

Low moments of the four-loop splitting functions in QCD

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Abstract

We have computed the four lowest even- N moments of all four splitting functions for the evolution of flavour-singlet parton densities of hadrons at the fourth order in the strong coupling constant α_s . The perturbative expansion of these moments, and hence of the splitting functions for momentum fractions $x \gtrsim 0.1$, is found to be well behaved with relative α_s -coefficients of order one and sub-percent effects on the scale derivatives of the quark and gluon distributions at $\alpha_s \lesssim 0.2$. More intricate computations, including other approaches such as the operator-product expansion, are required to cover the full x -range relevant to LHC analyses. Our results are presented analytically for a general gauge group for detailed checks and validations of such future calculations.

Fully consistent analyses of hard processes with initial-state hadrons at the (next-to) n -leading order (N n LO) of renormalization-group improved perturbative QCD require parton distributions functions (PDFs) evolved with the $(n+1)$ -loop splitting functions. Over the past years, N 2 LO (= NNLO) has become the standard approximation for many processes. Following pioneering computations of their lowest integer- N Mellin moments in refs. [1, 2], the corresponding 3-loop splitting functions were obtained in refs. [3, 4].

For certain benchmark cases, in particular Higgs-boson production at the LHC [5], N 2 LO calculations are not sufficiently accurate, hence the 4-loop splitting functions need to be calculated. These have been determined for the flavour non-singlet quark-quark case in ref. [6] – analytically in the limit of a large number of colours n_c , and numerically for the remaining contributions – and for the (next-to-)leading contributions for a large number of flavours n_f in ref. [7].

Here we present, as a first significant step towards at least approximate expressions for the 4-loop singlet splitting functions for use in phenomenological analyses, their lowest four even moments $N = 2 \dots, 8$ in the standard $\overline{\text{MS}}$ scheme, thus extending the computations of ref. [2] by one order in the strong coupling α_s . Following the approach of refs. [2, 4] our calculations are performed via physical quantities in deep-inelastic scattering, i.e., instead of working with 4-loop off-shell flavour-singlet operator matrix elements (OMEs) which, at this point, is still theoretically challenging. Our present results, obtained analytically for a general gauge group, should also be useful for checking and validating future OME computations of these quantities.

The evolution equations for the flavour-singlet quark and gluon PDFs of hadrons,

$$q_s(x, \mu_f^2) = \sum_{i=1}^{n_f} [q_i(x, \mu_f^2) + \bar{q}_i(x, \mu_f^2)] \quad \text{and} \quad g(x, \mu_f^2), \quad (1)$$

are

$$\frac{d}{d \ln \mu_f^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix}. \quad (2)$$

Here $\otimes$ represents the Mellin convolution in the momentum variable, and μ_f is the factorization scale. For the determination of the splitting functions P_{ik} , the renormalization scale can be identified with μ_f without loss of information. The even- N moments of splitting functions in eq. (2) are identical to the anomalous dimensions of twist-2 spin- N operators up to a conventional sign,

$$\gamma_{ik}(N, \alpha_s) = - \int_0^1 dx x^{N-1} P_{ik}(x, \alpha_s). \quad (3)$$

Their perturbative expansions can be written as

$$\gamma_{ik}(N, \alpha_s) = \sum_{n=0} a_s^{n+1} \gamma_{ik}^{(n)}(x) \quad \text{with} \quad a_s \equiv \frac{\alpha_s(\mu_f^2)}{4\pi}. \quad (4)$$

The quark-quark entry in eq. (3) can be expressed as $\gamma_{qq} = \gamma_{ns}^+ + \gamma_{ps}$ in terms of the non-singlet anomalous dimension γ_{ns}^+ for quark-antiquark sums addressed at four loops in ref. [6] and a pure-singlet contribution γ_{ps} which is suppressed at $N \gg 1$. At asymptotically large N the diagonal $\overline{\text{MS}}$ entries $\gamma_{kk}(N)$ in eq. (2) are governed by the (lightlike) cusp anomalous dimensions A_k [8], viz $\gamma_{kk}(N) = A_k \ln N + O(1)$, which are now fully known at four loops [9, 10].

The 4-loop contributions to the pure-singlet anomalous dimensions in eq. (4) at $N = 2, 4, 6$ are

