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Low moments of the four-loop splitting functions in QCD

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Abstract

We have computed the four lowest even-N moments of all four splitting functions for the evolution of flavour-singlet parton densities of hadrons at the fourth order in the strong coupling constant α_s. The perturbative expansion of these moments, and hence of the splitting functions for momentum fractions $x \gtrsim 0.1$, is found to be well behaved with relative α_s-coefficients of order one and sub-percent effects on the scale derivatives of the quark and gluon distributions at $\alpha_s \lesssim 0.2$. More intricate computations, including other approaches such as the operator-product expansion, are required to cover the full x-range relevant to LHC analyses. Our results are presented analytically for a general gauge group for detailed checks and validations of such future calculations.

Fully consistent analyses of hard processes with initial-state hadrons at the $(\text{next-to})^n$-leading order (N^{n}LO) of renormalization-group improved perturbative QCD require parton distributions functions (PDFs) evolved with the $(n+1)$-loop splitting functions. Over the past years, N^{2}LO (= NNLO) has become the standard approximation for many processes. Following pioneering computations of their lowest integer-N Mellin moments in refs. [1, 2], the corresponding 3-loop splitting functions were obtained in refs. [3, 4].

For certain benchmark cases, in particular Higgs-boson production at the LHC [5], N^{2}LO calculations are not sufficiently accurate, hence the 4-loop splitting functions need to be calculated. These have been determined for the flavour non-singlet quark-quark case in ref. [6] – analytically in the limit of a large number of colours n_c, and numerically for the remaining contributions – and for the (next-to-) leading contributions for a large number of flavours n_f in ref. [7].

Here we present, as a first significant step towards at least approximate expressions for the 4-loop singlet splitting functions for use in phenomenological analyses, their lowest four even moments $N = 2\ldots, 8$ in the standard $\overline{\text{MS}}$ scheme, thus extending the computations of ref. [2] by one order in the strong coupling α_s. Following the approach of refs. [2, 4] our calculations are performed via physical quantities in deep-inelastic scattering, i.e., instead of working with 4-loop off-shell flavour-singlet operator matrix elements (OMEs) which, at this point, is still theoretically challenging. Our present results, obtained analytically for a general gauge group, should also be useful for checking and validating future OME computations of these quantities.

The evolution equations for the flavour-singlet quark and gluon PDFs of hadrons,

$$q_{\rm s}(x,\mu_f^2) \;=\; \sum_{i=1}^{n_f} \left[q_i(x,\mu_f^2) + \bar{q}_i(x,\mu_f^2) \right] \quad \text{and} \quad g(x,\mu_f^2)\;, \tag{1}$$

are

$$\frac{d}{d\ln\mu_f^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} = \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} . \tag{2}$$

Here $\otimes$ represents the Mellin convolution in the momentum variable, and μ_f is the factorization scale. For the determination of the splitting functions $P_{\rm ik}$, the renormalization scale can be identified with μ_f without loss of information. The even-N moments of splitting functions in eq. (2) are identical to the anomalous dimensions of twist-2 spin-N operators up to a conventional sign,

$$\gamma_{\rm ik}(N,\alpha_{\rm s}) \;=\; -\int_0^1 dx\, x^{N-1}\, P_{\rm ik}(x,\alpha_{\rm s})\;. \tag{3}$$

Their perturbative expansions can be written as

$$\gamma_{\rm ik}\,(N,\alpha_{\rm s}) \;=\; \sum_{n=0} a_{\rm s}^{n+1}\, \gamma_{\rm ik}^{(n)}(x) \quad \text{with} \quad a_{\rm s} \equiv \frac{\alpha_{\rm s}(\mu_f^2)}{4\pi}\;. \tag{4}$$

The quark-quark entry in eq. (3) can be expressed as $\gamma_{\rm qq} = \gamma_{\rm ns}^{+} + \gamma_{\rm ps}$ in terms of the non-singlet anomalous dimension $\gamma_{\rm ns}^{+}$ for quark-antiquark sums addressed at four loops in ref. [6] and a pure-singlet contribution $\gamma_{\rm ps}$ which is suppressed at $N \gg 1$. At asymptotically large N the diagonal $\overline{\text{MS}}$ entries $\gamma_{\rm kk}(N)$ in eq. (2) are governed by the (lightlike) cusp anomalous dimensions $A_{\rm k}$ [8], viz $\gamma_{\rm kk}(N) = A_{\rm k} \ln N + O(1)$, which are now fully known at four loops [9, 10].

The 4-loop contributions to the pure-singlet anomalous dimensions in eq. (4) at $N = 2, 4, 6$ are

