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The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach

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Abstract

Within an effective field theory method to general relativity, we calculate the fifth-order post–Newtonian (5 PN) Hamiltonian dynamics also for the tail terms, extending earlier work on the potential contributions, working in harmonic coordinates. Here we calculate independently all (local) 5 PN contributions to the tail terms using the in–in formalism, on which we give a detailed account. The five expansion terms of the Hamiltonian in the effective one body (EOB) approach, $q_{82}, q_{63}, q_{44}, \bar{d}_5$ and a_6, can all be determined from the local contributions to periastron advance $K^{\rm loc,h}(\hat{E}, j)$, without further assumptions on the structure of the symmetric mass ratio, ν, of the expansion coefficients of the scattering angle χ_k. The $O(\nu^2)$ contributions to the 5 PN EOB parameters have been unknown in part before. We perform comparisons of our analytic results with the literature and also present numerical results on some observables.

1 Introduction

The discovery of gravitational wave signals from merging black holes and neutron stars [1] has been a recent milestone for general relativity and astrophysics. The different present and planned gravitational wave detectors are reaching higher and higher sensitivity [2], which requires more detailed predictions on the theoretical side than currently available. The motion of gravitating massive binary systems was studied perturbatively expanding in higher post–Newtonian (PN) orders right after the theory of general relativity has been found [3, 4]. Later on the corrections at 2 PN [5], 3 PN [6] and 4 PN [**?**, 7, 9–11] have been calculated using a variety of techniques. First results at 5 PN have been obtained in Refs. [12–17] and at 6 PN in Refs. [18–20].[1] In the Schwarzschild limit the contributions of $O(\nu^0)$ are obtained to all post–Newtonian orders and the terms of $O(\nu)$ are know to $O(G_N^{21.5})$ [25, 26] from the self–force formalism [27][2], where G_N denotes Newton's constant. The following kinematic variables are used

$$\nu = \frac{m_1 m_2}{M^2} \in \left[0, \tfrac{1}{4}\right], \quad M = m_1 + m_2, \quad m_{1,2} = \frac{1}{2} M \left[1 \pm \sqrt{1 - 4\nu}\right], \quad \mu = M\nu, \tag{1}$$

where m_1 and m_2 are the two gravitating masses.

In this paper we complete the calculation of the 5PN corrections of the conservative dynamics for binary systems using an effective field theory (EFT) method and present main physical results. The calculation needs to be performed in $D = 4 - 2\varepsilon$ dimensions because pole terms of $O(1/\varepsilon)$ are occurring in intermediate steps. One first considers gravity in $d = D - 1$ integer spatial dimensions for all contributing pieces in the Lagrangian and then performs an analytic continuation. The multipole moments are dealt with in configuration space, while the graviton dynamics is calculated in momentum–space, after the post–Newtonian expansion has been performed via a Fourier–transform. In the calculation of the contributions to the tail terms we follow the approach of Ref. [28]. The conservative Hamiltonian of the binary system, H, consists of the potential, $H_{\rm pot}$, and the tail terms, $H_{\rm tail}$, or can be likewise decomposed into the local, $H_{\rm loc}$, and non–local, $H_{\rm nl}$, contributions

$$H = H_{\rm pot} + H_{\rm tail} = H_{\rm loc} + H_{\rm nl}. \tag{2}$$

We calculate the 5PN Hamiltonian *ab initio* starting with the path integral for classical gravity, for which a post–Newtonian expansion is performed by applying an EFT description [4, 29]. Methods originally developed for Quantum Field Theories are applied. About 190.000 Feynman diagrams contribute. They are generated by `QGRAF` [30]. Main parts of the calculation are performed using `FORM` [31]. The reduction to a very small set of master integrals is performed using `Crusher` [32] using the integration-by-parts relations [33]. The 5PN potential terms have been calculated by us in Ref. [16].[3] There we also described how the singularities in the potential and tail terms [7, 15, 19, 28, 34–37] are canceling, together with an additional canonical transformation. The whole 5PN calculation is performed starting in the Lagrange formalism and finally deriving the Hamiltonian.[4] Also the non–local 5PN contributions were presented, see also [15]. What remained to be calculated beyond the contributions given in [16] is a series of local tail contributions. These

[1]There is also a lot of activity in calculating post–Minkowskian corrections, cf. [20], Ref. [12], and [21–24].

[2]Also in self–force calculations regularizations are applied. In the present paper we are not discussing this aspect, but assume that the final results derived are regularization–independent.

[3]We compared also to the factorizing contributions to the potential terms, which have been obtained in [17], version 2, very recently, to which the corresponding subset of our results is agreeing.

[4]As outlined in detail in Refs. [11, 16], the treatment of higher time derivatives has to be performed without applying the equation of motion. Note, that there are different approaches in the literature.

terms contribute at $O(\nu)$ and $O(\nu^2)$. They are due to the 1PN correction to the electric quadrupole moment, written symbolically, $EQ_{ij}Q_{ji}$, with E the energy, [37], the octupole moment, $EO_{ijk}O_{ijk}$, the magnetic quadrupole moment, $EJ_{ij}J_{ji}$, the angular momentum failed tail, $L_k\varepsilon_{ijk}Q_{il}Q_{jl}$, with $\vec{L}$ the angular momentum, and the memory terms, $Q_{ij}Q_{jk}Q_{ki}$, which have also been considered in Ref. [28].[5] These multipole contributions are calculated in D dimensions starting with harmonic coordinates. As has been outlined in [16] only the electric quadrupole, octupole, and magnetic quadrupole moments have poles of $O(1/\varepsilon)$ and receive logarithmic contributions. All other multipole moments can be calculated in $D = 4$ dimensions and form rational contributions to the 5PN Hamiltonian. We have performed an independent calculation of these and related contributions and performed a detailed comparison with [28].[6]

The Hamiltonian H in Eq. (2) is gauge dependent and singular. To compare different approaches one has to either relate the Hamiltonians by canonical transformations or to compare the predictions for the resulting observables. This applies to the case of harmonic coordinates in the present approach and effective one body coordinates in [15]. In Ref. [15] different methods and constraints implied by the ν-structure of the expansion coefficients of the scattering angle χ_k, (45), have been used to construct the Hamiltonian and all but two $O(\nu^2)$ parameters, $\bar{d}_5$ and a_6, were determined in this way for $H_{\rm EOB}$ up to 5PN.

In Ref. [16] we obtained all terms except the rational terms of $O(\nu^2)$, the calculation of which we had not yet completed, as far as local contributions to the tail terms were concerned. For $\bar{d}_5$ and a_6 we have, more than in Ref. [15], obtained the $O(\nu^2\pi^2)$ terms, which stem from the potential contributions. We remind that the magnetic quadrupole contribution necessitates a finite renormalization[7] due to the analytic continuation of ε_{ijk} in dimensional regularization, which implies the contribution $\delta_J H$, Eq. (61) in [16].[8] We observe the same finite renormalization in the $O(\nu)$ term for the binding energy and the local contribution to periastron advance $K^{\rm loc,h}(\hat{E}, j)$, cf. [16]. Here 'h' stands for harmonic coordinates.

In the present paper we calculate the 5PN $O(\nu^2)$ terms and perform a series of comparisons to the literature. In Section 2 we calculate the 5PN local tail terms. To compare with Ref. [15] we express our results for the local contributions obtained in the harmonic gauge in terms of effective one body (EOB) potentials in Section 3. To fix the five new EOB parameters at 5PN we are solely using the local contributions to periastron advance $K^{\rm loc,h}(\hat{E}, j)$, which is an observable [42] since the non–local terms are known in explicit form via an eccentricity expansion, cf. [15,16], and does not require any regularization.[9] Furthermore, we will not use assumptions on the ν dependence of the observables used but perform a direct calculation within the framework of an effective field theory approach. The binding energy and periastron advance in the circular case are used for consistency checks for two parameters. Finally, we also discuss the determination of the 5PN parameters using the expansion coefficients of the scattering angle and compare to the literature. We summarize phenomenological results in Section 4 and Section 5 contains the conclusions. Some technical aspects are given in the Appendices A–E on the Feynman rules, invariant functions, the in–in formalism, calculation of the tail terms and relations for the scattering angle.

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[5] For a definition of the multipole moments see e.g. [38].

[6] The result in [28] for the angular momentum failed tail still needs a sign change, as communicated to us by the authors.

[7] In renormalizable Quantum Field Theories the related problem is the so–called γ_5 problem, cf. [39], and the corresponding operation is called a finite renormalization which is well–known from numerous calculations.

[8] A slightly different D–dimensional representation than used in [16] has been presented later in Ref. [40], leading to the same contribution $\delta_J H$, however. There also multipole moments, vanishing in $d = 3$ dimensions have been discussed. They do not contribute in the present case [41].

[9] Regularizations may imply scheme dependencies.

2 The local tail terms of the pole–free Hamiltonian

The tail terms can be derived both classically and by using EFT methods applying methods from Quantum Field Theory. Both methods have to lead to the same result. There are two types of contributions to the tail terms, *i)* singular ones, which have to be calculated in D dimensions and *ii)* non–singular ones, which can be calculated in $D = 4$ space–time dimensions. The corresponding contributions are expressed in terms of a multipole expansion, known in gravity since long, starting with the quadrupole moment [43], the current quadrupole moment and the mass octupole moment [44][10].

The non–local tail terms, if viewed from a D dimensional calculation, arise as logarithmic corrections in the ε expansion. To 5PN their structure was derived by classical methods in [34,45], see also [15], and using EFT methods in [16,28,46]. Both methods lead to the same results for the electric quadrupole moment, $EQ_{ij}Q_{ji}$, [37], the octupole moment, $O_{ijk}O_{ijk}$, and the magnetic quadrupole moment, $EJ_{ij}J_{ji}$, which develop logarithmic and pole contributions in the EFT–approach. Their normalization coefficients are the same as for their imaginary part, contributing to dE/dt, [34], Eq. (4.16').[11]

The associated D–dimensional terms up to the constant parts were calculated in [16,28], see also [37,47], and do also agree. This concerns the 1PN correction to the electric quadrupole moment, $EQ_{ij}Q_{ji}$, [37], the octupole moment, $EO_{ijk}O_{ijk}$, and the magnetic quadrupole moment, $EJ_{ij}J_{ji}$. The angular momentum failed tail, $L_k\varepsilon_{ijk}Q_{il}Q_{jl}$, and the memory term, $Q_{ij}Q_{jk}Q_{ki}$, are non singular and contribute only local $O(\nu^2)$ terms. The latter two contributions were calculated in [28,48], however, not using the Schwinger–Keldysh formalism (also called in–in or closed time path formalism), which has been developed in Refs. [48–61]. The angular momentum failed tail had been calculated by us at the time of [16] obtaining the same result as in [28].

The action, by which the vertices of the multipole moments in the diagrams Ref. [28] are defined has been derived using group–theoretical methods in $D = 4$ dimensions in Ref. [36], Eq. (100), e.g., and reads

$$\begin{aligned} S_{\rm mp} &= -\sqrt{32\pi G_N}\Biggl\{\int_{-\infty}^{+\infty} dt[Eh_{00} + L^i\varepsilon_{ijk}\partial_j h_{0k} + \sum_{l=2}^{\infty}\frac{1}{l!}I^L\partial_{L-2}E_{k_l-1,k_l} \\ &\quad - \sum_{l=2}^{\infty}\frac{2l}{(l+1)!}J^L\partial_{L-2}B_{k_l-1,k_l} + ...\Biggr\}, \end{aligned} \tag{3}$$

with

$$h^{\mu\nu} = \sqrt{|g|}g^{\mu\nu} - \eta^{\mu\nu}, \tag{4}$$

$g^{\mu\nu}$ the metric, $|g|$ its determinant, $\eta^{\mu\nu}$ the Minkowski metric, and L a multi–index. The tensors E_{ij} and B_{ij} are given by

$$E_{ij} = R_{0i0j} \simeq \frac{1}{2}\left[h_{00,ij} + \ddot{h}_{ij} - \dot{h}_{0i,j} - \dot{h}_{0j,i}\right] + O(h_{\mu\nu}^2) \tag{5}$$

$$B_{ij} = \frac{1}{2}\varepsilon_{ikl}R_{0jkl} \simeq \frac{1}{4}\varepsilon_{ikl}\left[\dot{h}_{jk,l} - \dot{h}_{jl,k} + h_{0l,jk} - h_{0k,jl}\right] + O(h_{\mu\nu}^2), \tag{6}$$

[10]See [34] for details.

[11]For other classical calculations see [45]. To our knowledge, there is no other derivation yet for the failed angular momentum and memory term, but that by using EFT methods. We thank L. Blanchet for a corresponding remark.

$R^\mu_{\nu\rho\sigma}$ denotes the Riemann tensor and $\dot{x} = dx/dt$. Later we will also consider vertices of multipole moments and two gravitons, see also Appendix A.

The multipole–moment terms $K_L^{(k)} K_L^{(k)}$, $K = I, J, O$, and L the corresponding multi–index, have the following structure

$$[K_L^{(k)} K_L^{(k)}](\varepsilon) = [K_L^{(k)} K_L^{(k)}] + \varepsilon \overline{[K_L^{(k)} K_L^{(k)}]} + O(\varepsilon^2). \tag{7}$$

We calculate the corresponding contributions to the action, from which the Lagrangian can be obtained. A Legendre transformation leads to the following Hamiltonian for the finite contributions of the tail terms

$$\begin{aligned} H^{\text{loc,5PN}}_{\text{tail,finite}} &= \eta^{10} \Biggl\{ H^{\text{5PN}}_{\text{el.quad}} + G_N^2 \Biggl[-M \Biggl(\frac{82}{6615} O^{(4)}_{ikl} O^{(4)}_{ikl} - \frac{1}{378} \overline{O^{(4)}_{ikl} O^{(4)}_{ikl}} + \frac{268}{675} J^{(3)}_{ij} J^{(3)}_{ij} \\ &\quad - \frac{8}{45} \overline{J^{(3)}_{ij} J^{(3)}_{ij}} \Biggr) - \frac{8}{15} \varepsilon_{ijk} L_k Q^{(4)}_{il} Q^{(3)}_{jl} + H_{\text{mem.1}} + H_{\text{mem.2}} + H_{\text{mem.3}} \Biggr] \Biggr\}, \end{aligned} \tag{8}$$

where

$$X^{(k)} = \frac{d^k}{dt^k} X(t). \tag{9}$$

We performed the canonical transformation to eliminate the $1/\varepsilon$ pole terms in Ref. [16] before, where the corresponding terms have been given already. Here the multipole–moments have to be used in D dimensions, except for the last four terms in (8). One obtains

$$H^{\text{loc,5PN}}_{\text{tail,finite}} = \eta^{10} \Bigl\{ H^{\text{5PN}}_{\text{el.quad}} + H_{\text{oct.}} + H_{\text{mag.quad.}} + H_{\text{ang.}} + H_{\text{mem.1}} + H_{\text{mem.2}} + H_{\text{mem.3}} \Bigr\}. \tag{10}$$

The calculation is performed in the in–in formalism. For the discussion in Section 4 it is essential to quantify the local 5PN tail terms to the pole–free Hamiltonian.

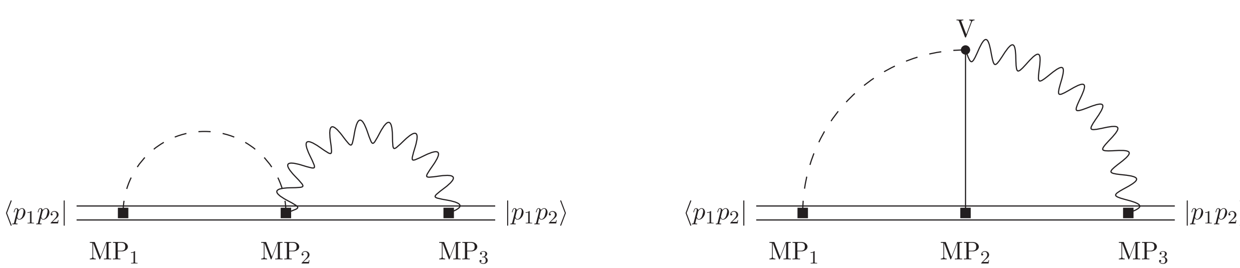


Figure 1: Schematic diagrams for the 5PN tail term contributions in the in–in formalism. The symbols MP_i denote the three multipole moments, V the triple vertex between the different contributing fields, the propagators in the decomposition of Ref. [4] of which have been depicted as dashed, full and curly lines, although some of them can denote the same field. Left: 2-graviton diagrams; Right: 3-graviton diagrams.

