

Logarithmic contributions to the polarized $O(\alpha_s^3)$ asymptotic massive Wilson coefficients and operator matrix elements in deeply inelastic scattering

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We compute the logarithmic contributions to the polarized massive Wilson coefficients for deep-inelastic scattering in the asymptotic region $Q^2 \gg m^2$ to three-loop order in the fixed-flavor number scheme and present the corresponding expressions for the polarized massive operator matrix elements needed in the variable flavor number scheme. The calculation is performed in the Larin scheme. For the massive operator matrix elements $A_{qq,Q}^{(3),PS}$ and $A_{qq,Q}^{(3),S}$ the complete results are presented. The expressions are given in Mellin- N space and in momentum fraction z space.

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I. INTRODUCTION

At leading twist the scaling violations of deeply-inelastic structure functions obtain corrections from the massless scale evolution of the parton densities, the massive Wilson coefficients, and target-mass corrections, Ref. [1], provided that the electroweak radiative corrections are properly removed [2–4]. At low scales of the virtuality $Q^2 = -q^2$, all kind of other dynamical contributions, such as dynamical higher twist effects, particular small x effects, with x the Bjorken variable, and particular hadronic contributions due to vector meson dominance etc. are present. For these reasons one usually applies respective cuts of the kind $Q^2 > 10 \text{ GeV}^2$, $W^2 = Q^2(1-x)/x > 15 \text{ GeV}^2$ [5] to eliminate these effects in order to obtain clean twist-two data for which a dedicated QCD analysis is going to be performed. The target mass effects can be accounted for analytically [1,6–8]. Furthermore, for a precision measurement of the strong coupling constant $\alpha_s(M_Z^2) = g_s^2/(4\pi)$, one should use targets with no (or only soft) nuclear binding, such as the case for p and d targets. In the latter case nuclear corrections still have to be performed.

Under these circumstances the largest source of scaling violations next to the massless higher order corrections are implied by the heavy quark corrections, in the form of

single and two-mass corrections. Both in the unpolarized and polarized cases the scaling violations due to the heavy flavor Wilson coefficients turn out to be different from those of the massless corrections in very wide kinematic regions covered by the present experiments. Therefore, one can not model these effects by just adding more quasi-massless flavors in massless higher order corrections. In future analysis of the polarized structure functions¹ at the EIC operating at high luminosity [10] these corrections are important for precision measurements of the strong coupling constant [11] and the charm quark mass, Ref. [12].

In the polarized case, the leading order corrections were derived in [13,14], the next-to-leading order (NLO) asymptotic corrections in [15–18], the complete NLO corrections in analytic form for the nonsinglet and pure singlet corrections in [15,19,20] and numerically in [21]. At three-loop order, the massive operator matrix elements (OMEs) for the nonsinglet $A_{qq,Q}^{(3),NS}$, pure singlet $A_{qq,Q}^{(3),PS}$, and the OME $A_{gg,Q}^{(3)}$ have been calculated in [22–24]. Furthermore, two-mass corrections to different OMEs at three-loop order were calculated in [24–27]. The three-loop heavy flavor nonsinglet contributions at leading twist to the structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$ were computed in [28]. Likewise, in the unpolarized case the leading order corrections were obtained in [29–34] at NLO in numerical form in [35] and in analytic form in the nonsinglet and pure

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¹For a review on the theory and phenomenology of polarized deep-inelastic structure functions see Ref. [9].

singlet cases in [19,36,37] and in the asymptotic case in [19,36,38–41]. At three-loop order a series of moments has been computed in [42] for all massive OMEs. A series of three-loop massive OMEs has been calculated in Refs. [22,43–47] and first analytic results for $A_{Qg}^{(3)}$ have been given in [48]. Two-mass three-loop corrections were calculated in Refs. [25,49,50].

In the present paper we calculate the three-loop logarithmic corrections to the polarized massive OMEs A_{ij} both in Mellin N and momentum fraction z -space in the single mass case. These OMEs appear as building blocks of the variable flavor number scheme (VFNS) at three-loop order, through which the matching of parton distribution functions at N_F and $N_F + 1$ massless heavy flavors can be obtained [38]. Because of the close values of the heavy quark masses m_c and m_b , two-mass decoupling has been considered and implemented in the VFNS in [18,25,51]. In the present paper we will consider the heavy quark masses in the on-shell scheme.² The OMEs determine also the polarized massive Wilson coefficients in the region of large virtualities $Q^2 \gg m_Q^2$, with m_Q the corresponding heavy quark mass $m_Q = m_c(m_b)$ of charm and bottom quarks, which we derive for the structure function $g_1(x, Q^2)$. It turns out that all functions can either be expressed in terms of harmonic sums [52,53] or harmonic polylogarithms [54]. The analytic continuation to z -space in Mellin space evolution have been studied in [55,56] in the case of harmonic sums. The expressions for the different OMEs and massive Wilson coefficients turn out to be rather long both in N and z space. As they are at present, they can be used to study the corrections at the logarithmic level and the expressions provide the frame for the final results.

The paper is organized as follows. In Sec. II, we summarize the basic formalism. It widely follows the representation in the unpolarized case [57] and resumes the representations of the structure functions and the relations for the single mass variable flavor number scheme. We then present the complete polarized massive operator matrix elements (OMEs) $A_{qq,Q}^{(3),\text{PS}}$ and $A_{qq,Q}^{(3),\text{S}}$ beyond the logarithmic terms in Sec. III. In Sec. IV the polarized OME $A_{Qg}^{(3),\text{S}}$ is given in N space, followed by the expressions for the OMEs $A_{gg,Q}^{(3),\text{S}}$ in Sec. V. The OMEs $A_{qq,Q}^{(3),\text{NS}}$, $A_{Qq}^{(3),\text{PS}}$ and $A_{gg,Q}^{(3),\text{S}}$ have already been published before in Refs. [22–24] in complete form. In Secs. VI–VIII we present the polarized three-loop Wilson coefficients L_q^{PS} , L_g^{S} , H_{Qq}^{PS} and H_{Qg}^{S} . The logarithmic contributions can be represented in terms of harmonic sums and a large number of polynomials. Section IX contains the conclusions. In Appendix A the corresponding expressions

for the OMEs in momentum fraction z space are presented, and in Appendix B the corresponding expressions for the polarized Wilson coefficients are presented. Here, the yet unknown constant parts of the polarized massive OMEs $a_{Qg}^{(3)}$ and $a_{gg,Q}^{(3)}$ and the yet missing three-loop polarized massless Wilson coefficients, are left as symbols. In Refs. [22,28] the flavor nonsinglet contributions were presented in the $\overline{\text{MS}}$ scheme to three-loop order. For a consistent treatment, we present the transformation for the massive OME $A_{qq,Q}^{\text{NS}}$ and the asymptotic massive Wilson coefficient L_q^{NS} in the Larin scheme in Appendix C.

II. THE FORMALISM

The explicit expressions showing the principal structure of the different massive OMEs have been derived in Refs. [42,57], as well as for the asymptotic massive Wilson coefficients. The different polarized massive OMEs obey the following expansion in the strong coupling constant

$$A_{ij}^{(k)} = \delta_{ij} + \sum_{l=1}^{\infty} a_s^l A_{ij}^{(l),k}, \quad (1)$$

with $a_s = \alpha_s/(4\pi)$, the indices $i, j = q, g$ label the partonic channels, and k denotes the different OMEs. Here, and in the following, we use the shorthand notations [42]

$$\hat{f}(N_F) = f(N_F + 1) - f(N_F) \quad (2)$$

$$\tilde{f}(N_F) = \frac{1}{N_F} f(N_F). \quad (3)$$

The heavy flavor Wilson coefficients, accounting for the single mass contributions to the structure function $g_1(x, Q^2)$, are given by

$$\begin{aligned} L_{q,(1)}^{\text{NS},Q}(N_F + 1) &= a_s^2 [A_{qq,Q}^{(2),\text{NS}}(N_F + 1) + \hat{C}_{q,(1)}^{(2),\text{NS}}(N_F)] \\ &\quad + a_s^3 [A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \\ &\quad + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(1)}^{(1),\text{NS}}(N_F + 1) \\ &\quad + \hat{C}_{q,(1)}^{(3),\text{NS}}(N_F)], \end{aligned} \quad (4)$$

$$\begin{aligned} L_{q,(1)}^{\text{PS}}(N_F + 1) &= a_s^3 [A_{qq,Q}^{(3),\text{PS}}(N_F + 1) \\ &\quad + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(1)}^{(1)}(N_F + 1) \\ &\quad + N_F \hat{C}_{q,(1)}^{(3),\text{PS}}(N_F)], \end{aligned} \quad (5)$$

²The transformation to the $\overline{\text{MS}}$ -scheme is straightforward.

$$L_{g,(1)}^S(N_F + 1) = a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + a_s^3 [A_{gg,Q}^{(3)}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),PS}(N_F + 1) + N_F \hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)], \quad (6)$$

$$H_{q,(1)}^{PS}(N_F + 1) = a_s^2 [A_{Qq}^{(2),PS}(N_F + 1) + \tilde{C}_{q,(2,L)}^{(2),PS}(N_F + 1)] + a_s^3 [A_{Qq}^{(3),PS}(N_F + 1) + \tilde{C}_{q,(2,L)}^{(3),PS}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qq}^{(2),PS}(N_F + 1) C_{q,(2,L)}^{(1),NS}(N_F + 1)], \quad (7)$$

$$H_{g,(1)}^S(N_F + 1) = a_s [A_{Qg}^{(1)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1)] + a_s^2 [A_{Qg}^{(2)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),NS}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1)] + a_s^3 [A_{Qg}^{(3)}(N_F + 1) + A_{Qg}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),NS}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \{C_{q,(2,L)}^{(2),NS}(N_F + 1) + \tilde{C}_{q,(2,L)}^{(2),PS}(N_F + 1)\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1)], \quad (8)$$

with $\beta_{0,Q} = -(4/3)T_F$. The QCD color factors are given by $C_A = N_c$, $C_F = (N_c^2 - 1)/(2N_c)$, $T_F = 1/2$ for $SU(N_c)$ and $N_c = 3$. N_F denotes the number of massless flavors. Here the massive OMEs A_{ij} depend on m^2/μ^2 and the massless Wilson coefficients depend on Q^2/μ^2 . Note that we extended the original notion in Refs. [38,42].

The argument $(N_F + 1)$ in the massive OMEs signals that these functions depend on N_F massless and one massive flavor, while the setting of N_F in the massless Wilson coefficients is a functional one. The massless Wilson coefficients are labeled by $C_i^{(l),k}$, where i refers

to the parton species and l to the expansion order in the strong coupling constant. The massless Wilson coefficients are known to two-loop order [20,58–61] and in the non-singlet case to three-loop order [62].

In the present paper we express all relations in the Larin scheme [63,64], which is a consistent scheme also in the massive case. The anomalous dimensions were calculated to two-loop order [65,66] and to three-loop order in [67,68].

The twist-two contributions to the structure function $g_1(x, Q^2)$ in the single mass case for pure virtual photon exchange³ are given by

$$\begin{aligned} \frac{1}{x} g_1(x, Q^2) = & \sum_{k=1}^{N_F} e_k^2 \left\{ L_{q,(2,L)}^{NS} \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes [\Delta f_k(x, \mu^2, N_F) + \Delta f_{\bar{k}}(x, \mu^2, N_F)] \right. \\ & + \frac{1}{N_F} L_{q,(2,L)}^{PS} \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta \Sigma(x, \mu^2, N_F) + \frac{1}{N_F} L_{g,(2,L)}^S \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta G(x, \mu^2, N_F) \Big\} \\ & + e_Q^2 \left[H_{q,(2,L)}^{PS} \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta \Sigma(x, \mu^2, N_F) + H_{g,(2,L)}^S \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Delta G(x, \mu^2, N_F) \right]. \quad (9) \end{aligned}$$

Here, the polarized quark and antiquark densities are denoted by Δf_k and $\Delta f_{\bar{k}}$, ΔG denotes the polarized gluon density, and the polarized singlet distribution is given by

$$\Delta \Sigma = \sum_{k=1}^{N_F} [\Delta f_k + \Delta f_{\bar{k}}]. \quad (10)$$

The structure function $g_2(x, Q^2)$ at twist-two is obtained by the Wandzura-Wilczek relation [6,7,69–71]

³For the structure of electroweak gauge boson exchange see e.g., [6,69].

$$g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{dz}{z} g_1(z, Q^2). \quad (11)$$

Besides the Wilson coefficients (4)–(8) the massive OMEs determine the relations of the matching of parton

distributions in the VFNS at large enough scales μ^2 , [38,42]. Here, the PDFs for $N_F + 1$ massless quarks are related to the former N_F massless quarks process independently. The corresponding relations to three-loop order read, (see also [38])⁴

$$\begin{aligned} \Delta f_k(N_F + 1, \mu^2) + \Delta f_{\bar{k}}(N_F + 1, \mu^2) &= A_{qq,Q}^{\text{NS}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes [\Delta f_k(N_F, \mu^2) + \Delta f_{\bar{k}}(N_F, \mu^2)] \\ &+ \tilde{A}_{qq,Q}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Delta \Sigma(N_F, \mu^2) + \tilde{A}_{qg,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Delta G(N_F, \mu^2) \end{aligned} \quad (12)$$

$$\Delta f_{Q+\bar{Q}}(N_F + 1, \mu^2) = A_{Qq}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Delta \Sigma(N_F, \mu^2) + A_{Qg}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Delta G(N_F, \mu^2) \quad (13)$$

$$\Delta G(N_F + 1, \mu^2) = A_{gq,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Delta \Sigma(N_F, \mu^2) + A_{gg,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Delta G(N_F, \mu^2) \quad (14)$$

$$\begin{aligned} \Delta \Sigma(N_F + 1, \mu^2) &= \left[A_{qq,Q}^{\text{NS}}\left(N_F, \frac{\mu^2}{m^2}\right) + N_F \tilde{A}_{qq,Q}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) + A_{Qq}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) \right] \otimes \Delta \Sigma(N_F, \mu^2) \\ &+ \left[N_F \tilde{A}_{qg,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) + A_{Qg}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \right] \otimes \Delta G(N_F, \mu^2). \end{aligned} \quad (15)$$

The N_F dependence of the OMEs is understood as functional and μ^2 denotes the matching scale, which for the heavy-to-light transitions are normally much larger than the mass scale m^2 , [72]. The corresponding matching equations in the two-mass case are given in Ref. [25].

The results of the calculations presented in the subsequent sections have been obtained making mutual use of the packages `HarmonicSums.m` [52–54,56,73,74] and `Sigma.m` [75,76], which is based on algorithms in difference-ring theory [77].

In the subsequent expressions we abbreviate the following logarithms by

$$L_Q = \ln\left(\frac{Q^2}{\mu^2}\right) \quad \text{and} \quad L_M = \ln\left(\frac{m^2}{\mu^2}\right), \quad (16)$$

⁴Here, we have corrected some typographical errors in (13)–(15) in [38], in accordance with the Appendix of Ref. [38].

where we set the renormalization and factorization scales equal $\mu \equiv \mu_F = \mu_R$.

III. $A_{qq,Q}^{(3),\text{PS}}$ AND $A_{qg,Q}^{(3),\text{S}}$

In what follows, the OMEs $A_{qq,Q}^{(3),\text{PS}}$ and $A_{qg,Q}^{(3),\text{S}}$ are presented in complete form analytically, where the $O(\epsilon^0)$ contribution to the unrenormalized OMEs, $a_{ij,Q}^{(3)}$, are given separately. Up to three-loop order the massive OMEs $A_{qq,Q}^{\text{PS}}$ and $A_{qg,Q}^{\text{S}}$ do not yet receive double mass contributions. The OMEs are expressed in terms of harmonic sums [52,53] $S_{\vec{a}}(N) \equiv S_{\vec{a}}$, which are defined by

$$\begin{aligned} S_{b,\vec{a}}(N) &= \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \\ S_{\emptyset} &= 1, b, a_i \in \mathbb{Z} \setminus \{0\}. \end{aligned} \quad (17)$$

The OME $A_{qq,Q}^{PS(3)}$ is given by

$$\begin{aligned}
 A_{qq,Q}^{PS(3)} = a_s^3 & \left\{ a_{qq,Q}^{PS(3)} + C_F N_F T_F^2 \left\{ -\frac{32L_M^3(N-1)(2+N)}{9N^2(1+N)^2} + \frac{32(N-1)(2+N)(98+369N+408N^2+164N^3)}{81N^2(1+N)^5} \right. \right. \\
 & + L_M^2 \left[-\frac{32(2+N)(3+4N-3N^2+8N^3)}{9N^3(1+N)^3} + \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1 \right] \\
 & + L_M \left[-\frac{32(2+N)P_1}{27N^4(1+N)^4} + \frac{64(2+N)(3+4N-3N^2+8N^3)}{9N^3(1+N)^3} S_1 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1^2 - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_2 \right] \\
 & + \left[-\frac{32(N-1)(2+N)(22+41N+28N^2)}{27N^2(1+N)^4} - \frac{16(N-1)(2+N)S_2}{3N^2(1+N)^2} \right] S_1 \\
 & + \frac{16(N-1)(2+N)(2+5N)}{9N^2(1+N)^3} (S_1^2 + S_2) - \frac{16(N-1)(2+N)}{9N^2(1+N)^2} (S_1^3 + 2S_3) \\
 & \left. + \left[\frac{16(2+N)(3+2N-6N^2+13N^3)}{9N^3(1+N)^3} - \frac{32(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] \zeta_2 + \frac{32(N-1)(2+N)}{9N^2(1+N)^2} \zeta_3 \right\} \right\}, \quad (18)
 \end{aligned}$$

with

$$P_1 = 86N^5 + 38N^4 + 40N^3 - 8N^2 - 15N - 9. \quad (19)$$

Here $\zeta_k, k \geq 2, k \in \mathbb{N}$ denotes the Riemann ζ function at integer arguments.

Likewise, the OME $A_{qq,Q}^{(3)}$ is obtained by

$$\begin{aligned}
 A_{qq,Q}^{(3)} = a_s^3 & \left\{ a_{qq,Q}^{(3)} + C_A N_F T_F^2 \left\{ -\frac{8(N-1)P_8}{81N^5(1+N)^5} + L_M^3 \left[-\frac{64(N-1)}{9N^2(1+N)^2} + \frac{32(N-1)}{9N(1+N)} S_1 \right] \right. \right. \\
 & + L_M^2 \left[\frac{8P_4}{9N^3(1+N)^3} + \frac{32(1+5N^2)}{9N(1+N)^2} S_1 - \frac{16(N-1)}{3N(1+N)} (S_1^2 + S_2) - \frac{32(N-1)}{3N(1+N)} S_{-2} \right] \\
 & + L_M \left[\frac{16P_7}{27N^4(1+N)^4} + \left[\frac{16(-1+44N+67N^2+94N^3)}{27N(1+N)^3} - \frac{16(N-1)S_2}{3N(1+N)} \right] S_1 \right. \\
 & - \frac{32(-2+5N^2)}{9N(1+N)^2} S_1^2 + \frac{16(N-1)}{9N(1+N)} S_1^3 - \frac{32(-2+6N+5N^2)}{9N(1+N)^2} S_2 + \frac{32(N-1)}{9N(1+N)} S_3 \\
 & - \frac{64(-2+5N)}{9N(1+N)} S_{-2} + \frac{64(N-1)}{3N(1+N)} S_{-3} + \frac{64(N-1)}{3N(1+N)} S_{2,1} \left. \right] \\
 & - \frac{16(N-1)(283+584N+328N^2)}{81N(1+N)^3} S_1 - \frac{8(N-1)}{3N(1+N)^2} S_1^2 + \frac{8(N-1)(1+2N)}{3N(1+N)^2} S_2 \\
 & + \left[-\frac{8P_3}{9N^3(1+N)^3} - \frac{32(-2+5N^2)}{9N(1+N)^2} S_1 + \frac{8(N-1)S_1^2}{3N(1+N)} \right. \\
 & + \frac{8(N-1)S_2}{3N(1+N)} + \frac{16(N-1)S_{-2}}{3N(1+N)} \left. \right] \zeta_2 + \left[\frac{64(N-1)}{9N^2(1+N)^2} - \frac{32(N-1)S_1}{9N(1+N)} \right] \zeta_3 \left. \right\} \\
 & + C_F N_F T_F^2 \left\{ -\frac{(N-1)P_{10}}{81N^6(1+N)^6} + L_M^3 \left[\frac{8(N-1)P_2}{9N^3(1+N)^3} - \frac{32(N-1)}{9N(1+N)} S_1 \right] \right. \\
 & + L_M^2 \left[\frac{4(N-1)P_6}{9N^4(1+N)^4} - \frac{32(N-1)(3+5N)}{9N^2(1+N)} S_1 + \frac{16(N-1)}{3N(1+N)} (S_1^2 + S_2) \right] \\
 & + L_M \left[\frac{4P_9}{27N^5(1+N)^5} + \left[-\frac{16(-24-52N+103N^2)}{27N^2(1+N)} - \frac{16(N-1)}{3N(1+N)} S_2 \right] S_1 \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{16(N-1)(3+10N)}{9N^2(1+N)} S_1^2 - \frac{16(N-1)}{9N(1+N)} S_1^3 + \frac{16(-3+5N+10N^2)}{9N^2(1+N)} S_2 + \frac{64(N-1)}{9N(1+N)} S_3 \Big] + \frac{5248(N-1)S_1}{81N(1+N)} \\
& - \frac{896(N-1)S_2}{27N(1+N)} + \frac{160(N-1)S_3}{9N(1+N)} - \frac{32(N-1)}{3N(1+N)} S_4 + \left[-\frac{4(N-1)P_5}{9N^4(1+N)^4} + \frac{16(N-1)(3+10N)}{9N^2(1+N)} S_1 \right. \\
& \left. - \frac{8(N-1)S_1^2}{3N(1+N)} - \frac{8(N-1)S_2}{N(1+N)} \right] \zeta_2 + \left[-\frac{8(N-1)P_2}{9N^3(1+N)^3} + \frac{32(N-1)S_1}{9N(1+N)} \right] \zeta_3 \Big\} \Big\}, \quad (20)
\end{aligned}$$

with the polynomials

$$P_2 = 3N^4 + 6N^3 - N^2 - 4N + 12, \quad (21)$$

$$P_3 = 6N^5 + 6N^4 - 67N^3 + 6N^2 + 43N - 18, \quad (22)$$

$$P_4 = 9N^5 + 9N^4 - 79N^3 + 15N^2 + 22N - 24, \quad (23)$$

$$P_5 = 18N^6 + 54N^5 + 5N^4 - 20N^3 + 95N^2 - 132N - 108, \quad (24)$$

$$P_6 = 33N^6 + 99N^5 + 41N^4 - 11N^3 + 86N^2 - 216N - 144, \quad (25)$$

$$P_7 = 99N^7 + 198N^6 - 410N^5 - 344N^4 + 128N^3 - 130N^2 - 39N + 90, \quad (26)$$

$$P_8 = 255N^8 + 1020N^7 - 532N^6 - 4536N^5 - 4344N^4 - 1138N^3 + 3N^2 + 36N + 108, \quad (27)$$

$$\begin{aligned}
P_9 = & 159N^9 + 477N^8 - 220N^7 - 710N^6 + 117N^5 - 1081N^4 + 2536N^3 \\
& + 1026N^2 - 1800N - 1080, \quad (28)
\end{aligned}$$

$$\begin{aligned}
P_{10} = & 1551N^{10} + 7755N^9 + 10982N^8 + 1910N^7 + 2427N^6 + 14975N^5 \\
& + 13952N^4 - 1488N^3 - 7488N^2 - 6912N - 2592. \quad (29)
\end{aligned}$$