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N{=}2) \;=\;& n_f C_F^3 \left(\frac{227938}{2187} + \frac{1952}{81}\zeta_3 + \frac{256}{9}\zeta_4 - \frac{640}{3}\zeta_5\right) \\
&+ n_f C_A C_F^2 \left(-\frac{162658}{6561} + \frac{8048}{27}\zeta_3 - \frac{1664}{9}\zeta_4 + \frac{320}{9}\zeta_5\right) \\
&+ n_f C_A^2 C_F \left(-\frac{410299}{6561} - \frac{26896}{81}\zeta_3 + \frac{1408}{9}\zeta_4 + \frac{4480}{27}\zeta_5\right) \\
&+ n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{1024}{9} + \frac{256}{9}\zeta_3 - \frac{2560}{9}\zeta_5\right) - n_f^2 C_F^2 \left(\frac{73772}{6561} + \frac{5248}{81}\zeta_3 - \frac{320}{9}\zeta_4\right) \\
&+ n_f^2 C_A C_F \left(\frac{160648}{6561} + 48\zeta_3 - \frac{320}{9}\zeta_4\right) + n_f^3 C_F \left(-\frac{1712}{729} + \frac{128}{27}\zeta_3\right) , \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N{=}4) \;=\;& n_f C_F^3 \left(\frac{1995890620891}{52488000000} - \frac{897403}{202500}\zeta_3 + \frac{18997}{2250}\zeta_4 - \frac{484}{15}\zeta_5\right) \\
&+ n_f C_A C_F^2 \left(\frac{209865827521}{26244000000} + \frac{6743539}{202500}\zeta_3 - \frac{29161}{750}\zeta_4 + \frac{242}{45}\zeta_5\right) \\
&+ n_f C_A^2 C_F \left(-\frac{55187654921}{3280500000} - \frac{3104267}{67500}\zeta_3 + \frac{34243}{1125}\zeta_4 + \frac{3164}{135}\zeta_5\right) \\
&+ n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{172231}{675} - \frac{5368}{25}\zeta_3 - \frac{3728}{45}\zeta_5\right) - n_f^2 C_F^2 \left(\frac{141522185707}{26244000000}\right. \\
&\quad \left. + \frac{1207}{135}\zeta_3 - \frac{242}{45}\zeta_4\right) + n_f^2 C_A C_F \left(\frac{9398360351}{1640250000} + \frac{57877}{10125}\zeta_3 - \frac{242}{45}\zeta_4\right) \\
&+ n_f^3 C_F \left(-\frac{46099151}{72900000} + \frac{484}{675}\zeta_3\right) , \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N{=}6) \;=\;& n_f C_F^3 \left(\frac{140565274663259489}{5403265623000000} - \frac{62727544}{24310125}\zeta_3 + \frac{343156}{77175}\zeta_4 - \frac{1936}{147}\zeta_5\right) \\
&+ n_f C_A C_F^2 \left(\frac{336481838777617}{360217708200000} + \frac{2111992}{324135}\zeta_3 - \frac{1389806}{77175}\zeta_4 + \frac{968}{441}\zeta_5\right) \\
&+ n_f C_A^2 C_F \left(-\frac{6194882229735067}{864522499680000} - \frac{2396237}{165375}\zeta_3 + \frac{41866}{3087}\zeta_4 + \frac{9544}{1323}\zeta_5\right) \\
&+ n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{64697569}{330750} - \frac{426976}{3675}\zeta_3 - \frac{39808}{441}\zeta_5\right) \\
&- n_f^2 C_F^2 \left(\frac{812984663253277}{270163281150000} + \frac{2594876}{694575}\zeta_3 - \frac{968}{441}\zeta_4\right) + n_f^2 C_A C_F \left(\frac{3092531515013}{964868861250}\right. \\
&\quad \left. + \frac{217432}{99225}\zeta_3 - \frac{968}{441}\zeta_4\right) + n_f^3 C_F \left(-\frac{19597073837}{61261515000} + \frac{1936}{6615}\zeta_3\right) . \qquad (7)
\end{aligned}
$$

The complete qq entries are obtained by adding the non-singlet contributions in app. B of ref. [6].