The structure of the diagrams contributing to the tail terms of the Hamiltonian in EFT is illustrated in Figure 1. The following velocity counting holds,

- each graviton propagator scales with v^{-2}

- each momentum integral yields a factor of v^3
- each post-Newtonian correction more adds a factor of v^2
- the graviton triple vertex is $\sim v^3$
- graviton coupling to $E \sim v$, $L_k \sim v^2$, $Q_{ij}, J_{ij} \sim v^3$, and $O_{ijk} \sim v^4$
- multipole moments $L_k, J_{ij} \sim v$
- the double graviton vertex to $Q_{ij} \sim v^4$.

The action (3) allows for a wide variety of triple multipole diagrams. The lowest contributing terms are those containing two electric dipole moments, because E and L_k are conserved in the stationary case we are considering. The diagram QEQ is of $O(v^{10})$ or of lowest order 4 PN. The diagrams QLQ, JEJ, OEO and QQQ are of $O(v^{12})$ and of lowest order 5PN. Other valid combinations are of higher than 5PN order or vanish, except the QQQ combination with two gravitons and one two–graviton Q_{ij} interaction.

In the following we list the local finite 5PN contributions from the electric quadrupole moment, $H^{5\mathrm{PN}}_{\mathrm{el.quad}}$, the octupole term, $H_{\mathrm{oct.}}$, the magnetic quadrupole moment, $H_{\mathrm{mag.quad.}}$, the angular momentum term $H_{\mathrm{ang.}}$, and the three contributions to the memory term, $H_{\mathrm{mem.1,2,3}}$,

$$
\begin{aligned}
H^{5\mathrm{PN}}_{\mathrm{el.quad}} &= \nu\Bigg[-(1-3\nu)\frac{1}{r^3}\left(\frac{392}{75}p^6-\frac{3508}{75}p^4(p.n)^2+\frac{124}{3}p^2(p.n)^4\right)\\
&+\frac{1}{r^4}\Bigg(-\frac{2(4573+32831\nu)p^4}{3675}-\frac{2(-598985+122233\nu)p^2(p.n)^2}{3675}\\
&+\frac{4(-286508+54209\nu)(p.n)^4}{3675}\Bigg)+\frac{1}{r^5}\Bigg(\left(\frac{696}{1225}+\frac{77428\nu}{3675}\right)p^2\\
&+\left(\frac{30904}{3675}+\frac{19168\nu}{735}\right)(p.n)^2\Bigg)+\frac{8(-244+137\nu)}{525r^6}\Bigg], \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
H_{\mathrm{oct.}} &= (1-4\nu)\nu\Bigg[\frac{1}{r^4}\left(\frac{17657p^4}{2205}-\frac{130058p^2(p.n)^2}{3675}+\frac{60409(p.n)^4}{2205}\right)\\
&+\frac{1}{r^5}\left(-\frac{2848p^2}{1575}+\frac{20848(p.n)^2}{11025}\right)+\frac{4}{7r^6}\Bigg], \qquad (12)
\end{aligned}
$$

$$
H_{\mathrm{mag.quad.}} = (1-4\nu)\nu\left[\frac{1}{r^4}\left(\frac{58p^4}{45}-\frac{604p^2(p.n)^2}{75}+\frac{1522(p.n)^4}{225}\right)-\frac{1}{r^5}\frac{488}{225}(p^2-(p.n)^2)\right], \qquad (13)
$$

$$
H_{\mathrm{ang.}} = \nu^2\left[\frac{1}{r^4}\left(-\frac{256p^4}{5}+\frac{1056p^2(p.n)^2}{5}-160(p.n)^4\right)+\frac{1}{r^5}\frac{256}{15}(p^2-(p.n)^2)\right]. \qquad (14)
$$

For all these terms we agree with the results given in Ref. [28] and [13], as well as for the 4 PN tail term QEQ.

In the EFT calculation the choice of the time dependence of the propagators in these 5PN cases turns out to be irrelevant in the explicit calculation, since the causal and the in–in formalism lead to the same result. The reason for this is that the internal multipole moment, either E or

L^k, is a conserved quantity, implying a δ–distribution for the energy components at their vertex. The associated propagator is therefore an (effective) space–like potential and the corresponding diagram only depends on a single energy variable, k_0. In the case of the potential terms, cf. [9,11], the reason for the same agreement is different. Here the propagators in Fourier–space are expanded as

$$\frac{1}{\vec{k}^2 - k_0^2 \pm i\varepsilon} = \frac{1}{\vec{k}^2} \sum_{l=0}^{\infty} \left(\frac{k_0^2}{\vec{k}^2}\right)^l, \tag{15}$$

in which the $i\varepsilon$–prescription does not play a role. When the in–in formalism is used for all these terms, it leads to the same results as using the usual (causal) path integral [62,63].

Indeed, a fundamental argument has been raised for the general use of the in–in formalism by B. DeWitt [58]. In using S–matrix theory the LSZ–formalism [64] requires a clear definition of the Hilbert spaces both for the initial state at $t = -\infty$ and for the final state at $t = +\infty$, made of (interaction) free states in both cases. This is fulfilled in elementary particle scattering processes, however, not in inspiraling processes like the merging of two large masses. While the initial state can be very well defined as

$$|p_1 p_2\rangle = |p_1\rangle |p_2\rangle, \tag{16}$$

with p_1 and p_2 the 4–momenta of the non–interacting masses at $t = -\infty$, the synonymous information on the final state of the merging process at $t = +\infty$ is not really known. The in–in formalism, however, requires only to know the initial state.[12]

The contributions to the tail term require retarded boundary conditions, as known from the Feynman–Wheeler formalism [67,68] in Quantum Electrodynamics. The Schwinger formalism using in–in states provides this description. As well–known, both the usual path–integral formalism [62,63], leading to causal Green's functions with T^*–ordering, and, analogously, the so-called in–in formalism, cf. [49,60], are exactly defined, also concerning the type of the contributing propagators. The in–in formalism has also applications in statistical physics, cf. [51,54]. A critical question concerns the unitarity of the respective formalism. As has been shown in [56] this is obeyed for the in–in formalism at least at two–loop order, the level necessary at 5PN. In the EFT approach the tail diagrams up 5PN read structurally as shown in Figure 1 using the in–in formalism: these are diagrams containing three multipole moments with single or double graviton interaction between two in–states. Also here the gravitons have at most self–couplings and all end at the worldline being connected to one of the multipole moment insertions. The only contributing diagrams at 5PN are two–loop diagrams. In the action, the 2–loop graviton exchange is integrated out. The necessary Feynman rules are listed in Appendix A. One then may read off the contributions to the conservative tail Lagrangian (Hamiltonian) from the action directly. The in–in formalism is described in detail in Appendix C.

In the following we calculate by a different method the contributions to the memory term, cf. [28], probably using different Feynman rules. The D dimensional tensor integrals are decomposed using the Passarino–Veltman representation [69] and the momentum integrals are performed using hypergeometric techniques [70,71] after a Feynman parameterization. In the final result the ε dependence cancels, as the case of $H_{\rm ang.}$, leaving the integrals over the energy components.

The results for the diagrams in Figure 1 are

$$H_{\rm mem.1} = \nu^2 \left[\frac{1}{r^4}\left(\frac{10p^4}{3} - \frac{967p^2(p.n)^2}{54} + \frac{83(p.n)^4}{6}\right) + \frac{1}{r^5}\left(-\frac{77p^2}{27} + \frac{97(p.n)^2}{27}\right) + \frac{11}{27r^6}\right]$$

[12]The method has some similarity to descriptions used in deep–inelastic scattering, like the forward Compton amplitude [65], referring to the optical theorem [66], using cutting methods in elementary particle physics.

(17)

for the term in the l.h.s. and we obtain for the two contributions of the second diagram

$$H_{\rm mem.2} = \nu^2\left[\frac{1}{r^4}\left(-4p^4+\frac{967p^2(p.n)^2}{45}-\frac{83(p.n)^4}{5}\right)+\frac{1}{r^5}\left(\frac{154p^2}{45}-\frac{194(p.n)^2}{45}\right)-\frac{22}{45r^6}\right], \tag{18}$$

$$H_{\rm mem.3} = \nu^2\left[\frac{1}{r^4}\left(\frac{16p^4}{315}-\frac{152p^2(p.n)^2}{105}+\frac{8(p.n)^4}{7}\right)+\frac{1}{r^5}\left(\frac{16p^2}{105}-\frac{256(p.n)^2}{315}\right)-\frac{32}{315r^6}\right]. \tag{19}$$

Details of the calculation of the $H_{\rm mem.}$ terms are given in Appendix D. The results in (18–19) differ from those given in [28] by a factor of -3 and the calculation of the contribution (17) is new. We have repeated the calculation of (18–19) in the same manner as described in [28][13] and agree up to the overall sign. However, our result is based on the derivation of the corresponding graph using the path integral in the in–in formalism and differs from that in [28]. There a specific choice for linking the advanced and retarded propagators has been made, which is not confirmed.

We also would like to mention that we have slightly modified the Hamiltonian given in [13], by treating higher time derivatives in the action, which are now eliminated by partial integration. This change, however, just corresponds to a canonical transformation, as we will show below in calculating observables which come out the same. The complete Hamiltonian, with tags on different contributions, is given in computer readable form in the ancillary file `HAMILTONIAN.m` to this paper.

3 Determining the EOB potentials from the Hamiltonian in harmonic coordinates

In Ref. [16] we have calculated the 5PN potential contributions to the Hamiltonian using dimensional regularization in $D = 4 - 2\varepsilon$ space–time dimensions, including the pole terms and the non–local contribution of the tail terms in complete form using harmonic coordinates. The sum of these terms does still contain a pole contribution $1/\varepsilon$. A canonical transformation leads to a pole–free Hamiltonian. In [16] we have left out finite rational 5PN terms of $O(\nu^2)$, i.e. local tail contributions which are purely rational. We have presented already all the π^2 terms of $O(\nu^2)$ since they stem from the potential terms.

A central point of our investigation is to compare to previous results given in Ref. [15], which have been presented based on a 5PN EOB Hamiltonian. For the local parts of both Hamiltonians one may either perform a canonical transformation, as done in Ref. [16], or determine the EOB potentials by an observable. We will choose the latter way and use the local contribution to periastron advance $K^{\rm loc,h}(\hat{E}, j)$. Corresponding consistency checks can be performed by additional observables, such as the binding energy and periastron advance for circular motion and also the expansion coefficients of the scattering angle.

3.1 The EOB parameters

The EOB Hamiltonian is given by [15]

$$H_{\rm EOB}^{\rm loc,eff} = \sqrt{A(1+AD\eta^2(p.n)^2+\eta^2(p^2-(p.n)^2)+Q)}, \tag{20}$$

[13] See the remark below Eq. (25) there.

which we consider up to the contributions to 5PN. Here $\eta = 1/c$ denotes the post–Newtonian expansion parameter, with c the velocity of light. The potentials A, D and Q are parameterized by[14]

$$\begin{aligned} A &= 1 + \sum_{k=1}^{6} a_k(\nu)\eta^{2k}u^k, \quad a_2 = 0, && (21)\\ D &= 1 + \sum_{k=2}^{5} d_k(\nu)\eta^{2k}u^k, && (22)\\ Q &= \eta^4 (p.n)^4 [q_{42}(\nu)\eta^4 u^2 + q_{43}(\nu)\eta^6 u^3 + q_{44}(\nu)\eta^8 u^4] + \eta^6 (p.n)^6 [q_{62}(\nu)\eta^4 u^2 + q_{63}(\nu)\eta^6 u^3] \\ &\quad + \eta^{12}(p.n)^8 u^2 q_{82}(\nu) \ . && (23) \end{aligned}$$

The known 4 PN Hamiltonians [7–11] in harmonic and ADM coordinates are connected by canonical transformations to $H_{\rm EOB}$, cf. [11],which imply the expansion parameters

$$\begin{aligned} {\rm N}, u\ &: a_1 && = -2, && (24)\\ 2{\rm PN}, u^2 &: d_2 && = 6\nu, && (25)\\ u^3 &: a_3 && = 2\nu, && (26)\\ 3{\rm PN}, u^2 &: q_{42} && = 8\nu - 6\nu^2, && (27)\\ u^3 &: d_3 && = 52\nu - 6\nu^2, && (28)\\ u^4 &: a_4 && = \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu, && (29)\\ 4{\rm PN}, u^2 &: q_{62} && = -\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3, && (30)\\ u^3 &: q_{43} && = 20\nu - 83\nu^2 + 10\nu^3, && (31)\\ u^4 &: d_4 && = \left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + \left(-260 + 123\pi^2\right)\nu^2. && (32) \end{aligned}$$

The expansion coefficients $a_6, \bar{d}_5, q_{44}, q_{63}$ and q_{82} emerge at 5PN. In [15] the following values were obtained[15]

$$\begin{aligned} 5{\rm PN}, u^2\ &: q_{82} = \frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4, && (33)\\ u^3\ &: q_{63} = \frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4, && (34)\\ u^4\ &: q_{44} = \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{9367}{15} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3, && (35)\\ u^5 &: \bar{d}_5 = \left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + \bar{d}_5^{\nu^2}\nu^2 + \left(\frac{1069}{3} - \frac{205}{16}\pi^2\right)\nu^3, && (36)\\ u^6 &: a_6 = \left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2}\nu^2 + 4\nu^3. && (37) \end{aligned}$$

In Ref. [13] we have calculated the π^2 contributions to $\bar{d}_5^{\nu^2}$ and $a_6^{\nu^2}$ which read

$$\bar{d}_5^{\pi^2\nu^2} = \frac{306545}{512}\pi^2\nu^2, \qquad (38)$$

[14]The 5PN level in EOB coordinates are of $O(\eta^{12})$.

[15]The reader should not be confused with the values given in [14], which are in the f–scheme of the authors. We compare to the h–scheme, cf. [15].

$$a_6^{\pi^2\nu^2} = \frac{25911}{256}\pi^2\nu^2. \tag{39}$$

All 5PN coefficients can be obtained from periastron advance $K^{\rm loc,h}(\hat{E},j)$. Using the Hamiltonian (2) in harmonic coordinates they are given by

$$q_{82} = \frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4, \tag{40}$$

$$q_{63} = \frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4, \tag{41}$$

$$q_{44} = \left(\frac{1580641}{3150} - \frac{93031\pi^2}{1536}\right)\nu + \left(-\frac{532676}{675} + \frac{31633\pi^2}{512}\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3, \tag{42}$$

$$\bar{d}_5 = \left(\frac{331054}{175} - \frac{63707\nu}{512}\pi^2\right)\nu + \left(-\frac{31244704}{4725} + \frac{306545}{512}\pi^2\right)\nu^2 + \left(\frac{1069}{3} - \frac{205}{16}\pi^2\right)\nu^3, \tag{43}$$

$$a_6 = \left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + \left(-\frac{1750163}{1575} + \frac{25911}{256}\pi^2\right)\nu^2 + 4\nu^3. \tag{44}$$

Here q_{82} obeys the sole equation implied by the $\hat{E}^4/j^2$ term and q_{63} and q_{44} the one for $\hat{E}^3/j^4$, etc. Values for $\bar{d}_5$ and a_6, which are consistent with (43) and (44), are also obtained from the binding energy and periastron advance in the circular case, which do not depend on the Q–potentials.