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=2) = & n_f C_F^3 \left(\frac{227938}{2187} + \frac{1952}{81} \zeta_3 + \frac{256}{9} \zeta_4 - \frac{640}{3} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(-\frac{162658}{6561} + \frac{8048}{27} \zeta_3 - \frac{1664}{9} \zeta_4 + \frac{320}{9} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{410299}{6561} - \frac{26896}{81} \zeta_3 + \frac{1408}{9} \zeta_4 + \frac{4480}{27} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{1024}{9} + \frac{256}{9} \zeta_3 - \frac{2560}{9} \zeta_5 \right) - n_f^2 C_F^2 \left(\frac{73772}{6561} + \frac{5248}{81} \zeta_3 - \frac{320}{9} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{160648}{6561} + 48 \zeta_3 - \frac{320}{9} \zeta_4 \right) + n_f^3 C_F \left(-\frac{1712}{729} + \frac{128}{27} \zeta_3 \right), \tag{5}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=4) = & n_f C_F^3 \left(\frac{1995890620891}{52488000000} - \frac{897403}{202500} \zeta_3 + \frac{18997}{2250} \zeta_4 - \frac{484}{15} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{209865827521}{26244000000} + \frac{6743539}{202500} \zeta_3 - \frac{29161}{750} \zeta_4 + \frac{242}{45} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{55187654921}{3280500000} - \frac{3104267}{67500} \zeta_3 + \frac{34243}{1125} \zeta_4 + \frac{3164}{135} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{172231}{675} - \frac{5368}{25} \zeta_3 - \frac{3728}{45} \zeta_5 \right) - n_f^2 C_F^2 \left(\frac{141522185707}{26244000000} \right. \\
& \quad \left. + \frac{1207}{135} \zeta_3 - \frac{242}{45} \zeta_4 \right) + n_f^2 C_A C_F \left(\frac{9398360351}{1640250000} + \frac{57877}{10125} \zeta_3 - \frac{242}{45} \zeta_4 \right) \\
& + n_f^3 C_F \left(-\frac{46099151}{72900000} + \frac{484}{675} \zeta_3 \right), \tag{6}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=6) = & n_f C_F^3 \left(\frac{140565274663259489}{5403265623000000} - \frac{62727544}{24310125} \zeta_3 + \frac{343156}{77175} \zeta_4 - \frac{1936}{147} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{336481838777617}{360217708200000} + \frac{2111992}{324135} \zeta_3 - \frac{1389806}{77175} \zeta_4 + \frac{968}{441} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{6194882229735067}{864522499680000} - \frac{2396237}{165375} \zeta_3 + \frac{41866}{3087} \zeta_4 + \frac{9544}{1323} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{64697569}{330750} - \frac{426976}{3675} \zeta_3 - \frac{39808}{441} \zeta_5 \right) \\
& - n_f^2 C_F^2 \left(\frac{812984663253277}{270163281150000} + \frac{2594876}{694575} \zeta_3 - \frac{968}{441} \zeta_4 \right) + n_f^2 C_A C_F \left(\frac{3092531515013}{964868861250} \right. \\
& \quad \left. + \frac{217432}{99225} \zeta_3 - \frac{968}{441} \zeta_4 \right) + n_f^3 C_F \left(-\frac{19597073837}{61261515000} + \frac{1936}{6615} \zeta_3 \right). \tag{7}
\end{aligned}$$

The complete qq entries are obtained by adding the non-singlet contributions in app. B of ref. [6].

The corresponding results for the off-diagonal splitting functions are given by

$$\begin{aligned}
\gamma_{\text{qg}}^{(3)}(N=2) = & n_f C_F^3 \left(\frac{16489}{729} + \frac{736}{81} \zeta_3 + \frac{256}{9} \zeta_4 - \frac{320}{3} \zeta_5 \right) \\
& + n_f C_A^3 \left(-\frac{88769}{729} + \frac{31112}{81} \zeta_3 - 132 \zeta_4 - \frac{3560}{27} \zeta_5 \right) - n_f C_A C_F^2 \left(\frac{1153727}{13122} - \frac{7108}{81} \zeta_3 \right. \\
& \left. + \frac{1136}{9} \zeta_4 - \frac{2000}{9} \zeta_5 \right) + n_f C_A^2 C_F \left(\frac{763868}{6561} - \frac{12808}{27} \zeta_3 + \frac{2068}{9} \zeta_4 + \frac{40}{9} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left(\frac{368}{9} - \frac{992}{9} \zeta_3 - \frac{2560}{9} \zeta_5 \right) - n_f^2 C_F^2 \left(\frac{110714}{6561} + \frac{272}{9} \zeta_3 - \frac{224}{9} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{249310}{6561} + \frac{5632}{81} \zeta_3 - \frac{440}{9} \zeta_4 \right) + n_f^2 C_A^2 \left(\frac{48625}{2187} - \frac{3572}{81} \zeta_3 \right. \\
& \left. + 24 \zeta_4 + \frac{160}{27} \zeta_5 \right) + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(-\frac{928}{9} - \frac{640}{9} \zeta_3 + \frac{2560}{9} \zeta_5 \right) \\
& + n_f^3 C_F \left(-\frac{8744}{2187} + \frac{128}{27} \zeta_3 \right) + n_f^3 C_A \left(\frac{3385}{2187} - \frac{176}{81} \zeta_3 \right), \tag{8}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{qg}}^{(3)}(N=4) = & n_f C_F^3 \left(-\frac{8103828487201}{104976000000} + \frac{5100751}{81000} \zeta_3 + \frac{154589}{4500} \zeta_4 - \frac{3158}{45} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{5121012352507}{26244000000} - \frac{48971263}{405000} \zeta_3 - \frac{143489}{750} \zeta_4 + \frac{951}{5} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{314624947013}{1312200000} - \frac{2024593}{9000} \zeta_3 + \frac{1674889}{4500} \zeta_4 + \frac{1237}{45} \zeta_5 \right) \\
& + n_f C_A^3 \left(\frac{143199094853}{1458000000} + \frac{11938031}{45000} \zeta_3 - \frac{26904}{125} \zeta_4 - \frac{17917}{135} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left(-\frac{12196}{135} - \frac{81008}{225} \zeta_3 + \frac{15976}{45} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(\frac{37295583467}{26244000000} - \frac{1400864}{50625} \zeta_3 + \frac{707}{45} \zeta_4 \right) + n_f^2 C_A C_F \left(\frac{217239001681}{13122000000} \right. \\
& \left. + \frac{4497112}{50625} \zeta_3 - \frac{103669}{2250} \zeta_4 \right) + n_f^2 C_A^2 \left(-\frac{7131194093}{4374000000} - \frac{12599759}{202500} \zeta_3 \right. \\
& \left. + \frac{7591}{250} \zeta_4 + \frac{664}{135} \zeta_5 \right) + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(-\frac{112424}{675} - \frac{2336}{75} \zeta_3 + \frac{10624}{45} \zeta_5 \right) \\
& + n_f^3 C_F \left(-\frac{312015851}{364500000} + \frac{6644}{3375} \zeta_3 \right) + n_f^3 C_A \left(\frac{338346151}{437400000} - \frac{5192}{2025} \zeta_3 \right), \tag{9}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{qg}}^{(3)}(N=6) = & n_f C_A^3 \left(\frac{49981299563948069}{345808999872000} + \frac{2383601783}{12965400} \zeta_3 - \frac{689907}{3430} \zeta_4 - \frac{159724}{1323} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{324177529264517279}{960580555200000} - \frac{1154450237}{9724050} \zeta_3 - \frac{28952417}{154350} \zeta_4 + \frac{9832}{441} \zeta_5 \right)
\end{aligned}$$