The $O(\epsilon^0)$ parts of the unrenormalized massive OMEs, $a_{qq,Q}^{(3),\text{PS}}$ and $a_{qq,Q}^{(3)}$ are given by

$$\begin{aligned}
a_{qq,Q}^{(3),\text{PS}} = & C_F T_F^2 N_F \left\{ \frac{32(2+N)R_2}{243N^5(1+N)^5} + \left(-\frac{64(2+N)R_1}{81N^4(1+N)^4} - \frac{128(N-1)(2+N)}{9N^2(1+N)^2} S_2 \right) S_1 \right. \\
& + \frac{64(2+N)(3+2N-6N^2+13N^3)}{27N^3(1+N)^3} [S_1^2 + S_2] - \frac{128(N-1)(2+N)}{27N^2(1+N)^2} [S_1^3 + 2S_3] \\
& + \left(\frac{16(2+N)(3+2N-6N^2+13N^3)}{9N^3(1+N)^3} - \frac{32(N-1)(2+N)}{3N^2(1+N)^2} S_1 \right) \zeta_2 \\
& \left. - \frac{224(N-1)(2+N)}{9N^2(1+N)^2} \zeta_3 \right\} \quad (30)
\end{aligned}$$

$$\begin{aligned}
a_{qg,Q}^{(3)} = & C_F T_F^2 N_F \left\{ \frac{112(N-1)R_3}{9N^3(1+N)^3} \zeta_3 - \frac{8(N-1)R_6}{9N^4(1+N)^4} \zeta_2 - \frac{16R_8}{243N^6(1+N)^6} + \frac{(N-1)(3+10N)}{N^2(1+N)} \right. \\
& \times \left(\frac{32}{81} S_1^3 + \frac{32}{27} S_1 S_2 + \frac{32}{9} S_1 \zeta_2 \right) + \frac{N-1}{N(1+N)} \left(-\frac{8}{27} S_1^4 - \frac{16}{9} S_1^2 S_2 - \frac{8}{9} S_2^2 - \frac{64}{27} S_1 S_3 \right. \\
& - \frac{80}{9} S_4 - 16 \left(\frac{1}{3} S_1^2 + S_2 \right) \zeta_2 - \frac{448}{9} S_1 \zeta_3 \left. \right) - \frac{32(420 + 2399N - 555N^2 - 2507N^3)}{243N^2(1+N)^2} S_1 \\
& - \frac{16(-24 - 164N + 215N^2)}{81N^2(1+N)} S_1^2 - \frac{16(-8 - 148N + 221N^2)}{27N^2(1+N)} S_2 + \frac{64(-3 + 5N)(1 + 8N)}{81N^2(1+N)} S_3 \left. \right\} \\
& + C_A T_F^2 N_F \left\{ -\frac{32R_4}{243N(1+N)^4} S_1 - \frac{16R_5}{9N^3(1+N)^3} \zeta_2 - \frac{32R_7}{243N^5(1+N)^5} \right. \\
& + \frac{-2 + 5N^2}{N(1+N)^2} \left(-\frac{64}{81} S_1^3 + \frac{64}{27} S_1 S_2 - \frac{128}{81} S_3 - \frac{256}{27} S_{2,1} - \frac{64}{9} S_1 \zeta_2 \right) \\
& + \frac{N-1}{N(1+N)} \left(\left(-\frac{320}{27} S_3 + \frac{128}{9} S_{2,1} \right) S_1 + \frac{8}{27} S_1^4 - \frac{16}{9} S_1^2 S_2 + \frac{8}{9} S_2^2 + \frac{112}{9} S_4 + \frac{256}{9} S_{-4} + \frac{256}{9} S_{3,1} - \frac{128}{9} S_{2,1,1} \right. \\
& + \left(\frac{16}{3} [S_2 + S_1^2] + \frac{32}{3} S_{-2} \right) \zeta_2 + \frac{448}{9} S_1 \zeta_3 \left. \right) + \frac{16(-59 - 68N + 125N^2 + 206N^3)}{81N(1+N)^3} S_1^2 \\
& - \frac{16(131 + 26N - 509N^2 - 332N^3)}{81N(1+N)^3} S_2 - \frac{256}{9(1+N)^2} S_3 - \frac{64(7 - 11N)(10 + 11N)}{81N(1+N)^2} S_{-2} \\
& \left. - \frac{256(-2 + 5N)}{27N(1+N)} S_{-3} - \frac{896(N-1)}{9N^2(1+N)^2} \zeta_3 \right\}, \tag{31}
\end{aligned}$$

with

$$R_1 = 142N^5 + 64N^4 + 2N^3 - 52N^2 - 15N - 9, \tag{32}$$

$$R_2 = 1648N^7 + 2444N^6 + 777N^5 - 1091N^4 - 106N^3 + 201N^2 + 72N + 27, \tag{33}$$

$$R_3 = 3N^4 + 6N^3 - N^2 - 4N + 12, \tag{34}$$

$$R_4 = 2228N^4 + 3601N^3 + 669N^2 - 2191N - 1055, \tag{35}$$

$$R_5 = 6N^5 + 6N^4 - 67N^3 + 6N^2 + 43N - 18, \tag{36}$$

$$R_6 = 18N^6 + 54N^5 + 5N^4 - 20N^3 + 95N^2 - 132N - 108, \tag{37}$$

$$R_7 = 2040N^9 + 6120N^8 - 4816N^7 - 16208N^6 - 1776N^5 + 7268N^4 + 2443N^3 + 756N^2 - 225N - 810, \tag{38}$$

$$\begin{aligned}
R_8 = & 2322N^{11} + 9288N^{10} + 5975N^9 - 11499N^8 - 7124N^7 + 4346N^6 + 522N^5 \\
& - 20360N^4 - 12027N^3 + 5193N^2 + 11880N + 4860. \tag{39}
\end{aligned}$$

IV. $A_{Qg}^{(3),S}$

The logarithmic contributions to the massive OME $A_{Qg}^{(3),S}$ are given by

$$\begin{aligned}
A_{Qg} = & -a_s \frac{4L_M(N-1)}{N(1+N)} T_F + a_s^2 \left\{ -\frac{16L_M^2(N-1)T_F^2}{3N(1+N)} \right. \\
& + C_F T_F \left\{ \frac{2P_{95}}{N^4(1+N)^4(2+N)} + L_M^2 \left[\frac{2(N-1)(2+3N+3N^2)}{N^2(1+N)^2} - \frac{8(N-1)S_1}{N(1+N)} \right] \right. \\
& + L_M \left[-\frac{4(N-1)P_{23}}{N^3(1+N)^3} + \frac{16(N-1)S_1}{N^2(1+N)} - \frac{8(N-1)S_1^2}{N(1+N)} + \frac{8(N-1)S_2}{N(1+N)} \right] \\
& + \left[\frac{4(-36-22N-2N^2+N^3)}{N^2(1+N)(2+N)} - \frac{4(N-1)S_2}{N(1+N)} \right] S_1 + \frac{4(-2+3N+3N^2)}{N^2(1+N)(2+N)} S_1^2 \\
& - \frac{4(N-1)S_1^3}{3N(1+N)} + \frac{4P_{16}}{N^2(1+N)^2(2+N)} S_2 + \frac{16(N-1)S_3}{3N(1+N)} \left. \right\} \\
& + C_A T_F \left\{ \frac{4P_{92}}{N^4(1+N)^4(2+N)} + L_M^2 \left[-\frac{16(N-1)}{N^2(1+N)^2} + \frac{8(N-1)S_1}{N(1+N)} \right] \right. \\
& + L_M \left[-\frac{8P_{43}}{N^3(1+N)^3} - \frac{32S_1}{N(1+N)^2} + \frac{8(N-1)S_1^2}{N(1+N)} + \frac{8(N-1)S_2}{N(1+N)} + \frac{16(N-1)S_{-2}}{N(1+N)} \right] \\
& + \left[-\frac{4P_{14}}{N(1+N)^3(2+N)} + \frac{12(N-1)S_2}{N(1+N)} \right] S_1 - \frac{4(5+4N+N^2)}{N(1+N)^2(2+N)} S_1^2 + \frac{4(N-1)S_1^3}{3N(1+N)} \\
& - \frac{4(-16+15N+24N^2+7N^3)}{N^2(1+N)^2(2+N)} S_2 + \frac{32(N-1)S_3}{3N(1+N)} + \left[\frac{16(N-1)}{N(1+N)^2} + \frac{16(N-1)S_1}{N(1+N)} \right] S_{-2} + \frac{8(N-1)S_{-3}}{N(1+N)} \\
& - \frac{16(N-1)S_{-2,1}}{N(1+N)} \left. \right\} + a_s^3 \left\{ a_{Qg}^{(3)} + C_A T_F^2 \left\{ \frac{8P_{106}}{81N^5(1+N)^5(2+N)} + L_M^3 \left[-\frac{448(N-1)}{9N^2(1+N)^2} + \frac{224(N-1)S_1}{9N(1+N)} \right] \right. \right. \\
& + L_M^2 \left[-\frac{8P_{53}}{9N^3(1+N)^3} + \frac{32(-23+5N^2)}{9N(1+N)^2} S_1 + \frac{16(N-1)S_1^2}{N(1+N)} + \frac{16(N-1)S_2}{N(1+N)} + \frac{32(N-1)S_{-2}}{N(1+N)} \right] \\
& + L_M \left[\frac{16P_{97}}{27N^4(1+N)^4(2+N)} + \left[\frac{16P_{39}}{27N(1+N)^3(2+N)} + \frac{80(N-1)S_2}{3N(1+N)} \right] S_1 - \frac{32(11+10N+13N^2+5N^3)}{9N(1+N)^2(2+N)} S_1^2 \right. \\
& + \frac{16(N-1)S_1^3}{3N(1+N)} - \frac{32P_{25}}{9N^2(1+N)^2(2+N)} S_2 + \frac{32(N-1)S_3}{N(1+N)} + \left[-\frac{64(4-3N+5N^2)}{9N(1+N)^2} + \frac{128(N-1)S_1}{3N(1+N)} \right] S_{-2} \\
& + \frac{128(N-1)S_{-3}}{3N(1+N)} + \frac{64(N-1)S_{2,1}}{3N(1+N)} - \frac{128(N-1)S_{-2,1}}{3N(1+N)} \left. \right] \\
& + \left[-\frac{16P_{60}}{81N(1+N)^4(2+N)} - \frac{32(-13+3N^2)}{3N(1+N)^2(2+N)} S_2 - \frac{640(N-1)S_3}{9N(1+N)} + \frac{256(N-1)S_{-2,1}}{3N(1+N)} \right] S_1 \\
& + \left[\frac{8P_{18}}{3N(1+N)^3(2+N)} - \frac{80(N-1)S_2}{3N(1+N)} \right] S_1^2 + \frac{32(5+4N+N^2)}{9N(1+N)^2(2+N)} S_1^3 \\
& - \frac{8(N-1)S_1^4}{9N(1+N)} + \frac{8P_{66}}{3N^3(1+N)^3(2+N)} S_2 - \frac{8(N-1)S_2^2}{3N(1+N)} + \frac{256(-6+8N+7N^2+N^3)}{9N^2(1+N)^2(2+N)} S_3 - \frac{48(N-1)S_4}{N(1+N)} \\
& + \left[\frac{128(3+N^2)}{3N(1+N)^3} - \frac{256(N-1)S_1}{3N(1+N)^2} - \frac{128(N-1)S_1^2}{3N(1+N)} - \frac{128(N-1)S_2}{3N(1+N)} \right] S_{-2} \\
& + \left[-\frac{128(N-1)}{3N(1+N)^2} - \frac{128(N-1)S_1}{3N(1+N)} \right] S_{-3} - \frac{64(N-1)S_{-4}}{3N(1+N)} + \frac{64(N-1)S_{3,1}}{3N(1+N)} \\
& + \frac{256(N-1)S_{-2,1}}{3N(1+N)^2} + \frac{128(N-1)S_{-2,2}}{3N(1+N)} + \frac{128(N-1)S_{-3,1}}{3N(1+N)} + \frac{64(N-1)S_{2,1,1}}{3N(1+N)} - \frac{256(N-1)S_{-2,1,1}}{3N(1+N)} \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{4P_{56}}{9N^3(1+N)^3} - \frac{160(-2+N)(2+N)S_1}{9N(1+N)^2} - \frac{40(N-1)S_1^2}{3N(1+N)} - \frac{40(N-1)S_2}{3N(1+N)} - \frac{80(N-1)S_{-2}}{3N(1+N)} \right] \zeta_2 \\
& + \left[\frac{448(N-1)}{9N^2(1+N)^2} - \frac{224(N-1)S_1}{9N(1+N)} \right] \zeta_3 \Big\} + C_A N_F T_F^2 \left\{ \frac{16P_{102}}{3N^5(1+N)^5(2+N)} + L_M^3 \left[-\frac{64(N-1)}{9N^2(1+N)^2} \right. \right. \\
& + \frac{32(N-1)S_1}{9N(1+N)} \Big] + L_M^2 \left[-\frac{8P_{48}}{9N^3(1+N)^3} - \frac{32(1+5N^2)}{9N(1+N)^2} S_1 + \frac{16(N-1)S_1^2}{3N(1+N)} \right. \\
& + \frac{16(N-1)S_2}{3N(1+N)} + \frac{32(N-1)S_{-2}}{3N(1+N)} \Big] + L_M \left[\frac{8P_{98}}{27N^4(1+N)^4(2+N)} + \left[\frac{16P_{36}}{27N(1+N)^3(2+N)} + \frac{32(N-1)S_2}{3N(1+N)} \right] S_1 \right. \\
& - \frac{16(7+8N+23N^2+10N^3)}{9N(1+N)^2(2+N)} S_1^2 + \frac{32(N-1)S_1^3}{9N(1+N)} - \frac{16P_{29}}{9N^2(1+N)^2(2+N)} S_2 + \frac{160(N-1)S_3}{9N(1+N)} \\
& + \left[-\frac{64(1+5N^2)}{9N(1+N)^2} + \frac{64(N-1)S_1}{3N(1+N)} \right] S_{-2} + \frac{32(N-1)S_{-3}}{N(1+N)} + \frac{64(N-1)S_{2,1}}{3N(1+N)} - \frac{64(N-1)S_{-2,1}}{3N(1+N)} \Big] \\
& + \left[-\frac{16P_{45}}{3N(1+N)^4(2+N)} - \frac{16(-13+3N^2)}{3N(1+N)^2(2+N)} S_2 - \frac{320(N-1)S_3}{9N(1+N)} + \frac{128(N-1)S_{-2,1}}{3N(1+N)} \right] S_1 \\
& + \left[\frac{8P_{13}}{3N(1+N)^3(2+N)} - \frac{40(N-1)S_2}{3N(1+N)} \right] S_1^2 + \frac{16(5+4N+N^2)}{9N(1+N)^2(2+N)} S_1^3 - \frac{4(N-1)S_1^4}{9N(1+N)} \\
& + \frac{8P_{62}}{3N^3(1+N)^3(2+N)} S_2 - \frac{4(N-1)S_2^2}{3N(1+N)} + \frac{128(-6+8N+7N^2+N^3)}{9N^2(1+N)^2(2+N)} S_3 - \frac{24(N-1)S_4}{N(1+N)} \\
& + \left[\frac{64(3+N^2)}{3N(1+N)^3} - \frac{128(N-1)S_1}{3N(1+N)^2} - \frac{64(N-1)S_1^2}{3N(1+N)} - \frac{64(N-1)S_2}{3N(1+N)} \right] S_{-2} + \left[-\frac{64(N-1)}{3N(1+N)^2} - \frac{64(N-1)S_1}{3N(1+N)} \right] S_{-3} \\
& - \frac{32(N-1)S_{-4}}{3N(1+N)} + \frac{32(N-1)S_{3,1}}{3N(1+N)} + \frac{128(N-1)S_{-2,1}}{3N(1+N)^2} + \frac{64(N-1)S_{-2,2}}{3N(1+N)} + \frac{64(N-1)S_{-3,1}}{3N(1+N)} + \frac{32(N-1)S_{2,1,1}}{3N(1+N)} \\
& - \frac{128(N-1)S_{-2,1,1}}{3N(1+N)} + \left[\frac{4P_{49}}{9N^3(1+N)^3} + \frac{16(7+5N^2)}{9N(1+N)^2} S_1 - \frac{16(N-1)S_1^2}{3N(1+N)} - \frac{16(N-1)S_2}{3N(1+N)} - \frac{32(N-1)S_{-2}}{3N(1+N)} \right] \zeta_2 \\
& + \left[\frac{64(N-1)}{9N^2(1+N)^2} - \frac{32(N-1)S_1}{9N(1+N)} \right] \zeta_3 \Big\} + C_F N_F T_F^2 \left\{ -\frac{4P_{111}}{3N^6(1+N)^6(2+N)} + L_M^3 \left[\frac{8(N-1)P_{20}}{9N^3(1+N)^3} - \frac{32(N-1)S_1}{9N(1+N)} \right] \right. \\
& + L_M^2 \left[-\frac{4(N-1)P_{72}}{9N^4(1+N)^4} + \frac{32(N-1)(3+5N)}{9N^2(1+N)} S_1 - \frac{16(N-1)S_1^2}{3N(1+N)} - \frac{16(N-1)S_2}{3N(1+N)} \right] \\
& + L_M \left[\frac{2P_{107}}{27N^5(1+N)^5(2+N)} + \left[-\frac{32(138+91N+58N^2+19N^3)}{27N^2(1+N)(2+N)} - \frac{32(N-1)S_2}{3N(1+N)} \right] S_1 \right. \\
& + \frac{32(-6-4N+11N^2+5N^3)}{9N^2(1+N)(2+N)} S_1^2 - \frac{32(N-1)S_1^3}{9N(1+N)} + \frac{16P_{63}}{3N^3(1+N)^3(2+N)} S_2 + \frac{224(N-1)S_3}{9N(1+N)} \Big] \\
& + \left[\frac{16P_{17}}{3N^2(1+N)^2(2+N)} - \frac{16(-2+3N+3N^2)}{3N^2(1+N)(2+N)} S_2 + \frac{32(N-1)S_3}{9N(1+N)} + \frac{64(N-1)S_{2,1}}{3N(1+N)} \right] S_1 \\
& + \left[-\frac{8(-36-22N-6N^2+N^3)}{3N^2(1+N)(2+N)} + \frac{8(N-1)S_2}{3N(1+N)} \right] S_1^2 - \frac{16(-2+3N+3N^2)}{9N^2(1+N)(2+N)} S_1^3 + \frac{4(N-1)S_1^4}{9N(1+N)} \\
& - \frac{8P_{94}}{3N^4(1+N)^4(2+N)} S_2 + \frac{4(N-1)S_2^2}{3N(1+N)} - \frac{16P_{65}}{9N^3(1+N)^3(2+N)} S_3 - \frac{8(N-1)S_4}{N(1+N)} + \frac{64S_{2,1}}{3N^2} + \frac{64(N-1)S_{3,1}}{3N(1+N)} \\
& - \frac{128(N-1)S_{2,1,1}}{3N(1+N)} + \left[-\frac{2(N-1)P_{84}}{9N^4(1+N)^4} - \frac{16(N-1)(6+5N)}{9N^2(1+N)} S_1 + \frac{16(N-1)S_1^2}{3N(1+N)} \right] \zeta_2 \\
& + \left[-\frac{8(N-1)P_{20}}{9N^3(1+N)^3} + \frac{32(N-1)S_1}{9N(1+N)} \right] \zeta_3 \Big\} + C_F T_F^2 \left\{ -\frac{2P_{112}}{3N^6(1+N)^6(2+N)} + L_M^3 \left[\frac{16(N-1)P_{27}}{9N^3(1+N)^3} - \frac{128(N-1)S_1}{9N(1+N)} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + L_M^2 \left[-\frac{4(N-1)P_{79}}{9N^4(1+N)^4} + \frac{32(N-1)(9+5N)}{9N^2(1+N)} S_1 - \frac{16(N-1)S_1^2}{N(1+N)} + \frac{16(N-1)S_2}{3N(1+N)} \right] \\
& + L_M \left[\frac{2P_{109}}{27N^5(1+N)^5(2+N)} + \left[-\frac{16(600+380N+134N^2+29N^3)}{27N^2(1+N)(2+N)} - \frac{16(N-1)S_2}{N(1+N)} \right] S_1 \right. \\
& + \frac{16(-18+N+31N^2+10N^3)}{9N^2(1+N)(2+N)} S_1^2 - \frac{16(N-1)S_1^3}{3N(1+N)} + \frac{16P_{19}}{3N^2(1+N)^2(2+N)} S_2 + \frac{32(N-1)S_3}{N(1+N)} \left. \right] \\
& + \left[\frac{32P_{17}}{3N^2(1+N)^2(2+N)} - \frac{32(-2+3N+3N^2)}{3N^2(1+N)(2+N)} S_2 + \frac{64(N-1)S_3}{9N(1+N)} + \frac{128(N-1)S_{2,1}}{3N(1+N)} \right] S_1 \\
& + \left[-\frac{16(-36-22N-6N^2+N^3)}{3N^2(1+N)(2+N)} + \frac{16(N-1)S_2}{3N(1+N)} \right] S_1^2 - \frac{32(-2+3N+3N^2)}{9N^2(1+N)(2+N)} S_1^3 + \frac{8(N-1)S_1^4}{9N(1+N)} \\
& - \frac{16P_{64}}{3N^3(1+N)^3(2+N)} S_2 + \frac{8(N-1)S_2^2}{3N(1+N)} - \frac{32P_{22}}{9N^2(1+N)^2(2+N)} S_3 - \frac{16(N-1)S_4}{N(1+N)} + \frac{128S_{2,1}}{3N^2} \\
& + \frac{128(N-1)S_{3,1}}{3N(1+N)} - \frac{256(N-1)S_{2,1,1}}{3N(1+N)} + \left[\frac{2(N-1)P_{82}}{9N^4(1+N)^4} - \frac{80(N-1)(3+N)S_1}{9N^2(1+N)} + \frac{40(N-1)S_1^2}{3N(1+N)} - \frac{8(N-1)S_2}{N(1+N)} \right] \zeta_2 \\
& + \left[-\frac{16(N-1)P_{27}}{9N^3(1+N)^3} + \frac{128(N-1)S_1}{9N(1+N)} \right] \zeta_3 \left. \right\} + C_A C_F T_F \left\{ \frac{P_{113}}{3N^6(1+N)^6(2+N)} + L_M^3 \left[-\frac{2(N-1)(2+3N+3N^2)}{9N^3(1+N)^3} \right. \right. \\
& \times (-48+11N+11N^2) - \frac{8(N-1)(60+7N+7N^2)}{9N^2(1+N)^2} S_1 + \frac{64(N-1)S_1^2}{3N(1+N)} \left. \right] \\
& + L_M^2 \left[-\frac{8S_1P_{52}}{9N^3(1+N)^3} + \frac{P_{61}}{9N^3(1+N)^3} + \frac{8(27-40N-12N^2+N^3)}{3N^2(1+N)^2} S_1^2 + \frac{32(N-1)S_1^3}{N(1+N)} + \frac{8(N-1)(9+N+N^2)}{3N^2(1+N)^2} S_2 \right. \\
& - \frac{16(N-1)S_3}{N(1+N)} - \frac{24(N-1)S_{-2}}{N(1+N)} - \frac{16(N-1)S_{-3}}{N(1+N)} + \frac{32(N-1)S_{-2,1}}{N(1+N)} \left. \right] + L_M \left[\frac{P_{100}}{54N^5(1+N)^5(2+N)} \right. \\
& + \left[\frac{8P_{90}}{27N^4(1+N)^3(2+N)} - \frac{8P_{34}}{3N^2(1+N)^2(2+N)} S_2 + \frac{128(N-1)S_3}{N(1+N)} - \frac{192(N-1)S_{2,1}}{N(1+N)} - \frac{64(N-1)S_{-2,1}}{N(1+N)} \right] S_1 \\
& + \left[-\frac{4P_{77}}{9N^3(1+N)^3(2+N)} + \frac{144(N-1)S_2}{N(1+N)} \right] S_1^2 + \frac{8P_{30}}{9N^2(1+N)^2(2+N)} S_1^3 + \frac{16(N-1)S_1^4}{N(1+N)} \\
& + \frac{4P_{71}}{3N^3(1+N)^3(2+N)} S_2 - \frac{32(N-1)S_2^2}{N(1+N)} - \frac{8(N-1)(216+41N+77N^2)}{9N^2(1+N)^2} S_3 \\
& + \left[-\frac{32P_{46}}{N^3(1+N)^3(2+N)} - \frac{16(N-1)(20+4N+N^2+3N^3)}{N^2(1+N)^2(2+N)} S_1 + \frac{160(N-1)S_1^2}{N(1+N)} \right] S_{-2} + \frac{48(N-1)S_{-2}^2}{N(1+N)} \\
& + \left[\frac{8(N-1)(3+N)(-2+3N)}{N^2(1+N)^2} + \frac{32(N-1)S_1}{N(1+N)} \right] S_{-3} + \frac{80(N-1)S_{-4}}{N(1+N)} + \frac{48(N-1)(4+N+N^2)}{N^2(1+N)^2} S_{2,1} \\
& + \frac{32(N-1)S_{3,1}}{N(1+N)} + \frac{16(N-1)(6-N+3N^2)}{N^2(1+N)^2} S_{-2,1} + \frac{32(N-1)S_{-2,2}}{N(1+N)} + \frac{96(N-1)S_{2,1,1}}{N(1+N)} - \frac{128(N-1)S_{-2,1,1}}{N(1+N)} \\
& + \left[-\frac{96(N-1)(-5+3N+3N^2)}{N^2(1+N)^2} - \frac{96(N-1)S_1}{N(1+N)} \right] \zeta_3 \left. \right] + \left[\frac{8P_{105}}{3N^5(1+N)^5(2+N)} + \frac{8P_{55}}{3N^2(1+N)^3(2+N)} S_2 \right. \\
& - \frac{8(N-1)S_2^2}{N(1+N)} + \frac{8P_{40}}{9N^2(1+N)^2(2+N)} S_3 - \frac{48(N-1)S_4}{N(1+N)} - \frac{16(36-11N+12N^2+11N^3)}{3N^2(1+N)^2} S_{2,1} - \frac{32(N-1)S_{3,1}}{N(1+N)} \\
& - \frac{32(N-1)(2-N+3N^2)}{N^2(1+N)^2} S_{-2,1} + \frac{64(N-1)S_{-2,2}}{N(1+N)} + \frac{64(N-1)S_{-3,1}}{N(1+N)} + \frac{160(N-1)S_{2,1,1}}{N(1+N)} - \frac{128(N-1)S_{-2,1,1}}{N(1+N)} \left. \right] S_1 \\
& + \left[-\frac{4P_{89}}{3N^3(1+N)^4(2+N)} + \frac{4P_{35}}{3N^2(1+N)^2(2+N)} S_2 - \frac{352(N-1)S_3}{3N(1+N)} - \frac{64(N-1)S_{2,1}}{N(1+N)} + \frac{128(N-1)S_{-2,1}}{N(1+N)} \right] S_1^2
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{8P_{74}}{9N^3(1+N)^3(2+N)} - \frac{48(N-1)S_2}{N(1+N)} \right] S_3^1 - \frac{2P_{12}}{9N^2(1+N)^2(2+N)} S_4^1 - \frac{8(N-1)S_1^5}{3N(1+N)} - \frac{4P_{96}}{3N^4(1+N)^4(2+N)} S_2 \\
& - \frac{2(N-1)(-15+N+N^2)}{3N^2(1+N)^2} S_2^2 + \frac{4P_{83}}{9N^3(1+N)^3(2+N)} S_3 + \frac{4(N-1)(-3+19N+19N^2)}{N^2(1+N)^2} S_4 \\
& + \left[-\frac{16P_{47}}{N^2(1+N)^4} + \left[\frac{32P_{26}}{N^2(1+N)^3} - \frac{64(N-1)S_2}{N(1+N)} \right] S_1 + \frac{16(N-1)^2(-2+3N)}{N^2(1+N)^2} S_1^2 - \frac{64(N-1)S_1^3}{N(1+N)} \right. \\
& + \frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_2 \left. \right] S_{-2} + \left[\frac{16(N-1)(2+9N+15N^2+6N^3)}{N^2(1+N)^3} + \frac{16(N-1)(2-N+3N^2)}{N^2(1+N)^2} S_1 \right. \\
& - \frac{64(N-1)S_1^2}{N(1+N)} \left. \right] S_{-3} + \left[\frac{8(N-1)(2+3N+3N^2)}{N^2(1+N)^2} - \frac{32(N-1)S_1}{N(1+N)} \right] S_{-4} - \frac{16(-24+11N+11N^2)}{3N^3(1+N)} S_{2,1} \\
& - \frac{8(N-1)(-42+31N+31N^2)}{3N^2(1+N)^2} S_{3,1} - \frac{32(N-1)(2+9N+15N^2+6N^3)}{N^2(1+N)^3} S_{-2,1} \\
& - \frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_{-2,2} - \frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_{-3,1} + \frac{8(N-1)(-102+35N+35N^2)}{3N^2(1+N)^2} S_{2,1,1} \\
& + \frac{32(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_{-2,1,1} + \left(\frac{192 \ln(2)(N-1)}{N(1+N)} + \frac{4S_1P_{59}}{9N^3(1+N)^3} + \frac{P_{86}}{18N^4(1+N)^4} \right. \\
& - \frac{8(27-40N-12N^2+N^3)}{3N^2(1+N)^2} S_1^2 - \frac{32(N-1)S_1^3}{N(1+N)} + \frac{12(N-1)^2(2+N)}{N^2(1+N)^2} S_2 + \frac{8(N-1)S_3}{N(1+N)} \\
& + \left[\frac{8(N-1)(1+3N+3N^2)}{N^2(1+N)^2} - \frac{16(N-1)S_1}{N(1+N)} \right] S_{-2} + \frac{8(N-1)S_{-3}}{N(1+N)} - \frac{16(N-1)S_{-2,1}}{N(1+N)} \left. \right) \zeta_2 \\
& + \left[-\frac{2(N-1)P_{42}}{9N^3(1+N)^3} + \frac{8(N-1)(60+7N+7N^2)}{9N^2(1+N)^2} S_1 - \frac{64(N-1)S_1^2}{3N(1+N)} \right] \zeta_3 \left. \right\} \\
& + C_F^2 T_F \left\{ \frac{P_{110}}{N^6(1+N)^6} + L_M^3 \left[-\frac{2(N-1)(2+3N+3N^2)^2}{3N^3(1+N)^3} + \frac{16(N-1)(2+3N+3N^2)}{3N^2(1+N)^2} S_1 - \frac{32(N-1)S_1^2}{3N(1+N)} \right] \right. \\
& + L_M^2 \left[\frac{(N-1)P_{73}}{N^4(1+N)^4} + \left[-\frac{8(N-1)P_{24}}{N^3(1+N)^3} + \frac{48(N-1)S_2}{N(1+N)} \right] S_1 + \frac{4(N-1)(2+N)(5+3N)}{N^2(1+N)^2} S_1^2 - \frac{16(N-1)S_1^3}{N(1+N)} \right. \\
& - \frac{12(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_2 + \frac{32(N-1)S_3}{N(1+N)} + \left[-\frac{32(N-1)}{N^2(1+N)^2} + \frac{64(N-1)S_1}{N(1+N)} \right] S_{-2} \\
& + \frac{32(N-1)S_{-3}}{N(1+N)} - \frac{64(N-1)S_{-2,1}}{N(1+N)} \left. \right] + L_M \left[\frac{P_{101}}{2N^5(1+N)^5(2+N)} + \left[\frac{8P_{93}}{N^4(1+N)^4(2+N)} \right. \right. \\
& + \frac{16P_{15}}{N^2(1+N)^2(2+N)} S_2 - \frac{96(N-1)S_3}{N(1+N)} + \frac{128(N-1)S_{2,1}}{N(1+N)} \left. \right] S_1 \\
& + \left[-\frac{8P_{67}}{N^3(1+N)^3(2+N)} - \frac{16(N-1)S_2}{N(1+N)} \right] S_1^2 + \frac{64(-1+N+N^2)}{N^2(1+N)(2+N)} S_1^3 - \frac{8(N-1)S_1^4}{N(1+N)} \\
& + \frac{4P_{70}}{N^3(1+N)^3(2+N)} S_2 - \frac{24(N-1)S_2^2}{N(1+N)} - \frac{8(N-1)(-6+11N+3N^2)}{N^2(1+N)^2} S_3 \\
& - \frac{80(N-1)S_4}{N(1+N)} + \left[\frac{32P_{44}}{N^2(1+N)^3(2+N)} - \frac{128(N-1)S_1}{N(1+N)^2} - \frac{64(N-1)S_2}{N(1+N)} \right] S_{-2} \\
& - \frac{32(N-1)S_{-2}^2}{N(1+N)} + \left[-\frac{64(N-1)^2}{N^2(1+N)^2} - \frac{128(N-1)S_1}{N(1+N)} \right] S_{-3} - \frac{160(N-1)S_{-4}}{N(1+N)} - \frac{64(N-1)S_{2,1}}{N^2(1+N)^2} + \frac{32(N-1)S_{3,1}}{N(1+N)} \\
& + \frac{128(N-1)S_{-2,1}}{N(1+N)^2} + \frac{64(N-1)S_{-2,2}}{N(1+N)} + \frac{128(N-1)S_{-3,1}}{N(1+N)} - \frac{96(N-1)S_{2,1,1}}{N(1+N)} + \frac{48(N-1)(-2+3N+3N^2)}{N^2(1+N)^2} \zeta_3 \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{4P_{103}}{N^5(1+N)^5(2+N)} - \frac{4P_{69}}{N^3(1+N)^3(2+N)} S_2 + \frac{4(N-1)S_2^2}{N(1+N)} - \frac{8P_{28}}{3N^2(1+N)^2(2+N)} S_3 - \frac{24(N-1)S_4}{N(1+N)} \right. \\
& - \frac{16(-6-9N-4N^2+3N^3)}{N^2(1+N)^2} S_{2,1} + \frac{64(N-1)S_{3,1}}{N(1+N)} - \frac{128(N-1)S_{2,1,1}}{N(1+N)} \left. \right] S_1 + \left[\frac{2P_{54}}{N^3(1+N)^2(2+N)} \right. \\
& - \frac{2P_{21}}{N^2(1+N)^2(2+N)} S_2 + \frac{32(N-1)S_3}{3N(1+N)} + \frac{64(N-1)S_{2,1}}{N(1+N)} \left. \right] S_1^2 + \left[-\frac{4P_{50}}{3N^3(1+N)^2(2+N)} + \frac{8(N-1)S_2}{N(1+N)} \right] S_1^3 \\
& + \frac{P_{11}}{3N^2(1+N)^2(2+N)} S_1^4 + \frac{4(N-1)S_1^5}{3N(1+N)} + \frac{2P_{91}}{N^4(1+N)^4(2+N)} S_2 - \frac{(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_2^2 \\
& + \frac{4P_{76}}{3N^3(1+N)^3(2+N)} S_3 + \frac{6(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_4 - \frac{16(2+3N+3N^2)}{N^3(1+N)} S_{2,1} \\
& - \frac{16(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_{3,1} + \frac{32(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_{2,1,1} \\
& + \left[-\frac{384\ln(2)(N-1)}{N(1+N)} + \frac{(N-1)P_{85}}{2N^4(1+N)^4} + \left[-\frac{8(N-1)P_{33}}{N^3(1+N)^3} - \frac{32(N-1)S_2}{N(1+N)} \right] S_1 \right. \\
& - \frac{4(N-1)(2+N)(5+3N)}{N^2(1+N)^2} S_1^2 + \frac{16(N-1)S_1^3}{N(1+N)} + \frac{8(N-1)(2+3N+3N^2)}{N^2(1+N)^2} S_2 - \frac{16(N-1)S_3}{N(1+N)} \\
& + \left[\frac{16(N-1)}{N^2(1+N)^2} - \frac{32(N-1)S_1}{N(1+N)} \right] S_{-2} - \frac{16(N-1)S_{-3}}{N(1+N)} + \frac{32(N-1)S_{-2,1}}{N(1+N)} \left. \right] \zeta_2 \\
& + \left\{ \frac{2(N-1)P_{41}}{3N^3(1+N)^3} - \frac{16(N-1)(2+3N+3N^2)}{3N^2(1+N)^2} S_1 + \frac{32(N-1)S_1^2}{3N(1+N)} \right\} \zeta_3 \Big\} \\
& + C_A^2 T_F \left\{ -\frac{4(-24+11N+11N^2)P_{102}}{3N^6(1+N)^6(2+N)} + L_M^3 \left[\frac{16(N-1)(-24+11N+11N^2)}{9N^3(1+N)^3} \right. \right. \\
& - \frac{8(N-1)(-48+11N+11N^2)}{9N^2(1+N)^2} S_1 - \frac{32(N-1)S_1^2}{3N(1+N)} \left. \right] + L_M^2 \left[\frac{16P_{87}}{9N^4(1+N)^4} + \left[\frac{8P_{57}}{9N^3(1+N)^3} - \frac{48(N-1)S_2}{N(1+N)} \right] S_1 \right. \\
& - \frac{4(24-83N+11N^3)}{3N^2(1+N)^2} S_1^2 - \frac{16(N-1)S_1^3}{N(1+N)} - \frac{4(N-1)(-72+11N+11N^2)}{3N^2(1+N)^2} S_2 - \frac{16(N-1)S_3}{N(1+N)} \\
& + \left[-\frac{8(N-1)(-48+11N+11N^2)}{3N^2(1+N)^2} - \frac{64(N-1)S_1}{N(1+N)} \right] S_{-2} - \frac{16(N-1)S_{-3}}{N(1+N)} + \frac{32(N-1)S_{-2,1}}{N(1+N)} \left. \right] \\
& + L_M \left[-\frac{4P_{108}}{27N^5(1+N)^5(2+N)} + \left[-\frac{8P_{99}}{27N^4(1+N)^4(2+N)} - \frac{8P_{32}}{3N^2(1+N)^2(2+N)} S_2 - \frac{160(N-1)S_3}{N(1+N)} \right. \right. \\
& + \frac{64(N-1)S_{2,1}}{N(1+N)} + \frac{320(N-1)S_{-2,1}}{N(1+N)} \left. \right] S_1 + \left[\frac{4P_{80}}{9N^3(1+N)^3(2+N)} - \frac{128(N-1)S_2}{N(1+N)} \right] S_1^2 \\
& - \frac{8(-184-119N+4N^2+11N^3)}{9N(1+N)^2(2+N)} S_1^3 - \frac{8(N-1)S_1^4}{N(1+N)} + \frac{4P_{81}}{9N^3(1+N)^3(2+N)} S_2 - \frac{8(N-1)S_2^2}{N(1+N)} \\
& - \frac{8(441-568N+55N^3)}{9N^2(1+N)^2} S_3 - \frac{16(N-1)S_4}{N(1+N)} + \left[\frac{16P_{78}}{9N^3(1+N)^3(2+N)} - \frac{16(54-113N+24N^2+11N^3)}{3N^2(1+N)^2} S_1 \right. \\
& - \frac{160(N-1)S_1^2}{N(1+N)} - \frac{64(N-1)S_2}{N(1+N)} \left. \right] S_{-2} - \frac{48(N-1)S_{-2}^2}{N(1+N)} + \left[-\frac{8(54-73N+11N^3)}{N^2(1+N)^2} - \frac{288(N-1)S_1}{N(1+N)} \right] S_{-3} \\
& - \frac{176(N-1)S_{-4}}{N(1+N)} - \frac{16(N-1)(24+11N+11N^2)}{3N^2(1+N)^2} S_{2,1} - \frac{64(N-1)S_{3,1}}{N(1+N)} + \frac{16(66-101N+11N^3)}{3N^2(1+N)^2} S_{-2,1} \\
& + \frac{224(N-1)S_{-2,2}}{N(1+N)} + \frac{256(N-1)S_{-3,1}}{N(1+N)} - \frac{384(N-1)S_{-2,1,1}}{N(1+N)} + \left[\frac{48(N-1)(-8+3N+3N^2)}{N^2(1+N)^2} + \frac{96(N-1)S_1}{N(1+N)} \right] \zeta_3 \Big]
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{4P_{104}}{3N^5(1+N)^5(2+N)} - \frac{4P_{68}}{3N^3(1+N)^3(2+N)} S_2 + \frac{4(N-1)S_2^2}{N(1+N)} + \frac{16P_{37}}{9N^2(1+N)^2(2+N)} S_3 + \frac{72(N-1)S_4}{N(1+N)} \right. \\
& - \frac{32(N-1)S_{3,1}}{N(1+N)} - \frac{32(N-1)(-24+23N+11N^2)}{3N^2(1+N)^2} S_{-2,1} - \frac{64(N-1)S_{-2,2}}{N(1+N)} - \frac{64(N-1)S_{-3,1}}{N(1+N)} - \frac{32(N-1)S_{2,1,1}}{N(1+N)} \\
& + \left. \frac{128(N-1)S_{-2,1,1}}{N(1+N)} \right] S_1 + \left[\frac{2P_{75}}{3N^2(1+N)^4(2+N)} + \frac{2P_{38}}{3N^2(1+N)^2(2+N)} S_2 + \frac{320(N-1)S_3}{3N(1+N)} - \frac{128(N-1)S_{-2,1}}{N(1+N)} \right] S_1^2 \\
& + \left[-\frac{4P_{51}}{9N^2(1+N)^3(2+N)} + \frac{40(N-1)S_2}{N(1+N)} \right] S_1^3 + \frac{P_{31}}{9N^2(1+N)^2(2+N)} S_1^4 + \frac{4(N-1)S_1^5}{3N(1+N)} \\
& - \frac{2(-24+11N+11N^2)P_{62}}{3N^4(1+N)^4(2+N)} S_2 + \frac{(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} S_2^2 \\
& - \frac{32(-24+11N+11N^2)(-6+8N+7N^2+N^3)}{9N^3(1+N)^3(2+N)} S_3 + \frac{6(N-1)(-24+11N+11N^2)}{N^2(1+N)^2} S_4 \\
& + \left[-\frac{16(3+N^2)(-24+11N+11N^2)}{3N^2(1+N)^4} + \left[\frac{32(-3+N)(-8+15N+5N^2)}{3N^2(1+N)^3} + \frac{64(N-1)S_2}{N(1+N)} \right] S_1 \right. \\
& + \frac{16(N-1)(-24+35N+11N^2)}{3N^2(1+N)^2} S_1^2 + \frac{64(N-1)S_1^3}{N(1+N)} + \frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} S_2 \left. \right] S_{-2} \\
& + \left[\frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^3} + \frac{16(N-1)(-24+23N+11N^2)}{3N^2(1+N)^2} S_1 + \frac{64(N-1)S_1^2}{N(1+N)} \right] S_{-3} \\
& + \left[\frac{8(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} + \frac{32(N-1)S_1}{N(1+N)} \right] S_{-4} - \frac{8(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} S_{3,1} \\
& - \frac{32(N-1)(-24+11N+11N^2)}{3N^2(1+N)^3} S_{-2,1} - \frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} S_{-2,2} \\
& - \frac{16(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} S_{-3,1} - \frac{8(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} S_{2,1,1} \\
& + \frac{32(N-1)(-24+11N+11N^2)}{3N^2(1+N)^2} S_{-2,1,1} + \left[-\frac{2P_{88}}{9N^4(1+N)^4} + \left[-\frac{4P_{58}}{9N^3(1+N)^3} + \frac{32(N-1)S_2}{N(1+N)} \right] S_1 \right. \\
& + \frac{4(24-83N+11N^3)}{3N^2(1+N)^2} S_1^2 + \frac{16(N-1)S_1^3}{N(1+N)} + \frac{4(N-1)(-48+11N+11N^2)}{3N^2(1+N)^2} S_2 + \frac{8(N-1)S_3}{N(1+N)} \\
& + \left[\frac{8(N-1)(-36+11N+11N^2)}{3N^2(1+N)^2} + \frac{48(N-1)S_1}{N(1+N)} \right] S_{-2} + \frac{8(N-1)S_{-3}}{N(1+N)} - \frac{16(N-1)S_{-2,1}}{N(1+N)} \left. \right] \zeta_2 \\
& + \left[-\frac{16(N-1)(-24+11N+11N^2)}{9N^3(1+N)^3} + \frac{8(N-1)(-48+11N+11N^2)}{9N^2(1+N)^2} S_1 + \frac{32(N-1)S_1^2}{3N(1+N)} \right] \zeta_3 \left. \right\} \\
& + T_F^3 \left[-\frac{64L_M^3(N-1)}{9N(1+N)} + \frac{64(N-1)\zeta_3}{9N(1+N)} \right] \Bigg\}, \tag{40}
\end{aligned}$$