3.2 The scattering angle

We will now study the expansion coefficients χ_k of the scattering angle[16]

$$\frac{\chi^{\rm loc} + \pi}{2} = \sum_{k=1}^{\infty} \frac{\chi_k^{\rm loc}}{j^k} = \mathsf{Reg}_{u_{\max}} \int_0^{u_{\max}} \frac{j\,dy}{(p.n)(y,j)}, \quad p.n \ := \ \sqrt{p_\infty^2 - j^2y^2 + W(y,p_\infty)}. \tag{45}$$

Some of the integrals are listed in Appendix E. Here the following kinematic variables are used, cf. [20, 73],

$$p_\infty = \sqrt{\gamma^2 - 1}, \tag{46}$$

$$\gamma = \frac{1}{m_1 m_2}[E_1 E_2 + p^2], \quad E_i = \sqrt{p^2 + m_i^2}, \tag{47}$$

$$j = \frac{J}{G_N m_1 m_2}, \tag{48}$$

$$\Gamma = \sqrt{1 + 2\nu(\gamma - 1)}, \tag{49}$$

E_i are the corresponding energies, p the cms momentum, and J the angular momentum. Note that in the definition of the scattering angle in Ref. [15], Eq. (9.1), the regularization operator has not been written, although being necessary, cf. e.g. Eq. (3.49) in [23] (in the energy gauge), since both the expansion in $1/j$ and the post–Newtonian expansion is finally carried out.

[16] It is most useful to perform the integral (45) directly using the $p.n$ gauge [23] both in harmonic and in EOB coordinates. It is advisable to use the necessary integrals from Ref. [72], since not all computer algebra systems do perform them correctly.

The quantity q_{63} can be extracted from the component $\chi_3^{\rm h,loc}$ of the scattering angle from the contribution $O(p_\infty^7)$

$$\begin{aligned}\chi_3^{\rm h,loc} &= -\frac{1}{3p_\infty^3}+\frac{4\eta^2}{p_\infty}+\eta^4(24-8\nu)p_\infty+\eta^6\left(\frac{64}{3}-36\nu+8\nu^2\right)p_\infty^3+\eta^8\left(-\frac{91}{5}\nu+34\nu^2\right.\\ &\left.-8\nu^3\right)p_\infty^5+\eta^{10}\left(\frac{96}{35}\nu+\frac{288}{35}\nu^2-\frac{108}{7}\nu^3+6\nu^4-\frac{1}{7}q_{63}\right)p_\infty^7. \end{aligned} \quad (50)$$

From

$$\begin{aligned}\chi_3^{\rm h,loc} &= \chi_3^{\rm Schw}-\frac{2\nu p_\infty}{\Gamma^2}\left[\frac{2}{3}\gamma(14\gamma^2+25)+\frac{4(4\gamma^4-12\gamma^2-3)}{p_\infty}\text{arcsinh}\left(\sqrt{\frac{\gamma-1}{2}}\right)\right]\\ &= -\frac{1}{3p_\infty^3}+\frac{4\eta^2}{p_\infty}+(24-8\nu)\eta^3p_\infty+\left(\frac{64}{3}-36\nu+8\nu^2\right)\eta^6p_\infty^3+\left(-\frac{91}{5}\nu+34\nu^2-8\nu^3\right)\\ &\times\eta^8p_\infty^5+\left(\frac{69}{70}\nu+\frac{51}{5}\nu^2-32\nu^3+8\nu^4\right)\eta^{10}p_\infty^7 \end{aligned} \quad (51)$$

and (50) one obtains q_{63} as in Eq. (41).

In Eq. (45) $(p.n)$ is calculated as in the case for periastron advance [13] and by setting

$$\hat{E}=\frac{\Gamma-1}{\nu} \quad (52)$$

and expanding to the respective post–Newtonian order. The operator $\mathsf{Reg}_{u_{\max}}$ removes all pole terms in ε, which are implied in the integral (45) by

$$y=u_{\max}:=\frac{1}{jp_\infty}\left[1+\sqrt{1+j^2p_\infty^2}\right]-\varepsilon, \quad (53)$$

where we choose $|\varepsilon|\ll 1$. In using (45) for the calculation of $\chi_k^{\rm loc}$, singular terms in ε in the corresponding integrals are disregarded. Here the expansion coefficients of the Schwarzschild scattering angle $\chi_k^{\rm Schw}$ are given by

$$\frac{\chi^{\rm Schw}}{2}=\sum_{k=1}^{\infty}\frac{1}{j^k}\chi_k^{\rm Schw}, \quad (54)$$

with

$$\begin{aligned}\chi_k^{\rm Schw} &= \mathsf{Reg}_\varepsilon\binom{-\frac{1}{2}}{k}\eta^k\int_0^{\sqrt{\eta^2p_\infty^2-\varepsilon}}dy\frac{[2y(1+y^2)]^k}{(p_\infty^2\eta^2-y^2)^{k+1/2}}, && k\ \text{ even}\\ &= \frac{(-1)^{k+1}2^{k-1}}{p_\infty^k}\binom{-\frac{1}{2}}{k}\frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2}-k)}{\Gamma(1-\frac{k}{2})}{}_2F_1\left[\begin{matrix}-k,\frac{k+1}{2}\\ 1-\frac{k}{2}\end{matrix};-\eta^2p_\infty^2\right], && k\ \text{ odd.}\end{aligned} \quad (55)$$

Reg_ε projects onto the constant part in the ε-expansion, which correspondingly implies a regularization scheme dependence. However, the relation

$$\chi_{2k}^{\rm h,loc}(p_\infty,\nu)=\frac{\pi}{2}K_{2k}^{\rm loc,h}(\hat{E}), \quad (56)$$

with

$$K^{\rm loc,h}(\hat{E},j)=\sum_{k=1}^{\infty}\frac{1}{j^{2k}}K_{2k}^{\rm loc,h}(\hat{E}), \quad (57)$$

cf. [22], can be used in the limit $\nu \to 0$, fixing the regularization scheme. In Eq. (56) it is understood, that $\hat{E}$ of (52) is used and an expansion in p_∞ to the respective PN order is performed. In the present case we use the choice (55). We label the different post–Newtonian contributions by the parameter η. For the first orders in the expansion of (54) one obtains

$$\chi_1^{\rm Schw} = \frac{1}{p_\infty} + 2p_\infty\eta^2, \tag{58}$$

$$\chi_2^{\rm Schw} = \pi\left(\frac{3}{2}\eta^2 + \frac{15}{8}p_\infty^2\eta^4\right), \tag{59}$$

$$\chi_3^{\rm Schw} = -\frac{1}{3p_\infty^3} + \frac{4\eta^2}{p_\infty} + 24p_\infty\eta^4 + \frac{64}{3}p_\infty^3\eta^6, \tag{60}$$

$$\chi_4^{\rm Schw} = \pi\left(\frac{105}{8}\eta^4 + \frac{315}{8}p_\infty^2\eta^6 + \frac{3465}{128}p_\infty^4\eta^8\right). \tag{61}$$

Higher–order expansion coefficients are listed in Appendix E. The coefficients q_{82} and q_{44} can also be extracted from the component $\chi_2^{\rm h}$ and $\chi_4^{\rm h}$ of the scattering angle, respectively, using relation (56) by expanding to $O(p_\infty^6)$. One obtains the values given in Eqs. (40, 42). For q_{44} we observe the difference

$$q_{44} - q_{44}^{\text{Ref. [15]}} = -\frac{111161}{675}\nu^2. \tag{62}$$

This coefficient is related to the $\hat{E}^3/j^4$ term of the periastron advance $K^{\rm loc,h}(\hat{E}, j)$. The potential contributions to this quantity are of $O(G_N^4)$ and have been first obtained in [16]. They have recently been confirmed by the $O(G_N^4)$ post–Minkowskian results of [74], cf. also [75, 76]. The $O(\nu^2\pi^2)$ contribution to this term is implied by q_{44}, since q_{82} and q_{63} are purely rational in ν. We also would like to mention that the π^2 terms of $\bar{d}_5$ and a_6, as determined using the scattering angle in [77], using the potential terms given in [16], are equivalently obtained by using Eqs. (56, 57), which has been presented before in Ref. [16].

The above expansion coefficients at 5 PN were all derived without referring to a specific ν dependence of the combination [23]

$$\Gamma^{k-1}\chi_k - \chi_k^{\rm Schw}. \tag{63}$$

The equations determining the rational $O(\nu^2)$ terms of $q_{44}, \bar{d}_5$ and a_6 contain also local tail contributions, which are finite and just of $O(\nu^2)$ together with the $O(\nu^2)$ from singular tail terms, which are fully predicted in relation to their $O(\nu)$ contributions, cf. [16]. This aspect will be also discussed in Section 4.

From the results obtained in the previous sections we determine now the $O(\nu^2)$ contributions to q_{44}, a_6 and $\bar{d}_5$. We first consider $\chi_4^{\rm total,h}$,

$$\chi_4^{\rm total,h} = \chi_4^{\rm nonloc,h} + \chi_4^{\rm loc,h}. \tag{64}$$

The non–local contribution, $\chi_4^{\rm nonloc,h}$, has been calculated in [15] and is given by

$$\chi_4^{\rm nonloc,h} = \nu\eta^8 p_\infty^4\pi\left\{-\frac{63}{4} - \eta^2\left(\frac{2753}{1120} - \frac{1071}{40}\nu\right)p_\infty^2 - \ln\left(\frac{p_\infty}{2}\right)\left[\frac{37}{5} + \eta^2\left(\frac{1357}{280} - \frac{111}{10}\nu\right)p_\infty^2\right]\right\} \tag{65}$$

Its transform to the f–scheme for $\Gamma^3\chi_4^{\rm f-h}$ is given in Eq. (8.6) there, to eliminate the term of $O(\pi\nu^2 p_\infty^6)$.[17] One has

$$\Gamma^3\chi_4^{\rm nonloc,h}=\nu\eta^8 p_\infty^4\pi\left\{-\frac{63}{4}-\frac{37}{5}\ln\left(\frac{p_\infty}{2}\right)+\eta^2 p_\infty^2\left[-\frac{2753}{1120}-\frac{1357}{280}\ln\left(\frac{p_\infty}{2}\right)+\frac{63}{20}\nu\right]\right\}. \quad (66)$$

The local contribution is obtained from Eq. (56) including all local contributions to the Hamiltonian, and reads

$$\begin{aligned}\chi_4^{\rm loc,h}=\pi\Biggl\{&\eta^4\left(\frac{105}{8}-\frac{15}{4}\nu\right)+\eta^6\left[\frac{315}{8}-\left(\frac{109}{2}-\frac{123}{256}\pi^2\right)\nu+\frac{45}{8}\nu^2\right]p_\infty^2\\ &+\eta^8\left[\frac{3465}{128}-\left(\frac{19597}{192}-\frac{33601}{16384}\pi^2\right)\nu+\left(\frac{4827}{64}-\frac{369}{512}\pi^2\right)\nu^2-\frac{225}{32}\nu^3\right]p_\infty^4\\ &+\eta^{10}\Biggl[-\left(\frac{1945583}{33600}-\frac{93031}{32768}\pi^2\right)\nu+\left(\frac{1809101}{14400}-\frac{94899}{32768}\pi^2\right)\nu^2\\ &-\left(\frac{2895}{32}-\frac{1845}{2048}\pi^2\right)\nu^3+\frac{525}{64}\nu^4\Biggr]p_\infty^6\Biggr\}, \end{aligned} \quad (67)$$

and one obtains

$$\begin{aligned}\Gamma^3\chi_4^{\rm loc,h}-\chi_4^{\rm Schw}=\pi\nu\Biggl[&-\frac{15}{4}\eta^4+\eta^6 p_\infty^2\left(-\frac{557}{16}+\frac{123}{256}\pi^2\right)+\eta^8 p_\infty^4\left(-\frac{4601}{96}+\frac{33601}{16384}\pi^2\right)\\ &+\eta^{10}p_\infty^6\left(-\frac{3978707}{134400}+\frac{93031}{32768}\pi^2+\frac{65801}{14400}\nu\right)\Biggr]. \end{aligned} \quad (68)$$

Forming the combination (63) and expanding to $O(\eta^{10})$ (5 PN) we obtain

$$\left[\Gamma^3\chi_4^{\rm total,h}-\chi_4^{\rm Schw}\right]-\left[\Gamma^3\chi_4^{\rm total,h}-\chi_4^{\rm Schw}\right]^{\rm Ref.\ [15]}=\frac{111161}{14400}\eta^{10}\pi\nu^2 p_\infty^6, \quad (69)$$

comparing with the result of Ref. [15], Eqs. (8.2, 8.6, 10.1c, 10.2d), and thus observe a breaking of the rule

$$\left[\Gamma^{k-1}\chi_k^{\rm total,h}-\chi_k^{\rm Schw}\right]=P_k(\nu) \quad (70)$$

with degree of $[(k-1)/2]$ of the polynomial $P_k(\nu)$.

On the other hand, one observes for $k\in[1,3]$, that (70) holds. It is trivially obeyed for $k=1,2$ because of the post–Minkowskian structure [78], see also [23], and is implied for $k=3$ by

$$\Gamma^3\chi_3^{\rm h,loc}-\chi_3^{\rm Schw}=\nu\left[-\frac{\eta^2}{3p_\infty}-\frac{47}{12}\eta^4 p_\infty-\frac{313}{24}\eta^6 p_\infty^3-\frac{749}{320}\eta^8 p_\infty^5-\frac{7519}{4480}\eta^{10}p_\infty^7\right]+O(p_\infty^9), \quad (71)$$

referring to the $O(G_N^3)$ post–Minkowskian Hamiltonian [24]. Note that for $k\leq 3$ no tail terms contribute and the scattering angle stems from the potential contributions only. The post–Minkowskian result of $O(G_N^3)$ [24] has been checked to 6 PN in [20].

[17] Eq. (7.24) of [15] contains typos. Both $a_{\rm 5PN}$ and $b_{\rm 5PN}$ need an additional factor p_∞^2. Otherwise Eq. (8.2) will not hold.

Finally we would like to discuss the hypothetical possibility to accommodate the result of the present paper on q_{44} with the one in [15]. In a later paper, [77], the authors of [15] claim to have still a missing term in Eq. (10.2) of [77] on radiation reaction, which is not yet quantified. It is of the type

$$\delta H_{\rm tail} = a\eta^{10}\nu^2\frac{(p.n)^4}{r^4}, \tag{72}$$

with $a \in \mathbb{R}$. Terms of the kind (72) can imply the mapping

$$q_{44}^{\rm h} \rightarrow q_{44}^{\text{Ref. [15]}} \tag{73}$$

here with the value

$$a = \frac{65801}{1350}, \tag{74}$$

to reach full agreement. It transforms $K(\hat{E},j)^{\rm 5PN}_{\rm loch}$ into [19], Eq. (F.5). This also applies e.g. to $\chi_k^{\rm loc},\ k = 4,5,6$, while $E^{\rm circ}$ and $K^{\rm circ}$ remain unaffected, see Section 4.