The corresponding results for the off-diagonal splitting functions are given by

$$
\begin{aligned}
\gamma^{(3)}_{\rm qg}(N\!=\!2) \;=\;& n_f C_F^3 \left(\frac{16489}{729}+\frac{736}{81}\zeta_3+\frac{256}{9}\zeta_4-\frac{320}{3}\zeta_5\right) \\
&+ n_f C_A^3 \left(-\frac{88769}{729}+\frac{31112}{81}\zeta_3-132\zeta_4-\frac{3560}{27}\zeta_5\right) - n_f C_A C_F^2 \left(\frac{1153727}{13122}-\frac{7108}{81}\zeta_3\right. \\
&\left.+\frac{1136}{9}\zeta_4-\frac{2000}{9}\zeta_5\right) + n_f C_A^2 C_F \left(\frac{763868}{6561}-\frac{12808}{27}\zeta_3+\frac{2068}{9}\zeta_4+\frac{40}{9}\zeta_5\right) \\
&+ n_f \frac{d_R^{abcd}d_A^{abcd}}{n_a}\left(\frac{368}{9}-\frac{992}{9}\zeta_3-\frac{2560}{9}\zeta_5\right) - n_f^2 C_F^2 \left(\frac{110714}{6561}+\frac{272}{9}\zeta_3-\frac{224}{9}\zeta_4\right) \\
&+ n_f^2 C_A C_F \left(\frac{249310}{6561}+\frac{5632}{81}\zeta_3-\frac{440}{9}\zeta_4\right) + n_f^2 C_A^2 \left(\frac{48625}{2187}-\frac{3572}{81}\zeta_3\right. \\
&\left.+24\zeta_4+\frac{160}{27}\zeta_5\right) + n_f^2 \frac{d_R^{abcd}d_R^{abcd}}{n_a}\left(-\frac{928}{9}-\frac{640}{9}\zeta_3+\frac{2560}{9}\zeta_5\right) \\
&+ n_f^3 C_F \left(-\frac{8744}{2187}+\frac{128}{27}\zeta_3\right) + n_f^3 C_A \left(\frac{3385}{2187}-\frac{176}{81}\zeta_3\right) , \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{(3)}_{\rm qg}(N\!=\!4) \;=\;& n_f C_F^3 \left(-\frac{8103828487201}{104976000000}+\frac{5100751}{81000}\zeta_3+\frac{154589}{4500}\zeta_4-\frac{3158}{45}\zeta_5\right) \\
&+ n_f C_A C_F^2 \left(\frac{5121012352507}{26244000000}-\frac{48971263}{405000}\zeta_3-\frac{143489}{750}\zeta_4+\frac{951}{5}\zeta_5\right) \\
&+ n_f C_A^2 C_F \left(-\frac{314624947013}{1312200000}-\frac{2024593}{9000}\zeta_3+\frac{1674889}{4500}\zeta_4+\frac{1237}{45}\zeta_5\right) \\
&+ n_f C_A^3 \left(\frac{143199094853}{1458000000}+\frac{11938031}{45000}\zeta_3-\frac{26904}{125}\zeta_4-\frac{17917}{135}\zeta_5\right) \\
&+ n_f \frac{d_R^{abcd}d_A^{abcd}}{n_a}\left(-\frac{12196}{135}-\frac{81008}{225}\zeta_3+\frac{15976}{45}\zeta_5\right) \\
&+ n_f^2 C_F^2 \left(\frac{37295583467}{26244000000}-\frac{1400864}{50625}\zeta_3+\frac{707}{45}\zeta_4\right) + n_f^2 C_A C_F \left(\frac{217239001681}{13122000000}\right. \\
&\left.+\frac{4497112}{50625}\zeta_3-\frac{103669}{2250}\zeta_4\right) + n_f^2 C_A^2 \left(-\frac{7131194093}{4374000000}-\frac{12599759}{202500}\zeta_3\right. \\
&\left.+\frac{7591}{250}\zeta_4+\frac{664}{135}\zeta_5\right) + n_f^2 \frac{d_R^{abcd}d_R^{abcd}}{n_a}\left(-\frac{112424}{675}-\frac{2336}{75}\zeta_3+\frac{10624}{45}\zeta_5\right) \\
&+ n_f^3 C_F \left(-\frac{312015851}{364500000}+\frac{6644}{3375}\zeta_3\right) + n_f^3 C_A \left(\frac{338346151}{437400000}-\frac{5192}{2025}\zeta_3\right) , \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{(3)}_{\rm qg}(N\!=\!6) \;=\;& n_f C_A^3 \left(\frac{49981299563948069}{345808999872000}+\frac{2383601783}{12965400}\zeta_3-\frac{689907}{3430}\zeta_4-\frac{159724}{1323}\zeta_5\right) \\
&+ n_f C_A C_F^2 \left(\frac{324177529264517279}{960580555200000}-\frac{1154450237}{9724050}\zeta_3-\frac{28952417}{154350}\zeta_4+\frac{9832}{441}\zeta_5\right)
\end{aligned}
$$

$$+ n_f C_A^2 C_F \left(-\frac{627686002393628869}{1729044999360000} - \frac{6170262713}{48620250}\zeta_3 + \frac{1096679}{3087}\zeta_4 + \frac{47774}{441}\zeta_5\right)$$
$$+ n_f C_F^3 \left(-\frac{2912197809548779709}{21613062492000000} + \frac{1026604067}{24310125}\zeta_3 + \frac{2582141}{77175}\zeta_4 + \frac{1328}{147}\zeta_5\right)$$
$$+ n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left(-\frac{23820479}{264600} - \frac{11627738}{33075}\zeta_3 + \frac{28624}{63}\zeta_5\right)$$
$$+ n_f^2 C_F^2 \left(\frac{1942638296203817}{540326562300000} - \frac{113578219}{4862025}\zeta_3 + \frac{28724}{2205}\zeta_4\right)$$
$$+ n_f^2 C_A C_F \left(\frac{3261418656515051}{216130624920000} + \frac{122909317}{1620675}\zeta_3 - \frac{600626}{15435}\zeta_4\right)$$
$$+ n_f^2 C_A^2 \left(-\frac{55264268415947}{6175160712000} - \frac{38177677}{720300}\zeta_3 + \frac{133186}{5145}\zeta_4 + \frac{5360}{1323}\zeta_5\right)$$
$$+ n_f^2 \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(-\frac{665983}{4725} - \frac{192736}{6615}\zeta_3 + \frac{85760}{441}\zeta_5\right)$$
$$+ n_f^3 C_F \left(-\frac{1262351231147}{2572983630000} + \frac{15268}{9261}\zeta_3\right) + n_f^3 C_A \left(\frac{34431246007}{55135363500} - \frac{8866}{3969}\zeta_3\right) . \qquad (10)$$

