_0 depends on the quantity under consideration. k denotes the logarithmic accuracy: $k = 0$ provides the leading-logarithmic (LL) terms, $k = 1$ the next-to-leading logarithmic (NLL) contributions etc.

For the flavour non-singlet quantities, the dominant contributions start at $p = 0$ with an offset of $n_0 = 2$ for the splitting functions $P^{\pm}$ and the coefficient function $C_L^{\pm}$, and of $n_0 = 1$ for the coefficient functions $C_2^{\pm}$ and $C_3^{\pm}$. The structure of the unfactorized n-th order partonic structure functions in dimensional regularization has been employed to perform a NNLL resummation of these small-x logarithms to all orders in full QCD for the splitting function P^+ and the coefficient functions C_2^+, C_3^+ and C_L^+; the former can be expressed in terms of modified Bessel functions. Using, in addition, the known structure of the singular terms for P^+ in Mellin N-space as $N \to 0$, the all-order resummation of the leading $\ln x$ terms has been pushed up to N^{7}LL accuracy in the large-n_c limit, where the functions P^+ and P^- coincide. For the coefficient functions C_2^+ and C_3^+ the large-n_c resummation has been extended to N^{3}LL accuracy. In all cases, explicit fixed-order expansions up to the fifth order in perturbation theory have been presented for future reference.

In the flavour-singlet sector of standard DIS, the dominant small-x contributions at the n-th order are proportional to one inverse power of x enhanced by single logarithms, i.e., the splitting functions $P_{\rm ik}$ and the singlet coefficient functions C_2 and C_L at n loops are of the form $x^{-1} \ln^m x$ with $m = 1, \ldots, n - n_0'$ in the small-x limit. Our study has not added any new information on these terms or their all-order resummation, which is a long-standing and prominent, yet in its full generality still open problem in QCD. Instead, we have considered the double logarithms appearing with powers x^p for even $p \geq 0$, which correspond to expansions in Mellin N-space around $(N + p) = 0$. In these cases, we can again use dimensional regularization and exploit the structure of the unfactorized partonic structure functions at the n-th order in perturbation theory. We have presented NNLL small-x ($p = 0$) predictions up to five loops in full QCD. We can compute order-by-order to a very high power of α_s in this framework, but we have not been able to find an all-order form which generates these results beyond the leading logarithms.

The results, certainly in the singlet sector, are not of immediate phenomenological relevance in DIS. However, they elucidate the analytic structure underlying the expressions at fixed order in perturbation theory. In addition, they provide important information for complete analytic computations of those quantities. In this regard, the results for the non-singlet splitting functions have already been used in the determination of the all-N expression of the large-n_c non-singlet splitting function at four loops, based on a limited number of Mellin moments together with constraints on its endpoint behaviour and its the functional form. This application also allowed for important independent checks on the methods employed in the present article for the study of the high-energy (single and) double logarithms, and in turn provided input coefficients for the N^{3}LL resummations.

While the intriguing structures of the resummations performed contribute to a much improved theoretical understanding, the chosen approach has also clear limitations. Most notably the leading $p = 0$ contributions to the non-singlet splitting function P^- and the main $\nu+\bar{\nu}$ charged-current structure function F_3 are not accessible beyond the large-n_c

limit. Progress in this direction will require new methods in addition to those developed and considered here. The closed-form all-order resummation of the small-x double logarithms in the singlet sector remains an open problem, pending the identification of the proper set of functions, which complement the modified Bessel functions found to suffice in the non-singlet sector. Finally, the systematic study of DIS with an exchanged scalar and the implications for the flavour-singlet coefficient functions is a subject we have touched upon only briefly, with a few results presented in the appendix. These issues deserve further thorough investigation, which we leave for the future.

An ancillary file with our results in FORM format is available from http://arXiv.org and as Supplementary Material to the present article.

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A Expansions about $N = 0$ to order α_s^3

Here we present the expansions of the fixed-order results used for the resummation of the $x^0 \ln^k x$ contributions to the even-N based splitting and coefficient functions. The N^{n}LL predictions are fixed by the corresponding N^{n}LO results, hence only the N^{-4} coefficients of the fourth-order splitting functions $P^{(3)}$ are missing for the N^{3}LL resummation of F_2, F_3 and F_ϕ. In view of a future determination of these quantities, the results below are given at N^{3}LL accuracy.

The corresponding expressions for the LO, NLO and N^{2}LO non-singlet splitting functions read

$$
\begin{aligned}
P^{+\,(0)} &= C_F \Big\{ 2\,N^{-1} \,+\, 1 \,+\, [2-4\zeta_2]\,N \,-\, [2-4\zeta_3]\,N^2 \Big\} \,+\, \mathcal{O}(N^3)\,, \\
P^{+\,(1)} &= C_F \Big\{ 4\,C_F N^{-3} \,+\, (4\,C_F - 2\beta_0)\,N^{-2} + \Big(\frac{20}{3}\,C_A - [4+8\zeta_2]\,C_F + \frac{22}{3}\,\beta_0 \Big) N^{-1} \\
&\quad + \Big(- \Big[\frac{17}{3} + 12\,\zeta_3\Big]\,C_A + \Big[\frac{19}{2} + 16\,\zeta_3\Big]\,C_F - \frac{29}{6}\,\beta_0 \Big) \Big\} \,+\, \mathcal{O}(N^1)\,, \\
P^{+\,(2)} &= C_F \Big\{ 16\,C_F^2\,N^{-5} \,+\, (24\,C_F^2 - 12\,C_F\beta_0)\,N^{-4} - \Big(60\zeta_2\,C_A^2 - \Big[\frac{80}{3} + 192\zeta_2\Big] C_F C_A \\
&\quad - [8-208\zeta_2]\,C_F^2 - \frac{64}{3}\,C_F\beta_0 - 2\beta_0^2 \Big) N^{-3} + \Big([14+48\,\zeta_2]\,C_A^2 \\
&\quad + [30+192\,\zeta_2+96\,\zeta_3]\,C_F^2 - \Big[-\frac{38}{3} + 216\,\zeta_2 + 48\,\zeta_3 \Big]\,C_A\,C_F - \frac{22}{3}\,\beta_0^2 \\
&\quad - \Big[\frac{50}{3} - 12\,\zeta_2\Big]\,C_A\,\beta_0 - \frac{44}{3}\,C_F\,\beta_0 \Big) N^{-2} \Big\} \,+\, \mathcal{O}(N^{-1})\,.
\end{aligned}
\tag{A.1}
$$

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The input coefficient functions for $\widehat{F}_2$ in eq. (2.7) in Laurent expansions analogous to eq. (2.23) are

