PROCEEDINGS OF SCIENCE

Four-loop scattering amplitudes journey into the forest

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In this document we present an overview of the analysis of the multiloop topologies that appear for the first time at four loops and the assembly of them in a general expression, the N^4MLT universal topology. Based on the fact that the LTD enables to open any scattering amplitude in terms of convolutions of known subtopologies, we go through the dual representation of the universal N^4MLT topology and the explicit causal representations of selected configurations written in terms of entangled thresholds. Additionally, we expose the application of a quantum algorithm as an alternative to identify the causal singular configurations of the N^4MLT multiloop Feynman diagrams.

**** The European Physical Society Conference on High Energy Physics (EPS-HEP2021), ****
**** 26-30 July 2021 ****
**** Online conference, jointly organized by Universität Hamburg and the research center DESY ****

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1. Introduction

A crucial challenge in perturbative Quantum Field Theory is the description of quantum fluctuations at high-energy scattering processes by the calculation of multi-loop scattering amplitudes. Aiming for improving the efficiency of these computations, we work in a framework based on the Loop-Tree Duality (LTD) [1–7].

The LTD is a novel method that open any loop diagram to a forest of non-disjoint trees. The most important LTD feature is the distinction between physical and unphysical singularities at integrand level [8, 9]. Besides this, LTD has others characteristics: in numerical implementations the number of integration variables is independent of the number of external legs [10–14], it provides asymptotic expansions [15], [16–18], and promising local renormalization approaches [19]. Furthermore, an important development was the propose, the four dimensional unsubtraction (FDU) [20–23].

In Ref. [7] is conjectured that LTD leads to a very compact and manifestly causal representations of scattering amplitudes to all orders. This statement was proven for a series of multiloop topologies, the maximal loop topology (MLT), next-to-maximal (NMLT) and next-to-next-to-maximal ($\mathrm{N^2MLT}$). Their analytic dual representations are implicitly free of unphysical singularities, and the causal structure is interpreted in terms of entangled causal thresholds [24]. The extension of the knowledge acquired in [7] and [24] is presented here with the analysis of the topologies that appear for the fist time at four loops.

In the causal representation context, a problem to be solve through the LTD framework, is the identification of all internal configurations that are causal compatible among the $N = 2^n$ potential solutions, where n is the number of internal Feynman propagators. This is a problem that can handled with a quantum computing approach, applying Grover's algorithm [25] for querying of multiple solutions [26] over unstructured databases.

2. Loop-tree duality

An arbitrary L-loop scattering amplitude with N external legs, $\{p_j\}_N$, is written in the Feynman representation as an integral in the Minkowski space of the L loop momenta, $\{\ell_s\}_L$,

$$\mathcal{A}_N^{(L)}(1,\dots,n) = \int_{\ell_1,\dots,\ell_L} \mathcal{N}(\{\ell_s\}_L, \{p_j\}_N)\, G_F(1,\dots,n) \tag{1}$$

where the integration measure in dimensional regularization [27, 28] reads like $\int_{\ell_s} = -\imath\mu^{4-d}\int \frac{d^d\ell_s}{(2\pi)^d}$, with d the number of space-time dimensions. The integrand in the Feynman representation presented in Eq. (1), is composed by a numerator given the specific theory and $G_F(1,\dots,n) = \prod_{i\in 1\cup\dots\cup n} (G_F(q_i))^{a_i}$ denoting the product of Feynman propagators of one set or the union of several sets, with a_i arbitrary powers. The Feynman propagator is written in terms of the on-shell energy component $G_F(q_i) = \left(q_{i,0}^2 - \left(q_{i,0}^{(+)}\right)^2\right)^{-1}$, with $q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - \imath 0}$.

To obtain the LTD representation, we integrate out one of the components per loop applying the Cauchy residue theorem. Considering the case of multiloop scattering amplitudes, the LTD representation is written in terms of nested residues [7, 29]

$$\mathcal{A}_D^{(L)}(1,\dots,r;r+1,\dots,n) = -2\pi\imath \sum_{i_r\in r} \mathrm{Res}(\mathcal{A}_D^{(L)}(1,\dots,r-1;r,\dots,n), \mathrm{Im}(\eta\cdot q_{i_r})<0), \tag{2}$$