and

$$\gamma_{\rm gq}^{(3)}(N=2) \;=\; -\gamma_{\rm qq}^{(3)}(N=2)\,, \qquad (11)$$

$$\gamma_{\rm gq}^{(3)}(N=4) \;=\; C_F^4 \left(-\frac{1438431824489}{17496000000} - \frac{21061493}{101250}\zeta_3 + \frac{259}{5}\zeta_4 + \frac{14408}{45}\zeta_5\right)$$
$$+ C_A C_F^3 \left(\frac{270563159561}{8748000000} + \frac{6105179}{101250}\zeta_3 + \frac{5917}{750}\zeta_4 - \frac{17488}{45}\zeta_5\right)$$
$$+ C_A^2 C_F^2 \left(\frac{1259255579057}{4374000000} + \frac{16267093}{67500}\zeta_3 - \frac{25621}{250}\zeta_4 + \frac{1484}{45}\zeta_5\right)$$
$$+ C_A^3 C_F \left(-\frac{632341192829}{2187000000} - \frac{1120409}{8100}\zeta_3 + \frac{16048}{375}\zeta_4 + \frac{8782}{135}\zeta_5\right)$$
$$+ \frac{d_R^{abcd} d_A^{abcd}}{n_c} \left(-\frac{12196}{135} - \frac{81008}{225}\zeta_3 + \frac{15976}{45}\zeta_5\right) + n_f C_F^3 \left(-\frac{316818132031}{3280500000}\right.$$
$$\left. + \frac{411629}{16875}\zeta_3 - \frac{4582}{225}\zeta_4 + \frac{352}{3}\zeta_5\right) + n_f C_A C_F^2 \left(\frac{569679966383}{6561000000} - \frac{13919446}{50625}\zeta_3\right.$$
$$\left. + \frac{12501}{125}\zeta_4 - \frac{176}{9}\zeta_5\right) + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(-\frac{112424}{675} - \frac{2336}{75}\zeta_3 + \frac{10624}{45}\zeta_5\right)$$
$$+ n_f C_A^2 C_F \left(\frac{2203719743}{52488000} + \frac{2857549}{11250}\zeta_3 - \frac{89599}{1125}\zeta_4 - \frac{11872}{135}\zeta_5\right)$$
$$+ n_f^2 C_F^2 \left(\frac{9798304643}{3280500000} + \frac{17096}{675}\zeta_3 - \frac{704}{45}\zeta_4\right) + n_f^2 C_A C_F \left(-\frac{1608863899}{328050000}\right.$$
$$\left. - \frac{39416}{2025}\zeta_3 + \frac{704}{45}\zeta_4\right) + n_f^3 C_F \left(\frac{3990397}{2733750} - \frac{704}{405}\zeta_3\right) , \qquad (12)$$

$$
\begin{aligned}
\gamma^{(3)}_{\rm gq}(N\!=\!6) \;\; = \;\; & C_F^4 \left(-\frac{27548846012571077}{225136067625000} - \frac{28516720088}{121550625}\zeta_3 + \frac{6416}{105}\zeta_4 + \frac{260192}{735}\zeta_5\right) \\
& + C_A C_F^3 \left(\frac{15370144370986843}{90054427050000} + \frac{23472335174}{121550625}\zeta_3 - \frac{1023364}{25725}\zeta_4 - \frac{370016}{735}\zeta_5\right) \\
& + C_A^2 C_F^2 \left(\frac{58564721355491371}{720435416400000} + \frac{10781187328}{121550625}\zeta_3 - \frac{1215814}{25725}\zeta_4 + \frac{373832}{2205}\zeta_5\right) \\
& + C_A^3 C_F \left(-\frac{133292466369681947}{864522499680000} - \frac{226736591}{2701125}\zeta_3 + \frac{667258}{25725}\zeta_4 + \frac{67288}{6615}\zeta_5\right) \\
& + \frac{d_R^{abcd}d_A^{abcd}}{n_c} \left(-\frac{23820479}{330750} - \frac{46510952}{165375}\zeta_3 + \frac{114496}{315}\zeta_5\right) \\
& + n_f C_F^3 \left(-\frac{75665018489451691}{1350816405750000} + \frac{187225352}{24310125}\zeta_3 - \frac{36352}{2205}\zeta_4 + \frac{1408}{21}\zeta_5\right) \\
& + n_f C_A C_F^2 \left(\frac{331099053590779}{6003628470000} - \frac{3771301108}{24310125}\zeta_3 + \frac{4877248}{77175}\zeta_4 - \frac{704}{63}\zeta_5\right) \\
& + n_f C_A^2 C_F \left(\frac{40511222207957}{3430644840000} + \frac{3610221368}{24310125}\zeta_3 - \frac{3604928}{77175}\zeta_4 - \frac{65344}{1323}\zeta_5\right) \\
& + n_f \frac{d_R^{abcd}d_R^{abcd}}{n_c} \left(-\frac{2663932}{23625} - \frac{770944}{33075}\zeta_3 + \frac{68608}{441}\zeta_5\right) \\
& + n_f^2 C_F^2 \left(\frac{27562736653631}{9648688612500} + \frac{266912}{19845}\zeta_3 - \frac{2816}{315}\zeta_4\right) - n_f^2 C_A C_F \left(\frac{301286343367}{110270727000}\right. \\
& \left. + \frac{944432}{99225}\zeta_3 - \frac{2816}{315}\zeta_4\right) + n_f^3 C_F \left(\frac{3574461862}{3281866875} - \frac{2816}{2835}\zeta_3\right) \, . \qquad (13)
\end{aligned}
$$

Finally the lowest three even moments (3) of the four-loop gluon-gluon splitting function read

$$
\gamma^{(3)}_{\rm gg}(N\!=\!2) \;\; = \;\; -\gamma^{(3)}_{\rm qg}(N\!=\!2)\,, \qquad (14)
$$