$$
\begin{aligned}
c_2^{+\,(1,0)} &= C_F\Big\{2N^{-2} + 3N^{-1} - [5 + 2\zeta_2] - [4 - 5\,\zeta_2 + 2\,\zeta_3]\,N\Big\} + \mathcal{O}(N^2)\,,\\
c_2^{+\,(1,1)} &= C_F\Big\{-2N^{-3} - 3N^{-2} + [5 + 3\zeta_2]N^{-1} - [10 - \frac{7}{2}\,\zeta_2]\Big\} + \mathcal{O}(N^1)\,,\\
c_2^{+\,(1,2)} &= C_F\Big\{2N^{-4} + 3N^{-3} - [5 + 3\zeta_2]N^{-2} + [10 - \frac{9}{2}\,\zeta_2 + \frac{14}{3}\,\zeta_3]\,N^{-1}\Big\} + \mathcal{O}(N^0)\,,\\
c_2^{+\,(2,0)} &= C_F\Big\{10\,C_F N^{-4} + (-5\beta_0 + 18C_F)N^{-3} + (10\,C_A - [17 + 24\zeta_2]C_F + 6\beta_0)N^{-2}\\
&\qquad + \Big(\Big[\frac{3}{2} - 8\,\zeta_2 + 56\,\zeta_3\Big]\,C_F - \Big[\frac{119}{9} + 12\,\zeta_3\Big]\,C_A - \Big[\frac{89}{18} - 4\,\zeta_2\Big]\,\beta_0\Big)\,N^{-1}\Big\}\\
&\quad + \mathcal{O}(N^0)\,,\\
c_2^{+\,(2,1)} &= C_F\Big\{-26\,C_F N^{-5} + (13\beta_0 - 50C_F)N^{-4} + \Big([47 + 68\zeta_2]C_F - \frac{70}{3}C_A - \frac{32}{3}\beta_0\Big)N^{-3}\\
&\qquad + \Big([34 + 24\,\zeta_3]\,C_A - [49 - 54\,\zeta_2 + 128\,\zeta_3]\,C_F + [10 - 14\,\zeta_2]\,\beta_0\Big)N^{-2}\Big\}\\
&\quad + \mathcal{O}(N^{-1})\,,\\
c_2^{+\,(3,0)} &= C_F\Big\{60\,C_F^2 N^{-6} + \Big(-\frac{182}{3}C_F\beta_0 + 134C_F^2\Big)N^{-5} + \Big(\Big[\frac{260}{3} + 384\zeta_2\Big]C_F C_A\\
&\qquad - 120\zeta_2 C_A^2 - [30 + 524\zeta_2]C_F^2 + \frac{46}{3}\beta_0^2 + \frac{5}{3}C_F\beta_0\Big)N^{-4}\\
&\qquad + \Big(\Big[\frac{112}{3} + 80\,\zeta_2\Big]\,C_A^2 + \Big[\frac{1315}{27} + \frac{266}{3}\,\zeta_2\Big]\,C_F\,\beta_0 - \Big[\frac{580}{9} - 24\,\zeta_2\Big]\,C_A\,\beta_0\\
&\qquad - \Big[\frac{113}{3} - \frac{598}{3}\,\zeta_2 - \frac{1292}{3}\,\zeta_3\Big]\,C_F^2 + \Big[\frac{950}{27} - 384\,\zeta_2 - 128\,\zeta_3\Big]\,C_A\,C_F\\
&\qquad - \frac{248}{9}\,\beta_0^2\Big)N^{-3}\Big\} + \mathcal{O}(N^{-2})\,.
\end{aligned}
\tag{A.2}
$$

The corresponding expansion coefficients for the longitudinal structure function are given by

$$
\begin{aligned}
c_L^{+\,(1,0)} &= C_F\Big\{4N^0 - 4N + 4N^2\Big\} + \mathcal{O}(N^3)\,,\\
c_L^{+\,(1,1)} &= C_F\Big\{4N^0 - [4 - 4\zeta_2]N\Big\} + \mathcal{O}(N^2)\,,\\
c_L^{+\,(1,2)} &= C_F[8 - 2\zeta_2]N^0 + \mathcal{O}(N)\,,\\
c_L^{+\,(2,0)} &= C_F\Big\{8\,C_F N^{-2} + (12C_F - 4\beta_0)N^{-1} + \Big(\frac{40}{3}C_A - [74 + 8\zeta_2]C_F + \frac{38}{3}\beta_0\Big)N^0\Big\}\\
&\quad + \mathcal{O}(N)\,,\\
c_L^{+\,(2,1)} &= C_F\Big\{-8\,C_F N^{-3} + (4\beta_0 - 4C_F)N^{-2} + \Big([70 + 20\zeta_2]C_F - \frac{40}{3}C_A - \frac{50}{3}\beta_0\Big)N^{-1}\Big\}\\
&\quad + \mathcal{O}(N^0)\,,
\end{aligned}
$$

$$c_L^{+\,(3,0)} = C_F\Big\{40\,C_F^2 N^{-4} + (64C_F^2 - 36C_F\beta_0)N^{-3} + \Big(\Big[\frac{200}{3} + 384\zeta_2\Big]C_F C_A - 120\zeta_2 C_A^2 - [168 + 416\zeta_2]C_F^2 + \frac{112}{3}C_F\beta_0 + 8\beta_0^2\Big)N^{-2}\Big\} + \mathcal{O}(N^{-1})\,, \tag{A.3}$$

and the input coefficient functions for the even-N based $\widehat{F}_3$ read

$$c_3^{+\,(1,0)} = C_F\Big\{2N^{-2} + N^{-1} - [7+2\zeta_2] - [2 - 5\,\zeta_2 + 2\,\zeta_3]\,N\Big\} + \mathcal{O}(N^2)\,,$$
$$c_3^{+\,(1,1)} = C_F\Big\{-2N^{-3} - N^{-2} + [1+3\zeta_2]N^{-1} - \Big[14 - \frac{3}{2}\,\zeta_2\Big]\Big\} + \mathcal{O}(N)\,,$$
$$c_3^{+\,(1,2)} = C_F\Big\{2N^{-4} + N^{-3} - [1+3\zeta_2]N^{-2} - \Big[\frac{3}{2}\,\zeta_2 - \frac{14}{3}\,\zeta_3\Big]\,N^{-1}\Big\} + \mathcal{O}(N^0)\,,$$
$$c_3^{+\,(2,0)} = C_F\Big\{10\,C_F N^{-4} + (10C_F - 5\beta_0)N^{-3} + (10\,C_A - [33 + 24\zeta_2]C_F + 10\beta_0)N^{-2} + \Big(\Big[\frac{29}{2} + 4\,\zeta_2 + 56\,\zeta_3\Big]\,C_F - \Big[\frac{179}{9} + 12\,\zeta_3\Big]\,C_A - \Big[\frac{131}{18} - 4\,\zeta_2\Big]\,\beta_0\Big)\,N^{-1}\Big\} + \mathcal{O}(N^0)\,,$$
$$c_3^{+\,(2,1)} = C_F\Big\{-26\,C_F N^{-5} + (13\beta_0 - 26C_F)N^{-4} + \Big([71 + 68\zeta_2]C_F - \frac{70}{3}C_A - \frac{68}{3}\beta_0\Big)N^{-3} + \Big([54 + 24\,\zeta_3]\,C_A - [120 - 8\,\zeta_2 + 128\,\zeta_3]\,C_F + [25 - 14\,\zeta_2]\,\beta_0\Big)\,N^{-2}\Big\} + \mathcal{O}(N^{-1})\,,$$
$$c_3^{+\,(3,0)} = C_F\Big\{60\,C_F^2 N^{-6} + \Big(90C_F^2 - \frac{182}{3}C_F\beta_0\Big)N^{-5} + \Big(\Big[\frac{260}{3} + 384\zeta_2\Big]C_F C_A - 120\zeta_2 C_A^2 - [142 + 524\zeta_2]C_F^2 + \frac{46}{3}\beta_0^2 + \frac{143}{3}C_F\beta_0\Big)N^{-4} + \Big(\Big[\frac{112}{3} + 140\,\zeta_2\Big]\,C_A^2 - \Big[\frac{47}{3} - \frac{1438}{3}\,\zeta_2 - \frac{1292}{3}\,\zeta_3\Big]\,C_F^2 - \Big[\frac{670}{27} + 576\,\zeta_2 + 128\,\zeta_3\Big]\,C_A\,C_F + \Big[\frac{1909}{27} + \frac{266}{3}\,\zeta_2\Big]\,C_F\,\beta_0 - \Big[\frac{580}{9} - 24\,\zeta_2\Big]\,C_A\,\beta_0 - \frac{356}{9}\,\beta_0^2\Big)\,N^{-3}\Big\} + \mathcal{O}(N^{-2})\,. \tag{A.4}$$