and the polynomials P_i read

$$P_{11} = -3N^4 - 54N^3 - 95N^2 - 12N + 36, \tag{41}$$

$$P_{12} = N^4 - 94N^3 - 256N^2 - 161N + 78, \tag{42}$$

$$P_{13} = N^4 + 2N^3 - 5N^2 - 12N + 2, \tag{43}$$

$$P_{14} = N^4 + 4N^3 - N^2 - 10N + 2, \tag{44}$$

$$P_{15} = N^4 + 10N^3 + 27N^2 + 30N + 4, \tag{45}$$

$$P_{16} = N^4 + 17N^3 + 43N^2 + 33N + 2, \tag{46}$$

$$P_{17} = 2N^4 - 4N^3 - 3N^2 + 20N + 12, \tag{47}$$

$$P_{18} = 2N^4 + 3N^3 - 12N^2 - 23N + 6, \tag{48}$$

$$P_{19} = 2N^4 + 39N^3 + 100N^2 + 73N + 2, \tag{49}$$

$$\begin{aligned}
P_{20} &= 3N^4 + 6N^3 - N^2 - 4N + 12, & (50) & & P_{47} &= 8N^5 + 7N^4 - 9N^3 + 7N^2 + 13N + 6, & (77) \\
P_{21} &= 3N^4 + 30N^3 + 47N^2 + 4N - 20, & (51) & & P_{48} &= 9N^5 + 9N^4 - 79N^3 + 15N^2 + 22N - 24, & (78) \\
P_{22} &= 3N^4 + 48N^3 + 123N^2 + 98N + 8, & (52) & & P_{49} &= 15N^5 + 15N^4 - 103N^3 + 33N^2 - 20N - 36, & (79) \\
P_{23} &= 5N^4 + 10N^3 + 8N^2 + 7N + 2, & (53) & & P_{50} &= 18N^5 - 15N^4 - 198N^3 - 381N^2 - 216N + 4, & (80) \\
P_{24} &= 5N^4 + 13N^3 + 14N^2 + 16N + 6, & (54) & & P_{51} &= 18N^5 + 47N^4 - 35N^3 - 141N^2 - 5N - 120, & (81) \\
P_{25} &= 5N^4 + 37N^3 + 82N^2 + 41N - 48, & (55) & & P_{52} &= 40N^5 + 73N^4 - 142N^3 - 163N^2 - 150N + 54, & (82) \\
P_{26} &= 6N^4 + 11N^3 - 6N^2 - N - 2, & (56) & & P_{53} &= 45N^5 + 45N^4 - 47N^3 + 27N^2 - 190N - 24, & (83) \\
P_{27} &= 6N^4 + 12N^3 + 7N^2 + N + 6, & (57) & & P_{54} &= 51N^5 + 89N^4 + 6N^3 - 66N^2 - 104N - 72, & (84) \\
P_{28} &= 9N^4 + 102N^3 + 245N^2 + 192N + 12, & (58) & & P_{55} &= 66N^5 + 336N^4 + 627N^3 + 415N^2 - 10N - 194, & (85) \\
P_{29} &= 10N^4 + 53N^3 + 92N^2 + 37N - 48, & (59) & & & & \\
P_{30} &= 11N^4 - 68N^3 - 263N^2 - 184N + 72, & (60) & & P_{56} &= 69N^5 + 69N^4 - 55N^3 + 51N^2 - 338N - 36, & (86) \\
P_{31} &= 11N^4 - 26N^3 - 227N^2 - 286N + 48, & (61) & & P_{57} &= 85N^5 + 85N^4 - 73N^3 + 197N^2 - 342N - 108, & (87) \\
P_{32} &= 11N^4 + 4N^3 - 239N^2 - 304N + 240, & (62) & & P_{58} &= 103N^5 + 103N^4 - 79N^3 + 317N^2 - 612N - 144, & (88) \\
P_{33} &= 13N^4 + 23N^3 + 4N^2 - 14N - 5, & (63) & & & & \\
P_{34} &= 13N^4 + 140N^3 + 365N^2 + 190N - 276, & (64) & & P_{59} &= 337N^5 + 403N^4 - 541N^3 - 583N^2 - 300N + 108, & (89) \\
P_{35} &= 17N^4 + 34N^3 + 82N^2 + 161N - 78, & (65) & & & & \\
P_{36} &= 29N^4 + 60N^3 + 149N^2 + 336N + 74, & (66) & & P_{60} &= 436N^5 + 1780N^4 + 2689N^3 + 2782N^2 & \\
& & & & & + 2167N - 134, & (90) \\
P_{37} &= 55N^4 + 86N^3 - 343N^2 - 422N + 384, & (67) & & P_{61} &= 489N^5 + 489N^4 - 1187N^3 - 57N^2 - 742N - 144, & (91) \\
P_{38} &= 55N^4 + 182N^3 - 175N^2 - 542N + 240, & (68) & & & & \\
P_{39} &= 76N^4 + 183N^3 + 196N^2 + 267N - 38, & (69) & & P_{62} &= N^6 + 18N^5 + 63N^4 + 84N^3 + 30N^2 - 64N - 16, & (92) \\
P_{40} &= 97N^4 + 494N^3 + 1079N^2 + 898N - 408, & (70) & & & & \\
P_{41} &= 153N^4 + 306N^3 + 165N^2 + 12N + 4, & (71) & & P_{63} &= N^6 + 23N^5 + 73N^4 + 85N^3 + 58N^2 + 24N - 24, & (93) \\
P_{42} &= 183N^4 + 366N^3 + 305N^2 + 122N + 96, & (72) & & & & \\
P_{43} &= N^5 + N^4 - 4N^3 + 3N^2 - 7N - 2, & (73) & & P_{64} &= 3N^6 + 30N^5 + 107N^4 + 124N^3 + 48N^2 + 20N + 8, & (94) \\
P_{44} &= 2N^5 + 6N^4 + 3N^3 + 11N + 2, & (74) & & & & \\
P_{45} &= 2N^5 + 10N^4 + 29N^3 + 64N^2 + 67N + 8, & (75) & & P_{65} &= 3N^6 + 51N^5 + 153N^4 + 185N^3 + 160N^2 & \\
& & & & & + 80N - 72, & (95) \\
P_{46} &= 3N^5 + 8N^4 + 6N^3 + 10N^2 + 7N + 2, & (76) & & P_{66} &= 4N^6 + 41N^5 + 126N^4 + 163N^3 + 58N^2 & \\
& & & & & - 128N - 32, & (96)
\end{aligned}$$