$$\begin{aligned}
& + n_f C_A^2 C_F \left(-\frac{627686002393628869}{1729044999360000} - \frac{6170262713}{48620250} \zeta_3 + \frac{1096679}{3087} \zeta_4 + \frac{47774}{441} \zeta_5 \right) \\
& + n_f C_F^3 \left(-\frac{2912197809548779709}{21613062492000000} + \frac{1026604067}{24310125} \zeta_3 + \frac{2582141}{77175} \zeta_4 + \frac{1328}{147} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left(-\frac{23820479}{264600} - \frac{11627738}{33075} \zeta_3 + \frac{28624}{63} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(\frac{1942638296203817}{540326562300000} - \frac{113578219}{4862025} \zeta_3 + \frac{28724}{2205} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{3261418656515051}{216130624920000} + \frac{122909317}{1620675} \zeta_3 - \frac{600626}{15435} \zeta_4 \right) \\
& + n_f^2 C_A^2 \left(-\frac{55264268415947}{6175160712000} - \frac{38177677}{720300} \zeta_3 + \frac{133186}{5145} \zeta_4 + \frac{5360}{1323} \zeta_5 \right) \\
& + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(-\frac{665983}{4725} - \frac{192736}{6615} \zeta_3 + \frac{85760}{441} \zeta_5 \right) \\
& + n_f^3 C_F \left(-\frac{1262351231147}{2572983630000} + \frac{15268}{9261} \zeta_3 \right) + n_f^3 C_A \left(\frac{34431246007}{55135363500} - \frac{8866}{3969} \zeta_3 \right). \quad (10)
\end{aligned}$$