Next we present the input quantities for the singlet cases. The 'diagonal' quantities can be separated into non-singlet and pure-singlet pieces. The quark cases in the following expressions are written in terms of the non-singlet parts presented above. Here we present the input expressions only to N^{2}LL accuracy. The singlet splitting functions, expanded about $N = 0$ are given by

$$P_{\rm qq}^{(0)} = P^{+\,(0)}\,,$$
$$P_{\rm qq}^{(1)} = P^{+\,(1)} + n_f C_F\Big\{-8N^{-3} - 4N^{-2} - 8N^{-1}\Big\} + \mathcal{O}(N^0)\,,$$
$$P_{\rm qq}^{(2)} = P^{+\,(2)} + n_f C_F\Big\{(64C_A - 64C_F\,)N^{-5} + \Big(\frac{232}{3}C_A - 24C_F - \frac{16}{3}n_f\Big)N^{-4} + \Big(\Big[\frac{404}{9} + 8\zeta_2\Big]C_A - [160 - 96\zeta_2]C_F + \frac{232}{9}n_f\Big)N^{-3}\Big\} + \mathcal{O}(N^{-2})\,,$$

$$
\begin{aligned}
P^{(0)}_{\mathrm{qg}} &= n_f\Big\{2N^{-1}-2+3N\Big\}+\mathcal{O}(N^2)\,,\\
P^{(1)}_{\mathrm{qg}} &= n_f\Big\{(4C_F-8C_A)N^{-3}-(6C_F+4C_A)N^{-2}+([28-8\zeta_2]C_F-8C_A)N^{-1}\Big\}+\mathcal{O}(N^0)\,,\\
P^{(2)}_{\mathrm{qg}} &= n_f\Big\{(64C_A^2-32C_AC_F+16C_F^2-32C_Fn_f)N^{-5}\\
&\qquad+\left(\frac{56}{3}C_A^2-\frac{44}{3}C_FC_A-12C_F^2+\frac{16}{3}C_An_f+\frac{152}{3}C_Fn_f\right)N^{-4}\\
&\qquad+\left(\left[\frac{1724}{9}-12\zeta_2\right]C_A^2-\left[\frac{1171}{9}-96\zeta_2\right]C_AC_F+[89-56\zeta_2]C_F^2+\frac{64}{9}C_An_f\right.\\
&\qquad\left.-\frac{1370}{9}C_Fn_f\right)N^{-3}\Big\}+\mathcal{O}(N^{-2})\,,\\
P^{(0)}_{\mathrm{gq}} &= C_F\Big\{-4N^{-1}-2-6N\Big\}+\mathcal{O}(N^2)\,,\\
P^{(1)}_{\mathrm{gq}} &= C_F\Big\{(16C_A-8C_F)N^{-3}+(16C_A-8C_F)N^{-2}\\
&\qquad+\left(14C_F+\frac{128}{9}n_f-\left[\frac{332}{9}-16\zeta_2\right]C_A\right)N^{-1}\Big\}+\mathcal{O}(N^0)\,,\\
P^{(2)}_{\mathrm{gq}} &= C_F\Big\{(-128C_A^2+64C_AC_F-32C_F^2+64C_Fn_f)N^{-5}\\
&\qquad+\left(-\frac{400}{3}C_A^2+\frac{376}{3}C_FC_A-48C_F^2-\frac{32}{3}C_An_f-\frac{16}{3}C_Fn_f\right)N^{-4}\\
&\qquad+\left(-\left[\frac{280}{9}+104\zeta_2\right]C_A^2-\frac{2446}{9}C_AC_F+[42+48\zeta_2]C_F^2-\frac{992}{9}C_An_f\right.\\
&\qquad\left.+\frac{2380}{9}C_Fn_f\right)N^{-3}\Big\}+\mathcal{O}(N^{-2})\,,\\
P^{(0)}_{\mathrm{gg}} &= -4C_AN^{-1}+\left(\frac{5}{3}C_A-\frac{2}{3}n_f\right)-[7+4\zeta_2]C_AN+\mathcal{O}(N^2)\,,\\
P^{(1)}_{\mathrm{gg}} &= (16C_A^2-8C_Fn_f)N^{-3}+\left(\frac{4}{3}C_A^2+\frac{8}{3}C_An_f+12C_Fn_f\right)N^{-2}\\
&\quad+\left(\left[\frac{74}{9}+16\zeta_2\right]C_A^2+\frac{76}{9}C_An_f-32C_Fn_f\right)N^{-1}+\mathcal{O}(N^0)\,,\\
P^{(2)}_{\mathrm{gg}} &= (-128C_A^3+128C_FC_An_f-32C_F^2n_f)N^{-5}\\
&\quad+\left(-16C_A^3-32C_A^2n_f-\frac{232}{3}C_AC_Fn_f+24C_F^2n_f+\frac{16}{3}C_Fn_f^2\right)N^{-4}\\
&\quad+\left(-\left[\frac{2612}{9}+160\zeta_2\right]C_A^3-\left[\frac{208}{3}+24\zeta_2\right]C_A^2n_f+\left[\frac{3548}{9}+96\zeta_2\right]C_AC_Fn_f\right.\\
&\quad\left.-[120-32\zeta_2]C_F^2n_f-\frac{16}{9}C_An_f^2+\frac{184}{9}C_Fn_f^2\right)N^{-3}+\mathcal{O}(N^{-2})\,.
\end{aligned}
\tag{A.5}
$$

The corresponding expansion coefficients of the D-dimensional coefficient functions for F_2 are

$$
\begin{aligned}
c_{2,\mathrm{q}}^{(1,l)} &= c_2^{+(1,l)}\,, \qquad (l=0,1,2)\,,\\
c_{2,\mathrm{q}}^{(2,0)} &= c_2^{+(2,0)} + n_f C_F\Big\{-20N^{-4} - 2N^{-3} - [56-16\zeta_2]N^{-2}\Big\} + \mathcal{O}(N^{-1})\\
c_{2,\mathrm{q}}^{(2,1)} &= c_2^{+\,(2,1)} + n_f C_F\Big\{52N^{-5} + 2N^{-4} + [160-56\zeta_2]N^{-3}\Big\} + \mathcal{O}(N^{-2})\\
c_{2,\mathrm{q}}^{(3,0)} &= c_2^{+(3,0)} + n_f C_F\Big\{240(C_A-C_F)N^{-6} + \Big(\frac{3416}{9}C_A - \frac{440}{3}C_F - \frac{368}{9}n_f\Big)N^{-5}\\
&\qquad + \Big(\Big[\frac{16984}{27} - \frac{320}{3}\zeta_2\Big]C_A - \Big[572 - \frac{1328}{3}\zeta_2\Big]C_F + \frac{1784}{27}n_f\Big)N^{-4}\Big\}\\
&\quad + \mathcal{O}(N^{-3})\,,
\end{aligned}
\tag{A.6}
$$