$$
\begin{aligned}
\gamma^{(3)}_{\rm gg}(N\!=\!4) \;\; = \;\; & C_A^4 \left(\frac{1502628149}{3375000} + \frac{1146397}{11250}\zeta_3 - \frac{504}{5}\zeta_5\right) + \frac{d_A^{abcd}d_A^{abcd}}{n_a} \left(\frac{21623}{150}\right. \\
& \left. + \frac{15596}{15}\zeta_3 - \frac{6048}{5}\zeta_5\right) + n_f \frac{d_R^{abcd}d_A^{abcd}}{n_a} \left(\frac{160091}{675} + \frac{80072}{225}\zeta_3 - \frac{48016}{45}\zeta_5\right) \\
& + n_f C_A^3 \left(-\frac{20580892841}{72900000} - \frac{12550223}{22500}\zeta_3 + \frac{8613}{25}\zeta_4 + \frac{4316}{27}\zeta_5\right) \\
& + n_f C_A^2 C_F \left(-\frac{4212122951}{41006250} + \frac{1170784}{5625}\zeta_3 - \frac{418198}{1125}\zeta_4 + \frac{17636}{45}\zeta_5\right) \\
& + n_f C_A C_F^2 \left(\frac{1913110089023}{26244000000} + \frac{39313783}{101250}\zeta_3 + \frac{26741}{750}\zeta_4 - \frac{3082}{5}\zeta_5\right) \\
& + n_f C_F^3 \left(\frac{34764568601}{2099520000} - \frac{958343}{40500}\zeta_3 - \frac{18997}{2250}\zeta_4 + \frac{908}{45}\zeta_5\right) \\
& + n_f^2 C_A^2 \left(-\frac{3250393649}{218700000} + \frac{2969291}{20250}\zeta_3 - \frac{1566}{25}\zeta_4 - \frac{1276}{135}\zeta_5\right)
\end{aligned}
$$

$$
\begin{aligned}
& + n_f^2 C_A C_F \left(\frac{136020246173}{3280500000} - \frac{1672751}{10125}\zeta_3 + \frac{15172}{225}\zeta_4 \right) - n_f^2 C_F^2 \left(\frac{275622924731}{26244000000} \right. \\
& \left. - \frac{253369}{10125}\zeta_3 + \frac{1078}{225}\zeta_4 \right) + n_f^2 \, \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(\frac{75788}{675} + \frac{3008}{15}\zeta_3 - \frac{20416}{45}\zeta_5 \right) \\
& + n_f^3 C_A \left(-\frac{20440457}{21870000} + \frac{1888}{405}\zeta_3 \right) + n_f^3 C_F \left(\frac{1780699}{24300000} - \frac{484}{675}\zeta_3 \right) , \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{(3)}_{\rm gg}(N\!=\!6) \;=\; & C_A^4 \left(\frac{14796034088334539}{23053933324800} + \frac{198201877}{777924}\zeta_3 - \frac{118210}{441}\zeta_5 \right) \\
& + \frac{d_A^{abcd} d_A^{abcd}}{n_a} \left(\frac{1255552}{2205} + \frac{2997592}{1323}\zeta_3 - \frac{472840}{147}\zeta_5 \right) \\
& + n_f C_A^3 \left(-\frac{352499691830939}{914838624000} - \frac{4467756563}{6482700}\zeta_3 + \frac{103356}{245}\zeta_4 + \frac{238544}{1323}\zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{174297079261544753}{864522499680000} + \frac{12199024283}{48620250}\zeta_3 - \frac{6710594}{15435}\zeta_4 + \frac{221576}{441}\zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{51836938615212157}{360217708200000} + \frac{459844342}{972405}\zeta_3 + \frac{1338986}{77175}\zeta_4 - \frac{334352}{441}\zeta_5 \right) \\
& + n_f C_F^3 \left(\frac{10457671535671561}{5403265623000000} - \frac{100551124}{8103375}\zeta_3 - \frac{343156}{77175}\zeta_4 + \frac{992}{147}\zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_A^{abcd}}{n_a} \left(\frac{9661697}{22050} + \frac{22351528}{33075}\zeta_3 - \frac{726848}{441}\zeta_5 \right) \\
& + n_f^2 C_A^2 \left(-\frac{2273514775943}{294055272000} + \frac{126516356}{694575}\zeta_3 - \frac{18792}{245}\zeta_4 - \frac{22000}{1323}\zeta_5 \right) \\
& + n_f^2 C_A C_F \left(\frac{122395144706959}{2205414540000} - \frac{133661648}{694575}\zeta_3 + \frac{173704}{2205}\zeta_4 \right) \\
& + n_f^2 C_F^2 \left(-\frac{61017705026527}{5403265623000} + \frac{2171164}{99225}\zeta_3 - \frac{4576}{2205}\zeta_4 \right) \\
& + n_f^2 \, \frac{d_R^{abcd} d_R^{abcd}}{n_a} \left(\frac{788419}{3675} + \frac{180272}{441}\zeta_3 - \frac{352000}{441}\zeta_5 \right) \\
& + n_f^3 C_A \left(-\frac{5226936307}{5250987000} + \frac{3224}{567}\zeta_3 \right) + n_f^3 C_F \left(-\frac{9085701773}{30630757500} - \frac{1936}{6615}\zeta_3 \right) . \qquad (16)
\end{aligned}
$$

For brevity, we here write down the results at $N = 8$ only numerically for the case of QCD:

$$
\begin{aligned}
\gamma^{(3)}_{\rm ps}(N\!=\!8) &= -24.014550\, n_f + 3.2351935\, n_f^2 - 0.0078892\, n_f^3 , && (17) \\
\gamma^{(3)}_{\rm qg}(N\!=\!8) &= 294.58768\, n_f - 135.37676\, n_f^2 - 3.6097756\, n_f^3 , && (18) \\
\gamma^{(3)}_{\rm gq}(N\!=\!8) &= -2803.6441 + 436.393057\, n_f + 1.8149462\, n_f^2 + 0.0735886\, n_f^3 , && (19) \\
\gamma^{(3)}_{\rm gg}(N\!=\!8) &= 62279.744 - 17150.6967\, n_f + 785.88061\, n_f^2 + 1.8933103\, n_f^3 . && (20)
\end{aligned}
$$