$$\begin{aligned}
P_{67} &= 5N^6 + 26N^5 + 77N^4 + 168N^3 + 159N^2 \\
&\quad + 19N - 22, \tag{97} \\
P_{68} &= 6N^6 + 75N^5 + 345N^4 + 719N^3 + 323N^2 \\
&\quad - 696N - 96, \tag{98} \\
P_{69} &= 18N^6 + 87N^5 + 199N^4 + 185N^3 \\
&\quad + 63N^2 + 44N + 20, \tag{99} \\
P_{70} &= 23N^6 + 39N^5 - 89N^4 - 219N^3 - 172N^2 \\
&\quad - 130N - 28, \tag{100} \\
P_{71} &= 25N^6 - 118N^5 - 662N^4 - 500N^3 + 421N^2 \\
&\quad + 186N - 264, \tag{101} \\
P_{72} &= 33N^6 + 99N^5 + 41N^4 - 11N^3 + 86N^2 \\
&\quad - 216N - 144, \tag{102} \\
P_{73} &= 33N^6 + 99N^5 + 137N^4 + 157N^3 + 62N^2 \\
&\quad + 8N - 16, \tag{103} \\
P_{74} &= 36N^6 + 48N^5 - 297N^4 - 977N^3 - 976N^2 \\
&\quad - 362N + 24, \tag{104} \\
P_{75} &= 37N^6 + 207N^5 + 753N^4 + 1771N^3 + 1598N^2 \\
&\quad - 118N + 48, \tag{105} \\
P_{76} &= 57N^6 + 297N^5 + 567N^4 + 615N^3 + 468N^2 \\
&\quad + 220N + 16, \tag{106} \\
P_{77} &= 80N^6 + 201N^5 - 775N^4 - 3495N^3 - 4405N^2 \\
&\quad - 2238N - 72, \tag{107} \\
P_{78} &= 94N^6 + 282N^5 + 79N^4 + 42N^3 + 286N^2 \\
&\quad - 585N - 18, \tag{108} \\
P_{79} &= 129N^6 + 387N^5 + 509N^4 + 349N^3 + 50N^2 \\
&\quad + 240N + 144, \tag{109} \\
P_{80} &= 170N^6 + 543N^5 + 221N^4 - 15N^3 + 425N^2 \\
&\quad - 864N - 288, \tag{110} \\
P_{81} &= 170N^6 + 873N^5 + 1547N^4 + 951N^3 - 1717N^2 \\
&\quad - 2976N + 864, \tag{111} \\
P_{82} &= 243N^6 + 729N^5 + 923N^4 + 583N^3 + 14N^2 \\
&\quad + 300N + 216, \tag{112} \\
P_{83} &= 321N^6 + 1353N^5 + 1521N^4 - 713N^3 - 2842N^2 \\
&\quad - 2216N + 96, \tag{113} \\
P_{84} &= 333N^6 + 999N^5 + 1075N^4 + 389N^3 - 68N^2 \\
&\quad + 384N + 216, \tag{114} \\
P_{85} &= 633N^6 + 1899N^5 + 1967N^4 + 697N^3 - 4N^2 \\
&\quad - 48N + 8, \tag{115} \\
P_{86} &= -891N^7 - 1782N^6 - 3712N^5 - 3058N^4 + 6775N^3 \\
&\quad + 7144N^2 + 276N - 144, \tag{116} \\
P_{87} &= 9N^7 + 18N^6 - 124N^5 - 109N^4 + 199N^3 \\
&\quad - 191N^2 + 138N + 72, \tag{117} \\
P_{88} &= 69N^7 + 138N^6 - 667N^5 - 541N^4 + 952N^3 \\
&\quad - 1277N^2 + 990N + 432, \tag{118} \\
P_{89} &= 95N^7 + 378N^6 + 853N^5 + 1832N^4 + 2190N^3 \\
&\quad + 364N^2 - 780N - 432, \tag{119} \\
P_{90} &= 251N^7 + 1335N^6 + 1745N^5 + 243N^4 - 529N^3 \\
&\quad - 4161N^2 - 5400N - 756, \tag{120} \\
P_{91} &= N^8 + 427N^7 + 2161N^6 + 4081N^5 + 3554N^4 \\
&\quad + 1404N^3 + 228N^2 + 64N + 16, \tag{121} \\
P_{92} &= 2N^8 + 10N^7 + 22N^6 + 36N^5 + 29N^4 + 4N^3 \\
&\quad + 33N^2 + 12N + 4, \tag{122} \\
P_{93} &= 2N^8 + 29N^7 + 135N^6 + 297N^5 + 333N^4 \\
&\quad + 204N^3 + 28N^2 - 44N - 24, \tag{123} \\
P_{94} &= 3N^8 + 33N^7 + 149N^6 + 267N^5 + 196N^4 + 104N^3 \\
&\quad + 64N^2 - 88N - 48, \tag{124} \\
P_{95} &= 12N^8 + 52N^7 + 60N^6 - 25N^4 - 2N^3 + 3N^2 \\
&\quad + 8N + 4, \tag{125} \\
P_{96} &= 36N^8 + 348N^7 + 1210N^6 + 2229N^5 + 2168N^4 \\
&\quad + 505N^3 - 424N^2 - 68N + 48, \tag{126}
\end{aligned}$$

$$P_{97} = 111N^8 + 480N^7 + 286N^6 - 468N^5 + 82N^4 + 246N^3 + 295N^2 + 228N + 252, \quad (127)$$

$$P_{98} = 201N^8 + 840N^7 + 565N^6 - 699N^5 - 344N^4 - 645N^3 - 314N^2 + 324N + 360, \quad (128)$$

$$P_{99} = 296N^8 + 1184N^7 + 2744N^6 + 5900N^5 + 4088N^4 + 476N^3 + 9477N^2 + 4725N + 702, \quad (129)$$

$$P_{100} = -7299N^{10} - 39375N^9 - 79900N^8 - 85198N^7 - 17323N^6 + 129917N^5 \\ + 137090N^4 + 25904N^3 + 12072N^2 + 30672N + 8640, \quad (130)$$

$$P_{101} = -149N^{10} - 793N^9 - 1404N^8 - 1170N^7 - 1341N^6 - 1221N^5 + 1710N^4 + 2800N^3 + 2256N^2 + 368N - 32, \quad (131)$$

$$P_{102} = 4N^{10} + 22N^9 + 45N^8 + 36N^7 - 11N^6 - 15N^5 + 25N^4 - 41N^3 - 21N^2 - 16N - 4, \quad (132)$$

$$P_{103} = 10N^{10} + 62N^9 + 403N^8 + 1523N^7 + 2997N^6 + 3197N^5 + 1812N^4 + 478N^3 + 46N^2 + 24N + 8, \quad (133)$$

$$P_{104} = 26N^{10} + 132N^9 + 159N^8 - 351N^7 - 877N^6 + 531N^5 + 1820N^4 - 300N^3 - 252N^2 - 192N - 48, \quad (134)$$

$$P_{105} = 28N^{10} + 139N^9 + 444N^8 + 803N^7 + 451N^6 + 3N^5 + 490N^4 + 219N^3 + 51N^2 - 60N - 12, \quad (135)$$

$$P_{106} = 435N^{10} + 2391N^9 + 6946N^8 + 11512N^7 + 4822N^6 - 7016N^5 - 5369N^4 - 6743N^3 - 2406N^2 - 1764N - 216, \quad (136)$$

$$P_{107} = 531N^{10} + 2799N^9 + 4124N^8 + 446N^7 - 3445N^6 - 5245N^5 + 4358N^4 + 18128N^3 - 1968N^2 \\ - 10800N - 4320, \quad (137)$$

$$P_{108} = 939N^{10} + 4893N^9 + 5386N^8 - 5198N^7 - 10400N^6 - 17636N^5 - 18137N^4 + 7177N^3 - 21672N^2 \\ - 14112N - 4104, \quad (138)$$

$$P_{109} = 1773N^{10} + 9153N^9 + 14204N^8 + 2930N^7 - 9151N^6 - 8431N^5 - 250N^4 + 13772N^3 + 1920N^2 \\ - 8928N - 4320, \quad (139)$$

$$P_{110} = -23N^{11} - 92N^{10} - 53N^9 + 322N^8 + 465N^7 - 348N^6 - 929N^5 - 384N^4 + 132N^3 + 102N^2 + 32N + 8, \quad (140)$$

$$P_{111} = 87N^{12} + 490N^{11} + 949N^{10} + 368N^9 - 1285N^8 - 2214N^7 - 1591N^6 - 126N^5 + 644N^4 - 86N^3 - 268N^2 \\ - 184N - 48, \quad (141)$$

$$P_{112} = 385N^{12} + 2182N^{11} + 4181N^{10} + 1458N^9 - 5589N^8 - 8414N^7 - 5041N^6 - 1754N^5 - 760N^4 - 176N^3 \\ + 152N^2 + 224N + 96, \quad (142)$$

$$P_{113} = 1623N^{12} + 9602N^{11} + 20093N^{10} + 15520N^9 - 3305N^8 - 13494N^7 - 5099N^6 + 9414N^5 + 10456N^4 \\ + 5270N^3 + 1624N^2 + 40N - 96. \quad (143)$$

$\mathbf{V} \cdot \mathbf{A}_{gg,Q}^{(3),S}$

For the logarithmic contributions to the OME $A_{gg,Q}^{(3),S}$ we obtain

$$\begin{aligned}
A_{gg,Q} = & \frac{4a_s L_M T_F}{3} + a_s^2 \left\{ \frac{16L_M^2 T_F^2}{9} + C_F T_F \left[\frac{4L_M P_{147}}{N^3(1+N)^3} + \frac{P_{166}}{N^4(1+N)^4} + \frac{4L_M^2(N-1)(2+N)}{N^2(1+N)^2} \right] \right. \\
& + C_A T_F \left\{ \frac{2P_{155}}{27N^3(1+N)^3} + L_M \left[\frac{16P_{134}}{9N^2(1+N)^2} - \frac{80}{9} S_1 \right] + L_M^2 \left[\frac{16}{3N(1+N)} - \frac{8}{3} S_1 \right] - \frac{4(47+56N)S_1}{27(1+N)} \right\} \Big\} \\
& + a_s^3 \left\{ \frac{64L_M^3}{27} T_F^3 + C_F T_F^2 \left\{ \frac{2P_{177}}{9N^5(1+N)^5} + \frac{80L_M^3(N-1)(2+N)}{9N^2(1+N)^2} + L_M^2 \left[\frac{8P_{153}}{9N^3(1+N)^3} + \frac{32(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] \right. \right. \\
& + L_M \left[-\frac{8P_{173}}{27N^4(1+N)^4} + \frac{32(N-1)(2+N)(-6-8N+N^2)}{9N^3(1+N)^3} S_1 + \frac{16(N-1)(2+N)S_1^2}{3N^2(1+N)^2} - \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} \right] \\
& + \left[-\frac{8P_{154}}{9N^3(1+N)^3} - \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] \zeta_2 - \frac{80(N-1)(2+N)\zeta_3}{9N^2(1+N)^2} \Big\} \\
& + C_F N_F T_F^2 \left\{ \frac{2P_{178}}{81N^5(1+N)^5} + \frac{64L_M^3(N-1)(2+N)}{9N^2(1+N)^2} + L_M \left[-\frac{4P_{171}}{9N^4(1+N)^4} - \frac{32(N-1)(2+N)(4+6N+N^2)}{3N^3(1+N)^3} S_1 \right. \right. \\
& + \frac{16(N-1)(2+N)S_1^2}{N^2(1+N)^2} - \frac{80(N-1)(2+N)S_2}{3N^2(1+N)^2} \Big\} \\
& + \left[\frac{32(N-1)(2+N)(22+41N+28N^2)}{27N^2(1+N)^4} + \frac{16(N-1)(2+N)S_2}{3N^2(1+N)^2} \right] S_1 - \frac{16(N-1)(2+N)(2+5N)}{9N^2(1+N)^3} S_1^2 \\
& + \frac{16(N-1)(2+N)S_1^3}{9N^2(1+N)^2} - \frac{16(N-1)(2+N)(2+5N)}{9N^2(1+N)^3} S_2 + \frac{32(N-1)(2+N)S_3}{9N^2(1+N)^2} \\
& + \left[\frac{4P_{162}}{9N^3(1+N)^3} + \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] \zeta_2 - \frac{64(N-1)(2+N)\zeta_3}{9N^2(1+N)^2} \Big\} \\
& + C_A^2 T_F \left\{ -\frac{4P_{174}}{243N^4(1+N)^4} + L_M^3 \left[-\frac{352}{27N(1+N)} + \frac{176}{27} S_1 \right] \right. \\
& + L_M^2 \left[-\frac{2P_{149}}{9N^3(1+N)^3} + \left[-\frac{8P_{140}}{9N^2(1+N)^2} + \frac{64}{3} S_2 \right] S_1 - \frac{128S_2}{3N(1+N)} + \frac{32}{3} S_3 \right. \\
& + \left. \left[-\frac{128}{3N(1+N)} + \frac{64}{3} S_1 \right] S_{-2} + \frac{32}{3} S_{-3} - \frac{64}{3} S_{-2,1} \right] \\
& + L_M \left[\frac{16S_2 P_{133}}{9N^2(1+N)^2} + \frac{16S_{-3} P_{139}}{9N^2(1+N)^2} - \frac{32S_{-2,1} P_{139}}{9N^2(1+N)^2} + \frac{8S_3 P_{142}}{9N^2(1+N)^2} + \frac{P_{183}}{81(N-1)N^5(1+N)^5(2+N)} \right. \\
& + \left. \left[-\frac{4P_{176}}{81(N-1)N^4(1+N)^4(2+N)} + \frac{640}{9} S_2 - \frac{32}{3} S_3 \right] S_1 + \left[-\frac{16P_{163}}{9(N-1)N^3(1+N)^3(2+N)} \right. \\
& + \frac{32P_{157}}{9(N-1)N^2(1+N)^2(2+N)} S_1 \Big] S_{-2} + \frac{32}{3} S_{-2}^2 + \left[\frac{64(-3+2N+2N^2)}{N^2(1+N)^2} - 64S_1 \right] \zeta_3 \Big\} \\
& - \frac{8(2339+4876N+2834N^2)}{243(1+N)^2} S_1 - \frac{44S_1^2}{9(1+N)} + \frac{44(1+2N)S_2}{9(1+N)} \\
& + \left[\frac{4P_{160}}{27N^3(1+N)^3} + \left(\frac{16(36+72N+N^2+2N^3+N^4)}{27N^2(1+N)^2} - \frac{32}{3} S_2 \right) S_1 \right. \\
& + \frac{64S_2}{3N(1+N)} - \frac{16}{3} S_3 + \left[\frac{64}{3N(1+N)} - \frac{32}{3} S_1 \right] S_{-2} - \frac{16}{3} S_{-3} + \frac{32}{3} S_{-2,1} \Big] \zeta_2 + \left[\frac{352}{27N(1+N)} - \frac{176}{27} S_1 \right] \zeta_3 \Big\}
\end{aligned}$$

$$\begin{aligned}
& + C_A N_F T_F^2 \left\{ \frac{16P_{172}}{243N^4(1+N)^4} + L_M \left[-\frac{16S_1 P_{145}}{81N^2(1+N)^2} - \frac{4P_{164}}{81N^3(1+N)^3} \right] + L_M^3 \left[\frac{128}{27N(1+N)} - \frac{64}{27} S_1 \right] \right. \\
& + \frac{32(283 + 584N + 328N^2)}{243(1+N)^2} S_1 + \frac{16S_1^2}{9(1+N)} - \frac{16(1+2N)S_2}{9(1+N)} + \left[-\frac{4P_{137}}{27N^2(1+N)^2} + \frac{160}{27} S_1 \right] \zeta_2 \\
& + \left[-\frac{128}{27N(1+N)} + \frac{64}{27} S_1 \right] \zeta_3 \left. \right\} + C_A T_F^2 \left\{ -\frac{8P_{167}}{81N^4(1+N)^4} + L_M \left[-\frac{8S_1 P_{141}}{9N^2(1+N)^2} - \frac{2P_{159}}{27N^3(1+N)^3} \right] \right. \\
& + L_M^2 \left[\frac{8P_{143}}{27N^2(1+N)^2} - \frac{640}{27} S_1 \right] + L_M^3 \left[\frac{448}{27N(1+N)} - \frac{224}{27} S_1 \right] + \frac{16(283 + 584N + 328N^2)}{81(1+N)^2} S_1 \\
& + \frac{8S_1^2}{3(1+N)} - \frac{8(1+2N)S_2}{3(1+N)} + \left[-\frac{4P_{144}}{27N^2(1+N)^2} + \frac{560}{27} S_1 \right] \zeta_2 + \left[-\frac{448}{27N(1+N)} + \frac{224}{27} S_1 \right] \zeta_3 \left. \right\} \\
& + C_F^2 T_F \left\{ \frac{8S_3 P_{136}}{3N^3(1+N)^3} + \frac{4S_2 P_{151}}{N^4(1+N)^4} + \frac{P_{180}}{N^6(1+N)^6} \right. \\
& + L_M^3 \left[-\frac{4(N-1)(2+N)(2+3N+3N^2)}{3N^3(1+N)^3} + \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] \\
& + L_M^2 \left[-\frac{8(N-1)(2+N)(-2-3N+N^3+2N^4)}{N^4(1+N)^4} + \frac{8(N-1)^2(2+N)(2+3N)}{N^3(1+N)^3} S_1 - \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} \right] \\
& + L_M \left[-\frac{4S_2 P_{138}}{N^3(1+N)^3} - \frac{2P_{181}}{(N-1)N^5(1+N)^5(2+N)} + \left[-\frac{8P_{152}}{N^4(1+N)^4} + \frac{24(N-1)(2+N)S_2}{N^2(1+N)^2} \right] S_1 \right. \\
& + \frac{4(-6-13N+3N^3)}{N^2(1+N)^3} S_1^2 - \frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2} + \frac{16(14+5N+5N^2)}{3N^2(1+N)^2} S_3 \\
& + \left[-\frac{32(10+N+N^2)}{(N-1)N(1+N)(2+N)} + \frac{256S_1}{N^2(1+N)^2} \right] S_{-2} + \frac{128S_{-3}}{N^2(1+N)^2} - \frac{32(N-1)(2+N)S_{2,1}}{N^2(1+N)^2} \\
& - \frac{256S_{-2,1}}{N^2(1+N)^2} - \frac{96(2+N+N^2)\zeta_3}{N^2(1+N)^2} \left. \right] + \left[-\frac{8P_{132}}{N^3(1+N)^3} + \frac{8(-2+3N+3N^2)}{N^3(1+N)^2} S_2 \right. \\
& - \frac{16(N-1)(2+N)S_3}{3N^2(1+N)^2} - \frac{32(N-1)(2+N)S_{2,1}}{N^2(1+N)^2} \left. \right] S_1 \\
& + \left[\frac{4(-36-22N-6N^2+N^3)}{N^3(1+N)^2} - \frac{4(N-1)(2+N)S_2}{N^2(1+N)^2} \right] S_1^2 \\
& + \frac{8(-2+3N+3N^2)}{3N^3(1+N)^2} S_1^3 - \frac{2(N-1)(2+N)S_1^4}{3N^2(1+N)^2} - \frac{2(N-1)(2+N)S_2^2}{N^2(1+N)^2} \\
& + \frac{12(N-1)(2+N)S_4}{N^2(1+N)^2} - \frac{32(2+N)S_{2,1}}{N^3(1+N)} - \frac{32(N-1)(2+N)S_{3,1}}{N^2(1+N)^2} \\
& + \frac{64(N-1)(2+N)S_{2,1,1}}{N^2(1+N)^2} + \left[128 \ln(2) - \frac{2P_{170}}{N^4(1+N)^4} \right. \\
& - \frac{4(N-1)(2+N)(-4-3N+3N^2)}{N^3(1+N)^3} S_1 - \frac{4(N-1)(2+N)S_1^2}{N^2(1+N)^2} \\
& + \frac{12(N-1)(2+N)S_2}{N^2(1+N)^2} \left. \right] \zeta_2 + \left[-\frac{4P_{158}}{3N^3(1+N)^3} - \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] \zeta_3 \left. \right\} \\
& + C_A C_F T_F \left\{ -\frac{4S_2 P_{148}}{N^4(1+N)^4} + \frac{P_{179}}{18N^6(1+N)^6} \right. \\
& + L_M^3 \left[-\frac{8(N-1)(2+N)(-12+11N+11N^2)}{9N^3(1+N)^3} - \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2} \right]
\end{aligned}$$

$$\begin{aligned}
& + L_M^2 \left[-\frac{8S_1 P_{150}}{3N^3(1+N)^3} - \frac{2P_{169}}{9N^4(1+N)^4} - \frac{16(N-1)(2+N)S_2}{N^2(1+N)^2} \right. \\
& - \left. \frac{32(N-1)(2+N)S_{-2}}{N^2(1+N)^2} \right] + L_M \left[\frac{4S_2 P_{135}}{N^3(1+N)^3} + \frac{8P_{182}}{27(N-1)N^5(1+N)^5(2+N)} \right. \\
& + \left[-\frac{8P_{175}}{9(N-1)N^4(1+N)^4(2+N)} - \frac{40(N-1)(2+N)S_2}{N^2(1+N)^2} \right] S_1 \\
& - \frac{4(-12-16N+5N^2+11N^3)}{3N^3(1+N)^2} S_1^2 + \frac{8(N-1)(2+N)S_1^3}{3N^2(1+N)^2} - \frac{16(26+5N+5N^2)}{3N^2(1+N)^2} S_3 \\
& + \left[-\frac{16P_{146}}{(N-1)N^2(1+N)^3(2+N)} + \frac{32P_{130}}{(N-1)N^2(1+N)^2(2+N)} S_1 \right] S_{-2} \\
& + \frac{16(-22+5N+5N^2)}{N^2(1+N)^2} S_{-3} + \frac{32(N-1)(2+N)S_{2,1}}{N^2(1+N)^2} - \frac{32(-14+N+N^2)S_{-2,1}}{N^2(1+N)^2} \\
& + \left[-\frac{32(-3+N)(4+N)}{N^2(1+N)^2} + 64S_1 \right] \zeta_3 + \left[-\frac{2P_{165}}{9N^2(1+N)^5} + \frac{8(-13+3N^2)}{N^2(1+N)^3} S_2 \right. \\
& + \frac{160(N-1)(2+N)S_3}{3N^2(1+N)^2} - \frac{64(N-1)(2+N)S_{-2,1}}{N^2(1+N)^2} \left. \right] S_1 + \left[-\frac{4P_{131}}{N^2(1+N)^4} \right. \\
& + \left. \frac{20(N-1)(2+N)S_2}{N^2(1+N)^2} \right] S_1^2 - \frac{8(5+4N+N^2)}{3N^2(1+N)^3} S_1^3 + \frac{2(N-1)(2+N)S_1^4}{3N^2(1+N)^2} \\
& + \frac{2(N-1)(2+N)S_2^2}{N^2(1+N)^2} - \frac{64(-6+8N+7N^2+N^3)}{3N^3(1+N)^3} S_3 + \frac{36(N-1)(2+N)S_4}{N^2(1+N)^2} \\
& + \left[-\frac{32(2+N)(3+N^2)}{N^2(1+N)^4} + \frac{64(N-1)(2+N)S_1}{N^2(1+N)^3} + \frac{32(N-1)(2+N)S_1^2}{N^2(1+N)^2} \right. \\
& + \left. \frac{32(N-1)(2+N)S_2}{N^2(1+N)^2} \right] S_{-2} + \left[\frac{32(N-1)(2+N)}{N^2(1+N)^3} + \frac{32(N-1)(2+N)S_1}{N^2(1+N)^2} \right] S_{-3} \\
& + \frac{16(N-1)(2+N)S_{-4}}{N^2(1+N)^2} - \frac{16(N-1)(2+N)S_{3,1}}{N^2(1+N)^2} - \frac{64(N-1)(2+N)S_{-2,1}}{N^2(1+N)^3} \\
& - \frac{32(N-1)(2+N)S_{-2,2}}{N^2(1+N)^2} - \frac{32(N-1)(2+N)S_{-3,1}}{N^2(1+N)^2} - \frac{16(N-1)(2+N)S_{2,1,1}}{N^2(1+N)^2} \\
& + \frac{64(N-1)(2+N)S_{-2,1,1}}{N^2(1+N)^2} + \left[-64\ln(2) - \frac{4S_1 P_{161}}{3N^3(1+N)^3} + \frac{4P_{168}}{9N^4(1+N)^4} \right. \\
& + \frac{4(N-1)(2+N)S_1^2}{N^2(1+N)^2} + \frac{12(N-1)(2+N)S_2}{N^2(1+N)^2} + \frac{24(N-1)(2+N)S_{-2}}{N^2(1+N)^2} \left. \right] \zeta_2 \\
& + \left[\frac{8P_{156}}{9N^3(1+N)^3} + \frac{16(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] \zeta_3 \left. \right\} - \frac{64}{27} T_F^3 \zeta_3 + a_{gg,Q}^{(3)} \left. \right\}. \tag{144}
\end{aligned}$$