and

$$\gamma_{\text{gq}}^{(3)}(N=2) = -\gamma_{\text{qq}}^{(3)}(N=2), \quad (11)$$

$$\begin{aligned}
\gamma_{\text{gq}}^{(3)}(N=4) = & C_F^4 \left(-\frac{1438431824489}{17496000000} - \frac{21061493}{101250} \zeta_3 + \frac{259}{5} \zeta_4 + \frac{14408}{45} \zeta_5 \right) \\
& + C_A C_F^3 \left(\frac{270563159561}{8748000000} + \frac{6105179}{101250} \zeta_3 + \frac{5917}{750} \zeta_4 - \frac{17488}{45} \zeta_5 \right) \\
& + C_A^2 C_F^2 \left(\frac{1259255579057}{4374000000} + \frac{16267093}{67500} \zeta_3 - \frac{25621}{250} \zeta_4 + \frac{1484}{45} \zeta_5 \right) \\
& + C_A^3 C_F \left(-\frac{632341192829}{2187000000} - \frac{1120409}{8100} \zeta_3 + \frac{16048}{375} \zeta_4 + \frac{8782}{135} \zeta_5 \right) \\
& + \frac{d_R^{abcd} d_A^{abcd}}{n_c} \left(-\frac{12196}{135} - \frac{81008}{225} \zeta_3 + \frac{15976}{45} \zeta_5 \right) + n_f C_F^3 \left(-\frac{316818132031}{3280500000} \right. \\
& \left. + \frac{411629}{16875} \zeta_3 - \frac{4582}{225} \zeta_4 + \frac{352}{3} \zeta_5 \right) + n_f C_A C_F^2 \left(\frac{569679966383}{6561000000} - \frac{13919446}{50625} \zeta_3 \right. \\
& \left. + \frac{12501}{125} \zeta_4 - \frac{176}{9} \zeta_5 \right) + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(-\frac{112424}{675} - \frac{2336}{75} \zeta_3 + \frac{10624}{45} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(\frac{2203719743}{52488000} + \frac{2857549}{11250} \zeta_3 - \frac{89599}{1125} \zeta_4 - \frac{11872}{135} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(\frac{9798304643}{3280500000} + \frac{17096}{675} \zeta_3 - \frac{704}{45} \zeta_4 \right) + n_f^2 C_A C_F \left(-\frac{1608863899}{328050000} \right. \\
& \left. - \frac{39416}{2025} \zeta_3 + \frac{704}{45} \zeta_4 \right) + n_f^3 C_F \left(\frac{3990397}{2733750} - \frac{704}{405} \zeta_3 \right), \quad (12)
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{gq}}^{(3)}(N=6) = & C_F^4 \left(-\frac{27548846012571077}{225136067625000} - \frac{28516720088}{121550625} \zeta_3 + \frac{6416}{105} \zeta_4 + \frac{260192}{735} \zeta_5 \right) \\
& + C_A C_F^3 \left(\frac{15370144370986843}{90054427050000} + \frac{23472335174}{121550625} \zeta_3 - \frac{1023364}{25725} \zeta_4 - \frac{370016}{735} \zeta_5 \right) \\
& + C_A^2 C_F^2 \left(\frac{58564721355491371}{720435416400000} + \frac{10781187328}{121550625} \zeta_3 - \frac{1215814}{25725} \zeta_4 + \frac{373832}{2205} \zeta_5 \right) \\
& + C_A^3 C_F \left(-\frac{133292466369681947}{864522499680000} - \frac{226736591}{2701125} \zeta_3 + \frac{667258}{25725} \zeta_4 + \frac{67288}{6615} \zeta_5 \right) \\
& + \frac{d_R^{abcd} d_A^{abcd}}{n_c} \left(-\frac{23820479}{330750} - \frac{46510952}{165375} \zeta_3 + \frac{114496}{315} \zeta_5 \right) \\
& + n_f C_F^3 \left(-\frac{75665018489451691}{1350816405750000} + \frac{187225352}{24310125} \zeta_3 - \frac{36352}{2205} \zeta_4 + \frac{1408}{21} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{331099053590779}{6003628470000} - \frac{3771301108}{24310125} \zeta_3 + \frac{4877248}{77175} \zeta_4 - \frac{704}{63} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(\frac{40511222207957}{3430644840000} + \frac{3610221368}{24310125} \zeta_3 - \frac{3604928}{77175} \zeta_4 - \frac{65344}{1323} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(-\frac{2663932}{23625} - \frac{770944}{33075} \zeta_3 + \frac{68608}{441} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(\frac{27562736653631}{9648688612500} + \frac{266912}{19845} \zeta_3 - \frac{2816}{315} \zeta_4 \right) - n_f^2 C_A C_F \left(\frac{301286343367}{110270727000} \right. \\
& \left. + \frac{944432}{99225} \zeta_3 - \frac{2816}{315} \zeta_4 \right) + n_f^3 C_F \left(\frac{3574461862}{3281866875} - \frac{2816}{2835} \zeta_3 \right). \tag{13}
\end{aligned}$$

Finally the lowest three even moments (3) of the four-loop gluon-gluon splitting function read

$$\gamma_{\text{gg}}^{(3)}(N=2) = -\gamma_{\text{qg}}^{(3)}(N=2), \tag{14}$$

$$\begin{aligned}
\gamma_{\text{gg}}^{(3)}(N=4) = & C_A^4 \left(\frac{1502628149}{3375000} + \frac{1146397}{11250} \zeta_3 - \frac{504}{5} \zeta_5 \right) + \frac{d_A^{abcd} d_A^{abcd}}{n_a} \left(\frac{21623}{150} \right. \\
& \left. + \frac{15596}{15} \zeta_3 - \frac{6048}{5} \zeta_5 \right) + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left(\frac{160091}{675} + \frac{80072}{225} \zeta_3 - \frac{48016}{45} \zeta_5 \right) \\
& + n_f C_A^3 \left(-\frac{20580892841}{72900000} - \frac{12550223}{22500} \zeta_3 + \frac{8613}{25} \zeta_4 + \frac{4316}{27} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{4212122951}{41006250} + \frac{1170784}{5625} \zeta_3 - \frac{418198}{1125} \zeta_4 + \frac{17636}{45} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{1913110089023}{26244000000} + \frac{39313783}{101250} \zeta_3 + \frac{26741}{750} \zeta_4 - \frac{3082}{5} \zeta_5 \right) \\
& + n_f C_F^3 \left(\frac{34764568601}{2099520000} - \frac{958343}{40500} \zeta_3 - \frac{18997}{2250} \zeta_4 + \frac{908}{45} \zeta_5 \right) \\
& + n_f^2 C_A^2 \left(-\frac{3250393649}{218700000} + \frac{2969291}{20250} \zeta_3 - \frac{1566}{25} \zeta_4 - \frac{1276}{135} \zeta_5 \right)
\end{aligned}$$