In eqs. (5) – (16) C_F and C_A are the standard colour factors with $C_F = 4/3$ and $C_A = n_c = 3$ in QCD. The terms with the quartic group invariants $d_A^{abcd} d_A^{abcd}$, $d_R^{abcd} d_A^{abcd}$ and $d_R^{abcd} d_R^{abcd}$ agree with the results of ref. [11] where these particular contributions were obtained to much higher values of N using OME calculations. All coefficients of the Riemann-ζ value $\zeta_4 = \pi^4/90$ agree with the all-N predictions in eqs. (9) – (12) of ref. [12] based on the 'no-π^2' conjecture of ref. [13]. The n_f^3 contributions to all four anomalous dimension are known for all N [7], see also refs. [14].

The above results lead to the numerical expansions

$$\begin{aligned}
\gamma_{qq}(2,3) &= 0.282942\,\alpha_s \left(1 + 0.736828\,\alpha_s + 0.517255\,\alpha_s^2 + 0.756972\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qq}(2,4) &= 0.282942\,\alpha_s \left(1 + 0.621883\,\alpha_s + 0.146133\,\alpha_s^2 + 0.362201\,\alpha_s^3 + \ldots\right) , \qquad (21)
\end{aligned}$$

$$\begin{aligned}
\gamma_{qq}(4,3) &= 0.555274\,\alpha_s \left(1 + 0.756202\,\alpha_s + 0.672283\,\alpha_s^2 + 0.701628\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qq}(4,4) &= 0.555274\,\alpha_s \left(1 + 0.680253\,\alpha_s + 0.427783\,\alpha_s^2 + 0.345861\,\alpha_s^3 + \ldots\right) , \qquad (22)
\end{aligned}$$

$$\begin{aligned}
\gamma_{qq}(6,3) &= 0.716450\,\alpha_s \left(1 + 0.725387\,\alpha_s + 0.685289\,\alpha_s^2 + 0.663440\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qq}(6,4) &= 0.716450\,\alpha_s \left(1 + 0.648931\,\alpha_s + 0.426442\,\alpha_s^2 + 0.324781\,\alpha_s^3 + \ldots\right) , \qquad (23)
\end{aligned}$$

$$\begin{aligned}
\gamma_{qq}(8,3) &= 0.832237\,\alpha_s \left(1 + 0.710075\,\alpha_s + 0.650750\,\alpha_s^2 + 0.643336\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qq}(8,4) &= 0.832237\,\alpha_s \left(1 + 0.632824\,\alpha_s + 0.423498\,\alpha_s^2 + 0.312139\,\alpha_s^3 + \ldots\right) \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{qg}(2,3) &= -0.159155\,\alpha_s \left(1 + 0.900404\,\alpha_s + 0.012215\,\alpha_s^2 - 0.055970\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qg}(2,4) &= -0.212207\,\alpha_s \left(1 + 0.900404\,\alpha_s - 0.102840\,\alpha_s^2 - 0.236731\,\alpha_s^3 + \ldots\right) , \qquad (25)
\end{aligned}$$

$$\begin{aligned}
\gamma_{qg}(4,3) &= -0.087535\,\alpha_s \left(1 - 0.280121\,\alpha_s - 0.893969\,\alpha_s^2 - 0.022754\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qg}(4,4) &= -0.116714\,\alpha_s \left(1 - 0.280121\,\alpha_s - 0.998634\,\alpha_s^2 + 0.129659\,\alpha_s^3 + \ldots\right) , \qquad (26)
\end{aligned}$$

$$\begin{aligned}
\gamma_{qg}(6,3) &= -0.062525\,\alpha_s \left(1 - 0.838938\,\alpha_s - 1.064575\,\alpha_s^2 + 0.145572\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qg}(6,4) &= -0.083367\,\alpha_s \left(1 - 0.838938\,\alpha_s - 1.150113\,\alpha_s^2 + 0.441744\,\alpha_s^3 + \ldots\right) , \qquad (27)
\end{aligned}$$

$$\begin{aligned}
\gamma_{qg}(8,3) &= -0.049728\,\alpha_s \left(1 - 1.255845\,\alpha_s - 1.091729\,\alpha_s^2 + 0.353099\,\alpha_s^3 + \ldots\right) , \\
\gamma_{qg}(8,4) &= -0.065430\,\alpha_s \left(1 - 1.255845\,\alpha_s - 1.160288\,\alpha_s^2 + 0.746929\,\alpha_s^3 + \ldots\right) \qquad (28)
\end{aligned}$$

for the upper row of the anomalous-dimension matrix, where the arguments of γ_{ik} are N and n_f; the values for $n_f = 5$ have been suppressed for brevity. The independent lower-row expansions – the values at $N = 2$ are fixed by the momentum sum-rule relations (11) and (14) – are given by

$$\begin{aligned}
\gamma_{gq}(4,3) &= -0.077809\,\alpha_s \left(1 + 1.165483\,\alpha_s + 1.163066\,\alpha_s^2 + 1.474368\,\alpha_s^3 + \ldots\right) , \\
\gamma_{gq}(4,4) &= -0.077809\,\alpha_s \left(1 + 1.115164\,\alpha_s + 0.823447\,\alpha_s^2 + 0.883269\,\alpha_s^3 + \ldots\right) , \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\gamma_{gq}(6,3) &= -0.044462\,\alpha_s \left(1 + 1.314556\,\alpha_s + 1.360970\,\alpha_s^2 + 1.726679\,\alpha_s^3 + \ldots\right) , \\
\gamma_{gq}(6,4) &= -0.044462\,\alpha_s \left(1 + 1.301901\,\alpha_s + 1.051619\,\alpha_s^2 + 1.126955\,\alpha_s^3 + \ldots\right) , \qquad (30)
\end{aligned}$$