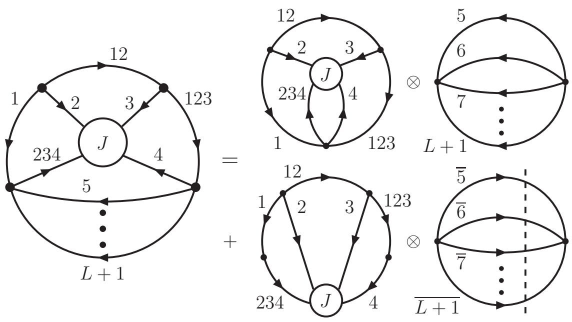

Figure 1: Diagrammatic representation for the factorized opening of the multiloop N^{4}MLT *universal topology*. Only the on-shell cut of the last MLT-like subtopology with reversed momentum flow is shown.

starting from the residue evaluation on the propagators that depend on the first loop momenta. All sets before the semicolon contain one on-shell propagator, the remaining sets located after the semicolon have all the propagators off shell. To evaluate the residue, we select the poles with negative imaginary components through the implementation of the future-like vector η indicating the component of the loop momenta to be integrated. In this case $\eta^\mu = (1, \mathbf{0})$, allowing to go in the integration domain from a Minkowsdy space to an Euclidean space.

3. N^{4}MLT universal topology

The topologies that first appear at four loops are describe by multiloop diagrams with $L+4$ and $L+5$ sets of propagators which correspond to the next-to-next-to-next-to maximal loop topology (N^{3}MLT) and next-to-next-to-next-to-next-to maximal loop topology (N^{4}MLT). The N^{4}MLT family is completely represented with three main topologies, which were checked with `QGRAF` [30] and include in a natural way all N^{k-1}MLT configurations with $k \leq 4$. A unified description of these topologies is achieved by interpreting them as the t-, s- and u-kinematic channels, respectively.

The three topologies differ only in one set of propagators, each of common sets depend on one characteristic loop momentum ℓ_s or in a linear combination of the loop momenta. The sets 23, 34, and 24 are considered as the distinctive key to each of the channels where the momenta of their propagators is taken as different linear combinations of ℓ_2, ℓ_3 and ℓ_4.

For joining the three N^{4}MLT channels in a single topology, a current J is defined as $23 \cup 34 \cup 24$ allowing to write the Feynman representation of the N^{4}MLT *universal topology*. The application of the nested residues in the universal topology gives the dual opening as,

$$\begin{aligned}
&\mathcal{A}^{(L)}_{\mathrm{N^4MLT}}(1, \ldots, L+1, 12, 123, 234, J) \\
&\qquad = \mathcal{A}^{(4)}_{\mathrm{N^4MLT}}(1, 2, 3, 4, 12, 123, 234, J) \otimes \mathcal{A}^{(L-4)}_{\mathrm{MLT}}(5, \ldots, L+1) \\
&\qquad + \mathcal{A}^{(3)}_{\mathrm{N^2MLT}}(1 \cup 234, 2, 3, 4 \cup 123, 12, J) \otimes \mathcal{A}^{(L-3)}_{\mathrm{MLT}}(\overline{5}, \ldots, \overline{L+1}) \,, \qquad (3)
\end{aligned}$$

expressed in a factorized form depicted in Fig. 1 and valid for any internal configuration. We call it the universal identity given the fact that is the only master expression needed to open any scattering amplitude of up to four loops to nondisjoint trees.

The terms $\mathcal{A}^{(L-4)}_{\mathrm{MLT}}(5, \ldots, L+1)$ and $\mathcal{A}^{(L-3)}_{\mathrm{MLT}}(\overline{5}, \ldots, \overline{L+1})$ on the r.h.s. of Eq. (3) are computed according to Ref. [7, 29]; the terms $\mathcal{A}^{(4)}_{\mathrm{N^4MLT}}$ and $\mathcal{A}^{(3)}_{\mathrm{N^2MLT}}$ considers all possible configurations with four and three on-shell conditions respectively, and are opened through a factorization identity which is written in terms of known subtopologies. These terms contemplate dual trees where all the propagators in J remain off shell and contributions that characterize the s, t or u channel, the explicit expressions are presented in [31], where all the results agree with the absence of nondisjoint trees.

4. Causal representations

The confirmation of the causal conjecture for the N^{4}MLT family, follows the strategy proposed in [24] and is applied to the multi-loop N^{3}MLT, t, s and u channels. The configuration used for each topology is: one internal propagator in each loop set, four external momenta for N^{3}MLT and six external particles for t, s and u channels. Also, only scalar integrals were considered given that they fully encode all the compatible causal matchings.

A manifestly causal expression is found after the straightforward application of the universal opening in Eq. (3) and adding all the dual terms together. Nevertheless, the calculated numerator is a lengthy polynomial in the on-shell and external energies, therefore, to derive a more appropriate causal expression we reinterpret it in terms of a number of entangled thresholds equal to the difference between the number of propagators and the number of loops as defined in Ref. [24].