$$\begin{aligned}
& + n_f^2 C_A C_F \left(\frac{136020246173}{3280500000} - \frac{1672751}{10125} \zeta_3 + \frac{15172}{225} \zeta_4 \right) - n_f^2 C_F^2 \left(\frac{275622924731}{26244000000} \right. \\
& \quad \left. - \frac{253369}{10125} \zeta_3 + \frac{1078}{225} \zeta_4 \right) + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(\frac{75788}{675} + \frac{3008}{15} \zeta_3 - \frac{20416}{45} \zeta_5 \right) \\
& + n_f^3 C_A \left(-\frac{20440457}{21870000} + \frac{1888}{405} \zeta_3 \right) + n_f^3 C_F \left(\frac{1780699}{24300000} - \frac{484}{675} \zeta_3 \right), \tag{15}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{gg}}^{(3)}(N=6) = & C_A^4 \left(\frac{14796034088334539}{23053933324800} + \frac{198201877}{777924} \zeta_3 - \frac{118210}{441} \zeta_5 \right) \\
& + \frac{d_A^{abcd} d_A^{abcd}}{n_a} \left(\frac{1255552}{2205} + \frac{2997592}{1323} \zeta_3 - \frac{472840}{147} \zeta_5 \right) \\
& + n_f C_A^3 \left(-\frac{352499691830939}{914838624000} - \frac{4467756563}{6482700} \zeta_3 + \frac{103356}{245} \zeta_4 + \frac{238544}{1323} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{174297079261544753}{864522499680000} + \frac{12199024283}{48620250} \zeta_3 - \frac{6710594}{15435} \zeta_4 + \frac{221576}{441} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{51836938615212157}{360217708200000} + \frac{459844342}{972405} \zeta_3 + \frac{1338986}{77175} \zeta_4 - \frac{334352}{441} \zeta_5 \right) \\
& + n_f C_F^3 \left(\frac{10457671535671561}{5403265623000000} - \frac{100551124}{8103375} \zeta_3 - \frac{343156}{77175} \zeta_4 + \frac{992}{147} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left(\frac{9661697}{22050} + \frac{22351528}{33075} \zeta_3 - \frac{726848}{441} \zeta_5 \right) \\
& + n_f^2 C_A^2 \left(-\frac{2273514775943}{294055272000} + \frac{126516356}{694575} \zeta_3 - \frac{18792}{245} \zeta_4 - \frac{22000}{1323} \zeta_5 \right) \\
& + n_f^2 C_A C_F \left(\frac{122395144706959}{2205414540000} - \frac{133661648}{694575} \zeta_3 + \frac{173704}{2205} \zeta_4 \right) \\
& + n_f^2 C_F^2 \left(-\frac{61017705026527}{5403265623000} + \frac{2171164}{99225} \zeta_3 - \frac{4576}{2205} \zeta_4 \right) \\
& + n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(\frac{788419}{3675} + \frac{180272}{441} \zeta_3 - \frac{352000}{441} \zeta_5 \right) \\
& + n_f^3 C_A \left(-\frac{5226936307}{5250987000} + \frac{3224}{567} \zeta_3 \right) + n_f^3 C_F \left(-\frac{9085701773}{30630757500} - \frac{1936}{6615} \zeta_3 \right). \tag{16}
\end{aligned}$$

For brevity, we here write down the results at $N = 8$ only numerically for the case of QCD:

$$\gamma_{\text{ps}}^{(3)}(N=8) = -24.014550 n_f + 3.2351935 n_f^2 - 0.0078892 n_f^3, \tag{17}$$

$$\gamma_{\text{qg}}^{(3)}(N=8) = 294.58768 n_f - 135.37676 n_f^2 - 3.6097756 n_f^3, \tag{18}$$

$$\gamma_{\text{gq}}^{(3)}(N=8) = -2803.6441 + 436.393057 n_f + 1.8149462 n_f^2 + 0.0735886 n_f^3, \tag{19}$$

$$\gamma_{\text{gg}}^{(3)}(N=8) = 62279.744 - 17150.6967 n_f + 785.88061 n_f^2 + 1.8933103 n_f^3. \tag{20}$$

In eqs. (5) – (16) C_F and C_A are the standard colour factors with $C_F = 4/3$ and $C_A = n_c = 3$ in QCD. The terms with the quartic group invariants $d_A^{abcd} d_A^{abcd}$, $d_R^{abcd} d_A^{abcd}$ and $d_R^{abcd} d_R^{abcd}$ agree with the results of ref. [11] where these particular contributions were obtained to much higher values of N using OME calculations. All coefficients of the Riemann- ζ value $\zeta_4 = \pi^4/90$ agree with the all- N predictions in eqs. (9) – (12) of ref. [12] based on the ‘no- π^2 ’ conjecture of ref. [13]. The n_f^3 contributions to all four anomalous dimension are known for all N [7], see also refs. [14].

The above results lead to the numerical expansions

$$\begin{aligned}\gamma_{qq}(2,3) &= 0.282942 \alpha_s (1 + 0.736828 \alpha_s + 0.517255 \alpha_s^2 + 0.756972 \alpha_s^3 + \dots), \\ \gamma_{qq}(2,4) &= 0.282942 \alpha_s (1 + 0.621883 \alpha_s + 0.146133 \alpha_s^2 + 0.362201 \alpha_s^3 + \dots),\end{aligned}\quad (21)$$

$$\begin{aligned}\gamma_{qq}(4,3) &= 0.555274 \alpha_s (1 + 0.756202 \alpha_s + 0.672283 \alpha_s^2 + 0.701628 \alpha_s^3 + \dots), \\ \gamma_{qq}(4,4) &= 0.555274 \alpha_s (1 + 0.680253 \alpha_s + 0.427783 \alpha_s^2 + 0.345861 \alpha_s^3 + \dots),\end{aligned}\quad (22)$$