$$\begin{aligned}
\gamma_{gq}(8,3) &= -0.031157\,\alpha_s \left(1 + 1.416509\,\alpha_s + 1.468523\,\alpha_s^2 + 1.899893\,\alpha_s^3 + \ldots\right) , \\
\gamma_{gq}(8,4) &= -0.031157\,\alpha_s \left(1 + 1.430863\,\alpha_s + 1.183046\,\alpha_s^2 + 1.318370\,\alpha_s^3 + \ldots\right) \qquad (31)
\end{aligned}$$

and

$$
\begin{aligned}
\gamma_{gg}(4,3) &= 1.161831\,\alpha_s\left(1+0.475446\,\alpha_s+0.333272\,\alpha_s^2+0.478025\,\alpha_s^3+\ldots\right),\\
\gamma_{gg}(4,4) &= 1.214882\,\alpha_s\left(1+0.383536\,\alpha_s+0.121966\,\alpha_s^2+0.240469\,\alpha_s^3+\ldots\right),
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\gamma_{gg}(6,3) &= 1.574497\,\alpha_s\left(1+0.489287\,\alpha_s+0.380902\,\alpha_s^2+0.429696\,\alpha_s^3+\ldots\right),\\
\gamma_{gg}(6,4) &= 1.627549\,\alpha_s\left(1+0.393705\,\alpha_s+0.169676\,\alpha_s^2+0.190156\,\alpha_s^3+\ldots\right),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\gamma_{gg}(8,3) &= 1.851503\,\alpha_s\left(1+0.497734\,\alpha_s+0.404644\,\alpha_s^2+0.398779\,\alpha_s^3+\ldots\right),\\
\gamma_{gg}(8,4) &= 1.904554\,\alpha_s\left(1+0.401746\,\alpha_s+0.194306\,\alpha_s^2+0.157133\,\alpha_s^3+\ldots\right).
\end{aligned}
\tag{34}
$$

Except for γ_{qg} and γ_{gg} at $N=8$ these numerical expansions have been presented before in ref. [15].

The results for the qq and gg cases at asymptotically (and unphysically) large values of N read

$$
\begin{aligned}
\gamma_{qq}(N,3) &= a_s\gamma_{qq}^{(0)}(N,3)\left(1+0.726574\,\alpha_s+0.734054\,\alpha_s^2+0.664730\,\alpha_s^3\right),\\
\gamma_{qq}(N,4) &= a_s\gamma_{qq}^{(0)}(N,4)\left(1+0.638154\,\alpha_s+0.509978\,\alpha_s^2+0.316848\,\alpha_s^3\right)
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
\gamma_{gg}(N,3) &= a_s\gamma_{gg}^{(0)}(N,3)\left(1+0.726574\,\alpha_s+0.734054\,\alpha_s^2+0.415609\,\alpha_s^3\right),\\
\gamma_{gg}(N,4) &= a_s\gamma_{gg}^{(0)}(N,4)\left(1+0.638154\,\alpha_s+0.509978\,\alpha_s^2+0.064476\,\alpha_s^3\right)
\end{aligned}
\tag{36}
$$

due to their relation to the cusp anomalous dimensions A_k. The quark and gluon results are identical up to the 'Casimir scaling' of the prefactors, $\gamma_{qq}^{(0)}(N,n_f)=4C_F$ and $\gamma_{gg}^{(0)}(N,n_f)=4C_A$, to three loops and are related by a generalized (not numerical, except in the large-n_c limit, due to the presence of the quartic group invariants) Casimir scaling [11, 16] at four loops.

The relative size of the N^{2}LO and N^{3}LO contributions in eqs. (21) – (36) is illustrated in fig. 1 for $n_f=4$ at $\alpha_s=0.2$: The N^{3}LO corrections amount to less than 1%, and less than 0.5% of the NLO results except for P_{gq}, the quantity with the lowest leading-order values, at $N\geq 4$. Unlike in the quark case, see also ref. [17] where also a first estimate of the five-loop contribution to A_q has been obtained, the N^{2}LO and N^{3}LO large-N limits in the gluon case do not, in general, roughly agree with values in the range $4\leq N\leq 8$ normalized as in eqs. (21) – (34).

The resulting low-N expansion for the evolution (2) of the singlet quark and gluon PDFs is illustrated in fig. 2 for the schematic but sufficiently realistic order-independent model input [4]

$$
\begin{aligned}
xq_s(x,\mu_0^2) &= 0.6\,x^{-0.3}(1-x)^{3.5}\left(1+5.0\,x^{0.8}\right),\\
xg(x,\mu_0^2) &= 1.6\,x^{-0.3}(1-x)^{4.5}\left(1-0.6\,x^{0.3}\right)
\end{aligned}
\tag{37}
$$

with $\alpha_s(\mu_0^2)=0.2$ and $n_f=4$. The N^{3}LO corrections are very small at the standard choice $\mu_r=\mu_f\equiv\mu_0$ of the renormalization scale. They lead to a reduction of the scale dependence to about 1% (full width) at $N\geq 4$ for the conventional range $\frac{1}{4}\mu_f^2\leq\mu_r^2\leq 4\mu_f^2$.