The causal representation of the multi-loop N^{3}MLT was analytically reconstructed Ref. [32–34] by matching all combinations of four thresholds that are causally compatible to each other. There are thirteen causal denominators which represent potential singular configuration and are constructed from sums of on-shell energies exclusively.

In the case of the N^{4}MLT family, it was consider all the entangled configurations involving five causal thresholds. The causal representation of the t-channel depends on the causal denominators already defined for the N^{3}MLT configuration and additional nine extra causal denominators that depend on $q^{(+)}_{23,0}$. For the s-channel, a clockwise rotation is applied to the t-channel; for the u-channel, besides a convenient substitution to the t-channel, three additional thresholds arise given the nonplanar context. All the details and specific results of the causal representation are in Ref. [31]

The main difference between the LTD and the LTD causal representations is the presence or absence of noncausal singularities. The straightforward application of the nested residue generates multiple threshold singularities, nevertheless, with a clever analytical rearrangement, the absence of noncausal singularities is achieved and leads to a causal representation which is more efficient and stable numerically in all the integration domain.

5. Application of Grover quantum algorithm

The idea to explore the application of quantum algorithms to Feynman loop integrals, arise duo the implicit connection between a Feynman propagator and a qubit. Feynman propagators have two possible on-shell states which can be encoded in qubit.

Given the nature of the problem of identifying the causal configurations of selected topologies, the algorithm selected to apply was Grover quantum algorithm. The standard Grover's querying algorithm over unstructured databases starts from a uniform superposition, with a total of $N = 2^n$

states which can be understood as the superposition of a winning state $|w\rangle$ and the orthogonal state $|q_\perp\rangle$,

$$|q\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \,, \qquad |q\rangle = \cos\theta \, |q_\perp\rangle + \sin\theta \, |w\rangle \,, \tag{4}$$

where $|w\rangle$ gathers all the causal solutions in an uniform superposition and $|q_\perp\rangle$ the noncausal states. The mixing angle is given by $\theta = \arcsin\sqrt{r/N}$, whit r the number of causal solutions. The algorithm requires two operators, the oracle and diffusion operators: $U_w = \boldsymbol{I} - 2|w\rangle\langle w|$ and $U_q = 2|q\rangle\langle q| - \boldsymbol{I}$, respectively. The oracle operator, U_w, flips the state $|x\rangle$ if $x \in w$, $U_w|x\rangle = -|x\rangle$ and leaves it unchanged otherwise $U_w|x\rangle = |x\rangle$ if $x \notin w$; in the case of the diffusion operator, U_q performs a reflection around the initial state $|q\rangle$ with the purpose of amplify the winning state probability. The iterative application of both operators t times leads to

$$(U_q U_w)^t |q\rangle = \cos\theta_t \, |q_\perp\rangle + \sin\theta_t \, |w\rangle, \quad \theta_t = (2t+1)\,\theta. \tag{5}$$

Based on Eq. (5), the target is to get a final state probability such that the probability associated to the orthogonal state is considerable much smaller than the probability of the causal solutions. To achieve this, is important to consider the range of θ because Grover's algorithm works well if $\theta \leq \pi/6$ ($r \leq N/4$), but does not provide the desired amplitude amplification for greater angles.

Studying the N^4MLT family we found a limitation, the causal configurations are nearly half of the total sates. To overcome this condition, we take advantage of the fact that given a causal solution, the configuration resulted by reversing all momentum flows is also a causal solution. Therefore, we can reduce the number of causal solutions by half, fixing one of the propagator set flow and assume that only one of its states contributes to the winning set.

The quantum algorithm proposed needs three registers for implementation, together with an extra qubit used as marker by the Grover's oracle. The first register, q_i, encodes the state of the propagators. The qubit q_i is in the state $|1\rangle$ if the momentum flow of the corresponding line is oriented as the initial assignment and in $|0\rangle$ if it is in the opposite direction. The second register stores the Boolean clauses, c_{ij}, probing whether or not two adjacent qubits are oriented in the same direction. These binary clauses are defined as $c_{ij} \equiv (q_i = q_j)$, $\bar{c}_{ij} \equiv (q_i \neq q_j)$ with $i, j \in \{0, \ldots, n-1\}$. The third register, $a_k(\{c_{ij}\}, \{\bar{c}_{ij}\})$, encodes the loop clauses that probe if all the qubits in each subloop form a cyclic circuit.