$$\begin{aligned}\gamma_{qq}(6,3) &= 0.716450 \alpha_s (1 + 0.725387 \alpha_s + 0.685289 \alpha_s^2 + 0.663440 \alpha_s^3 + \dots), \\ \gamma_{qq}(6,4) &= 0.716450 \alpha_s (1 + 0.648931 \alpha_s + 0.426442 \alpha_s^2 + 0.324781 \alpha_s^3 + \dots),\end{aligned}\quad (23)$$

$$\begin{aligned}\gamma_{qq}(8,3) &= 0.832237 \alpha_s (1 + 0.710075 \alpha_s + 0.650750 \alpha_s^2 + 0.643336 \alpha_s^3 + \dots), \\ \gamma_{qq}(8,4) &= 0.832237 \alpha_s (1 + 0.632824 \alpha_s + 0.423498 \alpha_s^2 + 0.312139 \alpha_s^3 + \dots)\end{aligned}\quad (24)$$

and

$$\begin{aligned}\gamma_{qg}(2,3) &= -0.159155 \alpha_s (1 + 0.900404 \alpha_s + 0.012215 \alpha_s^2 - 0.055970 \alpha_s^3 + \dots), \\ \gamma_{qg}(2,4) &= -0.212207 \alpha_s (1 + 0.900404 \alpha_s - 0.102840 \alpha_s^2 - 0.236731 \alpha_s^3 + \dots),\end{aligned}\quad (25)$$

$$\begin{aligned}\gamma_{qg}(4,3) &= -0.087535 \alpha_s (1 - 0.280121 \alpha_s - 0.893969 \alpha_s^2 - 0.022754 \alpha_s^3 + \dots), \\ \gamma_{qg}(4,4) &= -0.116714 \alpha_s (1 - 0.280121 \alpha_s - 0.998634 \alpha_s^2 + 0.129659 \alpha_s^3 + \dots),\end{aligned}\quad (26)$$

$$\begin{aligned}\gamma_{qg}(6,3) &= -0.062525 \alpha_s (1 - 0.838938 \alpha_s - 1.064575 \alpha_s^2 + 0.145572 \alpha_s^3 + \dots), \\ \gamma_{qg}(6,4) &= -0.083367 \alpha_s (1 - 0.838938 \alpha_s - 1.150113 \alpha_s^2 + 0.441744 \alpha_s^3 + \dots),\end{aligned}\quad (27)$$

$$\begin{aligned}\gamma_{qg}(8,3) &= -0.049728 \alpha_s (1 - 1.255845 \alpha_s - 1.091729 \alpha_s^2 + 0.353099 \alpha_s^3 + \dots), \\ \gamma_{qg}(8,4) &= -0.065430 \alpha_s (1 - 1.255845 \alpha_s - 1.160288 \alpha_s^2 + 0.746929 \alpha_s^3 + \dots)\end{aligned}\quad (28)$$

for the upper row of the anomalous-dimension matrix, where the arguments of γ_{ik} are N and n_f ; the values for $n_f = 5$ have been suppressed for brevity. The independent lower-row expansions – the values at $N = 2$ are fixed by the momentum sum-rule relations (11) and (14) – are given by

$$\begin{aligned}\gamma_{gq}(4,3) &= -0.077809 \alpha_s (1 + 1.165483 \alpha_s + 1.163066 \alpha_s^2 + 1.474368 \alpha_s^3 + \dots), \\ \gamma_{gq}(4,4) &= -0.077809 \alpha_s (1 + 1.115164 \alpha_s + 0.823447 \alpha_s^2 + 0.883269 \alpha_s^3 + \dots),\end{aligned}\quad (29)$$

$$\begin{aligned}\gamma_{gq}(6,3) &= -0.044462 \alpha_s (1 + 1.314556 \alpha_s + 1.360970 \alpha_s^2 + 1.726679 \alpha_s^3 + \dots), \\ \gamma_{gq}(6,4) &= -0.044462 \alpha_s (1 + 1.301901 \alpha_s + 1.051619 \alpha_s^2 + 1.126955 \alpha_s^3 + \dots),\end{aligned}\quad (30)$$

$$\begin{aligned}\gamma_{gq}(8,3) &= -0.031157 \alpha_s (1 + 1.416509 \alpha_s + 1.468523 \alpha_s^2 + 1.899893 \alpha_s^3 + \dots), \\ \gamma_{gq}(8,4) &= -0.031157 \alpha_s (1 + 1.430863 \alpha_s + 1.183046 \alpha_s^2 + 1.318370 \alpha_s^3 + \dots)\end{aligned}\quad (31)$$

and

$$\begin{aligned}\gamma_{\text{gg}}(4,3) &= 1.161831 \alpha_s (1 + 0.475446 \alpha_s + 0.333272 \alpha_s^2 + 0.478025 \alpha_s^3 + \dots) , \\ \gamma_{\text{gg}}(4,4) &= 1.214882 \alpha_s (1 + 0.383536 \alpha_s + 0.121966 \alpha_s^2 + 0.240469 \alpha_s^3 + \dots) ,\end{aligned}\quad (32)$$

$$\begin{aligned}\gamma_{\text{gg}}(6,3) &= 1.574497 \alpha_s (1 + 0.489287 \alpha_s + 0.380902 \alpha_s^2 + 0.429696 \alpha_s^3 + \dots) , \\ \gamma_{\text{gg}}(6,4) &= 1.627549 \alpha_s (1 + 0.393705 \alpha_s + 0.169676 \alpha_s^2 + 0.190156 \alpha_s^3 + \dots) ,\end{aligned}\quad (33)$$

$$\begin{aligned}\gamma_{\text{gg}}(8,3) &= 1.851503 \alpha_s (1 + 0.497734 \alpha_s + 0.404644 \alpha_s^2 + 0.398779 \alpha_s^3 + \dots) , \\ \gamma_{\text{gg}}(8,4) &= 1.904554 \alpha_s (1 + 0.401746 \alpha_s + 0.194306 \alpha_s^2 + 0.157133 \alpha_s^3 + \dots) .\end{aligned}\quad (34)$$

Except for γ_{qg} and γ_{gg} at $N = 8$ these numerical expansions have been presented before in ref. [15].