4 Phenomenological results

In the following we summarize the results of the present calculation for observables and present numerical results. We illustrate the result for the local 5PN contribution to the local contributions to general periastron advance $K^{\rm loc,h}(\hat{E},j)$ and the circular limits of the binding energy $E^{\rm circ}(j)$ and periastron advance $K^{\rm circ}(j)$ which are given by

$$\begin{aligned}
K(\hat{E},j)^{\rm 5PN}_{\rm loc,h} = \Bigg\{ & \left[\frac{15\nu^2}{16} - \frac{15}{4}\nu^3 + 3\nu^4\right]\frac{\hat{E}^4}{j^2} + \left[\frac{3465}{16} + \left(-\frac{12160657}{8400} + \frac{15829\pi^2}{256}\right)\nu + \left(\frac{1519151}{900}\right.\right. \\
& \left. - \frac{35569\pi^2}{1024}\right)\nu^2 + \left(\frac{1107\pi^2}{128} - \frac{7113}{8}\right)\nu^3 + 75\nu^4\Bigg]\frac{\hat{E}^3}{j^4} + \Bigg[\frac{315315}{32} + \left(-\frac{33023719}{840}\right. \\
& \left. + \frac{4899565\pi^2}{4096}\right)\nu + \left(-\frac{3289285\pi^2}{1024} + \frac{19930271}{360}\right)\nu^2 + \left(\frac{35055\pi^2}{256} - \frac{240585}{32}\right)\nu^3 \\
& \left. + \frac{1575}{8}\nu^4\right)\frac{\hat{E}^2}{j^6} + \Bigg[\frac{765765}{16} + \left(-\frac{30690127}{240} + \frac{16173395\pi^2}{8192}\right)\nu + \left(\frac{67543655}{432}\right. \\
& \left. - \frac{77646205\pi^2}{8192}\right)\nu^2 + \left(\frac{121975\pi^2}{512} - \frac{271705}{24}\right)\nu^3 + \frac{2205}{16}\nu^4\Bigg]\frac{\hat{E}}{j^8} + \Bigg[\frac{2909907}{64} \\
& + \left(-\frac{61358067}{640} + \frac{1096263\pi^2}{1024}\right)\nu + \left(\frac{169718789}{1920} - \frac{87068961\pi^2}{16384}\right)\nu^2 + \left(\frac{90405\pi^2}{1024}\right. \\
& \left. - \frac{127995}{32}\right)\nu^3 + \frac{3465}{128}\nu^4\Bigg]\frac{1}{j^{10}}\Bigg\}\eta^{10} + O(\eta^{12})
\end{aligned} \tag{75}$$

and

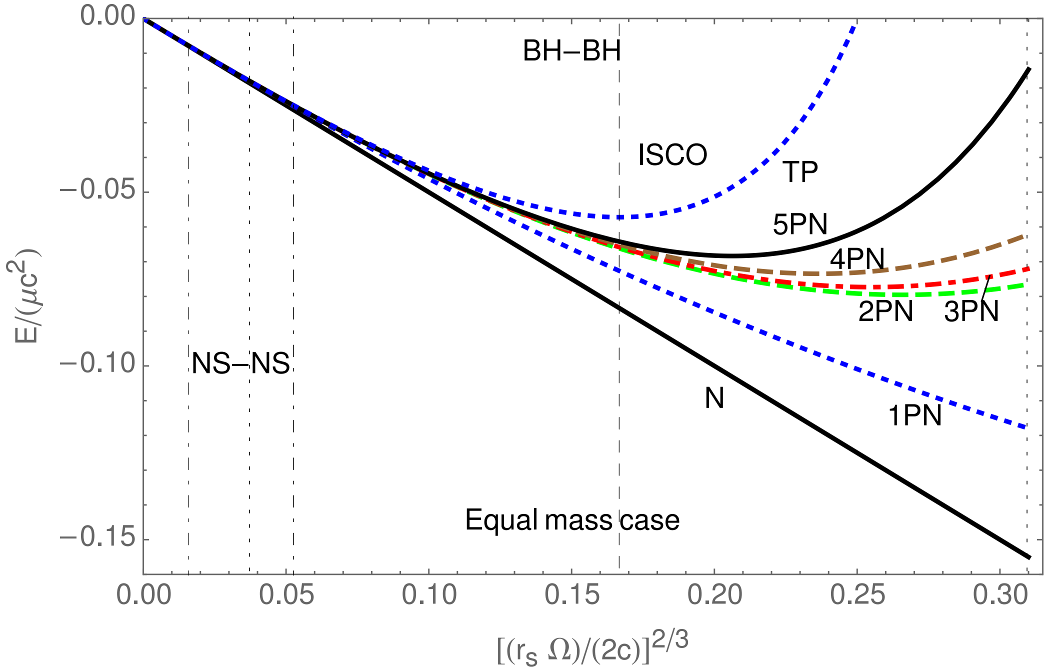


Figure 2: The binding energy in the quasi-circular case for equal masses. Lower full line: Newtonian case (N); Dotted line: 1PN; Dashed line: 2PN; Dash-dotted line: 3PN; Upper dashed line: 4PN; Upper full line: 5PN; Upper dotted line: test particle solution (TP). Dashed vertical line: the innermost stable circular orbit (ISCO) at 5PN. The other vertical lines mark the frequency spectrum for neutron star (NS) and black hole (BH) merging at LIGO.

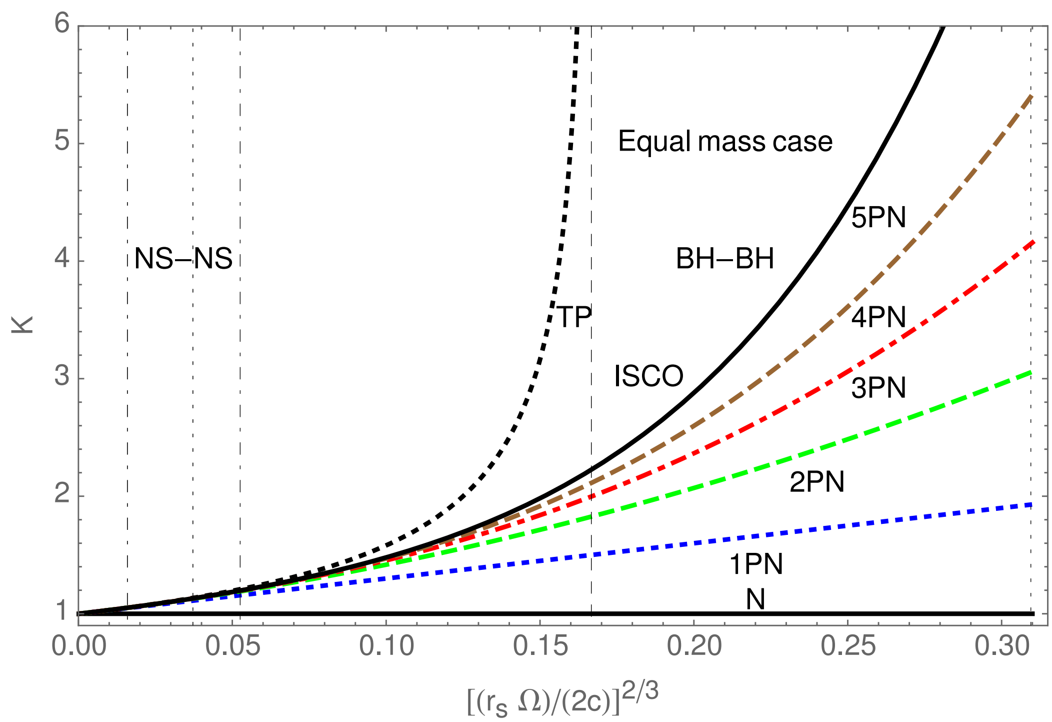


Figure 3: The periastron advance in the circular limit. The labeling of the curves is the same as in Figure 2.

$$\frac{E^{\rm circ,h}(j)}{\mu c^2}=-\frac{1}{2j^2}-\left(\frac{9}{8}+\frac{\nu}{8}\right)\frac{1}{j^4}\eta^2+\left(-\frac{81}{16}+\frac{7\nu}{16}-\frac{\nu^2}{16}\right)\frac{1}{j^6}\eta^4+\left[-\frac{3861}{128}+\left(\frac{8833}{384}-\frac{41\pi^2}{64}\right)\nu\right.$$
$$\left.+\frac{5\nu^2}{64}-\frac{5\nu^3}{128}\right]\frac{1}{j^8}\eta^6+\left[-\frac{53703}{256}+\left(\frac{989911}{3840}-\frac{6581\pi^2}{1024}\right)\nu+\left(-\frac{8875}{768}+\frac{41\pi^2}{128}\right)\nu^2\right.$$
$$\left.+\frac{3\nu^3}{128}-\frac{7\nu^4}{256}\right]\frac{1}{j^{10}}\eta^8+\left[-\frac{1648269}{1024}+\left(\frac{3747183493}{1612800}-\frac{31547\pi^2}{1536}\right)\nu+\left(-\frac{400383799}{403200}\right.\right.$$
$$\left.\left.+\frac{132979\pi^2}{2048}\right)\nu^2+\left(-\frac{3769}{3072}+\frac{41\pi^2}{512}\right)\nu^3+\frac{5\nu^4}{1024}-\frac{21\nu^5}{1024}\right]\frac{1}{j^{12}}\eta^{10}+\frac{E^{\rm circ}_{\rm nl}}{\mu c^2}+O\left(\eta^{12}\right), \tag{76}$$

$$K^{\rm circ,h}(j)=1+3\frac{1}{j^2}\eta^2+\left(\frac{45}{2}-6\nu\right)\frac{1}{j^4}\eta^4+\left[\frac{405}{2}+\left(-202+\frac{123}{32}\pi^2\right)\nu+3\nu^2\right]\frac{1}{j^6}\eta^6+\left[\frac{15795}{8}\right.$$
$$\left.+\left(-\frac{105991}{36}+\frac{185767}{3072}\pi^2\right)\nu+\left(\frac{2479}{6}-\frac{41}{4}\pi^2\right)\nu^2\right]\frac{1}{j^8}\eta^8+\left[\frac{161109}{8}+\left(-\frac{18144676}{525}\right.\right.$$
$$\left.\left.+\frac{488373}{2048}\pi^2\right)\nu+\left(\frac{15079451}{675}-\frac{1379075}{1024}\pi^2\right)\nu^2+\left(-\frac{1627}{6}+\frac{205}{32}\pi^2\right)\nu^3\right]\frac{1}{j^{10}}\eta^{10}$$
$$+K^{\rm nl}_{4+5\rm PN}(j)+O\left(\eta^{12}\right). \tag{77}$$

The non–local contributions to Eqs. (76) and (77) have been given in [16], Eqs. (44, 54, 55).

We illustrate the different post–Newtonian contributions to the binding energy, $E^{\rm circ}(x)$, and the periastron advance, $K^{\rm circ}(x)$ for circular orbits, setting $m_1 = m_2$, $(\nu = 1/4)$, with $x = (G_N M\Omega/c^3)^{2/3}$, $\Omega = \Omega_\phi$ the angular frequency, and r_s the Schwarzschild radius. The relation between j and x to 5PN is given by

$$j=\frac{1}{\sqrt{x}}\left\{1+\frac{1}{6}\eta^2x(9+\nu)+\frac{1}{24}\eta^4x^2\big(81-57\nu+\nu^2\big)+\eta^6x^3\left[\frac{135}{16}+\frac{1}{144}\big(-6889+246\pi^2\big)\nu\right.\right.$$
$$\left.+\frac{31\nu^2}{24}+\frac{7\nu^3}{1296}\right]+\eta^8x^4\left[\frac{2835}{128}+\nu\left(\frac{98869}{5760}-\frac{128\gamma_E}{3}-\frac{6455\pi^2}{1536}-\frac{256\ln(2)}{3}-\frac{64\ln(x)}{3}\right)\right.$$
$$\left.+\frac{5\big(71207-2706\pi^2\big)}{3456}\nu^2-\frac{215\nu^3}{1728}-\frac{55\nu^4}{31104}\right]+\eta^{10}x^5\left[\frac{15309}{256}+\left(\frac{59112343}{44800}+\frac{19952\gamma_E}{105}\right.\right.$$
$$\left.-\frac{126779\pi^2}{768}+\frac{47344\ln(2)}{105}-\frac{486\ln(3)}{7}+\frac{9976\ln(x)}{105}\right)\nu+\left(\frac{11116883}{5600}+\frac{2624\gamma_E}{15}\right.$$
$$\left.-\frac{289595\pi^2}{1536}+\frac{6976\ln(2)}{105}+\frac{1944\ln(3)}{7}+\frac{1312\ln(x)}{15}\right)\nu^2+\frac{1}{256}\nu^3\big(-25189+902\pi^2\big)$$
$$\left.\left.-\frac{55\nu^4}{768}-\frac{\nu^5}{768}\right]\right\}+O(\eta^{12}). \tag{78}$$

We also add the the test–particle lines (TP), which are given by [79]

$$\frac{E^{\rm circ}_{\rm TP}}{\mu c^2}=\frac{1-2x\eta^2}{\sqrt{1-3x\eta^2}}-1 \tag{79}$$

and

$$K_{\rm TP}^{\rm circ}=\frac{1}{\sqrt{1-6\eta^2 x}} \tag{80}$$

with

$$j_{\rm TP}=\frac{1}{\sqrt{x(1-3\eta^2 x)}}. \tag{81}$$

The different post–Newtonian corrections are all positive correcting lower order results. Even at 5PN the convergence is not yet perfect both for the binding energy and periastron advance, considering the range $x \in [0, 0.30]$, which is calling for effective resummations of these contributions.

Finally, the scattering angle to 5PN is given by

$$\chi_1^{\rm h,loc}=\frac{1}{p_\infty}+2\eta^2 p_\infty, \tag{82}$$

$$\chi_2^{\rm h,loc}=\pi\Bigg[\frac{3\eta^2}{2}+\frac{3}{8}\eta^4(5-2\nu)p_\infty^2+\frac{3}{16}\eta^6\nu(-4+3\nu)p_\infty^4+\eta^8\left(\frac{9\nu}{64}+\frac{27\nu^2}{64}-\frac{15\nu^3}{32}\right)p_\infty^6$$
$$+\eta^{10}\left(-\frac{15\nu}{256}-\frac{45\nu^2}{256}-\frac{15\nu^3}{64}+\frac{105\nu^4}{256}\right)p_\infty^8\Bigg], \tag{83}$$

$$\chi_3^{\rm h,loc}=-\frac{1}{3p_\infty^3}+\frac{4\eta^2}{p_\infty}+8\eta^4(3-\nu)p_\infty+\frac{4}{3}\eta^6\big(16-27\nu+6\nu^2\big)p_\infty^3-\frac{1}{5}\eta^8\nu\big(91-170\nu+40\nu^2\big)p_\infty^5$$
$$+\eta^{10}\left(\frac{69\nu}{70}+\frac{51\nu^2}{5}-32\nu^3+8\nu^4\right)p_\infty^7, \tag{84}$$

$$\chi_4^{\rm h,loc}=\pi\Bigg[\frac{15}{8}\eta^4(7-2\nu)+\eta^6 p_\infty^2\left(\frac{315}{8}-\frac{1}{256}\left(13952-123\pi^2\right)\nu+\frac{45}{8}\nu^2\right)$$
$$+\eta^8 p_\infty^4\left(\frac{3465}{128}-\frac{5016832-100803\pi^2}{49152}\nu+\frac{3}{512}\big(12872-123\pi^2\big)\nu^2-\frac{225}{32}\nu^3\right)$$
$$+\eta^{10}p_\infty^6\Bigg(\frac{-996138496+48841275\pi^2}{17203200}\nu+\left(\frac{1809101}{14400}-\frac{94899}{32768}\pi^2\right)\nu^2$$
$$-\frac{15\big(12352-123\pi^2\big)}{2048}\nu^3+\frac{525}{64}\nu^4\Bigg)\Bigg], \tag{85}$$

$$\chi_6^{\rm h,loc}=\pi\Bigg[\eta^6\left(\frac{1155}{8}-\left(\frac{625}{4}-\frac{615\pi^2}{256}\right)\nu+\frac{105}{16}\nu^2\right)+\eta^8 p_\infty^2\Bigg(\frac{45045}{64}-\frac{37556864-771585\pi^2}{24576}\nu$$
$$+\frac{5}{64}\big(7013-123\pi^2\big)\nu^2-\frac{525}{32}\nu^3\Bigg)+\eta^{10}p_\infty^4\Bigg(\frac{135135}{128}-\frac{8099529344-243724425\pi^2}{1720320}\nu$$
$$+\frac{10565677312-600102675\pi^2}{1474560}\nu^2-\frac{25\big(87872-1599\pi^2\big)}{2048}\nu^3+\frac{3675}{128}\nu^4\Bigg)\Bigg]. \tag{86}$$

If we consider the phenomenological addition, Eq. (72), the following changes are implied. Let X be one of the following observables and the change δX defined by

$$\delta X = X(H) - X(H+\delta H_{\rm tail}). \tag{87}$$

Then one obtains

$$\delta K(\hat{E},j)^{5\text{PN}}_{\text{loc,h}} = a\eta^{10}\nu^2\left[\frac{63}{16j^{10}} + \frac{105\hat{E}}{8j^8} + \frac{45\hat{E}^2}{4j^6} + \frac{3\hat{E}^3}{2j^4}\right], \tag{88}$$

$$\delta E^{\text{circ}}(j) = 0, \tag{89}$$

$$\delta K^{\text{circ}}(j) = 0, \tag{90}$$

$$\delta\chi_4^{\text{h,loc}} = a\pi\eta^{10}\nu^2\frac{3}{32}p_\infty^6, \quad \text{etc.}, \tag{91}$$

which, again, is a consequence of the transformation of q_{44} only. The expressions for $\chi_k^{\text{h}},\ k = 1,2,3$ remain the same. Yet no consistent solution demanding the constraint (70) and referring to the Hamiltonian for the tail term is obtained.