The causal quantum algorithm is implemented as follows. The initial uniform superposition is obtained by applying Hadamard gates to each of the qubits in the q-register, $|q\rangle = H^{\otimes n}|0\rangle$. The $|c\rangle$ and $|a\rangle$ registers are initialized to $|0\rangle$ while the qubit which is used as Grover's marker is initialized to the Bell state $|out_0\rangle = |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Each binary clause $\bar{c}_{ij}$ requires two CNOT gates operating between two qubits in the $|q\rangle$ register and one qubit in the $|c\rangle$ register. An extra XNOT gate acting on the corresponding qubit in $|c\rangle$ is needed to implement a c_{ij} binary clause.

The oracle is defined as $U_w|q\rangle|c\rangle|a\rangle|out_0\rangle = |q\rangle|c\rangle|a\rangle|out_0 \otimes f(a,q)\rangle$, with $|out_0 \otimes 0\rangle = |out_0\rangle$ and $|out_0 \otimes 1\rangle = -|out_0\rangle$. Therefore, if all the causal conditions are satisfied, $f(a,q) = 1$ and the corresponding states are marked; otherwise $f(a,q) = 0$. After the marking, the oracle operations in inverse order is applied. Then, the diffusion U_q is applied to the register $|q\rangle$. The diffuser used is the one described in the IBM Qiskit website (https://qiskit.org/). Also, we used the

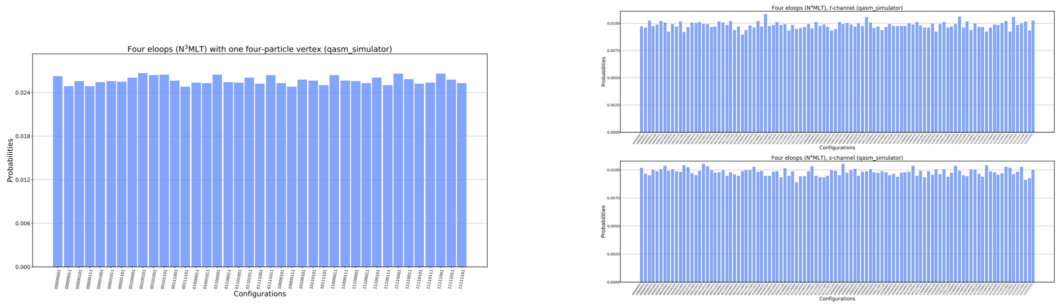

Figure 2: Probabilities of causal states for four-eloop configurations. On the left, N^{3}MLT; on the right from top to down, t- and s-channels of N^{4}MLT with $n_i = 1$. The number of selected states is 39/256, 102/512, and 102/512, respectively. The u-channel, with 115/512 causal states (classical computation) exceeds the current capacity of the IBM quantum simulator.

IBM's quantum simulator provided by the open source Qiskit framework, having as a current upper limit 32 qubits.

This algorithm requires one single iteration for all the cases analyzed. To illustrate the performance of the quantum algorithm, we show in Fig. 2 the probability of the winning states obtained in the IBM's Qiskit simulator. Despite the complexity of these four-eloop diagrams, all the causal configurations were successfully identified, as the algorithm noticeably enhanced their probabilities and suppressed those of the noncausal configurations.

6. Conclusions

We have presented a unify description and representation of any scattering amplitude up to four loops through the universal topology. The LTD was applied to this topology family to all orders, managing to get a LTD representation which exhibits a nested form in terms of simpler topologies.

An additional step have been taking in the LTD framework, the causal LTD representation is explicitly found and interpreted in terms of entangled causal thresholds. This representation allows to work with favorable conditions, the absence of noncuasal thresholds, enabling to get more efficient numerical evaluation of multiloop scattering amplitudes.

It was introduced an alternative procedure, based on quantum computing, to identify causal configurations of Feynman integrals at four loops. The problem has been restate for the proper application of Grover's quantum algorithm and the causal singular configurations have been efficiently unfolded for the N^{3}MLT, $t-$ and $s-$channels of N^{4}MLT topologies.

Acknowledgements

This work is supported by the Spanish Government (Agencia Estatal de Investigación) and ERDF funds from European Commission (Grant No. FPA2017-84445-P), Generalitat Valenciana (Grant No. PROMETEO/2017/053) and the COST Action CA16201 PARTICLEFACE. SRU acknowledges support from CONACyT and Universidad Autónoma de Sinaloa; RJHP from the Project No. A1-S-33202 (Ciencia Básica), Ciencia de Frontera 2021-2042 and Sistema Nacional de Investigadores; AERO from the Spanish Government (PRE2018-085925).

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