The polynomials P_i read

$$P_{130} = N^4 + 2N^3 - 7N^2 - 8N + 28, \tag{145}$$

$$P_{131} = N^4 + 2N^3 - 5N^2 - 12N + 2, \tag{146}$$

$$P_{132} = 2N^4 - 4N^3 - 3N^2 + 20N + 12, \tag{147}$$

$$P_{133} = 3N^4 + 6N^3 - 89N^2 - 92N + 12, \quad (148)$$

$$P_{134} = 3N^4 + 6N^3 + 16N^2 + 13N - 3, \quad (149)$$

$$P_{135} = 3N^4 + 32N^3 + 65N^2 - 16N - 60, \quad (150)$$

$$P_{136} = 3N^4 + 48N^3 + 123N^2 + 98N + 8, \quad (151)$$

$$P_{137} = 9N^4 + 18N^3 + 113N^2 + 104N - 24, \quad (152)$$

$$P_{138} = 11N^4 + 36N^3 + 43N^2 + 46N + 8, \quad (153)$$

$$P_{139} = 20N^4 + 40N^3 + 11N^2 - 9N + 54, \quad (154)$$

$$P_{140} = 23N^4 + 46N^3 + 23N^2 + 96N + 48, \quad (155)$$

$$P_{141} = 40N^4 + 74N^3 + 25N^2 - 9N + 16, \quad (156)$$

$$P_{142} = 40N^4 + 80N^3 + 73N^2 + 33N + 54, \quad (157)$$

$$P_{143} = 63N^4 + 126N^3 + 271N^2 + 208N - 48, \quad (158)$$

$$P_{144} = 99N^4 + 198N^3 + 463N^2 + 364N - 84, \quad (159)$$

$$P_{145} = 136N^4 + 254N^3 + 37N^2 - 81N + 144, \quad (160)$$

$$P_{146} = 3N^5 + 7N^4 - 29N^3 - 51N^2 - 2N - 8, \quad (161)$$

$$P_{147} = N^6 + 3N^5 + 5N^4 + N^3 - 8N^2 + 2N + 4, \quad (162)$$

$$P_{148} = N^6 + 18N^5 + 63N^4 + 84N^3 + 30N^2 - 64N - 16, \quad (163)$$

$$P_{149} = 3N^6 + 9N^5 - 163N^4 - 341N^3 + 164N^2 - 432N - 192, \quad (164)$$

$$P_{150} = 3N^6 + 9N^5 + 20N^4 + 25N^3 - 11N^2 - 46N - 12, \quad (165)$$

$$P_{151} = 3N^6 + 30N^5 + 107N^4 + 124N^3 + 48N^2 + 20N + 8, \quad (166)$$

$$P_{152} = 6N^6 + 23N^5 - 14N^4 - 121N^3 - 114N^2 - 20N + 8, \quad (167)$$

$$P_{153} = 15N^6 + 45N^5 + 49N^4 - 13N^3 - 64N^2 + 40N + 48, \quad (168)$$

$$P_{154} = 15N^6 + 45N^5 + 56N^4 + N^3 - 68N^2 + 29N + 42, \quad (169)$$

$$P_{155} = 15N^6 + 45N^5 + 374N^4 + 601N^3 + 161N^2 - 24N + 36, \quad (170)$$

$$P_{156} = 18N^6 + 54N^5 + 65N^4 + 40N^3 - 23N^2 - 34N + 24, \quad (171)$$

$$P_{157} = 20N^6 + 60N^5 + 11N^4 - 78N^3 - 13N^2 + 36N - 108, \quad (172)$$

$$P_{158} = 24N^6 + 72N^5 + 69N^4 + 18N^3 + N^2 + 4N + 4, \quad (173)$$

$$\begin{aligned}
P_{159} &= 27N^6 + 81N^5 - 1247N^4 - 2341N^3 - 720N^2 + 32N - 240, & (174) \\
P_{160} &= 27N^6 + 81N^5 + 148N^4 + 161N^3 + 253N^2 - 390N - 144, & (175) \\
P_{161} &= 30N^6 + 90N^5 + 79N^4 + 8N^3 + 23N^2 + 70N + 12, & (176) \\
P_{162} &= 63N^6 + 189N^5 + 157N^4 + 35N^3 + 80N^2 + 4N - 24, & (177) \\
P_{163} &= 95N^6 + 285N^5 + 92N^4 - 291N^3 - 97N^2 + 96N - 36, & (178) \\
P_{164} &= 297N^6 + 891N^5 - 461N^4 - 2119N^3 - 872N^2 - 96N - 432, & (179) \\
P_{165} &= 233N^7 + 1093N^6 + 1970N^5 + 1538N^4 - 167N^3 - 2143N^2 - 2412N - 288, & (180) \\
P_{166} &= -15N^8 - 60N^7 - 82N^6 - 44N^5 - 15N^4 - 4N^2 - 12N - 8, & (181) \\
P_{167} &= 3N^8 + 12N^7 + 2080N^6 + 5568N^5 + 4602N^4 + 1138N^3 - 3N^2 - 36N - 108, & (182) \\
P_{168} &= 15N^8 + 60N^7 + 242N^6 + 417N^5 + 344N^4 + 285N^3 + 185N^2 + 456N + 108, & (183) \\
P_{169} &= 33N^8 + 132N^7 - 82N^6 - 840N^5 - 571N^4 + 564N^3 + 308N^2 + 984N + 288, & (184) \\
P_{170} &= 40N^8 + 160N^7 + 205N^6 + 61N^5 + 18N^4 + 113N^3 + 75N^2 - 20N - 12, & (185) \\
P_{171} &= 67N^8 + 268N^7 + 194N^6 - 508N^5 - 533N^4 + 480N^3 + 616N^2 + 344N + 144, & (186) \\
P_{172} &= 126N^8 + 504N^7 - 1306N^6 - 5052N^5 - 4473N^4 - 1138N^3 + 3N^2 + 36N + 108, & (187) \\
P_{173} &= 219N^8 + 876N^7 + 1142N^6 + 288N^5 - 217N^4 + 240N^3 + 410N^2 + 366N + 180, & (188) \\
P_{174} &= 1386N^8 + 5544N^7 - 11270N^6 - 46284N^5 - 39915N^4 - 9422N^3 + 33N^2 + 396N + 1188, & (189) \\
P_{175} &= 15N^{10} + 75N^9 + 2N^8 - 469N^7 - 506N^6 + 524N^5 + 781N^4 + 26N^3 - 1192N^2 - 1128N - 432, & (190) \\
P_{176} &= 310N^{10} + 1748N^9 + 4811N^8 + 14192N^7 + 24974N^6 + 3194N^5 - 29393N^4 - 16866N^3 \\
&\quad + 8694N^2 + 7128N + 1944, & (191) \\
P_{177} &= 391N^{10} + 1955N^9 + 3622N^8 + 3046N^7 + 1595N^6 + 1327N^5 + 1152N^4 + 216N^3 - 288N^2 - 360N - 144, & (192) \\
P_{178} &= 1593N^{10} + 7965N^9 + 11578N^8 + 1594N^7 - 1379N^6 + 12793N^5 + 17152N^4 + 4432N^3 - 1728N^2 \\
&\quad - 2160N - 864, & (193) \\
P_{179} &= -3135N^{12} - 18810N^{11} - 42713N^{10} - 44692N^9 - 22145N^8 - 9290N^7 - 8167N^6 - 4136N^5 - 960N^4 + 11232N^3 \\
&\quad + 6720N^2 + 3360N + 576, & (194) \\
P_{180} &= -39N^{12} - 234N^{11} - 521N^{10} - 492N^9 - 85N^8 - 42N^7 - 883N^6 - 1660N^5 - 1324N^4 - 492N^3 \\
&\quad - 52N^2 + 48N + 16, & (195) \\
P_{181} &= N^{12} + 6N^{11} - 3N^{10} - 58N^9 - 21N^8 + 222N^7 + 609N^6 + 1144N^5 + 1122N^4 + 142N^3 - 180N^2 + 40N + 48, & (196)
\end{aligned}$$

$$P_{182} = 276N^{12} + 1656N^{11} + 3334N^{10} + 869N^9 - 6591N^8 - 7395N^7 + 7452N^6 + 13479N^5 + 3167N^4 + 7303N^3 + 1110N^2 - 5004N - 2376, \quad (197)$$

$$P_{183} = 2493N^{12} + 14958N^{11} + 42317N^{10} + 75910N^9 + 45511N^8 - 60782N^7 - 29777N^6 + 17194N^5 - 130384N^4 - 115536N^3 + 25776N^2 + 24192N + 5184. \quad (198)$$

Next we turn to the polarized massive Wilson coefficients in the asymptotic region $Q^2 \gg m^2$ in the single mass case.

VI. THE POLARIZED WILSON COEFFICIENT L_q^{PS} AND L_g^{S}

The massive Wilson coefficients (5) and (6) are related to the expansion coefficients of the massive OMEs and the massless Wilson coefficients. In presenting the Wilson coefficients for the structure function $g_1(x, Q^2)$ we leave the three-loop massive OMEs, calculated in the previous sections and Ref. [23] and the three-loop contribution to the polarized massless Wilson coefficients, symbolic. The latter depend also on the logarithmic terms L_Q (16). In the nonsinglet case the complete asymptotic Wilson coefficient to the structure function $g_1(x, Q^2)$ has already been calculated in Ref. [28].

For L_q^{PS} we obtain

$$\begin{aligned} L_q^{\text{PS}} = & \frac{1}{2} [1 - (-1)^N] \left\{ a_s^3 \left\{ C_F N_F T_F^2 \left\{ -\frac{32(N-1)^2(2+N)(22+41N+28N^2)}{27N^3(1+N)^4} \right. \right. \right. \\ & + L_M \left[-\frac{64(N-1)^2(2+N)(2+5N)}{9N^3(1+N)^3} + (N-1) \left[-\frac{64(2+N)(3+2N+2N^2)}{9N^3(1+N)^3} S_1 + \frac{64(2+N)S_1^2}{3N^2(1+N)^2} \right] \right. \\ & + L_M^2 \left[-\frac{32(N-1)^2(2+N)}{3N^3(1+N)^2} - \frac{32(N-1)(2+N)S_1}{3N^2(1+N)^2} \right] + (N-1) \left[L_Q \left[\frac{32L_M^2(2+N)}{3N^2(1+N)^2} + \frac{32(2+N)(22+41N+28N^2)}{27N^2(1+N)^4} \right. \right. \\ & + L_M \left[\frac{64(2+N)(2+5N)}{9N^2(1+N)^3} - \frac{64(2+N)S_1}{3N^2(1+N)^2} \right] - \frac{32(2+N)(2+5N)}{9N^2(1+N)^3} S_1 + \frac{16(2+N)S_1^2}{3N^2(1+N)^2} + \frac{16(2+N)S_2}{3N^2(1+N)^2} \left. \right. \\ & + \left[-\frac{32(2+N)(6+37N+35N^2+13N^3)}{27N^3(1+N)^4} - \frac{16(2+N)S_2}{3N^2(1+N)^2} \right] S_1 + \frac{16(2+N)(3+4N+7N^2)}{9N^3(1+N)^3} S_1^2 - \frac{16(2+N)S_1^3}{3N^2(1+N)^2} \left. \right. \\ & \left. \left. - \frac{16(N-1)^2(2+N)}{3N^3(1+N)^2} S_2 \right\} + A_{qq,Q}^{\text{PS}(3)} + N_F \hat{C}_q^{\text{PS}(3)}(L_Q, N_F) \right\}. \quad (199) \end{aligned}$$

L_g^{S} is given by

$$\begin{aligned} L_g^{\text{S}} = & \frac{1}{2} [1 - (-1)^N] \left\{ a_s^2 N_F T_F^2 \left\{ \frac{16L_M L_Q (N-1)}{3N(1+N)} + L_M \left[-\frac{16(N-1)^2}{3N^2(1+N)} - \frac{16(N-1)S_1}{3N(1+N)} \right] \right\} \right. \\ & + a_s^3 \left\{ N_F T_F^3 \left\{ \frac{64L_M^2 L_Q (N-1)}{9N(1+N)} + L_M^2 \left[-\frac{64(N-1)^2}{9N^2(1+N)} - \frac{64(N-1)S_1}{9N(1+N)} \right] \right\} \right. \\ & + C_A N_F T_F^2 \left\{ -\frac{8(N-1)^2 Q_9}{27N^5(1+N)^4} + L_Q \left[(N-1) \left[\frac{8Q_9}{27N^4(1+N)^4} + L_M^2 \left[\frac{64}{3N^2(1+N)^2} - \frac{32S_1}{3N(1+N)} \right] - \frac{16(47+56N)S_1}{27N(1+N)^2} \right] \right. \right. \\ & + L_M \left[\frac{32Q_5}{9N^3(1+N)^3} + (N-1) \left[\frac{32S_1^2}{3N(1+N)} - \frac{32S_2}{3N(1+N)} - \frac{64S_{-2}}{3N(1+N)} \right] - \frac{64(-9-2N+3N^2+2N^3)}{9N^2(1+N)^2} S_1 \left. \right] \left. \right. \\ & + L_M^2 \left[-\frac{64(N-1)^2}{3N^3(1+N)^2} + (N-1) \left[\frac{32(-3+N^2)S_1}{3N^2(1+N)^2} + \frac{32S_1^2}{3N(1+N)} \right] \right] + (N-1) \left[\frac{8S_1 Q_{10}}{27N^4(1+N)^4} \right. \\ & \left. \left. + L_M L_Q^2 \left[\frac{64}{3N^2(1+N)^2} - \frac{32S_1}{3N(1+N)} \right] + \frac{16(47+56N)S_1^2}{27N(1+N)^2} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + L_M \left[-\frac{16Q_{13}}{9(N-1)N^4(1+N)^4(2+N)^2} + (N-1) \left[-\frac{16S_1^3}{9N(1+N)} - \frac{64S_{2,1}}{3N(1+N)} \right] \right. \\
& + \left[-\frac{16Q_8}{9N^3(1+N)^3(2+N)} + \frac{80(N-1)S_2}{3N(1+N)} \right] S_1 + \frac{32(-9-10N+3N^2+7N^3)}{9N^2(1+N)^2} S_1^2 \\
& + \frac{32(3-2N+N^2+N^3)}{3N^2(1+N)^2} S_2 + \frac{32(2+5N+5N^2)}{9N(1+N)(2+N)} S_3 \\
& + \left[\frac{64Q_3}{3(N-1)N(1+N)^2(2+N)^2} + \frac{64(2+N+N^2)}{3N(1+N)(2+N)} S_1 \right] S_{-2} - \frac{64(-4+N+N^2)}{3N(1+N)(2+N)} S_{-3} \\
& - \frac{256S_{-2,1}}{3N(1+N)(2+N)} - \frac{32(2+N+N^2)}{N(1+N)(2+N)} \zeta_3 \left. \right\} + C_F N_F T_F^2 \left\{ \frac{4(N-1)^2 Q_{12}}{N^6(1+N)^5} \right. \\
& + (N-1) \left[\frac{4S_1 Q_{12}}{N^5(1+N)^5} + L_M L_Q^2 \left[\frac{8Q_2}{3N^3(1+N)^3} - \frac{32S_1}{3N(1+N)} \right] \right] \\
& + L_Q \left[\frac{16L_M^2(N-1)^2(2+N)}{N^3(1+N)^3} + (N-1) \left[-\frac{4Q_{12}}{N^5(1+N)^5} + L_M \left[\frac{16S_1 Q_1}{3N^3(1+N)^3} \right. \right. \right. \\
& - \left. \frac{16Q_6}{3N^4(1+N)^4} + \frac{64S_1^2}{3N(1+N)} - \frac{64S_2}{3N(1+N)} \right] \left. \right] \left. \right] + L_M^2 \left[-\frac{16(N-1)^3(2+N)}{N^4(1+N)^3} \right. \\
& - \left. \frac{16(N-1)^2(2+N)}{N^3(1+N)^3} S_1 \right] + L_M \left[\frac{8S_2 Q_4}{3N^3(1+N)^3} - \frac{8Q_{14}}{3(N-1)N^5(1+N)^5(2+N)^2} \right. \\
& + (N-1) \left[-\frac{8(-4+2N+3N^2)(3+2N+3N^2)}{3N^3(1+N)^3} S_1^2 - \frac{80S_1^3}{9N(1+N)} + \frac{64S_{2,1}}{3N(1+N)} \right] \\
& + \left[-\frac{32Q_{11}}{3N^4(1+N)^4(2+N)} + \frac{16(N-1)S_2}{N(1+N)} \right] S_1 - \frac{256(1+N+N^2)}{9N(1+N)(2+N)} S_3 \\
& + \left[\frac{64Q_7}{3(N-1)N^2(1+N)^2(2+N)^2} - \frac{512S_1}{3N(1+N)(2+N)} \right] S_{-2} - \frac{256S_{-3}}{3N(1+N)(2+N)} \\
& + \frac{512S_{-2,1}}{3N(1+N)(2+N)} + \frac{64(2+N+N^2)}{N(1+N)(2+N)} \zeta_3 \left. \right\} + A_{qg,Q}^{(3)} + N_F \hat{C}_g^{S(3)}(L_Q, N_F) \left. \right\}, \tag{200}
\end{aligned}$$

with the polynomials Q_i given by

$$Q_1 = 3N^4 + 2N^3 - N^2 - 12, \tag{201}$$

$$Q_2 = 3N^4 + 6N^3 - N^2 - 4N + 12, \tag{202}$$

$$Q_3 = N^5 + 3N^4 - 3N^3 - 9N^2 - 8N - 8, \tag{203}$$

$$Q_4 = 9N^5 - N^4 - 23N^3 - 15N^2 - 14N + 12, \tag{204}$$

$$Q_5 = 9N^5 + 9N^4 - 4N^3 + 15N^2 - 41N - 12, \tag{205}$$

$$Q_6 = N^6 - 19N^4 - 22N^3 + 22N^2 - 34N - 36, \tag{206}$$

$$Q_7 = N^6 + 3N^5 + 13N^4 + 21N^3 + 22N^2 + 12N - 24, \tag{207}$$

$$Q_8 = 13N^6 + 44N^5 + 71N^4 + 94N^3 - 90N^2 - 288N - 72, \tag{208}$$

$$Q_9 = 15N^6 + 45N^5 + 374N^4 + 601N^3 + 161N^2 - 24N + 36, \tag{209}$$

$$Q_{10} = 97N^6 + 161N^5 - 392N^4 - 807N^3 - 255N^2 + 24N - 36, \tag{210}$$

$$Q_{11} = N^8 + 3N^7 + 11N^6 + 20N^5 - 15N^4 - 17N^3 + 49N^2 - 20N - 36 \quad (211)$$

$$Q_{12} = 15N^8 + 60N^7 + 82N^6 + 44N^5 + 15N^4 + 4N^2 + 12N + 8, \quad (212)$$

$$Q_{13} = 24N^{10} + 102N^9 + 58N^8 - 210N^7 - 209N^6 + 23N^5 + 529N^4 + 1109N^3 + 234N^2 - 388N - 120, \quad (213)$$

$$Q_{14} = 8N^{12} + 50N^{11} + 122N^{10} + 98N^9 - 457N^8 - 1398N^7 - 1232N^6 - 634N^5 - 793N^4 + 388N^3 \\ + 1128N^2 - 16N - 336. \quad (214)$$

VII. THE POLARIZED WILSON COEFFICIENTS $H_{Qq}^{(3).PS}$

The next Wilson coefficient is $H_{Qq}^{(3).PS}$. It is given by

$$H_q^{PS} = \frac{1}{2} [1 - (-1)^N] \left\{ a_s^2 C_F T_F \left\{ \frac{8Q_{16}}{(N-1)N^4(1+N)^4(2+N)} + (2+N) \left[\frac{8L_M(1+2N+N^3)}{N^3(1+N)^3} \right. \right. \right. \\ + L_Q \left[-\frac{8(2+N-N^2+2N^3)}{N^3(1+N)^3} - \frac{8(N-1)S_1}{N^2(1+N)^2} \right] + (N-1) \left[-\frac{4L_M^2}{N^2(1+N)^2} + \frac{4L_Q^2}{N^2(1+N)^2} \right. \\ \left. \left. + \frac{4S_1^2}{N^2(1+N)^2} - \frac{12S_2}{N^2(1+N)^2} \right] + \frac{8(2+N-N^2+2N^3)}{N^3(1+N)^3} S_1 \right] - \frac{64}{(N-1)N(1+N)(2+N)} S_{-2} \Big\} \\ + a_s^3 \left\{ C_F^2 T_F (2+N) \left\{ \frac{8S_2 Q_{15}}{N^4(1+N)^4} + \frac{4(-1-3N-4N^2+4N^3)(-2+5N+6N^2+9N^3)}{N^6(1+N)^5} \right. \right. \\ + L_Q \left[-\frac{4(2+3N+3N^2)(-1-3N-4N^2+4N^3)}{N^5(1+N)^5} + L_M \left[\frac{8(2+3N+3N^2)(1+2N+N^3)}{N^4(1+N)^4} - \frac{32(1+2N+N^3)}{N^3(1+N)^3} S_1 \right] \right. \\ \left. + (N-1) \left[L_M^2 \left[-\frac{4(2+3N+3N^2)}{N^3(1+N)^3} + \frac{16S_1}{N^2(1+N)^2} \right] - \frac{8(2+3N+3N^2)}{N^3(1+N)^3} S_2 \right] \right. \\ \left. + \left[\frac{16(-1-3N-4N^2+4N^3)}{N^4(1+N)^4} + \frac{32(N-1)S_2}{N^2(1+N)^2} \right] S_1 \right] + L_M \left[-\frac{8(1+2N+N^3)(-2+5N+6N^2+9N^3)}{N^5(1+N)^4} \right. \\ \left. + \frac{8(-2+3N+3N^2)(1+2N+N^3)}{N^4(1+N)^4} S_1 + \frac{16(1+2N+N^3)}{N^3(1+N)^3} S_1^2 - \frac{16(1+2N+N^3)}{N^3(1+N)^3} S_2 \right] \\ \left. + (N-1) \left[L_M^2 \left[\frac{4(-2+5N+6N^2+9N^3)}{N^4(1+N)^3} - \frac{4(-2+3N+3N^2)}{N^3(1+N)^3} S_1 - \frac{8S_1^2}{N^2(1+N)^2} + \frac{8S_2}{N^2(1+N)^2} \right] + \frac{16S_2^2}{N^2(1+N)^2} \right] \right. \\ \left. + \left[-\frac{4(-2+3N+3N^2)(-1-3N-4N^2+4N^3)}{N^5(1+N)^5} - \frac{8(N-1)(-2+3N+3N^2)}{N^3(1+N)^3} S_2 \right] S_1 \right. \\ \left. + \left[-\frac{8(-1-3N-4N^2+4N^3)}{N^4(1+N)^4} - \frac{16(N-1)S_2}{N^2(1+N)^2} \right] S_1^2 \right\} + C_F T_F^2 (2+N) \left\{ -\frac{32(N-1)^2(22+41N+28N^2)}{27N^3(1+N)^4} \right. \\ + L_M \left[-\frac{64(N-1)^2(2+5N)}{9N^3(1+N)^3} + (N-1) \left[-\frac{64(3+2N+2N^2)}{9N^3(1+N)^3} S_1 + \frac{64S_1^2}{3N^2(1+N)^2} \right] \right. \\ \left. + L_M^2 \left[-\frac{32(N-1)^2}{3N^3(1+N)^2} \right. \right. \\ \left. \left. - \frac{32(N-1)S_1}{3N^2(1+N)^2} \right] + (N-1) \left[L_Q \left[\frac{32L_M^2}{3N^2(1+N)^2} + \frac{32(22+41N+28N^2)}{27N^2(1+N)^4} + L_M \left[\frac{64(2+5N)}{9N^2(1+N)^3} - \frac{64S_1}{3N^2(1+N)^2} \right] \right. \right. \\ \left. \left. - \frac{32(2+5N)S_1}{9N^2(1+N)^3} + \frac{16S_1^2}{3N^2(1+N)^2} + \frac{16S_2}{3N^2(1+N)^2} \right] + \left[-\frac{32(6+37N+35N^2+13N^3)}{27N^3(1+N)^4} - \frac{16S_2}{3N^2(1+N)^2} \right] S_1 \right. \\ \left. + \frac{16(3+4N+7N^2)}{9N^3(1+N)^3} S_1^2 - \frac{16S_1^3}{3N^2(1+N)^2} \right] - \frac{16(N-1)^2 S_2}{3N^3(1+N)^2} \Big\} + A_{Qq}^{PS(3)} + \tilde{C}_q^{PS(3)}(L_Q, N_F + 1) \Big\}, \quad (215)$$

where

$$Q_{15} = 9N^5 + 6N^4 - 12N^2 - 8N + 1, \quad (216)$$

$$Q_{16} = 3N^8 + 10N^7 - N^6 - 22N^5 - 14N^4 - 18N^3 - 30N^2 + 8. \quad (217)$$

The polarized massive OME $A_{Qg}^{(3),PS}$ has been calculated in Ref. [23].