The results for the qq and gg cases at asymptotically (and unphysically) large values of N read

$$\begin{aligned}\gamma_{\text{qq}}(N,3) &= a_s \gamma_{\text{qq}}^{(0)}(N,3) (1 + 0.726574 \alpha_s + 0.734054 \alpha_s^2 + 0.664730 \alpha_s^3) , \\ \gamma_{\text{qq}}(N,4) &= a_s \gamma_{\text{qq}}^{(0)}(N,4) (1 + 0.638154 \alpha_s + 0.509978 \alpha_s^2 + 0.316848 \alpha_s^3)\end{aligned}\quad (35)$$

and

$$\begin{aligned}\gamma_{\text{gg}}(N,3) &= a_s \gamma_{\text{gg}}^{(0)}(N,3) (1 + 0.726574 \alpha_s + 0.734054 \alpha_s^2 + 0.415609 \alpha_s^3) , \\ \gamma_{\text{gg}}(N,4) &= a_s \gamma_{\text{gg}}^{(0)}(N,4) (1 + 0.638154 \alpha_s + 0.509978 \alpha_s^2 + 0.064476 \alpha_s^3)\end{aligned}\quad (36)$$

due to their relation to the cusp anomalous dimensions A_k . The quark and gluon results are identical up to the ‘Casimir scaling’ of the prefactors, $\gamma_{\text{qq}}^{(0)}(N, n_f) = 4C_F$ and $\gamma_{\text{gg}}^{(0)}(N, n_f) = 4C_A$, to three loops and are related by a generalized (not numerical, except in the large- n_c limit, due to the presence of the quartic group invariants) Casimir scaling [11, 16] at four loops.

The relative size of the $N^2\text{LO}$ and $N^3\text{LO}$ contributions in eqs. (21) – (36) is illustrated in fig. 1 for $n_f = 4$ at $\alpha_s = 0.2$: The $N^3\text{LO}$ corrections amount to less than 1%, and less than 0.5% of the NLO results except for P_{gq} , the quantity with the lowest leading-order values, at $N \geq 4$. Unlike in the quark case, see also ref. [17] where also a first estimate of the five-loop contribution to A_q has been obtained, the $N^2\text{LO}$ and $N^3\text{LO}$ large- N limits in the gluon case do not, in general, roughly agree with values in the range $4 \leq N \leq 8$ normalized as in eqs. (21) – (34).

The resulting low- N expansion for the evolution (2) of the singlet quark and gluon PDFs is illustrated in fig. 2 for the schematic but sufficiently realistic order-independent model input [4]

$$\begin{aligned}xq_s(x, \mu_0^2) &= 0.6x^{-0.3}(1-x)^{3.5} (1 + 5.0x^{0.8}) , \\ xg(x, \mu_0^2) &= 1.6x^{-0.3}(1-x)^{4.5} (1 - 0.6x^{0.3})\end{aligned}\quad (37)$$

with $\alpha_s(\mu_0^2) = 0.2$ and $n_f = 4$. The $N^3\text{LO}$ corrections are very small at the standard choice $\mu_r = \mu_f \equiv \mu_0$ of the renormalization scale. They lead to a reduction of the scale dependence to about 1% (full width) at $N \geq 4$ for the conventional range $\frac{1}{4}\mu_f^2 \leq \mu_r^2 \leq 4\mu_f^2$.

To summarize, we have employed the theoretical framework of refs. [1–4] together with an optimized in-house version of the FORM [18] program FORCER for 4-loop propagator integrals [19] to compute the moments $N = 2, 4, 6$, and 8 of all $N^3\text{LO}$ flavour-singlet splitting functions. The numerical effect of these contributions is small, but more work is needed to arrive at sufficient ‘data’ for a $N^3\text{LO}$ analogue of the earlier approximate $N^2\text{LO}$ splitting functions of ref. [20].