and

$$
\begin{aligned}
c_{2,\mathrm{g}}^{(1,0)} &= n_f\Big\{2N^{-2} - 2N^{-1} + [6-2\zeta_2]\Big\} + \mathcal{O}(N)\,,\\
c_{2,\mathrm{g}}^{(1,1)} &= n_f\Big\{-2N^{-3} + 2N^{-2} - [6-3\zeta_2]N^{-1}\Big\} + \mathcal{O}(N^0)\,,\\
c_{2,\mathrm{g}}^{(1,2)} &= n_f\Big\{2N^{-4} - 2N^{-3} + [6-3\zeta_2]N^{-2}\Big\} + \mathcal{O}(N^{-1})\,,\\
c_{2,\mathrm{g}}^{(2,0)} &= n_f\Big\{(10C_F - 20C_A)N^{-4} - (2C_A + 3C_F)N^{-3} + ([-58+8\zeta_2]C_A\\
&\qquad + [16-16\zeta_2]C_F)N^{-2}\Big\} + \mathcal{O}(N^{-1})\,,\\
c_{2,\mathrm{g}}^{(2,1)} &= n_f\Big\{(52C_A - 26C_F)N^{-5} + (2C_A + 3C_F)N^{-4} + ([166-32\zeta_2]C_A\\
&\qquad - [20-44\zeta_2]C_F)N^{-3}\Big\} + \mathcal{O}(N^{-2})\,,\\
c_{2,\mathrm{g}}^{(3,0)} &= n_f\Big\{(240C_A^2 - 120C_FC_A + 60C_F^2 - 120C_Fn_f)N^{-6}\\
&\qquad + \Big(\frac{1436}{9}C_A^2 - \frac{1636}{9}C_FC_A + \frac{44}{3}C_F^2 - \frac{8}{9}C_An_f + \frac{1636}{9}C_Fn_f\Big)N^{-5}\\
&\qquad + \Big(\Big[\frac{27338}{27} - 56\zeta_2\Big]C_A^2 + \Big[-\frac{4589}{27} + \frac{656}{3}\zeta_2\Big]C_FC_A + \Big[\frac{178}{3} - \frac{524}{3}\zeta_2\Big]C_F^2\\
&\qquad + \frac{532}{27}C_An_f + \Big[-\frac{17782}{27} + 88\zeta_2\Big]C_Fn_f\Big)N^{-4}\Big\} + \mathcal{O}(N^{-3})\,.
\end{aligned}
\tag{A.7}
$$

The input coefficients $c_{\phi,\mathrm{i}}$ for $\widehat{F}_\phi$, which provides the resummation of P_{qg} and P_{gg}, are given by

$$
\begin{aligned}
c_{\phi,\mathrm{q}}^{(1,0)} &= C_F\Big\{-4N^{-2} - 4N^{-1} + [5+4\zeta_2]\Big\} + \mathcal{O}(N)\,,\\
c_{\phi,\mathrm{q}}^{(1,1)} &= C_F\Big\{4N^{-3} + 4N^{-2} + [1-6\zeta_2]N^{-1}\Big\} + \mathcal{O}(N^0)\,,
\end{aligned}
$$

$$
\begin{aligned}
c^{(1,2)}_{\phi,\mathrm{q}} &= C_F\Big\{ -4N^{-4} - 4N^{-3} - [1-6\zeta_2]N^{-2}\Big\} + \mathcal{O}(N^{-1})\,,\\
c^{(2,0)}_{\phi,\mathrm{q}} &= C_F\Big\{(40C_A - 20C_F)N^{-4} + \Big(\frac{344}{3}C_A - 28C_F - \frac{32}{3}n_f\Big)N^{-3}\\
&\qquad + ([16-16\zeta_2]C_A + [21+32\zeta_2]C_F + 12n_f)N^{-2}\Big\} + \mathcal{O}(N^{-1})\,,\\
c^{(2,1)}_{\phi,\mathrm{q}} &= C_F\Big\{(-104C_A + 52C_F)N^{-5} + (-328C_A + 76C_F + 32n_f)N^{-4}\\
&\qquad + \Big(\Big[-\frac{1196}{9} + 80\zeta_2\Big]C_A - [25+104\zeta_2]C_F - \frac{196}{9}n_f\Big)N^{-3}\Big\} + \mathcal{O}(N^{-2})\,,\\
c^{(3,0)}_{\phi,\mathrm{q}} &= C_F\Big\{(-480C_A^2 + 240C_FC_A - 120C_F^2 + 240C_Fn_f)N^{-6}\\
&\qquad + \Big(-\frac{13960}{9}C_A^2 + \frac{8960}{9}C_FC_A - 224C_F^2 + \frac{1072}{9}C_An_f - \frac{440}{9}C_Fn_f\Big)N^{-5}\\
&\qquad + \Big(\Big[-\frac{69928}{27} + \frac{208}{3}\zeta_2\Big]C_A^2 + \Big[\frac{2338}{27} - \frac{1120}{3}\zeta_2\Big]C_FC_A + \Big[\frac{308}{3} + 328\zeta_2\Big]C_F^2\\
&\qquad + \frac{6592}{27}C_An_f + \Big[\frac{17456}{27} - 176\zeta_2\Big]C_Fn_f - 32n_f^2\Big)N^{-4}\Big\} + \mathcal{O}(N^{-3}) \qquad \text{(A.8)}
\end{aligned}
$$

and

$$
\begin{aligned}
c^{(1,0)}_{\phi,\mathrm{g}} &= -4C_AN^{-2} - \Big(\frac{23}{3}C_A - \frac{2}{3}n_f\Big)N^{-1} + \Big(\Big[\frac{118}{9} + 4\zeta_2\Big]C_A - \frac{16}{9}n_f\Big) + \mathcal{O}(N)\,,\\
c^{(1,1)}_{\phi,\mathrm{g}} &= 4C_AN^{-3} + \Big(\frac{23}{3}C_A - \frac{2}{3}n_f\Big)N^{-2} + \Big(\Big[-\frac{64}{9} - 6\zeta_2\Big]C_A + \frac{16}{9}n_f\Big)N^{-1} + \mathcal{O}(N^0)\,,\\
c^{(1,2)}_{\phi,\mathrm{g}} &= -4C_AN^{-4} + \Big(-\frac{23}{3}C_A + \frac{2}{3}n_f\Big)N^{-3} + \Big(\Big[\frac{64}{9} + 6\zeta_2\Big]C_A - \frac{16}{9}n_f\Big)N^{-2} + \mathcal{O}(N^{-1})\,,\\
c^{(2,0)}_{\phi,\mathrm{g}} &= (40C_A^2 - 20C_Fn_f)N^{-4} + (78C_A^2 - 4C_An_f + 14C_Fn_f)N^{-3} + \Big(\frac{833}{9}C_A^2 - \frac{22}{9}C_An_f\\
&\qquad + [-34+16\zeta_2]C_Fn_f + \frac{8}{9}n_f^2\Big)N^{-2} + \mathcal{O}(N^{-1})\,,\\
c^{(2,1)}_{\phi,\mathrm{g}} &= (-104C_A^2 + 52C_Fn_f)N^{-5} + \Big(-\frac{698}{3}C_A^2 + \frac{44}{3}C_An_f - 30C_Fn_f\Big)N^{-4}\\
&\qquad + \Big(\Big[-\frac{2857}{9} + 32\zeta_2\Big]C_A^2 + \frac{142}{9}C_An_f + [94-56\zeta_2]C_Fn_f - \frac{8}{3}n_f^2\Big)N^{-3} + \mathcal{O}(N^{-2})\,,\\
c^{(3,0)}_{\phi,\mathrm{g}} &= -120\Big(4C_A^3 - C_Fn_f(4C_A - C_F)\Big)N^{-6} + \Big(-\frac{10000}{9}C_A^3 + \frac{352}{9}n_fC_A^2 + \frac{3140}{9}C_FC_An_f\\
&\qquad + \frac{44}{3}n_fC_F^2 - \frac{536}{9}n_f^2C_F\Big)N^{-5} + \Big(-\Big[\frac{59902}{27} + 224\zeta_2\Big]C_A^3 + \Big[\frac{560}{27} - 48\zeta_2\Big]n_fC_A^2\\
&\qquad + \Big[\frac{16622}{27} - \frac{256}{3}\zeta_2\Big]n_fC_FC_A - \Big[162 - \frac{616}{3}\zeta_2\Big]n_fC_F^2 - \frac{328}{27}n_f^2C_A + \frac{3508}{27}n_f^2C_F\Big)N^{-4}\\
&\qquad + \mathcal{O}(N^{-3})\,. \qquad \text{(A.9)}
\end{aligned}
$$