VIII. THE POLARIZED WILSON COEFFICIENT $H_{Qg}^{(3),S}$

Finally, $H_{Qg}^{(3),S}$, (8), is obtained by

$$\begin{aligned} H_g^S = & \frac{1}{2} [1 - (-1)^N] \left\{ a_s T_F \left\{ -\frac{4(N-1)^2}{N^2(1+N)} + (N-1) \left[-\frac{4L_M}{N(1+N)} + \frac{4L_Q}{N(1+N)} - \frac{4S_1}{N(1+N)} \right] \right\} \right. \\ & + a_s^2 \left\{ C_A T_F \left\{ -\frac{8Q_{78}}{(N-1)N^4(1+N)^4(2+N)^2} + L_M \left[-\frac{8Q_{42}}{N^3(1+N)^3} + (N-1) \left[\frac{8S_1^2}{N(1+N)} + \frac{8S_2}{N(1+N)} + \frac{16S_{-2}}{N(1+N)} \right] \right. \right. \right. \\ & - \frac{32S_1}{N(1+N)^2} \left. \right] + L_Q \left[\frac{8(-2+N)Q_{19}}{N^3(1+N)^3} + (N-1) \left[\frac{8S_1^2}{N(1+N)} - \frac{8S_2}{N(1+N)} - \frac{16S_{-2}}{N(1+N)} \right] \right. \\ & + \frac{16(3-N-N^2+N^3)}{N^2(1+N)^2} S_1 \left. \right] + (N-1) \left[L_Q^2 \left[\frac{16}{N^2(1+N)^2} - \frac{8S_1}{N(1+N)} \right] \right. \\ & + L_M^2 \left[-\frac{16}{N^2(1+N)^2} + \frac{8S_1}{N(1+N)} \right] - \frac{16S_{2,1}}{N(1+N)} \left. \right] + \left[-\frac{16Q_{55}}{N^3(1+N)^3(2+N)} + \frac{32(N-1)S_2}{N(1+N)} \right] S_1 \\ & - \frac{4(12-N+N^2+2N^3)}{N^2(1+N)(2+N)} S_1^2 + \frac{4Q_{21}}{N^2(1+N)^2(2+N)} S_2 + \frac{8(-2+3N+3N^2)}{N(1+N)(2+N)} S_3 \\ & + \left[\frac{16Q_{18}}{(N-1)N(1+N)(2+N)^2} + \frac{32S_1}{2+N} \right] S_{-2} - \frac{8(-2+N)(3+N)}{N(1+N)(2+N)} S_{-3} - \frac{16(2+N+N^2)}{N(1+N)(2+N)} S_{-2,1} - \frac{24(2+N+N^2)}{N(1+N)(2+N)} \zeta_3 \left. \right\} \\ & + C_F T_F \left\{ \frac{4Q_{79}}{(N-1)N^4(1+N)^4(2+N)^2} + (N-1) \left[L_M^2 \left[\frac{2(2+3N+3N^2)}{N^2(1+N)^2} - \frac{8S_1}{N(1+N)} \right] \right. \right. \\ & + L_Q^2 \left[\frac{2(2+3N+3N^2)}{N^2(1+N)^2} - \frac{8S_1}{N(1+N)} \right] + L_Q \left[-\frac{4Q_{27}}{N^3(1+N)^3} + L_M \left[-\frac{4(2+3N+3N^2)}{N^2(1+N)^2} + \frac{16S_1}{N(1+N)} \right] \right. \\ & + \frac{4(-6-N+3N^2)}{N^2(1+N)^2} S_1 + \frac{16S_1^2}{N(1+N)} - \frac{16S_2}{N(1+N)} \left. \right] + L_M \left[\frac{4Q_{27}}{N^3(1+N)^3} \right. \\ & - \frac{4(-6-N+3N^2)}{N^2(1+N)^2} S_1 - \frac{16S_1^2}{N(1+N)} + \frac{16S_2}{N(1+N)} \left. \right] - \frac{8S_1^3}{N(1+N)} + \frac{16S_{2,1}}{N(1+N)} \left. \right] \\ & + \left[\frac{4Q_{54}}{N^3(1+N)^3(2+N)} + \frac{8(N-1)S_2}{N(1+N)} \right] S_1 - \frac{2Q_{30}}{N^2(1+N)^2(2+N)} S_1^2 + \frac{2Q_{34}}{N^2(1+N)^2(2+N)} S_2 - \frac{16(2+N+N^2)}{N(1+N)(2+N)} S_3 \\ & + \left[\frac{16(10+N+N^2)}{(N-1)(2+N)^2} - \frac{128S_1}{N(1+N)(2+N)} \right] S_{-2} - \frac{64S_{-3}}{N(1+N)(2+N)} + \frac{128S_{-2,1}}{N(1+N)(2+N)} + \frac{48(2+N+N^2)}{N(1+N)(2+N)} \zeta_3 \left. \right\} \\ & + T_F^2 \left\{ (N-1) \left[-\frac{16L_M^2}{3N(1+N)} + \frac{16L_M L_Q}{3N(1+N)} \right] + L_M \left[-\frac{16(N-1)^2}{3N^2(1+N)} - \frac{16(N-1)S_1}{3N(1+N)} \right] \right\} \\ & + a_s^3 \left\{ C_F N_F T_F^2 \left\{ (N-1) \left[L_M L_Q^2 \left[-\frac{8Q_{24}}{3N^3(1+N)^3} + \frac{32S_1}{3N(1+N)} \right] + (L_M L_Q) \left[-\frac{16S_1 Q_{36}}{9N^3(1+N)^3} + \frac{8Q_{62}}{9N^4(1+N)^4} \right. \right. \right. \right. \\ & - \frac{32S_1^2}{3N(1+N)} + \frac{32S_2}{N(1+N)} \left. \right] + L_M \left[-\frac{2Q_{84}}{27N^5(1+N)^5(2+N)} + (N-1) \left[\frac{8S_1^2 Q_{36}}{9N^3(1+N)^3} - \frac{8S_2 Q_{39}}{9N^3(1+N)^3} \right. \right. \\ & + \left. \left[\frac{8Q_{64}}{27N^4(1+N)^4} - \frac{32S_2}{3N(1+N)} \right] S_1 + \frac{32S_1^3}{9N(1+N)} + \frac{352S_3}{9N(1+N)} - \frac{64S_{2,1}}{3N(1+N)} \right. \left. \left. \right] + \frac{256S_{-2}}{N^2(1+N)^2(2+N)} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& + C_A T_F^2 \left\{ -\frac{8(N-1)^2 Q_{58}}{27N^5(1+N)^4} + L_Q \left[(N-1) \left[\frac{8Q_{58}}{27N^4(1+N)^4} + L_M^2 \left[\frac{64}{3N^2(1+N)^2} - \frac{32S_1}{3N(1+N)} \right] - \frac{16(47+56N)S_1}{27N(1+N)^2} \right] \right. \right. \\
& + L_M \left[\frac{32Q_{48}}{9N^3(1+N)^3} + (N-1) \left[\frac{32S_1^2}{3N(1+N)} - \frac{32S_2}{3N(1+N)} - \frac{64S_{-2}}{3N(1+N)} \right] - \frac{64(-9-2N+3N^2+2N^3)}{9N^2(1+N)^2} S_1 \right] \\
& + L_M^2 \left[-\frac{64(N-1)^2}{3N^3(1+N)^2} + (N-1) \left[\frac{32(-3+N^2)S_1}{3N^2(1+N)^2} + \frac{32S_1^2}{3N(1+N)} \right] \right] \\
& + (N-1) \left[\frac{8S_1 Q_{63}}{27N^4(1+N)^4} + L_M L_Q^2 \left[\frac{64}{3N^2(1+N)^2} - \frac{32S_1}{3N(1+N)} \right] + \frac{16(47+56N)S_1^2}{27N(1+N)^2} \right] \\
& + L_M \left[-\frac{16Q_{82}}{9(N-1)N^4(1+N)^4(2+N)^2} + (N-1) \left[-\frac{16S_1^3}{9N(1+N)} - \frac{64S_{2,1}}{3N(1+N)} \right] \right. \\
& + \left[-\frac{16Q_{57}}{9N^3(1+N)^3(2+N)} + \frac{80(N-1)S_2}{3N(1+N)} \right] S_1 + \frac{32(-9-10N+3N^2+7N^3)}{9N^2(1+N)^2} S_1^2 + \frac{32(3-2N+N^2+N^3)}{3N^2(1+N)^2} S_2 \\
& + \frac{32(2+5N+5N^2)}{9N(1+N)(2+N)} S_3 + \left[\frac{64Q_{43}}{3(N-1)N(1+N)^2(2+N)^2} + \frac{64(2+N+N^2)}{3N(1+N)(2+N)} S_1 \right] S_{-2} - \frac{64(-4+N+N^2)}{3N(1+N)(2+N)} S_{-3} \\
& \left. - \frac{256S_{-2,1}}{3N(1+N)(2+N)} - \frac{32(2+N+N^2)}{N(1+N)(2+N)} \zeta_3 \right] \left. \right\} \\
& + C_F T_F^2 \left\{ \frac{4(N-1)^2 Q_{76}}{N^6(1+N)^5} + (N-1) \left[\frac{4S_1 Q_{76}}{N^5(1+N)^5} + L_M L_Q^2 \left[\frac{8(2+3N+3N^2)}{3N^2(1+N)^2} - \frac{32S_1}{3N(1+N)} \right] \right. \right. \\
& + L_Q \left[-\frac{4Q_{76}}{N^5(1+N)^5} + L_M^2 \left[-\frac{16Q_{22}}{3N^3(1+N)^3} + \frac{64S_1}{3N(1+N)} \right] + L_M \left[-\frac{16Q_{53}}{3N^4(1+N)^4} + \frac{16(-6-N+3N^2)}{3N^2(1+N)^2} S_1 \right. \right. \\
& + \left. \left. \frac{64S_1^2}{3N(1+N)} - \frac{64S_2}{3N(1+N)} \right] \right] + L_M^2 \left[-\frac{16S_1 Q_{23}}{3N^3(1+N)^3} + \frac{16Q_{49}}{3N^4(1+N)^3} - \frac{32S_1^2}{3N(1+N)} + \frac{32S_2}{3N(1+N)} \right] \\
& + L_M \left[-\frac{8Q_{86}}{3(N-1)N^5(1+N)^4(2+N)^2} + (N-1) \left[-\frac{8(-8+3N+9N^2)}{3N^2(1+N)^2} S_1^2 - \frac{80S_1^3}{9N(1+N)} + \frac{64S_{2,1}}{3N(1+N)} \right] \right. \\
& + \left[-\frac{32Q_{70}}{3N^4(1+N)^4(2+N)} + \frac{16(N-1)S_2}{N(1+N)} \right] S_1 + \frac{8(4-19N-10N^2+9N^3)}{3N^2(1+N)^2} S_2 - \frac{256(1+N+N^2)}{9N(1+N)(2+N)} S_3 \\
& + \left[\frac{64(10+N+N^2)}{3(N-1)(2+N)^2} - \frac{512S_1}{3N(1+N)(2+N)} \right] S_{-2} - \frac{256S_{-3}}{3N(1+N)(2+N)} + \frac{512S_{-2,1}}{3N(1+N)(2+N)} + \frac{64(2+N+N^2)}{N(1+N)(2+N)} \zeta_3 \left. \right\} \\
& + C_A C_F T_F \left\{ -\frac{4(-2+5N+6N^2+9N^3)Q_{72}}{N^6(1+N)^5(2+N)} + L_M^2 \left[(N-1) \left[\frac{16(-2+5N+6N^2+9N^3)}{N^4(1+N)^3} + \left[-\frac{8Q_{32}}{N^3(1+N)^3} - \frac{16S_2}{N(1+N)} \right] S_1 \right. \right. \right. \\
& + \left. \frac{16S_1^3}{N(1+N)} + \frac{32S_2}{N^2(1+N)^2} \right] + \frac{24(N-1)^2(2+N)}{N^2(1+N)^2} S_1^2 \left. \right] + L_Q \left[\frac{4(2+3N+3N^2)Q_{72}}{N^5(1+N)^5(2+N)} \right. \\
& + L_M \left[-\frac{2Q_{68}}{9N^4(1+N)^4} + (N-1) \left[-\frac{32S_1^3}{N(1+N)} - \frac{16(-1+4N+4N^2)}{N^2(1+N)^2} S_2 + \frac{32S_3}{N(1+N)} + \frac{48S_{-2}}{N(1+N)} \right. \right. \\
& + \left. \frac{32S_{-3}}{N(1+N)} - \frac{64S_{-2,1}}{N(1+N)} \right] + \left[\frac{4Q_{51}}{9N^3(1+N)^3} - \frac{32(N-1)S_2}{N(1+N)} \right] S_1 + \frac{16(-3+17N+10N^3)}{3N^2(1+N)^2} S_1^2 \\
& + (N-1) \left[L_M^2 \left[-\frac{16(2+3N+3N^2)}{N^3(1+N)^3} + \frac{8(10+3N+3N^2)}{N^2(1+N)^2} S_1 - \frac{32S_1^2}{N(1+N)} \right] - \frac{16S_1^4}{3N(1+N)} + \frac{32(2+3N+3N^2)}{3N^2(1+N)^2} S_3 \right. \\
& + \left[\frac{16(2+3N+3N^2)}{N^2(1+N)^3} + \frac{16(2-N+3N^2)}{N^2(1+N)^2} S_1 - \frac{64S_1^2}{N(1+N)} \right] S_{-2} + \left[\frac{8(2+3N+3N^2)}{N^2(1+N)^2} - \frac{32S_1}{N(1+N)} \right] S_{-3} \\
& \left. - \frac{16(2+3N+3N^2)}{N^2(1+N)^2} S_{-2,1} \right] + \left[-\frac{4Q_{74}}{N^4(1+N)^4(2+N)} + (N-1) \left[-\frac{128S_3}{3N(1+N)} + \frac{64S_{-2,1}}{N(1+N)} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{4Q_{33}}{N^2(1+N)^2(2+N)} S_2 \Big] S_1 + \left[\frac{4Q_{46}}{N^2(1+N)^3(2+N)} - \frac{48(N-1)S_2}{N(1+N)} \right] S_1^2 + \frac{4Q_{25}}{3N^2(1+N)^2(2+N)} S_1^3 \\
& - \frac{4(2+3N+3N^2)(-16+15N+24N^2+7N^3)}{N^3(1+N)^3(2+N)} S_2 \Big] + L_M \left[\frac{2S_2Q_{52}}{9N^3(1+N)^3} + \frac{Q_{85}}{54N^5(1+N)^5(2+N)} \right. \\
& + (N-1) \left[\left[-\frac{2Q_{41}}{9N^3(1+N)^3} - \frac{16S_2}{N(1+N)} \right] S_1^2 + \frac{16S_1^4}{N(1+N)} - \frac{8(-36+121N+121N^2)}{9N^2(1+N)^2} S_3 + \frac{32S_4}{N(1+N)} \right. \\
& + \frac{48S_{-2}^2}{N(1+N)} + \left[-\frac{32}{N^2(1+N)^2} + \frac{32S_1}{N(1+N)} \right] S_{-3} + \frac{80S_{-4}}{N(1+N)} + \frac{16(-6+11N+11N^2)}{3N^2(1+N)^2} S_{2,1} \\
& + \frac{96S_{3,1}}{N(1+N)} - \frac{32S_{-2,2}}{N(1+N)} - \frac{64S_{-3,1}}{N(1+N)} - \frac{96S_{2,1,1}}{N(1+N)} - \frac{24(1+3N)(2+3N)}{N^2(1+N)^2} \zeta_3 \Big] \\
& + \left[-\frac{2Q_{69}}{27N^4(1+N)^4} + (N-1) \left[-\frac{96S_3}{N(1+N)} + \frac{64S_{2,1}}{N(1+N)} + \frac{64S_{-2,1}}{N(1+N)} + \frac{192\zeta_3}{N(1+N)} \right] + \frac{32(1+5N^2)}{3N(1+N)^2} S_2 \right] S_1 \\
& + \frac{16(9-53N+8N^3)}{9N^2(1+N)^2} S_1^3 + \left[-\frac{16Q_{56}}{N^3(1+N)^3(2+N)} + (N-1) \left[\frac{48S_1}{N(1+N)} - \frac{32S_1^2}{N(1+N)} \right] \right] S_{-2} \Big] \\
& + (N-1) \left[L_M L_Q^2 \left[\frac{22(2+3N+3N^2)}{3N^2(1+N)^2} - \frac{88S_1}{3N(1+N)} \right] + \frac{8S_1^5}{3N(1+N)} - \frac{32(-2+5N+6N^2+9N^3)}{3N^3(1+N)^2} S_3 \right. \\
& + \left[-\frac{16(-2+5N+6N^2+9N^3)}{N^3(1+N)^3} + \left[-\frac{16Q_{31}}{N^3(1+N)^3} - \frac{32S_2}{N(1+N)} \right] S_1 + \frac{16(2+N)(-1+3N)}{N^2(1+N)^2} S_1^2 \right. \\
& + \frac{32S_1^3}{N(1+N)} - \frac{32S_2}{N(1+N)^2} \Big] S_{-2} + \left[-\frac{8(-2+5N+6N^2+9N^3)}{N^3(1+N)^2} + \frac{8(-2+3N+3N^2)}{N^2(1+N)^2} S_1 \right. \\
& + \frac{16S_1^2}{N(1+N)} - \frac{16S_2}{N(1+N)} \Big] S_{-3} + \frac{16(-2+5N+6N^2+9N^3)}{N^3(1+N)^2} S_{-2,1} \Big] \\
& + \left[\frac{4Q_{80}}{N^5(1+N)^5(2+N)} + (N-1) \left[-\frac{24S_2^2}{N(1+N)} + \frac{32(-2+3N+3N^2)}{3N^2(1+N)^2} S_3 - \frac{16(-2+3N+3N^2)}{N^2(1+N)^2} S_{-2,1} \right] \right. \\
& - \frac{4Q_{61}}{N^3(1+N)^3(2+N)} S_2 \Big] S_1 + \left[\frac{4Q_{71}}{N^4(1+N)^4(2+N)} + (N-1) \left[\frac{64S_3}{3N(1+N)} - \frac{32S_{-2,1}}{N(1+N)} \right] \right. \\
& + \frac{4Q_{29}}{N^2(1+N)^2(2+N)} S_2 \Big] S_1^2 + \left[-\frac{4Q_{59}}{3N^3(1+N)^3(2+N)} + \frac{64(N-1)S_2}{3N(1+N)} \right] S_1^3 \\
& + \frac{4(4-38N-29N^2+3N^4)}{3N^2(1+N)^2(2+N)} S_1^4 + \left[-\frac{4Q_{73}}{N^4(1+N)^4(2+N)} + (N-1) \left[-\frac{64S_3}{3N(1+N)} + \frac{32S_{-2,1}}{N(1+N)} \right] \right] S_2 \\
& + \frac{8(-16+15N+24N^2+7N^3)}{N^2(1+N)^2(2+N)} S_2^2 \Big\} \\
& + C_F^2 T_F \left\{ -\frac{2(-2+5N+6N^2+9N^3)Q_{75}}{N^6(1+N)^5(2+N)} + L_M^2 \left[(N-1) \left[-\frac{2(2+3N+3N^2)(-2+5N+6N^2+9N^3)}{N^4(1+N)^3} \right. \right. \right. \\
& + \left. \left[\frac{2Q_{38}}{N^3(1+N)^3} + \frac{16S_2}{N(1+N)} \right] S_1 - \frac{16S_1^3}{N(1+N)} - \frac{4(2+3N+3N^2)}{N^2(1+N)^2} S_2 \right] - \frac{12(N-1)^2(2+N)}{N^2(1+N)^2} S_1^2 \Big] \\
& + L_Q \left[\frac{2(2+3N+3N^2)Q_{75}}{N^5(1+N)^5(2+N)} + (N-1) \left[L_M^2 \left[\frac{2(2+3N+3N^2)^2}{N^3(1+N)^3} - \frac{16(2+3N+3N^2)}{N^2(1+N)^2} S_1 + \frac{32S_1^2}{N(1+N)} \right] \right. \right. \\
& + L_M \left[\frac{2Q_{60}}{N^4(1+N)^4} + \left[-\frac{4Q_{35}}{N^3(1+N)^3} - \frac{128S_2}{N(1+N)} \right] S_1 - \frac{64(2+N)S_1^2}{N^2(1+N)^2} + \frac{64S_1^3}{N(1+N)} + \frac{32(2+3N+3N^2)}{N^2(1+N)^2} S_2 \right. \\
& - \frac{64S_3}{N(1+N)} + \left[\frac{64}{N^2(1+N)^2} - \frac{128S_1}{N(1+N)} \right] S_{-2} - \frac{64S_{-3}}{N(1+N)} + \frac{128S_{-2,1}}{N(1+N)} \Big] + \frac{16S_1^4}{3N(1+N)} + \frac{16(2+3N+3N^2)}{3N^2(1+N)^2} S_3 \Big]
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{4Q_{77}}{N^4(1+N)^4(2+N)} - \frac{4Q_{28}}{N^2(1+N)^2(2+N)} S_2 - \frac{64(N-1)S_3}{3N(1+N)} \right] S_1 \\
& + \left[-\frac{4Q_{45}}{N^3(1+N)^2(2+N)} + \frac{16(N-1)S_2}{N(1+N)} \right] S_1^2 - \frac{4Q_{26}}{3N^2(1+N)^2(2+N)} S_1^3 + \frac{4(2+3N+3N^2)Q_{20}}{N^3(1+N)^3(2+N)} S_2 \\
& + (N-1) \left[L_M L_Q^2 \left[-\frac{2(2+3N+3N^2)^2}{N^3(1+N)^3} + \frac{16(2+3N+3N^2)}{N^2(1+N)^2} S_1 - \frac{32S_1^2}{N(1+N)} \right] \right. \\
& \left. - \frac{8S_1^5}{3N(1+N)} - \frac{16(-2+5N+6N^2+9N^3)}{3N^3(1+N)^2} S_3 \right] \\
& + L_M \left[\frac{Q_{83}}{2N^5(1+N)^5(2+N)} + (N-1) \left[-\frac{2S_2Q_{40}}{N^3(1+N)^3} + \left[\frac{2Q_{37}}{N^3(1+N)^3} + \frac{112S_2}{N(1+N)} \right] S_1^2 \right. \right. \\
& \left. \left. - \frac{24S_1^4}{N(1+N)} - \frac{40S_2^2}{N(1+N)} + \frac{8(-2+9N+9N^2)}{N^2(1+N)^2} S_3 - \frac{48S_4}{N(1+N)} - \frac{96S_{-2}^2}{N(1+N)} \right. \right. \\
& \left. \left. + \left[\frac{64}{N^2(1+N)^2} - \frac{64S_1}{N(1+N)} \right] S_{-3} - \frac{160S_{-4}}{N(1+N)} - \frac{16(-2+3N+3N^2)}{N^2(1+N)^2} S_{2,1} - \frac{160S_{3,1}}{N(1+N)} \right. \right. \\
& \left. \left. + \frac{64S_{-2,2}}{N(1+N)} + \frac{128S_{-3,1}}{N(1+N)} + \frac{96S_{2,1,1}}{N(1+N)} + \frac{288\zeta_3}{N(1+N)} \right] + \left[\frac{2Q_{66}}{N^4(1+N)^4} + (N-1) \left[\frac{32(-4+2N+3N^2)}{N^2(1+N)^2} S_2 \right. \right. \right. \\
& \left. \left. + \frac{96S_3}{N(1+N)} - \frac{64S_{2,1}}{N(1+N)} - \frac{128S_{-2,1}}{N(1+N)} - \frac{192\zeta_3}{N(1+N)} \right] \right] S_1 - \frac{16(N-1)^2(4+3N)}{N^2(1+N)^2} S_1^3 \\
& + \left[\frac{32Q_{44}}{N^2(1+N)^3(2+N)} + (N-1) \left[-\frac{64S_1}{N^2(1+N)^2} + \frac{128S_1^2}{N(1+N)} - \frac{64S_2}{N(1+N)} \right] \right] S_{-2} \\
& + \left[\frac{2Q_{81}}{N^5(1+N)^5(2+N)} + (N-1) \left[\frac{8S_2^2}{N(1+N)} + \frac{16(-2+3N+3N^2)}{3N^2(1+N)^2} S_3 \right] + \frac{8Q_{47}}{N^2(1+N)^3(2+N)} S_2 \right] S_1 \\
& + \left[\frac{4Q_{65}}{N^3(1+N)^4(2+N)} - \frac{4Q_{17}}{N^2(1+N)^2(2+N)} S_2 + \frac{32(N-1)S_3}{3N(1+N)} \right] S_1^2 + \left[\frac{4Q_{50}}{3N^3(1+N)^2(2+N)} - \frac{16(N-1)S_2}{3N(1+N)} \right] S_1^3 \\
& - \frac{4(-8-5N+N^2)(-2+3N+3N^2)}{3N^2(1+N)^2(2+N)} S_1^4 + \left[-\frac{4Q_{67}}{N^3(1+N)^4(2+N)} - \frac{32(N-1)S_3}{3N(1+N)} \right] S_2 - \frac{8Q_{20}}{N^2(1+N)^2(2+N)} S_2^2 \Big\} \\
& + T_F^3 \left\{ \frac{64L_M^2 L_Q(N-1)}{9N(1+N)} + L_M^2 \left[-\frac{64(N-1)^2}{9N^2(1+N)} - \frac{64(N-1)S_1}{9N(1+N)} \right] \right\} + A_{Qg}^{(3)} + \tilde{C}_g^{S(3)}(L_Q, N_F + 1) \Big\}, \quad (218)
\end{aligned}$$