To summarize, we have employed the theoretical framework of refs. [1–4] together with an optimized in-house version of the FORM [18] program FORCER for 4-loop propagator integrals [19] to compute the moments $N=2$, 4, 6, and 8 of all N^{3}LO flavour-singlet splitting functions. The numerical effect of these contributions is small, but more work is needed to arrive at sufficient 'data' for a N^{3}LO analogue of the earlier approximate N^{2}LO splitting functions of ref. [20].

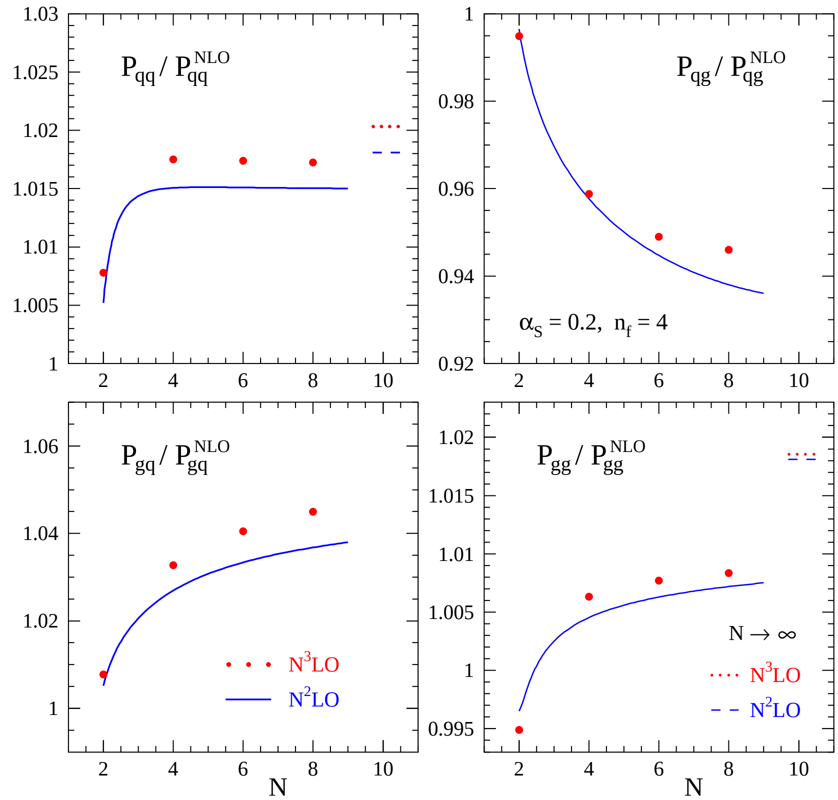

Figure 1: Moments of the splitting functions (2) at NNLO (lines) and N^{3}LO (even-N points) at $\alpha_s = 0.2$ and $n_f = 4$, normalized to the NLO results. Also shown are the qq and gg large-N limits.

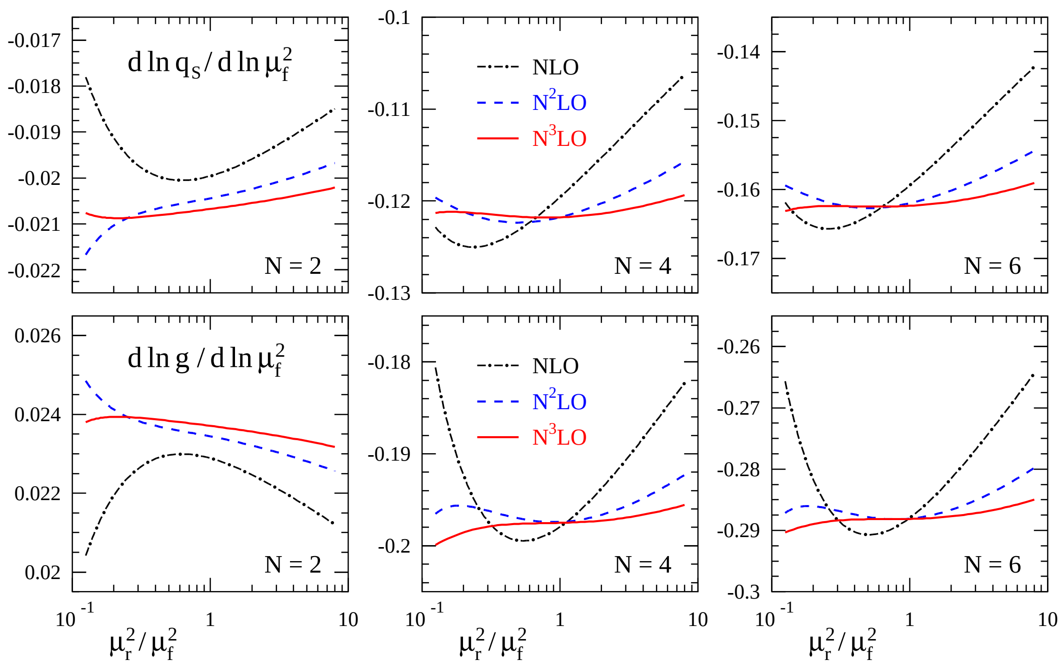

Figure 2: The dependence of the logarithmic factorization-scale derivatives of the singlet PDFs on the renormalization scale μ_r at $N = 2$ (where the very small scaling violations of q_s and g are related by the momentum sum rule), $N = 4$ and $N = 6$ for the initial distributions (37).

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