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Finally the input coefficients of $c_{L,\mathrm{q}}$ and $c_{L,\mathrm{g}}$ read

$$\begin{aligned}
c^{(1,l)}_{L,\mathrm{q}} &= c^{+(1,l)}_{L}\,, \qquad (l=0,1,2)\,,\\
c^{(2,0)}_{L,\mathrm{q}} &= c^{+(2,0)}_{L} + n_f\, C_F\Big\{ -16N^{-2} + \Big(\frac{144}{9} + 16\zeta_2\Big)\Big\} + \mathcal{O}(N)\,,\\
c^{(2,1)}_{L,\mathrm{q}} &= c^{+(2,1)}_{L} + n_f\, C_F\Big\{ 16N^{-3} - \frac{96}{3}N^{-2} + \Big[\frac{648}{9} - 40\zeta_2\Big]N^{-1}\Big\} + \mathcal{O}(N^0)\,,\\
c^{(3,0)}_{L,\mathrm{q}} &= c^{+(3,0)}_{L} + n_f\, C_F\Big\{ (160C_A - 160C_F)N^{-4} + \Big(\frac{496}{3}C_A - 16C_F - \frac{64}{3}n_f\Big)N^{-3}\\
&\qquad\qquad + \Big(\Big[\frac{400}{9} - 112\zeta_2\Big]C_A - [80-256\zeta_2]C_F + \frac{512}{9}n_f\Big)N^{-2}\Big\}\\
&\qquad + \mathcal{O}(N^{-1})\,,
\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}
c^{(1,0)}_{L,\mathrm{g}} &= n_f\Big\{4 - 6N + 7N^2\Big\} + \mathcal{O}(N^3)\,,\\
c^{(1,1)}_{L,\mathrm{g}} &= n_f\Big\{8 - [12-4\zeta_2]N\Big\} + \mathcal{O}(N^2)\,,\\
c^{(1,2)}_{L,\mathrm{g}} &= n_f\Big\{[16-2\zeta_2]\Big\} + \mathcal{O}(N)\,,\\
c^{(2,0)}_{L,\mathrm{g}} &= n_f\Big\{ -(16C_A - 8C_F)N^{-2} - 8C_F N^{-1} + ([16+16\zeta_2]C_A - [4+8\zeta_2]C_F)\Big\} + \mathcal{O}(N)\,,\\
c^{(2,1)}_{L,\mathrm{g}} &= n_f\Big\{ (16C_A - 8C_F)N^{-3} - (32C_A - 16C_F)N^{-2}\\
&\qquad + ([72-40\zeta_2]C_A - [12-20\zeta_2]C_F)N^{-1}\Big\} + \mathcal{O}(N^0)\,,\\
c^{(3,0)}_{L,\mathrm{g}} &= n_f\Big\{ (160C_A^2 - 80C_F C_A + 40C_F^2 - 80n_f C_F)N^{-4}\\
&\qquad + \Big(\frac{56}{3}C_A^2 - \frac{152}{3}C_F C_A - 20C_F^2 + \frac{16}{3}n_f C_A + \frac{464}{3}n_f C_F\Big)N^{-3}\\
&\qquad + \Big(\Big[\frac{3640}{9} - 120\zeta_2\Big]C_A^2 + \Big[\frac{308}{9} + 144\zeta_2\Big]C_F C_A - [16+96\zeta_2]C_F^2\\
&\qquad + \frac{80}{9}n_f C_A - \Big[\frac{3416}{9} - 64\zeta_2\Big]n_f C_F\Big)N^{-2}\Big\} + \mathcal{O}(N^{-1})\,.
\end{aligned} \tag{A.11}$$

B Hypergeometric functions for the non-singlet coefficient functions

The hypergeometric functions relevant for the non-singlet x-space coefficient functions are

$$\begin{aligned}
m\,a \int_0^1 dx\, x^{N-1} \ln\frac{1}{x}\; {}_1F_2\Big(\frac{m}{4}{+}1; 2, \frac{3}{2}; a\ln^2\frac{1}{x}\Big) &= \Big(1-\frac{4a}{N^2}\Big)^{-m/4} - 1\,,\\
m\,a \int_0^1 dx\, x^{N-1}\; {}_1F_2\Big(\frac{m}{4}{+}1; 2, \frac{1}{2}; a\ln^2\frac{1}{x}\Big) &= N\Big\{\Big(1-\frac{4a}{N^2}\Big)^{-m/4} - 1\Big\},\\
\frac{1}{2}m(m{+}4)\,a^2 \int_0^1 dx\, x^{N-1} \ln\frac{1}{x}\; {}_1F_2\Big(\frac{m}{4}{+}2; 3, \frac{3}{2}; a\ln^2\frac{1}{x}\Big) &= N^2\Big\{\Big(1-\frac{4a}{N^2}\Big)^{-m/4} - 1 - \frac{m\,a}{N^2}\Big\}
\end{aligned} \tag{B.1}$$

with $a = 2\,C_F\,a_{\mathrm{s}}$.

C Leading-logarithmic coefficient functions for F_ϕ

In this final appendix, we present some analytic results for the scalar-exchange coefficient functions $C_{\phi,\mathrm{q}}$ and $C_{\phi\,g}$. Using the decomposition

$$C_{\phi,\mathrm{g}}(N) \;=\; C_\phi^{+}(N) + C_{\phi,\mathrm{g}}^{\mathrm{ps}}(N)\,, \tag{C.1}$$

which is analogous to that of the gluon-gluon splitting function in the second line of eq. (5.1), the F_ϕ counterparts of the LL expressions (5.18)–(5.21) for C_2 and C_L are given by

$$C_\phi^{+\,(n)}(N) = \frac{(-4\,C_A)^n}{N^{2n}}\,\mathcal{D}_n\,, \tag{C.2}$$

$$C_{\phi,\mathrm{g}}^{\mathrm{ps}\,(\mathrm{n})}(N) = \mathcal{D}_n\,\frac{2^n}{N^{2n}}\sum_{i=0}^{\lfloor\frac{n-2}{2}\rfloor}\;\sum_{k=0}^{n-2-2i}(-2)^{i+1+k}(n_f C_F)^{i+1}C_A^k C_F^{\rho'}\binom{k+i+1}{k}\binom{\rho'+i}{\rho'}, \tag{C.3}$$

and

$$C_{\phi,\mathrm{q}}^{(n)}(N) = -C_F\mathcal{D}_n\,\frac{2^{n+1}}{N^{2n}}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\;\sum_{k=0}^{n-1-2i}(-2)^{i+k}(n_f C_F)^{i}C_A^k C_F^{\delta'}\binom{k+i}{k}\binom{\delta'+i}{\delta'}, \tag{C.4}$$

where δ' and ρ' have been given below eq. (5.21).

Unlike the splitting functions and coefficient functions for F_2 and F_L, the coefficient functions for ϕ-exchange DIS exhibit double logarithms also in the BFKL-limit, i.e., they include contributions of the form $\alpha_s^{\,n}\,x^{-1}\ln^{2n-n_0-k}x$. We have considered these in what might be called 'extended quantum gluodynamics' (eQGD), i.e., QCD in the limit $C_F = 0$, and found an analogue to eq. (2.17) for the expansion about $N = 1$ which generates the resummation of these double logarithms.