with the polynomials Q_i

$$Q_{17} = N^4 - 22N^3 - 79N^2 - 72N - 4, \quad (219)$$

$$Q_{18} = N^4 + 3N^3 - 4N^2 - 8N - 4, \quad (220)$$

$$Q_{19} = N^4 + 3N^3 - 2N^2 + 3N + 3, \quad (221)$$

$$Q_{20} = N^4 + 17N^3 + 43N^2 + 33N + 2, \quad (222)$$

$$Q_{21} = 2N^4 - N^3 - 24N^2 - 17N + 28, \quad (223)$$

$$Q_{22} = 3N^4 + 6N^3 + 2N^2 - N + 6, \quad (224)$$

$$Q_{23} = 3N^4 + 6N^3 + 4N^2 + N - 6, \quad (225)$$

$$Q_{24} = 3N^4 + 6N^3 + 11N^2 + 8N - 12, \quad (226)$$

$$Q_{25} = 3N^4 + 18N^3 + 47N^2 + 56N - 4, \quad (227)$$

$$Q_{26} = 3N^4 + 42N^3 + 71N^2 + 8N - 28, \quad (228)$$

$$Q_{27} = 4N^4 + 5N^3 + 3N^2 - 4N - 4, \quad (229)$$

$$Q_{28} = 7N^4 + 74N^3 + 171N^2 + 128N + 4, \quad (230)$$

$$Q_{29} = 9N^4 + 6N^3 - 55N^2 - 44N + 44, \quad (231)$$

$$Q_{30} = 9N^4 + 6N^3 - 35N^2 - 16N + 20, \quad (232)$$

$$Q_{31} = 9N^4 + 12N^3 + 8N^2 + 5N - 2, \quad (233)$$

$$Q_{32} = 9N^4 + 15N^3 + 17N^2 + 9N - 6, \quad (234)$$

$$Q_{33} = 9N^4 + 46N^3 + 93N^2 + 48N - 76, \quad (235)$$

$$Q_{34} = 11N^4 + 42N^3 + 47N^2 + 32N + 12, \quad (236)$$

$$Q_{35} = 25N^4 + 26N^3 - 3N^2 - 52N - 36, \quad (237)$$

$$Q_{36} = 29N^4 + 58N^3 + 5N^2 - 24N + 36, \quad (238)$$

$$Q_{37} = 43N^4 + 86N^3 + 107N^2 - 8N - 56, \quad (239)$$

$$Q_{38} = 45N^4 + 78N^3 + 53N^2 + 12N - 12, \quad (240)$$

$$Q_{39} = 85N^4 + 170N^3 + 61N^2 - 24N + 36, \quad (241)$$

$$Q_{40} = 111N^4 + 198N^3 + 135N^2 + 24N - 32, \quad (242)$$

$$Q_{41} = 763N^4 + 1418N^3 + 985N^2 + 978N + 72, \quad (243)$$

$$Q_{42} = N^5 + N^4 - 4N^3 + 3N^2 - 7N - 2, \quad (244)$$

$$Q_{43} = N^5 + 3N^4 - 3N^3 - 9N^2 - 8N - 8, \quad (245)$$

$$Q_{44} = 2N^5 + 6N^4 + N^3 - 6N^2 + 11N + 10, \quad (246)$$

$$Q_{45} = 4N^5 - 13N^4 - 114N^3 - 241N^2 - 144N + 4, \quad (247)$$

$$Q_{46} = 4N^5 + 13N^4 - 19N^3 - 69N^2 - 15N - 10, \quad (248)$$

$$Q_{47} = 5N^5 + 39N^4 + 118N^3 + 171N^2 + 93N + 2, \quad (249)$$

$$Q_{48} = 9N^5 + 9N^4 - 4N^3 + 15N^2 - 41N - 12, \quad (250)$$

$$Q_{49} = 9N^5 + 15N^4 + 8N^3 + 3N^2 + 7N - 6, \quad (251)$$

$$Q_{50} = 15N^5 + 36N^4 - 97N^3 - 366N^2 - 264N + 16, \quad (252)$$

$$Q_{51} = 439N^5 + 439N^4 - 937N^3 - 367N^2 - 582N - 144, \quad (253)$$

$$Q_{52} = 815N^5 + 923N^4 - 1349N^3 - 491N^2 - 258N - 216, \quad (254)$$

$$Q_{53} = N^6 - 7N^4 - 4N^3 + 16N^2 - 10N - 12, \quad (255)$$

$$Q_{54} = 2N^6 + 5N^5 - 22N^4 - 95N^3 - 114N^2 - 24N + 16, \quad (256)$$

$$Q_{55} = 2N^6 + 5N^5 - 3N^4 - 7N^3 + 2N^2 - 11N - 8, \quad (257)$$

$$Q_{56} = 11N^6 + 30N^5 + 9N^4 - 22N^3 - 10N^2 + 2N + 4, \quad (258)$$

$$Q_{57} = 13N^6 + 44N^5 + 71N^4 + 94N^3 - 90N^2 - 288N - 72, \quad (259)$$

$$Q_{58} = 15N^6 + 45N^5 + 374N^4 + 601N^3 + 161N^2 - 24N + 36, \quad (260)$$

$$Q_{59} = 15N^6 + 57N^5 + 47N^4 - N^3 + 12N^2 - 38N + 4, \quad (261)$$

$$Q_{60} = 21N^6 + 45N^5 + 55N^4 - 13N^3 - 12N^2 - 8N + 8, \quad (262)$$

$$Q_{61} = 25N^6 + 85N^5 + 119N^4 + 75N^3 - 118N^2 - 102N + 44, \quad (263)$$

$$Q_{62} = 57N^6 + 153N^5 + 233N^4 + 163N^3 - 70N^2 + 120N + 144, \quad (264)$$

$$Q_{63} = 97N^6 + 161N^5 - 392N^4 - 807N^3 - 255N^2 + 24N - 36, \quad (265)$$

$$Q_{64} = 247N^6 + 795N^5 + 555N^4 - 71N^3 + 210N^2 - 360N - 432, \quad (266)$$

$$Q_{65} = 15N^7 + 28N^6 - 116N^5 - 453N^4 - 575N^3 - 221N^2 + 114N + 88, \quad (267)$$

$$Q_{66} = 21N^7 - 52N^6 - 86N^5 - 60N^4 + 17N^3 + 176N^2 + 24N - 40, \quad (268)$$

$$Q_{67} = 21N^7 + 220N^6 + 713N^5 + 1132N^4 + 1010N^3 + 486N^2 + 38N - 52, \quad (269)$$

$$Q_{68} = 753N^7 + 1308N^6 - 44N^5 - 1118N^4 - 1549N^3 - 766N^2 - 600N - 288, \quad (270)$$

$$Q_{69} = 3479N^7 + 7444N^6 - 5160N^5 - 13414N^4 - 8111N^3 - 5478N^2 + 1368N + 864, \quad (271)$$

$$Q_{70} = N^8 + 3N^7 + 5N^6 + 5N^5 - 9N^4 - 8N^3 + 19N^2 - 8N - 12, \quad (272)$$

$$Q_{71} = N^8 + 14N^7 + 88N^6 + 229N^5 + 202N^4 + 47N^3 + 77N^2 + 14N + 8, \quad (273)$$

$$Q_{72} = 2N^8 + 10N^7 + 22N^6 + 36N^5 + 29N^4 + 4N^3 + 33N^2 + 12N + 4, \quad (274)$$

$$Q_{73} = 4N^8 - 43N^7 - 277N^6 - 500N^5 - 308N^4 + 25N^3 + 245N^2 + 102N - 24, \quad (275)$$

$$Q_{74} = 11N^8 + 55N^7 + 99N^6 + 119N^5 + 90N^4 + 2N^3 + 136N^2 + 48N + 16, \quad (276)$$

$$Q_{75} = 12N^8 + 52N^7 + 60N^6 - 25N^4 - 2N^3 + 3N^2 + 8N + 4, \quad (277)$$

$$Q_{76} = 15N^8 + 60N^7 + 82N^6 + 44N^5 + 15N^4 + 4N^2 + 12N + 8, \quad (278)$$

$$Q_{77} = 21N^8 + 101N^7 + 193N^6 + 321N^5 + 528N^4 + 550N^3 + 302N^2 + 88N + 8, \quad (279)$$

$$Q_{78} = N^{10} + 3N^9 - 15N^8 - 56N^7 - 8N^6 + 90N^5 + 60N^4 + 67N^3 + 86N^2 - 12N - 24, \quad (280)$$

$$Q_{79} = 5N^{10} + 23N^9 + 31N^8 - N^7 + 54N^6 + 268N^5 + 342N^4 + 98N^3 - 60N^2 - 8N + 16, \quad (281)$$

$$Q_{80} = 15N^{10} + 87N^9 + 154N^8 + 96N^7 + 18N^6 - 64N^5 + 47N^4 + 153N^3 - 22N^2 - 12N - 8, \quad (282)$$

$$Q_{81} = 18N^{10} + 162N^9 + 740N^8 + 2296N^7 + 4511N^6 + 5341N^5 + 3593N^4 + 1065N^3 - 130N^2 - 148N - 8, \quad (283)$$

$$Q_{82} = 24N^{10} + 102N^9 + 58N^8 - 210N^7 - 209N^6 + 23N^5 + 529N^4 + 1109N^3 + 234N^2 - 388N - 120, \quad (284)$$

$$Q_{83} = 29N^{10} + 229N^9 + 620N^8 + 1434N^7 + 2173N^6 + 505N^5 - 86N^4 + 712N^3 + 704N^2 - 16N - 160, \quad (285)$$

$$Q_{84} = 1371N^{10} + 6171N^9 + 10220N^8 + 5678N^7 - 9493N^6 - 17113N^5 - 9154N^4 - 10864N^3 - 10656N^2 + 1872N + 4320, \quad (286)$$

$$Q_{85} = 20283N^{10} + 92379N^9 + 127804N^8 + 11278N^7 - 154181N^6 - 222809N^5 - 170666N^4 - 111392N^3 - 62568N^2 + 5040N + 8640, \quad (287)$$

$$Q_{86} = 8N^{11} + 42N^{10} + 44N^9 - 102N^8 - 415N^7 - 539N^6 - 195N^5 - 241N^4 - 414N^3 + 268N^2 + 104N - 96. \quad (288)$$

IX. CONCLUSIONS

We have calculated the logarithmic contributions to the massive operator matrix elements A_{ij} and the massive Wilson coefficients in the asymptotic region in the polarized case to three-loop order, also referring to previous results in the literature. These quantities provide important corrections to the deep-inelastic scattering structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$, as well as the matching functions in the polarized variable flavor number scheme to three-loop order. Here the OMEs $A_{qq,Q}^{(3),PS}$ and $A_{gg,Q}^{(3),S}$ are new quantities, which have not been calculated in complete form. At present, except for the massless flavor nonsinglet Wilson coefficients [62], the polarized massless three-loop Wilson coefficients are not yet known. Therefore, we have given them only symbolically in the corresponding expressions. Similarly, the calculation of the constant part of the unrenormalized polarized massive OMEs $A_{Qg}^{(3)}$ and $A_{gg,Q}^{(3)}$ is still underway. However, the logarithmic contributions are already available for use. These contributions, unlike the case for the nonlogarithmic parts, can all be represented in terms of harmonic sums in Mellin N space or by harmonic polylogarithms in momentum fraction z space. Finally, one has to know these expressions to three-loop order, because this is required by the experimental accuracy expected in future high-luminosity deep-inelastic scattering experiments. Likewise, the matrix elements for the polarized variable flavor number scheme are needed to describe the decoupling of heavy flavors at polarized hadron colliders, to give full account of these QCD corrections. Working in the Larin scheme provides an alternative to other schemes

in matching the massive OMEs with the massless Wilson coefficients and polarized parton densities, which would also be described in this scheme. This implies a modification for their evolution, but maintains their universality and leads to the same predictions of all observables. The analytic expressions for the massive operator matrix elements and Wilson coefficients in the asymptotic regions are lengthy, as expected for single differential three-loop quantities (see also [62]). For this reason we also provide computer-readable files as attachments to this paper.

In addition to the single mass corrections, on which we have concentrated in the present paper, there are double mass corrections due to charm and bottom quarks of the polarized massive OMEs starting with three-loop order. They have been calculated for all OMEs except $A_{Qg}^{(3)}$ in Refs. [25–27], in complete analytic form and the calculation of $A_{Qg}^{(3)}$ maintaining the leading terms in the mass expansion m_c^2/m_b^2 is underway. We also presented the nonsinglet OME and the associated Wilson coefficient in the Larin scheme since it previously has been given in the $\overline{MS}$ scheme only [22,28].

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APPENDIX A: THE z -SPACE EXPRESSIONS FOR THE MASSIVE OMEs

The massive OMEs in z space decompose into the three contributions A_δ, A_+ and A_{reg} . Their convolutions with regular functions, such as parton distribution functions, are given by [78]

$$A_\delta(z) \otimes f(z) = f(1) \quad (\text{A1})$$

$$A_+(z) \otimes f(z) = \int_z^1 \frac{dy}{y} A_+(y) \left[f\left(\frac{z}{y}\right) - f(z) \right] - f(z) \int_0^z dy A_+(y) \quad (\text{A2})$$

$$A_{\text{reg}}(z) \otimes f(z) = \int_z^1 \frac{dy}{y} A_{\text{reg}}(y) f\left(\frac{z}{y}\right). \quad (\text{A3})$$

The following terms can all be expressed by harmonic polylogarithms [54]. They correspond to words out of letters of the three-letter alphabet

$$\mathfrak{A} = \left\{ f_0(x) = \frac{1}{x}, f_1(x) = \frac{1}{1-x}, f_{-1}(x) = \frac{1}{1+x} \right\} \quad (\text{A4})$$

and perform algebraic reductions, Ref. [73]. The harmonic polylogarithms are given by

$$H_{b,\vec{a}}(x) = \int_0^x dy f_a(y) H_{\vec{a}}(y), \quad H_\emptyset = 1, \quad a, a_i \in \{0, 1, -1\}. \quad (\text{A5})$$

With the exception of $A_{gg,Q}^{(3)}$ and $A_{qq,Q}^{(3),\text{NS}}$, all OMEs shown in the following have only regular contributions. For the OME $A_{qq,Q}^{(3),\text{PS}}$ they read

$$\begin{aligned} A_{qq,Q}^{PS}(z) = a_s^3 \Big\{ & C_F N_F T_F^2 \left\{ L_M^3 \left[\frac{160}{9} (z-1) - \frac{64}{9} (1+z) H_0 \right] \right. \\ & + L_M^2 \left[-\frac{32}{9} (z-1)(-1+15H_1) + (1+z) \left(-\frac{32}{3} H_0^2 + \frac{64}{3} H_{0,1} - \frac{64}{3} \zeta_2 \right) - \frac{32}{9} (7+z) H_0 \right] \\ & + L_M \left[(1+z) \left(-\frac{176}{9} H_0^2 - \frac{32}{9} H_0^3 + \frac{128}{3} H_{0,0,1} - \frac{128}{3} H_{0,1,1} \right) \right. \\ & - \frac{32}{27} (55+31z) H_0 + (z-1) \left(\frac{5312}{27} - \frac{64}{9} H_1 + \frac{160}{3} H_1^2 \right) + \frac{64}{9} (7+z) H_{0,1} \\ & + \left[-\frac{64}{9} (7+z) - \frac{128}{3} (1+z) H_0 \right] \zeta_2 \Big] + (1+z) \left(-\frac{64}{3} H_{0,1,1,1} + \frac{448}{15} \zeta_2^2 \right) \\ & + \frac{128}{81} (49+82z) H_0 + (z-1) \left(-\frac{10880}{81} + \frac{3712}{27} H_1 - \frac{64}{9} H_1^2 + \frac{80}{9} H_1^3 \right) - \frac{128}{27} (11+14z) H_{0,1} \\ & + \frac{64}{9} (2+5z) H_{0,1,1} + \left[\frac{16}{27} (103+97z) + (1+z) \left(\frac{176}{9} H_0 + \frac{16}{3} H_0^2 - \frac{64}{3} H_{0,1} \right) + \frac{160}{3} (z-1) H_1 \right] \zeta_2 \\ & \left. + \left[-\frac{32}{9} (-1+15z) + \frac{64}{9} (1+z) H_0 \right] \zeta_3 \right\} + a_{qq,Q}^{PS(3)} \Big\}. \quad (\text{A6}) \end{aligned}$$

The contributions to $A_{qq,Q}^{(3)}$ are given by

$$\begin{aligned} A_{qq,Q} = a_s^3 \Big\{ & C_F N_F T_F^2 \left\{ \frac{1}{27} (58501 - 59018z) + L_M^3 \left[\frac{8}{3} (-41+42z) + (-1+2z) \left(\frac{16}{3} H_0^2 - \frac{32}{9} H_1 \right) - \frac{32}{9} (13+19z) H_0 \right] \right. \\ & + L_M^2 \left[\frac{4}{3} (-285+296z) + (-1+2z) \left(\frac{32}{3} H_0^3 + \frac{16}{3} H_1^2 + \frac{32}{3} H_{0,1} - \frac{32}{3} \zeta_2 \right) - \frac{8}{9} (425+2z) H_0 - 16(7+6z) H_0^2 \right. \\ & - \frac{32}{9} (1+4z) H_1 \Big] + L_M \left[\frac{212}{9} (-69+70z) + (-1+2z) \left(\frac{20}{3} H_0^4 - \frac{16}{9} H_1^3 + \frac{32}{3} H_{0,0,1} - \frac{32}{3} H_{0,1,1} + \frac{32}{3} \zeta_3 \right) \right. \\ & \left. + \left[-\frac{8}{27} (4103+1565z) + \frac{64}{3} (z-1) H_1 - \frac{32}{3} (-1+2z) H_{0,1} \right] H_0 + \left[\frac{4}{3} (-343+10z) + \frac{16}{3} (-1+2z) H_1 \right] H_0^2 \right. \\ & \left. \left. + \left[-\frac{8}{27} (4103+1565z) + \frac{64}{3} (z-1) H_1 - \frac{32}{3} (-1+2z) H_{0,1} \right] H_0 + \left[\frac{4}{3} (-343+10z) + \frac{16}{3} (-1+2z) H_1 \right] H_0^2 \right. \right. \\ & \left. \left. + \left[-\frac{8}{27} (4103+1565z) + \frac{64}{3} (z-1) H_1 - \frac{32}{3} (-1+2z) H_{0,1} \right] H_0 + \left[\frac{4}{3} (-343+10z) + \frac{16}{3} (-1+2z) H_1 \right] H_0^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{16}{9}(50+23z)H_0^3 - \frac{16}{27}(-28+131z)H_1 + \frac{32}{9}(-2+7z)H_1^2 + \frac{64}{9}(1+4z)H_{0,1} - \frac{64}{9}(-2+7z)\zeta_2 \\
& + (-1+2z)\left(-\frac{4}{15}H_0^5 + \frac{5248}{81}H_1 - \frac{896}{27}H_{0,1} + \frac{160}{9}H_{0,0,1} - \frac{32}{3}H_{0,0,0,1} + \frac{16}{3}\zeta_2^2\right) \\
& + \left[\frac{16}{81}(7333+4195z) + (-1+2z)\left(\frac{896}{27}H_1 - \frac{160}{9}H_{0,1} + \frac{32}{3}H_{0,0,1}\right)\right]H_0 \\
& + \left[\frac{8}{27}(1331+140z) + (-1+2z)\left(\frac{80}{9}H_1 - \frac{16}{3}H_{0,1}\right)\right]H_0^2 + \left[-\frac{16}{27}(-101+28z) + \frac{16}{9}(-1+2z)H_1\right]H_0^3 \\
& + \frac{4}{9}(13+7z)H_0^4 + \left[-\frac{4}{3}(-299+305z) + (-1+2z)\left(-8H_0^3 - \frac{8}{3}H_1^2 - \frac{32}{3}H_{0,1}\right)\right. \\
& + \left.\left[\frac{4}{9}(757+178z) + \frac{16}{3}(-1+2z)H_1\right]H_0 + \frac{8}{3}(34+25z)H_0^2 + \frac{32}{9}(-2+7z)H_1\right]\zeta_2 \\
& + \left[-\frac{8}{3}(-41+42z) + (-1+2z)\left(-\frac{16}{3}H_0^2 + \frac{32}{9}H_1\right) + \frac{32}{9}(13+19z)H_0\right]\zeta_3 \Big\} \\
& + C_A N_F T_F^2 \left\{ -\frac{8}{27}(-4028+4113z) + L_M^3 \left[\frac{64}{3}(z-1) - \frac{64}{9}(1+z)H_0 + \frac{32}{9}(-1+2z)H_1 \right] \right. \\
& + L_M^2 \left[\frac{8}{3}(-65+68z) + \left[-\frac{16}{9}(47+38z) - \frac{32}{3}(1+2z)H_{-1} \right] H_0 - \frac{32}{3}H_0^2 + \frac{32}{9}(1+4z)H_1 \right. \\
& - \left. \frac{16}{3}(-1+2z)H_1^2 + \frac{32}{3}(1+2z)H_{0,-1} - \frac{32}{3}\zeta_2 \right] + L_M \left[\frac{16}{9}(-514+547z) \right. \\
& + (-1+2z)\left(\frac{16}{9}H_1^3 + \frac{32}{3}H_{0,0,1} + 32H_{0,1,1}\right) + \left. \left[-\frac{32}{27}(463+211z) + (-1+2z)\left(\frac{16}{3}H_1^2 - \frac{32}{3}H_{0,1}\right) \right. \right. \\
& - \left. \frac{64}{9}(2+7z)H_{-1} + \frac{64}{3}(1+2z)H_{0,-1} \right] H_0 + \left. \left[-\frac{8}{9}(133+72z) + \frac{16}{3}(-1+2z)H_1 - \frac{32}{3}(1+2z)H_{-1} \right] H_0^2 \right. \\
& + \frac{16}{9}(-5+6z)H_0^3 + \left. \left[\frac{16}{27}(-1+95z) - \frac{64}{3}(-1+2z)H_{0,1} \right] H_1 \right. \\
& - \frac{32}{9}(-2+7z)H_1^2 + \frac{64}{3}zH_{0,1} + \frac{64}{9}(2+7z)H_{0,-1} - \frac{64}{3}(1+2z)H_{0,0,-1} \\
& + \left. \left[-\frac{64}{9}(2+3z) + \frac{32}{3}(-1+2z)H_1 \right] \zeta_2 + 32\zeta_3 \right] + \left[\frac{8}{81}(5836+4771z) - \frac{16}{3}(-1+2z)H_1 \right] H_0 \\
& - \frac{4}{27}(-649+128z)H_0^2 + \frac{16}{27}(17+2z)H_0^3 - \frac{4}{9}(-1+2z)H_0^4 - \frac{16}{81}(-283+611z)H_1 \\
& - \frac{8}{3}(z-1)H_1^2 - \frac{16}{9}(3+10z)H_{0,1} + \frac{32}{3}zH_{0,1,1} \\
& + \left[-\frac{8}{9}(-231+205z) + \left[\frac{8}{9}(97+100z) + \frac{16}{3}(1+2z)H_{-1} \right] H_0 - \frac{8}{3}(-3+2z)H_0^2 \right. \\
& - \left. \frac{32}{9}(-2+7z)H_1 + \frac{8}{3}(-1+2z)H_1^2 - \frac{16}{3}(1+2z)H_{0,-1} \right] \zeta_2 + \frac{16}{3}\zeta_2^2 \\
& + \left. \left[-\frac{32}{3}(-2+3z) + \frac{64}{9}(1+z)H_0 - \frac{32}{9}(-1+2z)H_1 \right] \zeta_3 \right\} + a_{qq,Q}^{(3)} \Big\} \tag{A7}
\end{aligned}$$