5 Conclusions

We have calculated the 5PN Hamiltonian in the harmonic gauge using an EFT method both for the potential and the tail terms. For the memory term we obtain a different result comparing to [28] and find one more contributing diagram. From the local contributions to periastron advance $K^{\text{loc,h}}(\hat{E},j)$ it has been possible to derive the five 5PN EOB parameters $q_{82}, q_{63}, q_{44}, \bar{d}_5$ and a_6. The rational contributions of $O(\nu^2)$ to $q_{44}, \bar{d}_5$ and a_6 do also depend on the non–singular multipole moments $L_k\varepsilon_{ijk}Q_{il}Q_{jl}$ and $Q_{ij}Q_{jk}Q_{ki}$, which enter as $D=4$ dependent quantities and are of $O(\nu^2)$. We are not aware of other contributions to the tail term in the EFT approach.

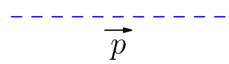
Our results on the observables $K^{\text{loc,h}}(\hat{E},j)$, the total circular peristron advance $K^{\text{tot}}(j)$, the circular binding energy E^{circ} and the contributions to the scattering angle $\chi_k,\quad k\in[1,6]$, do agree with the results of Ref. [15], in the case of the parameters given there, except of the rational $O(\nu^2)$ term of q_{44}. Furthermore, we have also newly obtained the rational contributions of $O(\nu^2)$ to $\bar{d}_5$ and a_6 for the first time. The $O(\nu^2\pi^2)$ contributions to q_{44} were obtained in [14] and those to $\bar{d}_5$ and a_6 in [16] before. These terms stem from the potential contributions. For them the relation (70) holds. Our calculation of the 5PN Hamiltonian ab initio leads to a violation of relation (70) for the $O(\nu^2)$ rational term in q_{44}. As has been outlined in Section 4, the results of [14] could be obtained by envoking the additional term (72, 74) changing only q_{44}, but leaving $\bar{d}_5$ and a_6 invariant. We presented numerical results to 5PN for E^{circ} and K^{circ}.

We are aware of the fact that one may perform resummations of the exact results obtained for the Hamiltonian dynamics to a certain post–Newtonian order, see e.g. [73,80], even with a matching to results from numerical gravity, to be performed in a gauge invariant way. However, it is well–known that resummations of this kind, also applied in phenomenological implementations of exact quantum field theoretic calculations sometimes, cannot accommodate for exact results, since some of the higher order corrections are necessarily not included. The latter ones may be of the same order or even larger than the resummed terms. One example can be found in Ref. [81].

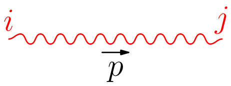
The level of the 5PN corrections to Hamiltonian dynamics exhibits a complexity, which currently could only be solved using the effective field theory approach. The latter has been originally developed for renormalizable quantum field theories. There refined algorithms for the automated computer algebaric derivation of all dynamical contributions and very efficient algorithms for term reduction exist [32,33]. Likewise, many methods to compute the contributing integrals analytically have been developed [71]. By these methods the problem on hand could be solved. Any theory having a path integral representation, including classical mechanics, can be formulated in this way. Here the path integral [62] is a consequence of the variational principles of mechanics [82], the basis of any dynamical physical law. Symbioses of different fields in science lead to progress in the present case. Future calculations will use similar technologies at higher post-Newtonian orders. Given the fast growth of complexity, however, even more refined technologies have to be developed to solve these problems.

A The Feynman rules and integrals

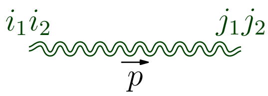
In the following we list the Feynman rules, which are necessary to calculate the convergent contributions to the tail terms. We first present the Feynman rules in the standard case [4] and turn then to those in the in–in formalism.

ϕ : 

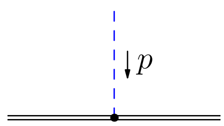
$$= -\frac{i}{2c_d} D(p)\,, \tag{92}$$

A : 

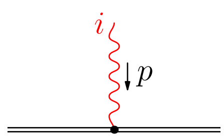
$$= \frac{i\delta_{ij}}{2} D(p)\,, \tag{93}$$

σ : 

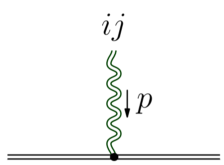
$$= -\frac{i}{2} D(p)[\delta_{i_1j_1}\delta_{i_2j_2} + \delta_{i_1j_2}\delta_{i_2j_1} + (2-c_d)\delta_{i_1i_2}\delta_{j_1j_2}]\,, \tag{94}$$



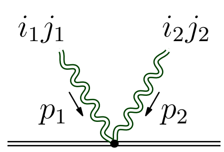
$$= -\frac{i}{\Lambda}\left(E + \frac{1}{2}Q_{ij}p_ip_j + \frac{i}{6}O_{ijk}p_ip_jp_k\right), \tag{95}$$



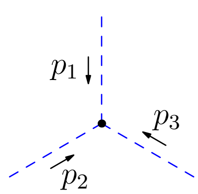
$$= \frac{i}{\Lambda}\left(-\frac{i}{2}L_j\varepsilon_{ijk}p_k + \frac{1}{2}Q_{ij}p_0p_j + \frac{1}{3}J_{jk}\varepsilon_{ijl}p_kp_l + \frac{i}{6}O_{ijk}p_jp_kp_0\right), \tag{96}$$



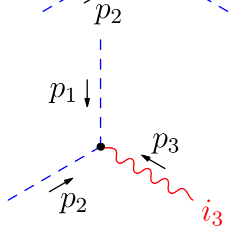
$$= \frac{i}{\Lambda}\left(-\frac{1}{4}Q_{ij}p_0^2 + \frac{1}{6}(J_{ik}\varepsilon_{jkl} + J_{jk}\varepsilon_{ikl})p_lp_0 + \frac{i}{12}O_{ijk}p_kp_0^2\right), \tag{97}$$



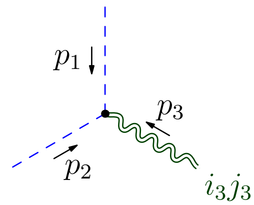
$$= -\tfrac{i}{16\Lambda^2}p_{10}p_{20}(\delta_{i_1i_2}Q_{j_1j_2} + \delta_{i_1j_2}Q_{i_2j_1} + \delta_{i_2j_1}Q_{i_1j_2} + \delta_{j_1j_2}Q_{i_1i_2} + \dots)\,, \tag{98}$$



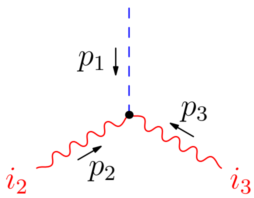
$$= \frac{2ic_d^2}{\Lambda}(p_{10}p_{20} + p_{10}p_{30} + p_{20}p_{30})\,, \tag{99}$$



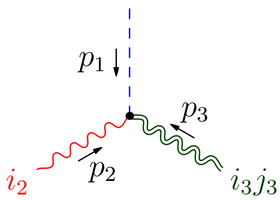
$$= -\frac{2ic_d}{\Lambda}(p_{10}p_{2i_3} + p_{1i_3}p_{20})\,, \tag{100}$$



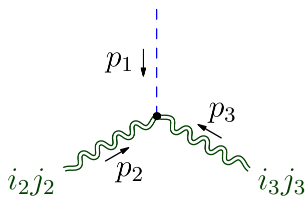
$$= \frac{ic_d}{\Lambda}(-p_{1_3i}p_{2j_3} - p_{1j_3}p_{2i_3} + \delta_{i_3j_3}p_1 \cdot p_2)\,, \tag{101}$$



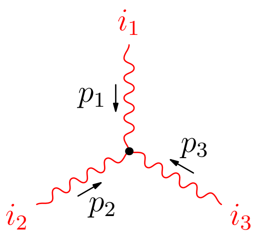
$$= \frac{2ic_d}{\Lambda}(-p_{2i_2}p_{3i_3} + p_{2i_3}p_{3i_2} - 2\delta_{i_2i_3}p_{2i}p_{3i})\,, \tag{102}$$



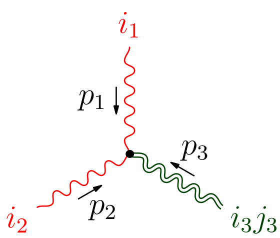
$$\begin{aligned} = \tfrac{ic_d}{\Lambda}\big(& -\delta_{i_3i_2}p_{10}p_{2i_3} + \delta_{i_3i_2}p_{1j_3}p_{20} + \delta_{i_3j_3}p_{10}p_{2i_2} \\ & - \delta_{i_3j_3}p_{1i_2}p_{20} - \delta_{i_2j_3}p_{10}p_{2i_3} + \delta_{i_2j_3}p_{1i_3}p_{20}\big)\,, \end{aligned} \tag{103}$$



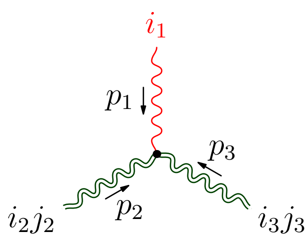
$$= \frac{ic_d}{2\Lambda}p_{20}p_{30}(\delta_{i_2i_3}\delta_{j_2j_3} - \delta_{i_2j_2}\delta_{i_3j_3} + \delta_{i_2j_3}\delta_{i_3j_2})\,, \tag{104}$$



$$\begin{aligned} = \tfrac{2i}{\Lambda}\big(& -\delta_{i_1i_2}p_{10}p_{2i_3} - \delta_{i_1i_2}p_{10}p_{3i_3} + \delta_{i_1i_2}p_{1i_3}p_{20} + \delta_{i_1i_2}p_{1i_3}p_{30} \\ & + \delta_{i_1i_2}p_{20}p_{3i_3} + \delta_{i_1i_2}p_{2i_3}p_{30} + \delta_{i_1i_3}p_{10}p_{2i_2} + \delta_{i_1i_3}p_{10}p_{3i_2} \\ & - \delta_{i_1i_3}p_{1i_2}p_{20} + \delta_{i_1i_3}p_{1i_2}p_{30} - \delta_{i_1i_3}p_{20}p_{3i_2} + \delta_{i_1i_3}p_{2i_2}p_{30} \\ & + \delta_{i_2i_3}p_{10}p_{2i_1} + \delta_{i_2i_3}p_{10}p_{3i_1} + \delta_{i_2i_3}p_{1i_1}p_{20} - \delta_{i_2i_3}p_{1i_1}p_{30} \\ & + \delta_{i_2i_3}p_{20}p_{3i_1} - \delta_{i_2i_3}p_{2i_1}p_{30}\big)\,, \end{aligned} \tag{105}$$

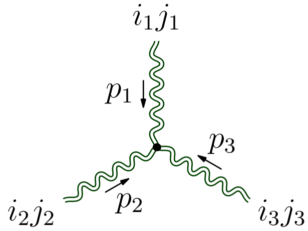


$$\begin{aligned} = \tfrac{i}{\Lambda}\big(& p_1 \cdot p_2(-\delta_{i_1i_2}\delta_{i_3j_3} + \delta_{i_1i_3}\delta_{i_2j_3} + \delta_{i_2i_3}\delta_{i_1j_3}) \\ & + \delta_{i_1i_2}(p_{1i_3}p_{2j_3} + p_{1j_3}p_{2i_3}) + \delta_{i_1i_3}(p_{1i_2}p_{2j_3} - p_{1j_3}p_{2i_2}) \\ & - \delta_{i_2i_3}(p_{1i_1}p_{2j_3} - p_{1j_3}p_{2i_1}) + \delta_{i_1j_3}(p_{1i_2}p_{2i_3} - p_{1i_3}p_{2i_2}) \\ & - \delta_{i_2j_3}(p_{1i_1}p_{2i_3} - p_{1i_3}p_{2i_1}) + \delta_{i_3j_3}(p_{1i_1}p_{2i_2} - p_{1i_2}p_{2i_1})\big)\,, \end{aligned} \tag{106}$$



$$= \frac{i}{8\Lambda}(V^{A\sigma\sigma}_{i_1,i_2j_2,i_3j_3} + V^{A\sigma\sigma}_{i_1,i_2j_2,j_3i_3} + V^{A\sigma\sigma}_{i_1,j_2i_2,i_3j_3} + V^{A\sigma\sigma}_{i_1,j_2i_2,j_3i_3}) \tag{107}$$

$$\begin{aligned} V^{A\sigma\sigma}_{i_1,i_2j_2,i_3j_3} = {} & \delta_{i_2i_3}\delta_{j_2j_3}[4p_{20}p_{2i_1} + 2(p_{1i_1}p_{20} + p_{10}p_{2i_1})] + \delta_{i_3j_3}[-2p_{20}(\delta_{i_1i_2}p_{1j_2} \\ & + \delta_{i_2j_2}p_{2i_1}) - \delta_{i_2j_2}(p_{1i_1}p_{20} + p_{10}p_{2i_1})] + \delta_{i_1i_2}[2\delta_{i_3j_3}p_{10}p_{2j_2} \\ & + \delta_{i_3j_2}(4p_{1j_3}p_{20} - 4p_{10}p_{2j_3})] + \delta_{i_1i_3}[\delta_{i_2j_3}(-4p_{1j_2}p_{20} \\ & + 4p_{10}p_{2j_2}) + \delta_{i_2j_2}(2p_{1j_3}p_{20} - 2p_{10}p_{2j_3})] \end{aligned} \tag{108}$$



$$= \frac{i}{32\Lambda}\left(\tilde{V}^{\sigma\sigma\sigma}_{i_1j_1,i_2j_2,i_3j_3} + \tilde{V}^{\sigma\sigma\sigma}_{j_1i_1,i_2j_2,i_3j_3}\right) \tag{109}$$

$$\tilde{V}^{\sigma\sigma\sigma}_{i_1j_1,i_2j_2,i_3j_3} = V^{\sigma\sigma\sigma}_{i_1j_1,i_2j_2,i_3j_3} + V^{\sigma\sigma\sigma}_{i_1j_1,j_2i_2,i_3j_3} + V^{\sigma\sigma\sigma}_{i_1j_1,i_2j_2,j_3i_3} + V^{\sigma\sigma\sigma}_{i_1j_1,j_2i_2,j_3i_3} \tag{110}$$

$$\begin{aligned}
V^{\sigma\sigma\sigma}_{i_1j_1,i_2j_2,i_3j_3} = {} & (p_1^2 + p_1\cdot p_2 + p_2^2)\Big(-\delta_{i_2j_2}\big(2\delta_{i_1i_3}\delta_{j_1j_3} - \delta_{i_1j_1}\delta_{i_3j_3}\big) \\
& \quad + 2\big[\delta_{i_1i_2}\big(4\delta_{j_1i_3}\delta_{j_2j_3} - \delta_{j_1j_2}\delta_{i_3j_3}\big) - \delta_{i_1j_1}\delta_{i_2i_3}\delta_{j_2j_3}\big]\Big) \\
& + 2\Big\{4\big(p_{1j_3}p_{2j_1} - p_{1j_1}p_{2j_3}\big)\delta_{i_1i_2}\delta_{j_2i_3} \\
& \quad + 2\big[\big(p_{1i_1} + p_{2i_1}\big)p_{2j_1}\delta_{i_2i_3}\delta_{j_2j_3} - p_{1i_3}p_{2j_3}\delta_{i_1i_2}\delta_{j_1j_2}\big] \\
& \quad + \delta_{i_2j_2}\big[p_{1i_3}p_{2j_3}\delta_{i_1j_1} + 2\big(p_{1j_3}p_{2j_1} - p_{1j_1}p_{2j_3}\big)\delta_{i_1i_3} \\
& \quad - \big(p_{1i_1} + p_{2i_1}\big)p_{2j_1}\delta_{i_3j_3}\big] \\
& \quad + p_{2j_2}\Big(4p_{1j_1}\delta_{i_1i_3}\delta_{i_2j_3} + p_{1i_2}\big(2\delta_{i_1i_3}\delta_{j_1j_3} - \delta_{i_1j_1}\delta_{i_3j_3}\big) \\
& \qquad + 2\big[\delta_{i_1i_2}\big(p_{1j_1}\delta_{i_3j_3} - 2p_{1j_3}\delta_{j_1i_3}\big) - p_{1j_3}\delta_{i_1j_1}\delta_{i_2i_3}\big]\Big) \\
& \quad + p_{1j_2}\Big(p_{1i_2}\big(2\delta_{i_1i_3}\delta_{j_1j_3} - \delta_{i_1j_1}\delta_{i_3j_3}\big) - 4p_{2j_1}\delta_{i_1i_3}\delta_{i_2j_3} \\
& \qquad + 2\big[p_{2j_3}\delta_{i_1j_1}\delta_{i_2i_3} + \delta_{i_1i_2}\big(2p_{2j_3}\delta_{j_1i_3} - p_{2j_1}\delta_{i_3j_3}\big)\big]\Big)\Big\},
\end{aligned} \tag{111}$$

with

$$c_d = \frac{2(d-1)}{d-2}, \tag{112}$$

$\Lambda^{-1} = \sqrt{32\pi G_N}$ and $d = 3 - 2\varepsilon$. The scalar propagators $D(p)$ are left generic and will be specified either as causal, retarded, or advanced propagators, cf. Section B. Furthermore, in the specific in–in calculations below one has to replace in (97, 98) for the electric quadrupole moment the vertices as given in (134–136). Futhermore, we list the contributing field combinations in Table 1.