In terms of

$$S'' \;=\; (1-4\,\xi'')^{-1} \quad \text{with} \quad \xi'' \;=\; C_A a_\mathrm{s}/\overline{N}^{\,2} \quad \text{where} \quad \overline{N} \;\equiv\; N-1\,, \tag{C.5}$$

the resummed coefficient function, including the finite NNLL contributions at $\mathcal{O}(a_\mathrm{s})$, is found to be

$$\begin{aligned} C_\phi(\overline{N})\big|_{C_F=0} = {}& (S''-1) + \frac{1}{12\,C_A}\,\overline{N}\Big\{44\,C_A(S''-1) + (3\,\beta_0 - 22\,C_A)(S''^2-1) - 3\,\beta_0(S''^3-1)\Big\} \\ & + \frac{1}{3\,C_A}\,a_\mathrm{s}\Big\{C_A\Big(5\,\beta_0 + 4\,C_A[1-3\,\zeta_2]\Big) + C_A\Big(C_A\Big[\frac{653}{6} - 12\,\zeta_2\Big] - \beta_0\Big)(S''-1) \\ & \qquad + C_A\Big(\frac{23}{2}\,\beta_0 - C_A\Big[\frac{201}{2} + 12\,\zeta_2\Big]\Big)(S''^2-1) \\ & \qquad + \Big(\frac{121}{3}\,C_A^2 - 22\,C_A\,\beta_0 + \frac{3}{4}\,\beta_0^2\Big)(S''^3-1) \\ & \qquad + \beta_0\Big(\frac{33}{2}\,C_A - 3\,\beta_0\Big)(S''^4-1) + \frac{9}{4}\,\beta_0^2(S''^5-1)\Big\}, \end{aligned} \tag{C.6}$$

where the two terms in the first line provide the LL and NLL contributions, and the remaining four lines the NNLL result. This resummation does, of course, also return the corresponding x^{-1} double-logarithms in $P_{\mathrm{gg}}\big|_{C_F=0}$. These are found to vanish, as they have to, see refs. [51–53].

References

[1] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, *The next next-to-leading QCD approximation for nonsinglet moments of deep inelastic structure functions*, *Nucl. Phys. B* **427** (1994) 41 [INSPIRE].

[2] S.A. Larin, P. Nogueira, T. van Ritbergen and J.A.M. Vermaseren, *The three loop QCD calculation of the moments of deep inelastic structure functions*, *Nucl. Phys. B* **492** (1997) 338 [hep-ph/9605317] [INSPIRE].

[3] I. Bierenbaum, J. Blümlein and S. Klein, *Mellin Moments of the $O(\alpha_s^3)$ Heavy Flavor Contributions to unpolarized Deep-Inelastic Scattering at $Q^2 \gg m^2$ and Anomalous Dimensions*, *Nucl. Phys. B* **820** (2009) 417 [arXiv:0904.3563] [INSPIRE].

[4] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Massless Propagators, $R(s)$ and Multiloop QCD*, *Nucl. Part. Phys. Proc.* **261-262** (2015) 3 [arXiv:1501.06739] [INSPIRE].

[5] S. Moch, J.A.M. Vermaseren and A. Vogt, *The three loop splitting functions in QCD: The nonsinglet case*, *Nucl. Phys. B* **688** (2004) 101 [hep-ph/0403192] [INSPIRE].

[6] A. Vogt, S. Moch and J.A.M. Vermaseren, *The three-loop splitting functions in QCD: The singlet case*, *Nucl. Phys. B* **691** (2004) 129 [hep-ph/0404111] [INSPIRE].

[7] S. Moch, J.A.M. Vermaseren and A. Vogt, *The longitudinal structure function at the third order*, *Phys. Lett. B* **606** (2005) 123 [hep-ph/0411112] [INSPIRE].

[8] J.A.M. Vermaseren, A. Vogt and S. Moch, *The third-order QCD corrections to deep-inelastic scattering by photon exchange*, *Nucl. Phys. B* **724** (2005) 3 [hep-ph/0504242] [INSPIRE].

[9] S. Moch, J.A.M. Vermaseren and A. Vogt, *Third-order QCD corrections to the charged-current structure function F_3*, *Nucl. Phys. B* **813** (2009) 220 [arXiv:0812.4168] [INSPIRE].

[10] B. Ruijl, T. Ueda, J.A.M. Vermaseren, J. Davies and A. Vogt, *First Forcer results on deep-inelastic scattering and related quantities*, *PoS* **LL2016** (2016) 071 [arXiv:1605.08408] [INSPIRE].

[11] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Low moments of the four-loop splitting functions in QCD*, *Phys. Lett. B* **825** (2022) 136853 [arXiv:2111.15561] [INSPIRE].

[12] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Non-singlet structure functions at four loops*, to appear.

[13] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Singlet structure functions at four loops*, to appear.

[14] P.A. Baikov and K.G. Chetyrkin, *New four loop results in QCD*, *Nucl. Phys. B Proc. Suppl.* **160** (2006) 76 [INSPIRE].

[15] V.N. Velizhanin, *Four loop anomalous dimension of the second moment of the non-singlet twist-2 operator in QCD*, *Nucl. Phys. B* **860** (2012) 288 [arXiv:1112.3954] [INSPIRE].

[16] V.N. Velizhanin, *Four-loop anomalous dimension of the third and fourth moments of the nonsinglet twist-2 operator in QCD*, *Int. J. Mod. Phys. A* **35** (2020) 2050199 [arXiv:1411.1331] [INSPIRE].

[17] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4 in a General Gauge Theory*, *Phys. Rev. Lett.* **104** (2010) 132004 [arXiv:1001.3606] [INSPIRE].

[18] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn and J. Rittinger, *Adler Function, Sum Rules and Crewther Relation of Order $\mathcal{O}(\alpha_s^4)$: the Singlet Case*, *Phys. Lett. B* **714** (2012) 62 [arXiv:1206.1288] [INSPIRE].

[19] G.P. Korchemsky, *Asymptotics of the Altarelli-Parisi-Lipatov Evolution Kernels of Parton Distributions*, *Mod. Phys. Lett. A* **4** (1989) 1257 [INSPIRE].

[20] S. Albino and R.D. Ball, *Soft resummation of quark anomalous dimensions and coefficient functions in $\overline{MS}$ factorization*, *Phys. Lett. B* **513** (2001) 93 [hep-ph/0011133] [INSPIRE].

[21] Y.L. Dokshitzer, G. Marchesini and G.P. Salam, *Revisiting parton evolution and the large-x limit*, *Phys. Lett. B* **634** (2006) 504 [hep-ph/0511302] [INSPIRE].

[22] S. Moch, J.A.M. Vermaseren and A. Vogt, *Higher-order corrections in threshold resummation*, *Nucl. Phys. B* **726** (2005) 317 [hep-ph/0506288] [INSPIRE].

[23] G. Das, S.-O. Moch and A. Vogt, *Soft corrections to inclusive deep-inelastic scattering at four loops and beyond*, *JHEP* **03** (2020) 116 [arXiv:1912.12920] [INSPIRE].

[24] V. Ravindran, *Higher-order threshold effects to inclusive processes in QCD*, *Nucl. Phys. B* **752** (2006) 173 [hep-ph/0603041] [INSPIRE].

[25] G.F. Sterman, *Summation of Large Corrections to Short Distance Hadronic Cross-Sections*, *Nucl. Phys. B* **281** (1987) 310 [INSPIRE].

[26] S. Catani and L. Trentadue, *Resummation of the QCD Perturbative Series for Hard Processes*, *Nucl. Phys. B* **327** (1989) 323 [INSPIRE].

[27] L. Magnea, *All Order Summation and Two Loop Results for the Drell-Yan Cross-section*, *Nucl. Phys. B* **349** (1991) 703 [INSPIRE].