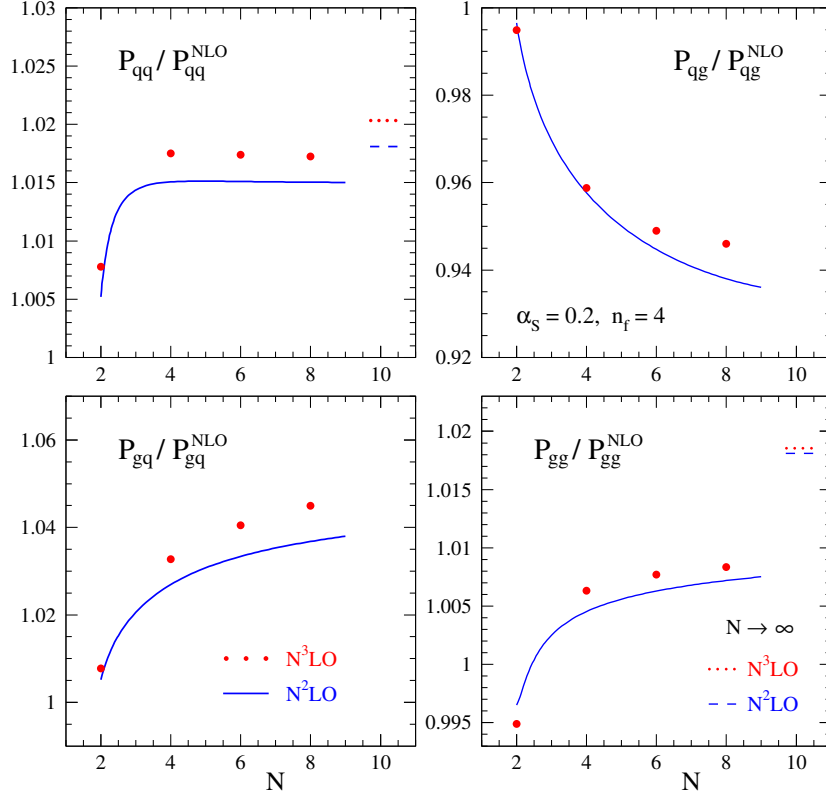


Figure 1: Moments of the splitting functions (2) at NNLO (lines) and $N^3\text{LO}$ (even- N points) at $\alpha_s = 0.2$ and $n_f = 4$, normalized to the NLO results. Also shown are the qq and gg large- N limits.

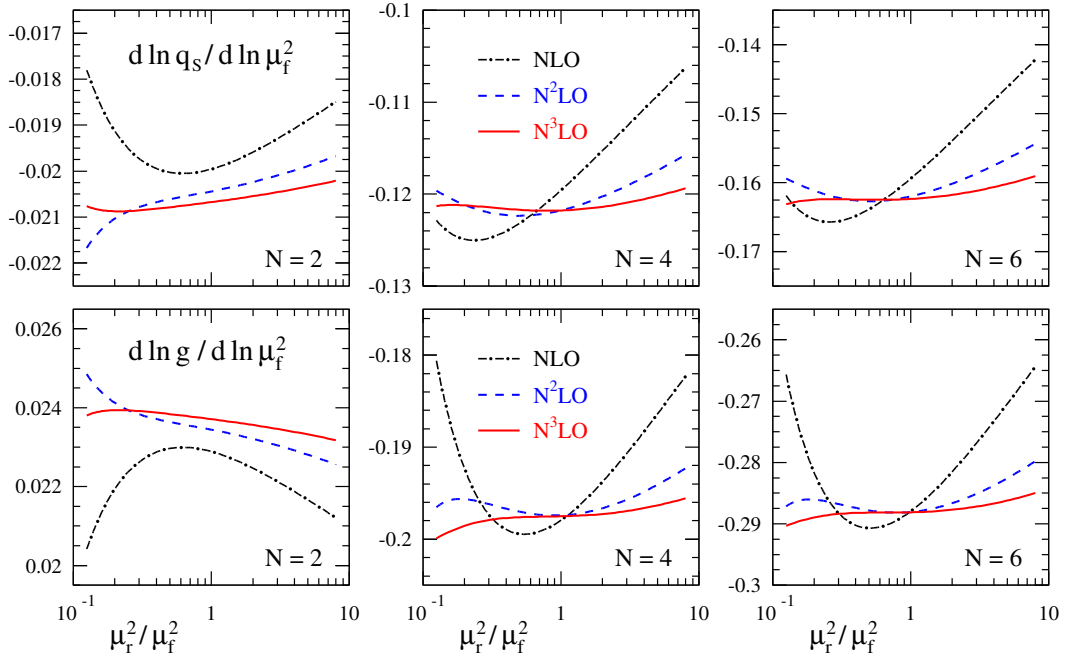


Figure 2: The dependence of the logarithmic factorization-scale derivatives of the singlet PDFs on the renormalization scale μ_r at $N = 2$ (where the very small scaling violations of q_s and g are related by the momentum sum rule), $N = 4$ and $N = 6$ for the initial distributions (37).

Acknowledgements

This work has been supported by the *European Research Council* (ERC) via the grants 320651 (*HEPGAME*) and 694712 (*PertQCD*), by the *European Cooperation in Science and Technology* (*COST*) via *COST Action CA16201 PARTICLEFACE*, by the *Deutsche Forschungsgemeinschaft* (DFG) through the Research Unit FOR 2926, *Next Generation pQCD for Hadron Structure: Preparing for the EIC*, project number 40824754 and DFG grant MO 1801/4-1, by the *Swiss National Science Foundation* (SNSF) grant 179016, by the *JSPS KAKENHI* grants 19K03831 and 21K03583, and by the *UK Science & Technology Facilities Council* (STFC) grants ST/L000431/1 and ST/T000988/1. Some of our computations were carried out on the Dutch national e-infrastructure with the support of the SURF Cooperative and the PDP Group at Nikhef, and on the `ulggcd` computer cluster in Liverpool which was funded by the STFC grant ST/H008837/1.

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