Eq.	contributions
(99)	$++-, ---$
(100)	$++-, +-+, ---$
(101)	$++-, +-+, ---$
(102)	$++-, -++, ---$
(103)	$++-, +-+, -++, ----$
(104)	$++-, -++, ---$
(105)	$++-, ---$
(106)	$++-, +-+, ---$
(107)	$++-, -++, ---$
(109)	$++-, ---$

Table 1: The contributing field combinations in the in–in formalism.

Here the bulk vertices have to be rescaled by $1/\sqrt{2}$. The free–field combinations are defined in Eq. (130).

The D–dimensional integral over an Euclidean momentum k_E is given by, cf. e.g. [83],

$$\int \frac{d^D\vec{k}_E}{(2\pi)^D}\frac{(\vec{k}_E^2)^r}{(\vec{k}_E^2+R^2)^m} = \frac{(-1)^{r-m}}{(4\pi)^{D/2}}\frac{\Gamma(r+D/2)}{\Gamma(D/2)}\Gamma(m-r-D/2)(R^2)^{D/2+r-m}. \tag{113}$$

B Invariant functions

In the following we summarize different functions, related to the scalar field operators $\Phi(x_1)$ and $\Phi(x_2)$, and $x = x_1 - x_2$, leading to the different kind of propagators, [84], which are distribution–valued in part [85, 86]. The corresponding contours for the defining integrals are shown in Figure 4.

We start with the commutator

$$i\Delta(x) = [\Phi(x_1), \Phi(x_2)]_- = -i\int \frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{1}{\omega_k}\frac{e^{-ikx}}{k^2-m^2}, \tag{114}$$

with $\omega_k = \sqrt{\vec{k}^2+m^2}$, denoting the Jordan–Pauli function [87], also called Hadamard function [88]. It has the contour integral representation

$$\Delta(x)=\int_{C_0}\frac{d^Dk}{(2\pi)^D}\frac{e^{ik.x}}{k^2-m^2} = \Delta_+(x)+\Delta_-(x) = \Delta_{\rm ret}(x)-\Delta_{\rm adv}(x), \tag{115}$$

showing also the relation to other quantities

(116)

$$\Delta_1(x)=\int_{C_1}\frac{d^Dk}{(2\pi)^D}\frac{e^{ik.x}}{k^2-m^2} = \Delta_+(x)-\Delta_-(x) \tag{117}$$

$$\Delta_\pm(x)=\int_{C_\pm}\frac{d^Dk}{(2\pi)^D}\frac{e^{ik.x}}{k^2-m^2} = \frac{1}{2}\left[\Delta(x)\pm\Delta_1(x)\right], \tag{118}$$

$$\Delta_c(x)=-i\int_{C}\frac{d^Dk}{(2\pi)^D}\frac{e^{ik.x}}{k^2-m^2} = -i\left[\Theta(t)\Delta_+(x)+\Theta(-t)\Delta_-(x)\right],$$

(119)

$$\Delta_D(x)=-i\left[\Theta(t)\Delta_-(x)+\Theta(-t)\Delta_+(x)\right],$$

(120)

$$\Delta_{\rm ret}(x)=\int_{C_{\rm ret}}\frac{d^Dk}{(2\pi)^D}\frac{e^{ik.x}}{k^2-m^2} = \Theta(t)\Delta(x), \tag{121}$$

$$\Delta_{\rm adv}(x)=\int_{C_{\rm adv}}\frac{d^Dk}{(2\pi)^D}\frac{e^{ik.x}}{k^2-m^2} = -\Theta(-t)\Delta(x), \tag{122}$$

with $\Theta(t)$ the Heaviside function.

The causal Green's function $\Delta_c(x)$ [89] is also called Feynman function $\Delta_F(x)$ and the Dyson function $\Delta_D(x)$ is also called anti–causal Green's function.

In momentum space the causal, retaded and advance propagators read

$$\Delta_c(k) = \frac{1}{k^2-m^2+i0}, \tag{123}$$

$$\Delta_{\rm ret}(k) = \frac{1}{k^2-m^2+(2k_0)i0}, \tag{124}$$

$$\Delta_{\text{adv}}(k) = \frac{1}{k^2 - m^2 - (2k_0)i0}, \tag{125}$$

where the distribution relations [85, 86, 90]

$$\frac{1}{x \pm i0} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \tag{126}$$

hold, with $\mathcal{P}$ Cauchy's principal value.

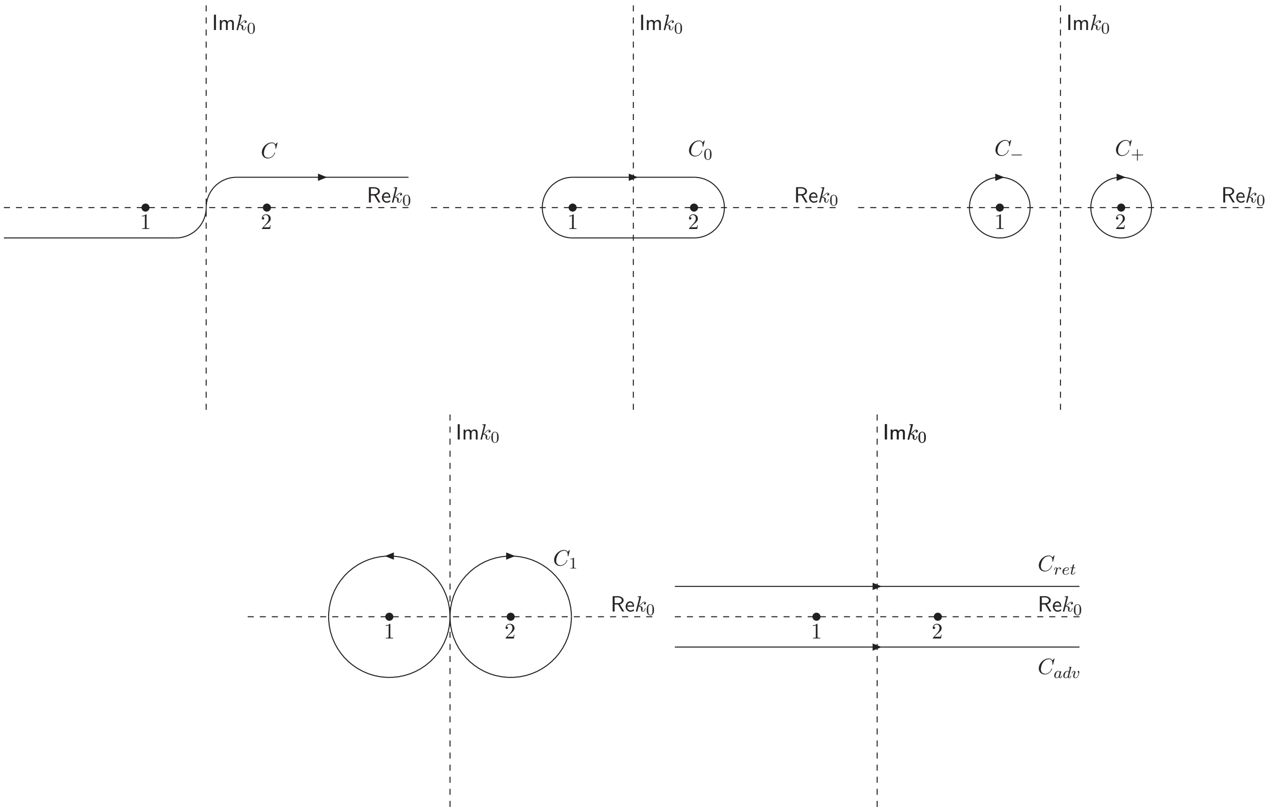


Figure 4: The integration paths in the complex k_0 plane for the different types of propagator functions. The paths correspond to the integrals given in the text and correspond to the causal Green's function (C), the Jordan–Pauli function (C_0), the two Wightman functions ($C_\pm$), the anticommutator (C_1), and the retarded and advanced Green's functions ($C_{\text{ret(adv)}}$), where C_1 denotes a single path.

C The in-in formalism

The in–in formalism for binary systems in classical gravity refers to a well–defined initial state at $t = -\infty$, which is also defined to be the final state, and one integrates over the time paths t_F and t_B

$$t_F = (-\infty, +\infty), \qquad t_B = (+\infty, -\infty), \tag{127}$$

which are linked together, with $t_F = t_1$ and $t_B = t_2$ labeling the time path. One should note that different authors use a quite different notation. For definiteness we refer to the one by Keldysh [51] also given in [54]. In the following 2–dimensional representations the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{128}$$

are of use. One has

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \frac{1}{\sqrt{2}}(1+i\sigma_2)\begin{pmatrix}\Psi_-\\ \Psi_+\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\Psi_+ + \Psi_-\\ \Psi_+ - \Psi_-\end{pmatrix} \tag{129}$$

for the fields, coordinates, and Schwinger–parameters, as well as later also the multipole moments, leaving the action invariant. The free fields are doubled to allow for a single time, i.e. we set

$$\Phi(\vec{x}, t_i) = \Phi_i(x, t) \tag{130}$$

and use (129) to form the $\pm$ combinations.

We first consider the path integral for the general motion of the gravitating two–body system, not specifying yet either to the near or far zones. In the in–in formalism it is given by [60]

$$\begin{aligned}\exp\left[iW(\{\vec{j}_{k,1}\},\{\vec{j}_{k,2}\},J_1,J_2)\right] = \int \prod_{k=1}^{2} \mathcal{D}\vec{x}_{k,1}\mathcal{D}\vec{x}_{k,1} \exp\Bigg\{ & i\sum_{k=1}^{2}\left(S_{pp}(\vec{x}_{k,1}) - S_{pp}(\vec{x}_{k,2})\right) \\ & + i\sum_{k=1}^{2}\int dt\left(\vec{j}_{k,1}.\vec{x}_{k,1} - \vec{j}_{k,2}.\vec{x}_{k,2}\right) \\ & + i\int d^Dx \mathcal{L}_{\rm int}\left[\{\vec{x}_{k,1}\},\{\vec{x}_{k,2}\},-i\delta_{J_1},-i\delta_{J_2}\right]\Bigg\} Z_0\left[J_1,J_2\right].\end{aligned} \tag{131}$$

Here $\vec{x}_k$ denote the positions of the gravitating masses, $\vec{j}_k$ the associated Schwinger parameters, S_{pp} the point particle action, Eq. (10) [13], $\mathcal{L}_{\rm int}$ the interaction Lagrangian density, J_i are the Schwinger parameters to Φ_i, with

$$\delta_{J_k} \equiv \frac{\delta}{\delta J_k}, \tag{132}$$

and Z_0 denotes the free gravitational field part of the path integral.

As has been shown in [91], Eq. (2.22), the functional derivative of the coarse grained effective action $S_{\rm CGEA}$

$$\left.\frac{\delta S_{\rm CGEA}}{\delta x_{k-}}\right|_{x_1=x_2=\bar{x}} = 0, \tag{133}$$

which implies that in the tail terms being dealt with below the contributing functions are at most $\propto x_{k-}$. This is implied by the $\mathbb{Z}_-$ operator applied to the path integrals for the tail term below.

We are now specifying to the calculation of tail term contributions in the far zone. For this purpose the multipole expansion has to be carried out and the multipole moments appear in the effective interaction Lagrangian, cf. (3), as new entities.

Let us consider the vertex functions, with $Q_{1(2)}$ the electric quadrupole moments with indices suppressed, which contribute to the tail terms shown in Figure 1 and the graviton self–interaction vertex. The symbols $h_{1(2)}$ denote the respective field couplings given by $O_k(\Phi_{1(2)})$ as different linear differential operators acting on the decomposition of the gravitational field into 10 fields Φ, as described in [4]. One obtains

$$V_{Qh} = Q_1h_1 - Q_2h_2 = Q_-h_+ + Q_+h_-, \tag{134}$$

$$V_{Qh^2} = Q_1h_1^2 - Q_2h_2^2 = \frac{1}{\sqrt{2}}\left[2h_+h_-Q_+ + (h_+^2 + h_-^2)Q_-\right], \tag{135}$$

$$V_{h^3} = \left[h_1^3 - h_2^3\right] \; = \; \frac{1}{\sqrt{2}}\left[3h_+^2 h_- + h_-^3\right]. \tag{136}$$

This notation is symbolic, but sufficient to derive the corresponding Feynman diagrams by functional differentiation for the Schwinger parameters. It is understood that the respective Feynman rules given in Appendix A are used in the final result. From Eq. (134) one obtains for the electric quadrupole moments at leading order (0PN)

$$Q_{ij,1(2)} = \sum_{a=1}^{2} m_a \left(x_{a,1(2),i}x_{a,1(2),j} - \delta_{ij}\frac{x_{a,1(2),k}x_{a,1(2),k}}{d}\right), \tag{137}$$

the $\pm$ projections

$$Q_{ij,-} \;=\; \sum_{a=1}^{2} \frac{m_a}{\sqrt{2}}\left(x_{a,-,i}x_{a,+,j} + x_{a,+,i}x_{a,-,j} - 2\frac{\delta_{ij}}{d}x_{a,-,k}x_{a,+,k}\right) \tag{138}$$

$$Q_{ij,+} \;=\; \sum_{a=1}^{2} \frac{m_a}{\sqrt{2}}\left[x_{a,+,i}x_{a,+,j} + x_{a,-,j}x_{a,-,j} - \frac{\delta_{ij}}{d}\left(x_{a,+,k}x_{a,+,k} + x_{a,-,k}x_{a,-,k}\right)\right], \tag{139}$$

with $x_l \equiv x_l(t)$.