[28] S. Catani and L. Trentadue, *Comment on QCD exponentiation at large x*, *Nucl. Phys. B* **353** (1991) 183 [INSPIRE].

[29] S. Catani, M.L. Mangano, P. Nason and L. Trentadue, *The resummation of soft gluons in hadronic collisions*, *Nucl. Phys. B* **478** (1996) 273 [hep-ph/9604351] [INSPIRE].

[30] H. Contopanagos, E. Laenen and G.F. Sterman, *Sudakov factorization and resummation*, *Nucl. Phys. B* **484** (1997) 303 [hep-ph/9604313] [INSPIRE].

[31] S. Moch and A. Vogt, *Threshold Resummation of the Structure Function F(L)*, *JHEP* **04** (2009) 081 [arXiv:0902.2342] [INSPIRE].

[32] S. Moch and A. Vogt, *On non-singlet physical evolution kernels and large-x coefficient functions in perturbative QCD*, *JHEP* **11** (2009) 099 [arXiv:0909.2124] [INSPIRE].

[33] G. Soar, S. Moch, J.A.M. Vermaseren and A. Vogt, *On Higgs-exchange DIS, physical evolution kernels and fourth-order splitting functions at large x*, *Nucl. Phys. B* **832** (2010) 152 [arXiv:0912.0369] [INSPIRE].

[34] G. Grunberg, *Large-x structure of physical evolution kernels in Deep Inelastic Scattering*, *Phys. Lett. B* **687** (2010) 405 [arXiv:0911.4471] [INSPIRE].

[35] A. Vogt, *Leading logarithmic large-x resummation of off-diagonal splitting functions and coefficient functions*, *Phys. Lett. B* **691** (2010) 77 [arXiv:1005.1606] [INSPIRE].

[36] A.A. Almasy, G. Soar and A. Vogt, *Generalized double-logarithmic large-x resummation in inclusive deep-inelastic scattering*, *JHEP* **03** (2011) 030 [arXiv:1012.3352] [INSPIRE].

JHEP08(2022)135

[37] A.A. Almasy, N.A. Lo Presti and A. Vogt, *Generalized threshold resummation in inclusive DIS and semi-inclusive electron-positron annihilation*, *JHEP* **01** (2016) 028 [arXiv:1511.08612] [INSPIRE].

[38] A.H. Mueller, *On the Multiplicity of Hadrons in QCD Jets*, *Phys. Lett. B* **104** (1981) 161 [INSPIRE].

[39] A.H. Mueller, *Multiplicity and Hadron Distributions in QCD Jets: Nonleading Terms*, *Nucl. Phys. B* **213** (1983) 85 [INSPIRE].

[40] A. Vogt, *Resummation of small-x double logarithms in QCD: semi-inclusive electron-positron annihilation*, *JHEP* **10** (2011) 025 [arXiv:1108.2993] [INSPIRE].

[41] C.H. Kom, A. Vogt and K. Yeats, *Resummed small-x and first-moment evolution of fragmentation functions in perturbative QCD*, *JHEP* **10** (2012) 033 [arXiv:1207.5631] [INSPIRE].

[42] P. Bolzoni, B.A. Kniehl and A.V. Kotikov, *Gluon and quark jet multiplicities at N^3LO+NNLL*, *Phys. Rev. Lett.* **109** (2012) 242002 [arXiv:1209.5914] [INSPIRE].

[43] P. Bolzoni, B.A. Kniehl and A.V. Kotikov, *Average gluon and quark jet multiplicities at higher orders*, *Nucl. Phys. B* **875** (2013) 18 [arXiv:1305.6017] [INSPIRE].

[44] A. Vogt et al., *Progress on double-logarithmic large-x and small-x resummations for (semi-)inclusive hard processes*, *PoS* **LL2012** (2012) 004 [arXiv:1212.2932] [INSPIRE].

[45] B. Ruijl, T. Ueda and J.A.M. Vermaseren, *Forcer, a FORM program for the parametric reduction of four-loop massless propagator diagrams*, *Comput. Phys. Commun.* **253** (2020) 107198 [arXiv:1704.06650] [INSPIRE].

[46] J. Davies, A. Vogt, B. Ruijl, T. Ueda and J.A.M. Vermaseren, *Large-N_f contributions to the four-loop splitting functions in QCD*, *Nucl. Phys. B* **915** (2017) 335 [arXiv:1610.07477] [INSPIRE].

[47] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*, *JHEP* **10** (2017) 041 [arXiv:1707.08315] [INSPIRE].

[48] V.N. Velizhanin, *Generalised double-logarithmic equation in QCD*, *Mod. Phys. Lett. A* **32** (2017) 1750213 [arXiv:1412.7143] [INSPIRE].

[49] W.L. van Neerven and A. Vogt, *NNLO evolution of deep inelastic structure functions: The singlet case*, *Nucl. Phys. B* **588** (2000) 345 [hep-ph/0006154] [INSPIRE].

[50] W.L. van Neerven and A. Vogt, *Nonsinglet structure functions beyond the next-to-next-to-leading order*, *Nucl. Phys. B* **603** (2001) 42 [hep-ph/0103123] [INSPIRE].

[51] T. Jaroszewicz, *Gluonic Regge Singularities and Anomalous Dimensions in QCD*, *Phys. Lett. B* **116** (1982) 291 [INSPIRE].

[52] S. Catani, F. Fiorani and G. Marchesini, *Small x Behavior of Initial State Radiation in Perturbative QCD*, *Nucl. Phys. B* **336** (1990) 18 [INSPIRE].

[53] S. Catani and F. Hautmann, *High-energy factorization and small-x deep inelastic scattering beyond leading order*, *Nucl. Phys. B* **427** (1994) 475 [hep-ph/9405388] [INSPIRE].

[54] D.J. Gross and F. Wilczek, *Ultraviolet Behavior of Nonabelian Gauge Theories*, *Phys. Rev. Lett.* **30** (1973) 1343 [INSPIRE].

[55] H.D. Politzer, *Reliable Perturbative Results for Strong Interactions?*, *Phys. Rev. Lett.* **30** (1973) 1346 [INSPIRE].

JHEP08(2022)135

[56] W.E. Caswell, *Asymptotic Behavior of Nonabelian Gauge Theories to Two Loop Order*, *Phys. Rev. Lett.* **33** (1974) 244 [INSPIRE].

[57] D.R.T. Jones, *Two Loop Diagrams in Yang-Mills Theory*, *Nucl. Phys. B* **75** (1974) 531 [INSPIRE].

[58] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025 [INSPIRE].

[59] M. Tentyukov and J.A.M. Vermaseren, *The multithreaded version of FORM*, *Comput. Phys. Commun.* **181** (2010) 1419 [hep-ph/0702279] [INSPIRE].

[60] J. Kuipers, T. Ueda, J.A.M. Vermaseren and J. Vollinga, *FORM version 4.0*, *Comput. Phys. Commun.* **184** (2013) 1453 [arXiv:1203.6543] [INSPIRE].

[61] W.L. van Neerven and E.B. Zijlstra, *Order α_s^2 contributions to the deep inelastic Wilson coefficient*, *Phys. Lett. B* **272** (1991) 127 [INSPIRE].

[62] E.B. Zijlstra and W.L. van Neerven, *Contribution of the second order gluonic Wilson coefficient to the deep inelastic structure function*, *Phys. Lett. B* **273** (1991) 476 [INSPIRE].

[63] E.B. Zijlstra and W.L. van Neerven, *Order α_s^2 correction to the structure function $F_3(x, Q^2)$ in deep inelastic neutrino-hadron scattering*, *Phys. Lett. B* **297** (1992) 377 [INSPIRE].