The contribution of the interaction terms to the path integral read

$$\exp\left[-iS_{\rm int}\right] = \exp\left[-i\int d^D z\left(V_{Qh}(z) + V_{Qh^2}(z) + V_{h^3}(z)\right)\right], \tag{140}$$

where the fields $\Phi_\pm$ are replaced by the folloing functional derivative

$$\Phi_\pm \to i\frac{\delta}{\delta J_\pm}. \tag{141}$$

Equivalently on may use

$$h_\pm \to i\frac{\delta}{\delta J_\pm}\frac{\partial O_k(\Phi_\pm)}{\partial \Phi_\pm} \equiv i\delta_{J_\pm}\frac{\partial O_k(\Phi_\pm)}{\partial\Phi_\pm}. \tag{142}$$

For the free–field part the path integral can be integrated to [92, 93]

$$\exp[-\frac{i}{2}\mathbf{J}^T\hat{G}\mathbf{J}] \quad \text{with} \quad \mathbf{J} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \tag{143}$$

and

$$\hat{G} = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}, \tag{144}$$

where

$$\Delta_{11} \;=\; -i\Theta(t)\Delta_- - i\Theta(-t)\Delta_+, \tag{145}$$

$$\Delta_{12} \;=\; -i\Delta_+ \tag{146}$$

$$\Delta_{21} \;=\; -i\Delta_- \tag{147}$$

$$\Delta_{22} \;=\; -i\Theta(-t)\Delta_- - i\Theta(t)\Delta_+. \tag{148}$$

The following relation holds

$$\Delta_{11} + \Delta_{22} \;=\; \Delta_{12} + \Delta_{21}. \tag{149}$$

Eq. (144) can be rewritten by

$$\hat{G} = \frac{1}{2} \begin{pmatrix} \Delta_{\rm ret} + \Delta_{\rm adv} + \Delta_{\rm C} & -\Delta_{\rm ret} + \Delta_{\rm adv} + \Delta_{\rm C} \\ \Delta_{\rm ret} - \Delta_{\rm adv} + \Delta_{\rm C} & -\Delta_{\rm ret} - \Delta_{\rm adv} + \Delta_{\rm C} \end{pmatrix}, \tag{150}$$

where

$$\Delta_{\rm adv} \;=\; \frac{1}{2}\left[\Delta_{11} - \Delta_{21} + \Delta_{12} - \Delta_{22}\right], \tag{151}$$

$$\Delta_{\rm ret} \;=\; \frac{1}{2}\left[\Delta_{11} - \Delta_{12} + \Delta_{21} - \Delta_{22}\right], \tag{152}$$

$$\Delta_{\rm C} \;=\; \frac{1}{2}\left[\Delta_{11} + \Delta_{22} + \Delta_{12} + \Delta_{21}\right]. \tag{153}$$

After functional differentiation retarded and advanced propagators appear in different directions, which one may synchronize using

$$\Delta_{\rm adv,ret}(x_j - x_i) \;=\; \Delta_{\rm ret,adv}(x_i - x_j). \tag{154}$$

One further considers the transformed matrix $\tilde{G}$ [51] and the transformed vectors

$$\tilde{G} = \frac{1}{\sqrt{2}}(1 - i\sigma_2)\hat{G}\frac{1}{\sqrt{2}}(1 + i\sigma_2) \;=\; \begin{pmatrix} 0 & \Delta_{\rm adv} \\ \Delta_{\rm ret} & \Delta_{\rm C} \end{pmatrix}, \tag{155}$$

appearing in the combination

$$\tilde{\mathbf{J}}^T \tilde{G} \tilde{\mathbf{J}} = \mathbf{J}^T \hat{G} \mathbf{J}, \tag{156}$$

with

$$\tilde{\mathbf{J}} = \frac{1}{\sqrt{2}}(1 - i\sigma_2)\mathbf{J} = \begin{pmatrix} J_- \\ J_+ \end{pmatrix} \quad \text{and} \quad J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm J_2). \tag{157}$$

This leads to the free propagator contribution (143)

$$\exp\left[-iS_{\rm free}\right]=\exp\Biggl[-\frac{i}{2}\int dx \int dy \,\bigl(J_-(x)\Delta_{\rm adv}(x-y)J_+(y) + J_+(x)\Delta_{\rm ret}(x-y)J_-(y) + J_+(x)\Delta_{\rm C}(x-y)J_+(y)\bigr)\Biggr]. \tag{158}$$

In deriving the Feynman diagrams of Figure 1 we consider the connected Green's function from the beginning [63], since the disconnected diagrams are canceled by the denominator function and apply the $\mathbb{Z}_-$ operator,

$$\mathbb{Z}_-\left[\exp\left[-iS_{\rm int}(\delta J_\pm)\right]\exp\left[-iS_{\rm free}(J_\pm)\right]\right]_{\rm conn.} \tag{159}$$

Within the in–in formalism one ends up with representations in which the multipole moments, M_I, appear in terms of their projections $M_{I,\pm}$. At 5PN the tail contributions calculated in Appendix C depend on the electric quadrupole moment only. The $\mathbb{Z}_-$ operator projects onto contributions containing one multipole moment Q^{ij}_- only. For the translation from $Q^{ij}_\pm$ to Q^{ij} see Ref. [91].

D Calculation of tail term diagrams

All contributing Feynman diagrams are of two–loop order with maximally three propagators. We employ integration-by-parts [32, 33]

$$\int d^d k \int d^d q \frac{\partial}{\partial p_\mu} \frac{r_\mu}{k^2 q^2 (k+q)^2} = 0, \quad p, r \in \{k, q\}. \tag{160}$$

One obtains

$$I_{111} = \int d^d k \int d^d q \frac{1}{k^2 q^2 (k+q)^2} = -\left(\frac{1}{4\varepsilon} + \frac{1}{2}\right) \left[-\frac{I_{1,1,0}}{l_1^0 l_2^0} + \frac{I_{1,0,1}}{l_1^0 (l_1^0 + l_2^0)} + \frac{I_{1,0,1}}{l_2^0 (l_1^0 + l_2^0)} \right]. \tag{161}$$

The integrals $I_{i_1 i_2 i_3}$ are given by

$$I_{i_1 i_2 i_3} = \int d^d k \int d^d q \frac{1}{(k^2)^{i_1} (q^2)^{i_2} ((k+q)^2)^{i_3}}. \tag{162}$$

There are also other possibilities to calculate the three–propagator integrals, like hypergeometric methods [70, 71] and/or the use of one Mellin–Barnes integral [97]. In all these represenations one has to maintain the distribution chatacter of these integrals in all intermediary steps, which can be technically demanding. The IBP method, on the other hand, maintains the propagator structure in all these respects and is therefore the method of choice in the following.

In the following we present the explicit calculation of the diagrams shown in Figure 1. The 5PN diagram at the r.h.s. is obtained as the closed Green's function[18]

$$\begin{aligned} I_1 = \mathbb{Z}_- \Bigg[\tfrac{1}{3!} \prod_{l=1}^{3} \Big[-i \int d^D w_l (V_{Qh}(w_l) + V_{Qh^2}(w_l)) \Big] \tfrac{1}{2!} \prod_{k=1}^{2} \Bigg[-\frac{i}{2} \Bigg(\int dy_k^D dz_k^D J_-(y_k) \Delta_{\rm adv}(y_k - z_k) \\ \times J_+(z_k) + J_+(y_k) \Delta_{\rm ret}(y_k - z_k) J_-(z_k) + J_+(y_k) \Delta_{\rm C}(y_k - z_k) J_+(z_k) \Bigg) \Bigg] \Bigg], \end{aligned} \tag{163}$$

i.e. integrating over the coordinates x_1 to x_3 and applying the $\mathbb{Z}_-$ operator, leading to

$$\begin{aligned} I_1 = \Bigg[\frac{(-i)^3 i^4}{3!} \int dx_1^D \int dx_2^D \int dx_3^D \Big[& \delta_{J_-(x_1)} Q_+(x_1) V_+(x_1) \delta^2_{J_+(x_2)} Q_-(x_2) V_-(x_2) \delta_{J_-(x_3)} Q_+(x_3) V_+(x_3) \\ & + \delta^2_{J_+(x_1)} Q_-(x_1) V_-(x_1) \delta_{J_-(x_2)} Q_+(x_2) V_+(x_2) \delta_{J_-(x_3)} Q_+(x_3) V_+(x_3) \\ & + \delta_{J_-(x_1)} Q_+(x_1) V_+(x_1) \delta_{J_-(x_2)} Q_+(x_2) V_+(x_2) \delta^2_{J_+(x_3)} Q_-(x_3) V_-(x_3) \\ + 2 \big[& \delta_{J_-(x_1)} Q_+(x_1) V_+(x_1) \delta_{J_-(x_2)} \delta_{J_+(x_2)} Q_+(x_2) V_+(x_2) \delta_{J_+(x_3)} Q_-(x_3) V_-(x_3) \\ & + \delta_{J_-(x_1)} \delta_{J_+(x_1)} Q_+(x_1) V_+(x_1) \delta_{J_-(x_2)} Q_+(x_2) V_+(x_2) \delta_{J_+(x_3)} Q_-(x_3) V_-(x_3) \\ & + \delta_{J_+(x_1)} Q_-(x_1) V_-(x_1) \delta_{J_-(x_2)} \delta_{J_+(x_2)} Q_+(x_2) V_+(x_2) \delta_{J_-(x_3)} Q_+(x_3) V_+(x_3) \\ & + \delta_{J_-(x_1)} \delta_{J_+(x_1)} Q_+(x_1) V_+(x_1) \delta_{J_+(x_2)} Q_-(x_2) V_-(x_2) \delta_{J_-(x_3)} Q_+(x_3) V_+(x_3) \\ & + \delta_{J_-(x_1)} Q_+(x_1) V_+(x_1) \delta_{J_+(x_2)} Q_-(x_2) V_-(x_2) \delta_{J_-(x_3)} \delta_{J_+(x_3)} Q_+(x_3) V_+(x_3) \\ & + \delta_{J_+(x_1)} Q_-(x_1) V_-(x_1) \delta_{J_-(x_2)} Q_+(x_2) V_-(x_2) \delta_{J_-(x_3)} \delta_{J_+(x_3)} Q_+(x_3) V_+(x_3) \big] \Big] \\ \times \tfrac{1}{2!} \prod_{k=4}^{5} \Bigg[-\frac{i}{2} \Bigg(\int dx_k^D dy_k^D J_-(x_k) \Delta_{\rm adv}(x_k - y_k) J_+(y_k) \end{aligned}$$

[18] We have provided the corresponding `mathematica` code to S. Foffa and R. Sturani prior to publication of the present paper.

$$+J_+(x_k)\Delta_{\rm ret}(x_k-y_k)J_-(y_k)+J_+(x_k)\Delta_{\rm C}(x_k-y_k)J_+(y_k))]\Bigg]_{\rm conn.}, \tag{164}$$

$$=-2\int dx_1^D\int dx_2^D\int dx_3^D\Delta_{\rm adv}(x_1-x_2)Q_+(x_1)V_+(x_1)$$
$$\times\left[\Delta_{\rm adv}(x_2-x_3)Q_+(x_2)V_+(x_2)Q_-(x_3)V_-(x_3)+\frac{\Delta_{\rm ret}(x_2-x_3)}{2}Q_-(x_2)V_-(x_2)Q_+(x_3)V_+(x_3)\right]. \tag{165}$$

Here $V_\pm(x_i)$ denote the numerator functions at the respective vertices, while $G_{r(a)}(x_i-x_j)$ are the retarded and advanced propagators in configuration space. In the diagrams of Figure 1 the propagator $\Delta_{\rm C}(x-y)$ does not contribute.

The Fourier transform from momentum to configuration space and its inverse are defined by [85, 86]

$$\mathbf{F}[f](r)=\int\frac{d^Dk}{(2\pi)^D}\exp[ik.r]f(k) \tag{166}$$
$$\mathbf{F}^{-1}[g](k)=\int d^Dr\exp[-ik.r]g(r). \tag{167}$$

One obtains the following contribution to the action in momentum space[19]

$$\mathsf{S}_1=-\frac{8}{3}\sqrt{2}\pi^2G_N^2\int_{-\infty}^{+\infty}\frac{dl_1^0}{2\pi}\int_{-\infty}^{+\infty}\frac{dl_2^0}{2\pi}\int\frac{d^d\mathbf{l}_1}{(2\pi)^d}\int\frac{d^d\mathbf{l}_2}{(2\pi)^d}\frac{\tilde{Q}_+^{ij}(l_1^0)}{|\mathbf{l}_1|^2-(l_1^0+i\delta)^2}$$
$$\times\left[\frac{P_{11}[l_1^0,l_2^0]\tilde{Q}_+^{jk}(-l_1^0-l_2^0)\tilde{Q}_-^{ki}(l_2^0)}{|\mathbf{l_2}|^2-(l_2^0+i\delta)^2}+\frac{1}{2}\frac{P_{12}[l_1^0,l_2^0]\tilde{Q}_-^{jk}(-l_1^0-l_2^0)\tilde{Q}_+^{ki}(l_2^0)}{|\mathbf{l_2}|^2-(l_2^0-i\delta)^2}\right], \tag{168}$$

with

$$-\int\frac{d^d\mathbf{l}_1}{(2\pi)^d}\frac{1}{|\mathbf{l}_1|^2-(l_1^0-i\delta)^2}=\frac{i}{4\pi}l_1^0, \tag{169}$$
$$-\int\frac{d^d\mathbf{l}_1}{(2\pi)^d}\frac{1}{|\mathbf{l}_1|^2-(l_1^0+i\delta)^2}=-\frac{i}{4\pi}l_1^0 \tag{170}$$

for $d=3$ represented by an analytic continuation in case [96]. Eq. (168) has been obtained after tensor decomposition and the use of master integrals, such that the three momenta appear only in the propagators.

We use

$$\tilde{Q}_\pm(k_0)=\int_{-\infty}^{+\infty}dt\exp[-ik_0t]Q_\pm(t) \tag{171}$$

leading to

$$\mathsf{S}_1=-\frac{\sqrt{2}}{6}G_N^2\int_{-\infty}^{+\infty}\frac{dl_1^0}{2\pi}\int_{-\infty}^{+\infty}\frac{dl_2^0}{2\pi}l_1^0l_2^0\tilde{Q}_+(l_1^0)$$
$$\times\left[P_{11}[l_1^0,l_2^0]\tilde{Q}_+(-l_1^0-l_2^0)\tilde{Q}_-(l_2^0)-\frac{1}{2}P_{12}[l_1^0,l_2^0]\tilde{Q}_-(-l_1^0-l_2^0)\tilde{Q}_+(l_2^0)\right], \tag{172}$$

[19] The spatial coordinates of the multipole moments are kept fixed.

with

$$P_{11}(x,y) = P_{12}(x,y) = x^3y^3, \tag{173}$$

resulting in[20]

$$\mathsf{S}_1 = -\frac{\sqrt{2}}{12} G_N^2 \int_{-\infty}^{+\infty} dt \ \mathrm{tr}\left\{2Q_+(t)Q_+^{(4)}(t)Q_-^{(4)}(t) - Q_-(t)(Q_+^{(4)}(t))^2\right\}. \tag{174}$$

.

Let us also remark on the mathematical structure in the causal case for completeness, which does not contribute in the present case. Here the derivation of the final result requires a different technique. One has

$$\tilde{\mathsf{S}}_1^{\rm c} = -8\pi^2 G_N^2 \int_{-\infty}^{+\infty} \frac{dl_1^0}{2\pi} \int_{-\infty}^{+\infty} \frac{dl_2^0}{2\pi} \int \frac{d^{D-1}\mathbf{l}_1}{(2\pi)^{D-1}} \int \frac{d^{D-1}\mathbf{l}_2}{(2\pi)^{D-1}} P_2[l_1^0, l_2^0] \frac{\tilde{Q}(l_1^0)\tilde{Q}(l_2^0)\tilde{Q}(-l_1^0 - l_2^0)}{[(l_1^0)^2 - \mathbf{l}_1^2 + i\delta][(l_2^0)^2 - \mathbf{l}_2^2 + i\delta]}. \tag{175}$$

This integral leads to terms $((l_k^0)^2 + i\delta)^{1/2}$ which are Schwartz $\mathcal{S}'$ distributions [85]. One has, cf. [86],

$$(z + i\delta)^\lambda = z_+^\lambda + \exp(+i\pi\lambda) z_-^\lambda \tag{176}$$
$$(z - i\delta)^\lambda = z_+^\lambda + \exp(-i\pi\lambda) z_-^\lambda, \tag{177}$$

with $z, \lambda \in \mathbb{R}$ and $\lambda \neq 0, -1, -2, -3, ...$ and

$$z_+^\lambda = \theta(+z) z^\lambda \tag{178}$$
$$z_-^\lambda = \theta(-z)(-z)^\lambda. \tag{179}$$

Due to the polynomial P_2 in (175) we have to consider Fourier-transforms of $f \in \mathcal{S}'$ distributions of the kind

$$\mathbf{F}^{-1}[x^m f(x)](\sigma) = (-i)^m \frac{\partial^m}{\partial \sigma^m} \mathbf{F}^{-1}[f(x)](\sigma), \quad m \in \mathbb{N}. \tag{180}$$

Since $(l_1^0)^2 > 0$ one has

$$\left((l_1^0)^2 + i\delta\right)^{1/2} = \left[\sqrt{(l_1^0)^2}\right]_+ = |l_1^0|. \tag{181}$$

We consider now the one-dimensional Fourier transform of the distribution $z_\pm^\lambda$, cf. [86].

$$\mathbf{F}^{-1}[z_+^\lambda](t) = i \frac{\Gamma(\lambda+1)}{2\pi} \frac{\exp[i\lambda\pi/2]}{(-t + i\delta)^{\lambda+1}}, \tag{182}$$
$$\mathbf{F}^{-1}[z_-^\lambda](t) = -i \frac{\Gamma(\lambda+1)}{2\pi} \frac{\exp[-i\lambda\pi/2]}{(-t - i\delta)^{\lambda+1}} \tag{183}$$

and one has

$$\mathbf{F}^{-1}[|z|](t) = \mathbf{F}^{-1}[z_+](t) + \mathbf{F}^{-1}[z_-](t) = -\frac{1}{2\pi}\left[\frac{1}{(t+i\delta)^2} + \frac{1}{(t-i\delta)^2}\right]. \tag{184}$$

[20] This term would vanish for $P_{11}(x,y) = P_{12}(x,y)/2$.

Only the first term in the r.h.s. contributes to the contour integral using the residue theorem.

As an example we consider

$$P_2(l_1^0, l_2^0) = (l_1^0)^2 (l_2^0)^2 [l_1^0 + l_2^0]^2. \tag{185}$$

According to (180) the polynomial implies the term

$$-\frac{1}{4\pi^2}\left[\frac{1440}{(t_1 - t + i\delta)^6 (t_2 - t + i\delta)^4} + \frac{1152}{(t_1 - t + i\delta)^5 (t_2 - t + i\delta)^5}\right]. \tag{186}$$

Therefore we have

$$\begin{aligned}
\tilde{\mathsf{S}}_1^{\rm c} =& -8\pi^2 G_N^2 \int_{-\infty}^{+\infty} dt Q(t) \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \frac{(-1)}{16\pi^2}\frac{(-1)}{4\pi^2} \\
& \times \left\{\frac{1440}{(t_1 - t + i\delta)^6 (t_2 - t + i\delta)^4} + \frac{1152}{(t_1 - t + i\delta)^5 (t_2 - t + i\delta)^5}\right\} Q(t_1) Q(t_2) \qquad (187) \\
=& 16\pi^2 G_N^2 \frac{(-1)^2}{64\pi^4}(2\pi i)^2 \int_{-\infty}^{+\infty} dt Q(t) \left\{\left[\frac{d^5}{dt^5}Q(t)\right]\left[\frac{d^3}{dt^3}Q(t)\right] + \left[\frac{d^4}{dt^4}Q(t)\right]^2\right\} \\
=& -G_N^2 \int_{-\infty}^{+\infty} dt Q(t) \left\{\left[\frac{d^5}{dt^5}Q(t)\right]\left[\frac{d^3}{dt^3}Q(t)\right] + \left[\frac{d^4}{dt^4}Q(t)\right]^2\right\} \qquad (188) \\
=& -\frac{G_N^2}{2} \int_{-\infty}^{+\infty} dt \left[\frac{d^2}{dt^2}Q(t)\right]\left[\frac{d^3}{dt^3}Q(t)\right]^2, \qquad (189)
\end{aligned}$$

using the residue theorem, [95], since $Q(t)$ is bounded and obeys a Taylor expansion. The result is given in (188). In (189) it has been further assumed, that $Q(t)$ and the derivatives of $Q(t)$ vanish in the limit $t \to \pm\infty$. (189) is again a Riemann integral.

We finally turn to the memory term, Figure 1 r.h.s. The corresponding Green's function is obtained by carrying out the functional derivations in I_2

$$\begin{aligned}
I_2 = & \Bigg[\frac{i^{10}}{3!} \int dx_1^D \int dx_2^D \int dx_3^D \int dx_4^D \Big[\delta_{J_-(x_1)} Q_+(x_1) V_+(x_1) \delta_{J_-(x_2)} Q_+(x_2) V_+(x_2) \\
& \times \delta_{J_+(x_3)} Q_-(x_3) V_-(x_3) \\
& + \delta_{J_-(x_1)} Q_+(x_1) V_+(x_1) \delta_{J_+(x_2)} Q_-(x_2) V_-(x_2) \delta_{J_-(x_3)} Q_+(x_3) V_+(x_3) \\
& + \delta_{J_+(x_1)} Q_-(x_1) V_-(x_1) \delta_{J_-(x_2)} Q_+(x_2) V_+(x_2) \delta_{J_-(x_3)} Q_+(x_3) V_+(x_3)\Big] \delta^2_{J_+(x_4)} \delta_{J_-(x_4)} v(x_4) \\
& \times \frac{1}{3!} \prod_{k=5}^{8} \Bigg[-\frac{i}{2} \int dx_k^D dy_k^D \, (J_-(x_k) \Delta_{\rm adv}(x_k - y_k) J_+(y_k) \\
& + J_+(x_k) \Delta_{\rm ret}(x_k - y_k) J_-(y_k) + J_+(x_k) \Delta_{\rm C}(x_k - y_k) J_+(y_k))\Bigg]\Bigg]_{\rm conn.}
\end{aligned} \tag{190}$$

and setting the Schwinger parameters to zero. Here the function v refers to the contributing triple bulk vertices, including their combinatorics, and also account for the propagator numerators as also $V_\pm$ at the multipole vertices. Unlike the case for I_1, here various fields are contributing. By analogous operations as in the case of S_1 one obtains

$$\begin{aligned}
\mathsf{S}_2 =& G_N^2 \sqrt{2} \frac{1}{10} \int_{-\infty}^{+\infty} dt \, {\rm tr}\Bigg\{\frac{4}{7}\left[3 Q_+^{(2)}(t) Q_+^{(3)}(t) Q_-^{(3)}(t) - 2 Q_-^{(2)}(t) Q_+^{(3)}(t) Q_+^{(3)}(t)\right] \\
& + \left[2 Q_+(t) Q_+^{(4)}(t) Q_-^{(4)}(t) - Q_-(t) Q_+^{(4)}(t) Q_+^{(4)}(t)\right]\Bigg\}.
\end{aligned} \tag{191}$$

Finally one obtains from these terms the contributions (17–19).

E Scattering angle integrals

In the following we list integrals, which appear in the calculation of the scattering angle, using relations given in [72].

At the Newtonian level one obtains the well–known integral

$$I_0 = \int_0^{x_{\max}} dy \frac{1}{\sqrt{1 - x^2 + \dfrac{2x}{j p_\infty}}} = -\frac{\pi}{2} + \arcsin\left(\frac{1}{\sqrt{1 + j^2 p_\infty^2}}\right). \tag{192}$$

It implies the value $\pi/2$ in (45). Its η–expansion delivers the ν and η–independent terms of χ_{2k+1} with

$$I_0^{(\mathrm{reg})} = -\frac{\pi}{2} + \frac{1}{p_\infty j} - \frac{1}{3 p_\infty^3 j^3} + \frac{1}{5 p_\infty^5 j^5} + O\left(\frac{1}{j^7}\right). \tag{193}$$

The $O(\eta^2)$ term reads

$$I_2 = \eta^2 \left[\frac{2 p_\infty}{j} + \frac{3\pi}{2 j^2} + \frac{4}{j^3 p_\infty} - \frac{2}{j^5 p_\infty^3} - \frac{10}{7 j^9 p_\infty^7} + \frac{8}{5 j^7 p_\infty^5} + \frac{4}{3 j^{11} p_\infty^9}\right] + O\left(\frac{1}{p_\infty^{11}}\right) + O\left(\frac{1}{\sqrt{\varepsilon}}\right). \tag{194}$$

The $O(\eta^4)$ is given by

$$\begin{aligned} I_4 = \eta^4 \Bigg[& \pi \left(\frac{15}{8} - \frac{3}{4}\nu\right) \frac{1}{j^2} + (24 - 8\nu) p_\infty \frac{1}{j^3} + \pi \left(\frac{105}{8} - \frac{15}{4}\nu\right) \frac{1}{j^4} + (32 - 8\nu) \frac{1}{j^5 p_\infty} \\ & - \left(16 - \frac{16}{5}\nu\right) \frac{1}{j^7 p_\infty^3} + \left(\frac{96}{7} - \frac{16}{7}\nu\right) \frac{1}{j^9 p_\infty^5} - \left(\frac{40}{3} - \frac{40}{21}\nu\right) \frac{1}{j^{11} p_\infty^7} \Bigg] + O\left(\frac{1}{p_\infty^9}\right) + O\left(\frac{1}{\varepsilon^{3/2}}\right). \end{aligned} \tag{195}$$

Similar structures are obtained for the higher order terms in η. Here the singularity in ε becomes stronger by one unit going from I_{2k} to I_{2k+2}. One easily sees that these integrals form the first terms of the coefficients χ_k given in Eq. (82–86).

We also list a series of higher expansion coefficients for χ_k^{Schw} for convenience, which result from (55),

$$\chi_5^{\mathrm{Schw}} = \frac{1}{5 p_\infty} - \frac{2\eta^2}{p_\infty^3} + \frac{32\eta^4}{p_\infty} + 320 p_\infty \eta^6 + 640 p_\infty^3 \eta^8 + \frac{1792}{5} p_\infty^5 \eta^{10}, \tag{196}$$

$$\chi_6^{\mathrm{Schw}} = \pi \left(\frac{1155}{8}\eta^6 + \frac{45045}{64} p_\infty^2 \eta^8 + \frac{135135}{128} p_\infty^4 \eta^{10} + \frac{255255}{512} p_\infty^6 \eta^{12}\right), \tag{197}$$

$$\chi_7^{\mathrm{Schw}} = -\frac{1}{7 p_\infty^7} + \frac{8\eta^2}{5 p_\infty^5} - \frac{16\eta^4}{p_\infty^3} + \frac{320\eta^6}{p_\infty} + 4480 p_\infty \eta^8 + 14336 p_\infty^3 \eta^{10} + \frac{86016}{5} p_\infty^5 \eta^{12} + \frac{49152}{7} p_\infty^7 \eta^{14}, \tag{198}$$

$$\chi_8^{\mathrm{Schw}} = \pi \left(\frac{225225}{128}\eta^8 + \frac{765765}{64} p_\infty^2 \eta^{10} + \frac{14549535}{512} p_\infty^4 \eta^{12} + \frac{14549535}{512} p_\infty^6 \eta^{14} + \frac{334639305}{32768} p_\infty^8 \eta^{16}\right), \tag{199}$$

$$\chi_9^{\mathrm{Schw}} = \frac{1}{9 p_\infty^9} - \frac{10\eta^2}{7 p_\infty^7} + \frac{96\eta^4}{7 p_\infty^5} - \frac{448\eta^6}{3 p_\infty^3} + \frac{3584\eta^8}{p_\infty} + 64512 p_\infty \eta^{10} + 286720 p_\infty^3 \eta^{12} + 540672 p_\infty^5 \eta^{14}$$

$$+\frac{3244032}{7}p_\infty^7\eta^{16}+\frac{9371648}{63}p_\infty^9\eta^{18}, \tag{200}$$

$$\chi_{10}^{\rm Schw}=\pi\left(\frac{2909907}{128}\eta^{10}+\frac{101846745}{512}p_\infty^2\eta^{12}+\frac{334639305}{512}p_\infty^4\eta^{14}+\frac{8365982625}{8192}p_\infty^6\eta^{16}\right.$$
$$\left.+\frac{25097947875}{32768}p_\infty^8\eta^{18}+\frac{29113619535}{131072}p_\infty^{10}\eta^{20}\right), \tag{201}$$

$$\chi_{11}^{\rm Schw}=-\frac{1}{11p_\infty^{11}}+\frac{4\eta^2}{3p_\infty^9}-\frac{40\eta^4}{3p_\infty^7}+\frac{128\eta^6}{p_\infty^5}-\frac{1536\eta^8}{p_\infty^3}+\frac{43008\eta^{10}}{p_\infty}+946176p_\infty\eta^{12}+5406720p_\infty^3\eta^{14}$$
$$+14057472p_\infty^5\eta^{16}+18743296p_\infty^7\eta^{18}+\frac{37486592p_\infty^9}{3}\eta^{20}+\frac{109051904p_\infty^{11}}{33}\eta^{22}, \tag{202}$$

$$\chi_{12}^{\rm Schw}=\pi\left(\frac{156165009}{512}\eta^{12}+\frac{1673196525}{512}p_\infty^2\eta^{14}+\frac{225881530875}{16384}p_\infty^4\eta^{16}+\frac{242613496125}{8192}p_\infty^6\eta^{18}\right.$$
$$\left.+\frac{4512611027925}{131072}p_\infty^8\eta^{20}+\frac{2707566616755}{131072}p_\infty^{10}\eta^{22}+\frac{10529425731825}{2097152}p_\infty^{12}\eta^{24}\right), \tag{203}$$

$$\chi_{13}^{\rm Schw}=\frac{1}{13p_\infty^{13}}-\frac{14\eta^2}{11p_\infty^{11}}+\frac{448\eta^4}{33p_\infty^9}-\frac{128\eta^6}{p_\infty^7}+\frac{1280\eta^8}{p_\infty^5}-\frac{16896\eta^{10}}{p_\infty^3}+\frac{540672\eta^{12}}{p_\infty}+14057472p_\infty\eta^{14}$$
$$+98402304p_\infty^3\eta^{16}+328007680p_\infty^5\eta^{18}+599785472p_\infty^7\eta^{20}+\frac{1853882368p_\infty^9}{3}\eta^{22}$$
$$+\frac{3707764736p_\infty^{11}}{11}\eta^{24}+\frac{10838081536p_\infty^{13}}{143}\eta^{26}, \tag{204}$$

$$\chi_{14}^{\rm Schw}=\pi\left(\frac{2151252675}{512}\eta^{14}+\frac{436704293025p_\infty^2}{8192}\eta^{16}+\frac{4512611027925p_\infty^4}{16384}\eta^{18}\right.$$
$$+\frac{49638721307175p_\infty^6}{65536}\eta^{20}+\frac{157941385977375p_\infty^8}{131072}\eta^{22}+\frac{1168766256232575p_\infty^{10}}{1048576}\eta^{24}$$
$$\left.+\frac{1168766256232575p_\infty^{12}}{2097152}\eta^{26}+\frac{977947275623175p_\infty^{14}}{8388608}\eta^{28}\right). \tag{205}$$

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Note added. After completion of this paper two preprints appeared [100], in which the representation of the electric quadrupole moment have been given in d dimensions, without the previously used additional Hadamard regularization. This new representation has no impact on the results of the present calculation.

S. Foffa communicated to us a note in preparation [101], in which it is shown that the previously necessary finite renormalization of the magnetic quadrupole term JEJ, using a representation containing Levi-Civita symbols, can be avoided by utilizing a dual representation, free of them, giving the same result.

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