[64] T. Matsuura and W.L. van Neerven, *Second Order Logarithmic Corrections to the Drell-Yan Cross-section*, *Z. Phys. C* **38** (1988) 623 [INSPIRE].

[65] T. Matsuura, S.C. van der Marck and W.L. van Neerven, *The Calculation of the Second Order Soft and Virtual Contributions to the Drell-Yan Cross-Section*, *Nucl. Phys. B* **319** (1989) 570 [INSPIRE].

[66] A. Daleo, A. Gehrmann-De Ridder, T. Gehrmann and G. Luisoni, *Antenna subtraction at NNLO with hadronic initial states: initial-final configurations*, *JHEP* **01** (2010) 118 [arXiv:0912.0374] [INSPIRE].

[67] S. Moch and M. Rogal, *Charged current deep-inelastic scattering at three loops*, *Nucl. Phys. B* **782** (2007) 51 [arXiv:0704.1740] [INSPIRE].

[68] S. Moch, M. Rogal and A. Vogt, *Differences between charged-current coefficient functions*, *Nucl. Phys. B* **790** (2008) 317 [arXiv:0708.3731] [INSPIRE].

[69] J. Davies, A. Vogt, S. Moch and J.A.M. Vermaseren, *Non-singlet coefficient functions for charged-current deep-inelastic scattering to the third order in QCD*, *PoS* **DIS2016** (2016) 059 [arXiv:1606.08907] [INSPIRE].

[70] E.B. Zijlstra and W.L. van Neerven, *Order-α_s^2 corrections to the polarized structure function $g_1(x, Q^2)$*, *Nucl. Phys. B* **417** (1994) 61 [*Erratum ibid.* **426** (1994) 245] [*Erratum ibid.* **773** (2007) 105] [*Erratum ibid.* **501** (1997) 599] [INSPIRE].

[71] A. Vogt, S. Moch, M. Rogal and J.A.M. Vermaseren, *Towards the NNLO evolution of polarised parton distributions*, *Nucl. Phys. B Proc. Suppl.* **183** (2008) 155 [arXiv:0807.1238] [INSPIRE].

[72] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three-Loop Splitting Functions in QCD: The Helicity-Dependent Case*, *Nucl. Phys. B* **889** (2014) 351 [arXiv:1409.5131] [INSPIRE].

[73] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements*, *Nucl. Phys. B* **971** (2021) 115542 [arXiv:2107.06267] [INSPIRE].

[74] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The three-loop polarized singlet anomalous dimensions from off-shell operator matrix elements*, *JHEP* **01** (2022) 193 [arXiv:2111.12401] [INSPIRE].

JHEP08(2022)135

[75] D.J. Broadhurst, A.L. Kataev and C.J. Maxwell, *Comparison of the Gottfried and Adler sum rules within the large N_c expansion*, *Phys. Lett. B* **590** (2004) 76 [hep-ph/0403037] [INSPIRE].

[76] R. Kirschner and L.n. Lipatov, *Double Logarithmic Asymptotics and Regge Singularities of Quark Amplitudes with Flavor Exchange*, *Nucl. Phys. B* **213** (1983) 122 [INSPIRE].

[77] J. Blümlein and A. Vogt, *On the behavior of nonsinglet structure functions at small x*, *Phys. Lett. B* **370** (1996) 149 [hep-ph/9510410] [INSPIRE].

[78] J. Blümlein and A. Vogt, *On the resummation of $\alpha \ln^2 x$ terms for nonsinglet structure functions in QED and QCD*, *Acta Phys. Polon. B* **27** (1996) 1309 [hep-ph/9603450] [INSPIRE].

[79] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series amd Products*, 6th ed., Academic Press (2000).

[80] M. Jamin and R. Miravitllas, *Absence of even-integer ζ-function values in Euclidean physical quantities in QCD*, *Phys. Lett. B* **779** (2018) 452 [arXiv:1711.00787] [INSPIRE].

[81] J. Davies and A. Vogt, *Absence of π^2 terms in physical anomalous dimensions in DIS: Verification and resulting predictions*, *Phys. Lett. B* **776** (2018) 189 [arXiv:1711.05267] [INSPIRE].

[82] P.A. Baikov and K.G. Chetyrkin, *The structure of generic anomalous dimensions and no-π theorem for massless propagators*, *JHEP* **06** (2018) 141 [arXiv:1804.10088] [INSPIRE].

[83] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys. A* **15** (2000) 725 [hep-ph/9905237] [INSPIRE].

[84] J.A.M. Vermaseren, *Harmonic sums, Mellin transforms and integrals*, *Int. J. Mod. Phys. A* **14** (1999) 2037 [hep-ph/9806280] [INSPIRE].

[85] J. Blümlein and S. Kurth, *Harmonic sums and Mellin transforms up to two loop order*, *Phys. Rev. D* **60** (1999) 014018 [hep-ph/9810241] [INSPIRE].

[86] A. Grozin, J.M. Henn, G.P. Korchemsky and P. Marquard, *The three-loop cusp anomalous dimension in QCD and its supersymmetric extensions*, *JHEP* **01** (2016) 140 [arXiv:1510.07803] [INSPIRE].

[87] J.M. Henn, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *A planar four-loop form factor and cusp anomalous dimension in QCD*, *JHEP* **05** (2016) 066 [arXiv:1604.03126] [INSPIRE].

[88] A. Grozin, *Leading and next-to-leading large-N_f terms in the cusp anomalous dimension and quark-antiquark potential*, *PoS* **LL2016** (2016) 053 [arXiv:1605.03886] [INSPIRE].

[89] J. Henn, A.V. Smirnov, V.A. Smirnov, M. Steinhauser and R.N. Lee, *Four-loop photon quark form factor and cusp anomalous dimension in the large-N_c limit of QCD*, *JHEP* **03** (2017) 139 [arXiv:1612.04389] [INSPIRE].

[90] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *The n_f^2 contributions to fermionic four-loop form factors*, *Phys. Rev. D* **96** (2017) 014008 [arXiv:1705.06862] [INSPIRE].

[91] A.A. Almasy, S. Moch and A. Vogt, *On the Next-to-Next-to-Leading Order Evolution of Flavour-Singlet Fragmentation Functions*, *Nucl. Phys. B* **854** (2012) 133 [arXiv:1107.2263] [INSPIRE].

[92] H. Chen, T.-Z. Yang, H.X. Zhu and Y.J. Zhu, *Analytic Continuation and Reciprocity Relation for Collinear Splitting in QCD*, *Chin. Phys. C* **45** (2021) 043101 [arXiv:2006.10534] [INSPIRE].

[93] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions*, Dover, New York, U.S.A. (1965).

[94] J.A. Gracey, *Anomalous dimension of nonsinglet Wilson operators at O (1 /N(f)) in deep inelastic scattering*, Phys. Lett. B **322** (1994) 141 [hep-ph/9401214] [INSPIRE].

[95] L. Mankiewicz, M. Maul and E. Stein, *Perturbative part of the non-singlet structure function F_2 in the large N_f limit*, Phys. Lett. B **404** (1997) 345 [hep-ph/9703356] [INSPIRE].

[96] F. Herzog, S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Five-loop contributions to low-N non-singlet anomalous dimensions in QCD*, Phys. Lett. B **790** (2019) 436 [arXiv:1812.11818] [INSPIRE].

[97] J. Blümlein and W.L. van Neerven, *Less singular terms and small x evolution in a soluble model*, Phys. Lett. B **450** (1999) 412 [hep-ph/9811519] [INSPIRE].

[98] S. Moch and A. Vogt, *On third-order timelike splitting functions and top-mediated Higgs decay into hadrons*, Phys. Lett. B **659** (2008) 290 [arXiv:0709.3899] [INSPIRE].