and the functions $a_{qq,Q}^{(3)}(z)$ and $a_{qg,Q}^{(3)}(z)$ are

$$\begin{aligned}
a_{qq,Q}^{(3),\text{PS}}(z) = C_F T_F^2 N_F \Big\{ & (1-z) \left(\frac{85504}{243} - \frac{25472}{81} H_1 + \frac{320}{27} H_1^2 - \frac{640}{27} H_1^3 - \frac{160}{3} H_1 \zeta_2 \right) \\
& + (1+z) \left[\left(\frac{54592}{243} - 64 \zeta_3 \right) H_0 + \frac{2288}{81} H_0^2 + \frac{176}{81} H_0^3 + \frac{8}{27} H_0^4 - \frac{9152}{81} H_{0,1} \right. \\
& - \frac{704}{27} H_{0,0,1} + \frac{1408}{27} H_{0,1,1} - \frac{128}{9} H_{0,0,0,1} + \frac{256}{9} H_{0,0,1,1} - \frac{512}{9} H_{0,1,1,1} \\
& \left. + \left(\frac{1232}{27} H_0 + \frac{112}{9} H_0^2 - \frac{64}{3} H_{0,1} \right) \zeta_2 + \frac{704}{15} \zeta_2^2 \right] + \frac{16}{81} (617 + 527z) \zeta_2 + \frac{32}{27} (-127 + 83z) \zeta_3 \Big\} \quad (\text{A8})
\end{aligned}$$

$$\begin{aligned}
a_{gg,Q}^{(3)}(z) = C_F T_F^2 N_F \Big\{ & -\frac{16}{81} (-49511 + 50285z) - (1-2z) \left(-\frac{8}{3} H_0^5 + \frac{32}{27} H_0^3 H_1 - \frac{8}{27} H_1^4 - \frac{32}{9} H_0^2 H_{0,1} + \frac{64}{9} H_0 H_{0,0,1} \right. \\
& - \frac{64}{9} H_{0,0,0,1} - \frac{64}{9} H_{0,1,1,1} \Big) + \frac{16}{243} (107935 + 46408z) H_0 + \frac{8}{81} (21955 + 3853z) H_0^2 - \frac{8}{81} (-3997 + 1118z) H_0^3 \\
& + \frac{8}{27} (160 + 61z) H_0^4 + \left(\frac{32}{243} (-1559 + 4066z) + \frac{896}{81} (-5 + 13z) H_0 + \frac{64}{27} (-2 + 7z) H_0^2 \right) H_1 \\
& - \frac{80}{81} (-28 + 71z) H_1^2 + \frac{64}{81} (-2 + 7z) H_1^3 + \left(-\frac{1792}{81} (-5 + 13z) - \frac{128}{27} (-2 + 7z) H_0 \right) H_{0,1} \\
& + \frac{128}{27} (-2 + 7z) [H_{0,0,1} + H_{0,1,1}] + \left(-\frac{8}{81} (-7513 + 6779z) - (1-2z) \left(-16 H_0^3 + \frac{32}{3} H_0 H_1 - \frac{16}{3} H_1^2 - \frac{64}{3} H_{0,1} \right) \right. \\
& \left. + \frac{8}{9} (757 + 178z) H_0 + \frac{16}{3} (34 + 25z) H_0^2 + \frac{64}{9} (-2 + 7z) H_1 \right) \zeta_2 - \frac{608}{45} (1-2z) \zeta_2^2 \\
& + \left(\frac{16}{27} (-2567 + 2590z) + (-1+2z) \left(\frac{224}{3} H_0^2 - \frac{448}{9} H_1 \right) - \frac{448}{9} (13 + 19z) H_0 \right) \zeta_3 \Big\} \\
& + C_A T_F^2 N_F \Big\{ -\frac{4352}{81} (-104 + 109z) - (1-2z) \left(-\frac{32}{9} H_0^3 H_1 - \frac{16}{9} H_0^2 H_1^2 + \frac{32}{27} H_0 H_1^3 + \frac{8}{27} H_1^4 \right. \\
& + \left(\frac{32}{3} H_0^2 - \frac{64}{9} H_1^2 \right) H_{0,1} - \frac{64}{9} H_{0,1}^2 + \left(-\frac{64}{3} H_0 + \frac{128}{9} H_1 \right) H_{0,0,1} + \left(\frac{64}{9} H_0 + \frac{128}{9} H_1 \right) H_{0,1,1} + \frac{64}{3} H_{0,0,0,1} \\
& - \frac{64}{9} H_{0,0,1,1} - \frac{64}{9} H_{0,1,1,1} \Big) + (1+2z) \left(\frac{128}{27} H_{-1} H_0^3 - \frac{128}{9} H_0^2 H_{0,-1} - \frac{256}{9} H_{0,0,0,-1} + \frac{256}{27} H_{0,0,-1} (2 + 3H_0) \right) \\
& + \left(\frac{32}{243} (23638 + 13471z) + \frac{64}{81} (70 + 191z) H_{-1} \right) H_0 + \left(\frac{16}{81} (3423 + 680z) + \frac{128}{27} (2 + 7z) H_{-1} \right) H_0^2 \\
& + \frac{80}{81} (85 + 8z) H_0^3 - \frac{8}{27} (-15 + 26z) H_0^4 + \left(-\frac{32}{243} (-1055 + 3283z) - \frac{32}{9} (-4 + 11z) H_0 - \frac{64}{27} (-2 + 7z) H_0^2 \right) H_1 \\
& + \left(\frac{16}{81} (-59 + 265z) - \frac{64}{27} (-2 + 7z) H_0 \right) H_1^2 - \frac{64}{81} (-2 + 7z) H_1^3 + \left(-\frac{128}{27} (3 + 26z) + \frac{64}{27} (-4 + 17z) H_0 \right. \\
& + \frac{256}{27} (-2 + 7z) H_1 \Big) H_{0,1} + \left(-\frac{64}{81} (70 + 191z) - \frac{128}{27} (4 + 11z) H_0 \right) H_{0,-1} - \frac{64}{27} (-4 + 35z) H_{0,0,1} \\
& - \frac{128}{9} (-2 + 3z) H_{0,1,1} + \left(-\frac{16}{81} (-2359 + 1311z) - (1-2z) \left(-\frac{64}{9} H_0 H_1 + \frac{80}{9} H_1^2 + \frac{64}{9} H_{0,1} \right) \right. \\
& + (1+2z) \left(\frac{32}{3} H_{-1} H_0 - \frac{32}{3} H_{0,-1} \right) + \frac{16}{9} (97 + 120z) H_0 - \frac{16}{3} (-3 + 2z) H_0^2 - \frac{320}{27} (-2 + 7z) H_1 \Big) \zeta_2 \\
& \left. + \frac{1376}{45} \zeta_2^2 + \left(\frac{128}{9} (-23 + 22z) - \frac{896}{9} (1+z) H_0 - \frac{320}{9} (1-2z) H_1 \right) \zeta_3 \right\}. \quad (\text{A9})
\end{aligned}$$

The single mass contributions to the OME $A_{Qg}^{(3)}$ read

$$\begin{aligned}
A_{Qg} = & -4a_s L_M (-1+2z) T_F + a_s^2 \left\{ -\frac{16}{3} L_M^2 (-1+2z) T_F^2 + C_F T_F \left\{ 2(-11+23z) + L_M^2 [6 + (-1+2z)(-4H_0 - 8H_1)] \right. \right. \\
& + L_M [-4(2+3z) + (-1+2z)(-4H_0^2 - 8H_1^2 + 16\zeta_2)] + [-4(-1+16z) - 16(-1+2z)H_1] H_0 - 32(z-1)H_1 \\
& + (-1+2z) \left((5+2z+4H_1)H_0^2 + \frac{2}{3}H_0^3 - \frac{4}{3}H_1^3 + 24H_{0,0,1} - 8H_{0,1,1} - 8\zeta_3 \right) \\
& + (-2(8+25z) + 4(-11+8z+2z^2)H_1 - 16(-1+2z)H_{0,1})H_0 - 4(-16+15z)H_1 - 4(z-1)(3+z)H_1^2 \\
& \left. \left. - 4(-29+12z+4z^2)H_{0,1} + 8(-9+2z+z^2)\zeta_2 \right\} \right. \\
& + C_A T_F \left\{ 4(-51+53z) + L_M^2 [48(z-1) - 16(1+z)H_0 + 8(-1+2z)H_1] + (1+z)(-16H_{0,-1} - 64H_{0,0,1}) \right. \\
& + (-1+2z) \left(\frac{4}{3}H_1^3 + 8H_{0,1,1} \right) + (1+2z) \left(-8H_{-1}^2 H_0 - \frac{4}{3}H_0^3 + 8H_{0,0,-1} - 16H_{0,-1,-1} \right) \\
& + L_M [8(-12+11z) + (1+2z)(16H_{-1}H_0 + 8H_0^2 - 16H_{0,-1}) - 8(1+8z)H_0 + 32(z-1)H_1 + 8(-1+2z)H_1^2 + 16\zeta_2] \\
& + [-4(14+37z) - 4(z-1)(25+z)H_1 - 4(-1+2z)H_1^2 + 32(1+z)H_{0,1} - 8(1+2z)H_{0,-1}]H_0 \\
& - 2(3+z^2)H_0^2 - 4H_1 + 2(z-1)(5+z)H_1^2 + [(1+2z)(4H_0^2 + 16H_{0,-1} - 8\zeta_2) + 16(1+z)H_0]H_{-1} \\
& \left. \left. + 4(-25+28z+2z^2)H_{0,1} - 4(-4+4z+z^2)\zeta_2 + 8(7+8z)\zeta_3 \right\} \right\} \\
& + a_s^3 \left\{ T_F^3 (-1+2z) \left(-\frac{64L_M^3}{9} + \frac{64\zeta_3}{9} \right) + C_A T_F^2 \left\{ -\frac{8}{27}(-7758+7613z) \right. \right. \\
& + L_M^3 \left[\frac{448}{3}(z-1) - \frac{448}{9}(1+z)H_0 + \frac{224}{9}(-1+2z)H_1 \right] + (1+z) \left(-\frac{128}{3}H_{0,0,-1} + \frac{256}{3}H_{0,-1,-1} \right) \\
& + (-1+2z) \left(-\frac{8}{9}H_1^4 + \frac{64}{3}H_{0,1}^2 + \frac{320}{3}H_{0,0,1,1} + \frac{64}{3}H_{0,1,1,1} \right) + (1+2z) \left(-\frac{128}{9}H_{-1}^3 H_0 - \frac{64}{3}H_{0,0,0,-1} + \frac{256}{3}H_{0,0,1,-1} \right. \\
& \left. \left. + \frac{256}{3}H_{0,0,-1,1} - \frac{128}{3}H_{0,0,-1,-1} + \frac{128}{3}H_{0,-1,0,1} + \frac{256}{3}H_{0,-1,-1,-1} \right) \right. \\
& + L_M^2 \left[\frac{8}{3}(-151+136z) + (1+2z)(32H_{-1}H_0 - 32H_{0,-1}) - \frac{16}{9}(59+134z)H_0 + \frac{32}{3}(1+4z)H_0^2 \right. \\
& \left. \left. + \frac{32}{9}(-23+28z)H_1 + 16(-1+2z)H_1^2 + 32\zeta_2 \right] \right. \\
& + L_M \left[\frac{16}{9}(-812+849z) + (-1+2z) \left(\frac{16}{3}H_1^3 + \frac{160}{3}H_{0,1,1} \right) + (1+2z) \left(-\frac{64}{3}H_{-1}^2 H_0 - \frac{128}{3}H_{0,-1,-1} \right) \right. \\
& + \left[-\frac{32}{27}(589+544z) - \frac{32}{3}(z-1)(25+z)H_1 - \frac{16}{3}(-1+2z)H_1^2 + 32(3+2z)H_{0,1} \right] H_0 \\
& + \left[-\frac{8}{9}(151+72z+6z^2) + \frac{16}{3}(-1+2z)H_1 \right] H_0^2 + \frac{16}{9}(-7+2z)H_0^3 \\
& + \left[\frac{304}{27}(-1+5z) + (-1+2z) \left(-\frac{64H_{0,1}}{3} + \frac{32\zeta_2}{3} \right) \right] H_1 + \frac{16}{9}(-11-2z+3z^2)H_1^2 \\
& + \left[(1+2z) \left(\frac{128H_{0,-1}}{3} - \frac{64\zeta_2}{3} \right) - \frac{64}{9}(-4+z)H_0 \right] H_{-1} + \frac{32}{3}(-25+30z+2z^2)H_{0,1} + \frac{64}{9}(-4+z)H_{0,-1} \\
& \left. \left. - \frac{32}{3}(17+14z)H_{0,0,1} - \frac{32}{9}(-8+18z+3z^2)\zeta_2 + \frac{32}{3}(17+16z)\zeta_3 \right] + \left[\frac{8}{81}(7672+14815z) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + (-1 + 2z) \left(\frac{64}{9} H_1^3 - \frac{320}{3} H_{0,1,1} \right) + (1 + 2z) \left(\frac{64}{3} H_{0,0,-1} - \frac{128}{3} H_{0,1,-1} - \frac{128}{3} H_{0,-1,1} + \frac{128}{3} H_{0,-1,-1} \right) \\
& + \left[-\frac{16}{3} (63 - 66z + 4z^2) + 64(-1 + 2z) H_{0,1} \right] H_1 + \frac{64}{3} (z - 1) H_1^2 + \frac{32}{3} (-27 + 8z + z^2) H_{0,1} \\
& + \frac{128}{3} (1 + z) H_{0,-1} - \frac{128}{3} (3 + 2z) H_{0,0,1} + \frac{8}{9} (115 + 244z) \zeta_2 + \frac{64}{9} (19 + 31z) \zeta_3 \Big] H_0 \\
& + \left[-\frac{4}{27} (-577 + 1244z + 72z^2) - \frac{16}{3} (z - 1)(25 + z) H_1 - \frac{16}{3} (-1 + 2z) H_1^2 + \frac{128}{3} (1 + z) H_{0,1} - \frac{32}{3} (1 + 2z) H_{0,-1} \right. \\
& \left. - \frac{8}{3} (3 + 14z) \zeta_2 \right] H_0^2 - \frac{16}{27} (-2 + z)(4 + 3z) H_0^3 - \frac{4}{9} (1 + 6z) H_0^4 + \left[-\frac{16}{81} (-67 + 503z) \right. \\
& \left. + (-1 + 2z) \left(-\frac{320}{3} H_{0,0,1} - \frac{64}{3} H_{0,1,1} + \frac{160}{9} \zeta_3 \right) - \frac{160}{9} (-4 + 5z) \zeta_2 \right] H_1 + \left[\frac{8}{3} (3 - 5z + 4z^2) - \frac{40}{3} (-1 + 2z) \zeta_2 \right] H_1^2 \\
& - \frac{16}{9} (z - 1)(5 + z) H_1^3 + \left[(1 + z) \left(-\frac{64}{3} H_0^2 - \frac{256}{3} H_{0,-1} + \frac{128}{3} \zeta_2 \right) + (1 + 2z) \left(\frac{32}{9} H_0^3 - \frac{256}{3} H_{0,0,1} + \frac{128}{3} H_{0,0,-1} \right. \right. \\
& \left. \left. - \frac{256}{3} H_{0,-1,-1} + 64 \zeta_3 \right) + \left[-128(1 + z) + (1 + 2z) \left(\frac{128}{3} H_{0,1} - \frac{128}{3} H_{0,-1} - \frac{80}{3} \zeta_2 \right) \right] H_0 \right] H_{-1} \\
& + \left[(1 + 2z) \left(\frac{32}{3} H_0^2 + \frac{128}{3} H_{0,-1} - \frac{64}{3} \zeta_2 \right) + \frac{128}{3} (1 + z) H_0 \right] H_{-1}^2 + \frac{16}{9} (189 - 238z + 24z^2) H_{0,1} \\
& + \left[128(1 + z) + \frac{80}{3} (1 + 2z) \zeta_2 \right] H_{0,-1} - \frac{32}{3} (-29 - 8z + z^2) H_{0,0,1} - \frac{32}{3} (-4 + 7z + z^2) H_{0,1,1} + 128 H_{0,0,0,1} \\
& - \frac{4}{9} (-561 + 620z + 48z^2) \zeta_2 - \frac{432}{5} \zeta_2^2 + \frac{32}{3} (14 - 43z + z^2) \zeta_3 \Big\} \\
& + C_A N_F T_F^2 \left\{ -\frac{32}{3} (-53 + 51z) + L_M^3 \left[\frac{64}{3} (z - 1) - \frac{64}{9} (1 + z) H_0 + \frac{32}{9} (-1 + 2z) H_1 \right] \right. \\
& + (1 + z) \left(-\frac{64}{3} H_{0,0,-1} + \frac{128}{3} H_{0,-1,-1} \right) + (-1 + 2z) \left(-\frac{4}{9} H_1^4 + \frac{32}{3} H_{0,1}^2 + \frac{160}{3} H_{0,0,1,1} + \frac{32}{3} H_{0,1,1,1} \right) \\
& + (1 + 2z) \left(-\frac{64}{9} H_{-1}^3 H_0 - \frac{4}{9} H_0^4 - \frac{32}{3} H_{0,0,0,-1} + \frac{128}{3} H_{0,0,1,-1} + \frac{128}{3} H_{0,0,-1,1} - \frac{64}{3} H_{0,0,-1,-1} \right. \\
& \left. + \frac{64}{3} H_{0,-1,0,1} + \frac{128}{3} H_{0,-1,-1,-1} \right) + L_M^2 \left[-\frac{8}{3} (-65 + 68z) + (1 + 2z) \left(\frac{32}{3} H_{-1} H_0 - \frac{32}{3} H_{0,-1} \right) \right. \\
& \left. + \frac{16}{9} (47 + 38z) H_0 + \frac{32}{3} H_0^2 - \frac{32}{9} (1 + 4z) H_1 + \frac{16}{3} (-1 + 2z) H_1^2 + \frac{32}{3} \zeta_2 \right] \\
& + L_M \left[\frac{8}{9} (-778 + 845z) + (-1 + 2z) \left(\frac{32}{9} H_1^3 + \frac{128}{3} H_{0,1,1} \right) + (1 + 2z) \left(-\frac{32}{3} H_{-1}^2 H_0 - \frac{32}{3} H_{0,0,-1} - \frac{64}{3} H_{0,-1,-1} \right) \right. \\
& \left. + \left[-\frac{8}{27} (1319 + 866z) - \frac{16}{3} (z - 1)(25 + z) H_1 + \frac{32}{3} (5 + 2z) H_{0,1} + \frac{32}{3} (1 + 2z) H_{0,-1} \right] H_0 \right. \\
& \left. + \left[-\frac{8}{9} (108 + 68z + 3z^2) + \frac{16}{3} (-1 + 2z) H_1 \right] H_0^2 + \frac{16}{9} (-5 + 2z) H_0^3 \right. \\
& \left. + \left[-\frac{16}{27} (-37 + 8z) + (-1 + 2z) \left(-\frac{64 H_{0,1}}{3} + \frac{32 \zeta_2}{3} \right) \right] H_1 + \frac{8}{9} (-7 - 16z + 3z^2) H_1^2 \right. \\
& \left. + \left[(1 + 2z) \left(-\frac{16}{3} H_0^2 + \frac{64}{3} H_{0,-1} - \frac{32}{3} \zeta_2 \right) - \frac{64}{9} (-1 + 4z) H_0 \right] H_{-1} + \frac{16}{3} (-25 + 30z + 2z^2) H_{0,1} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{9}(-1 + 4z)H_{0,-1} - 32(3 + 2z)H_{0,0,1} - \frac{16}{9}(-4 + 18z + 3z^2)\zeta_2 + \frac{64}{3}(5 + 4z)\zeta_3 \Big] \\
& + \left[\frac{16}{3}(17 + 93z) + (-1 + 2z) \left(\frac{32}{9}H_1^3 - \frac{160}{3}H_{0,1,1} \right) + (1 + 2z) \left(\frac{32}{3}H_{0,0,-1} - \frac{64}{3}H_{0,1,-1} \right. \right. \\
& \left. \left. - \frac{64}{3}H_{0,-1,1} + \frac{64}{3}H_{0,-1,-1} \right) + \left[-\frac{32}{3}(-16 + z)(z - 1) + 32(-1 + 2z)H_{0,1} \right] H_1 \right. \\
& + \frac{32}{3}(z - 1)H_1^2 + \frac{16}{3}(-27 + 8z + z^2)H_{0,1} + \frac{64}{3}(1 + z)H_{0,-1} - \frac{64}{3}(3 + 2z)H_{0,0,1} \\
& \left. - \frac{16}{9}(22 + 7z)\zeta_2 + \frac{64}{9}(7 + 13z)\zeta_3 \right] H_0 + \left[-\frac{8}{3}(2 + 31z + 2z^2) - \frac{8}{3}(z - 1)(25 + z)H_1 \right. \\
& \left. - \frac{8}{3}(-1 + 2z)H_1^2 + \frac{64}{3}(1 + z)H_{0,1} - \frac{16}{3}(1 + 2z)H_{0,-1} - \frac{8}{3}(3 + 2z)\zeta_2 \right] H_0^2 \\
& - \frac{8}{9}(3 + z^2)H_0^3 + \left[\frac{32}{3}(-2 + z) + (-1 + 2z) \left(-\frac{160}{3}H_{0,0,1} - \frac{32}{3}H_{0,1,1} + \frac{160}{9}\zeta_3 \right) \right. \\
& \left. - \frac{16}{9}(-7 + 2z)\zeta_2 \right] H_1 + \left[\frac{8}{3}(1 - 2z + 2z^2) - \frac{16}{3}(-1 + 2z)\zeta_2 \right] H_1^2 - \frac{8}{9}(z - 1)(5 + z)H_1^3 \\
& + \left[(1 + z) \left(-\frac{32}{3}H_0^2 - \frac{128}{3}H_{0,-1} + \frac{64}{3}\zeta_2 \right) + (1 + 2z) \left(\frac{16}{9}H_0^3 - \frac{128}{3}H_{0,0,1} + \frac{64}{3}H_{0,0,-1} \right. \right. \\
& \left. \left. - \frac{128}{3}H_{0,-1,-1} + 32\zeta_3 \right) + \left[-64(1 + z) + (1 + 2z) \left(\frac{64}{3}H_{0,1} - \frac{64}{3}H_{0,-1} - \frac{32}{3}\zeta_2 \right) \right] H_0 \right] H_{-1} \\
& + \left[(1 + 2z) \left(\frac{16}{3}H_0^2 + \frac{64}{3}H_{0,-1} - \frac{32}{3}\zeta_2 \right) + \frac{64}{3}(1 + z)H_0 \right] H_{-1}^2 + \frac{32}{3}(16 - 19z + 2z^2)H_{0,1} \\
& + \left[64(1 + z) + \frac{32}{3}(1 + 2z)\zeta_2 \right] H_{0,-1} - \frac{16}{3}(-29 - 8z + z^2)H_{0,0,1} - \frac{16}{3}(-4 + 8z + z^2)H_{0,1,1} \\
& + 64H_{0,0,0,1} - \frac{4}{3}(89 - 62z + 8z^2)\zeta_2 - \frac{608}{15}\zeta_2^2 + \frac{16}{3}(4 - 32z + z^2)\zeta_3 \Big\} \\
& + C_F T_F^2 \left\{ -\frac{22}{3}(-339 + 374z) + L_M^3 \left[\frac{16}{3}(-19 + 21z) + (-1 + 2z) \left(\frac{16}{3}H_0^2 - \frac{128}{9}H_1 \right) \right. \right. \\
& \left. \left. - \frac{16}{9}(23 + 44z)H_0 \right] + L_M^2 \left[\frac{4}{3}(-287 + 244z) + (-1 + 2z) \left(\frac{32}{3}H_0^3 - 16H_1^2 - \frac{32}{3}H_{0,1} + 32\zeta_2 \right) \right. \right. \\
& \left. \left. + \left[-\frac{8}{9}(433 + 94z) - \frac{64}{3}(-1 + 2z)H_1 \right] H_0 - \frac{320}{3}(1 + z)H_0^2 - \frac{32}{9}(-13 + 8z)H_1 \right] \right. \\
& + L_M \left[\frac{2}{9}(-7895 + 8486z) + (-1 + 2z) \left(\frac{20}{3}H_0^4 - \frac{16}{3}H_1^3 + \frac{256}{3}H_{0,0,1} - 32H_{0,1,1} - \frac{32}{3}\zeta_3 \right) \right. \\
& \left. + \left[-\frac{8}{27}(4285 + 1843z) + \frac{64}{9}(-22 + 20z + 3z^2)H_1 - 64(-1 + 2z)H_{0,1} \right] H_0 \right. \\
& + \left[\frac{4}{9}(-1069 + 128z + 24z^2) + \frac{64}{3}(-1 + 2z)H_1 \right] H_0^2 - \frac{8}{9}(103 + 40z)H_0^3 \\
& \left. - \frac{16}{27}(-260 + 289z)H_1 - \frac{32}{9}(-7 - z + 3z^2)H_1^2 - \frac{64}{9}(-47 + 19z + 6z^2)H_{0,1} \right. \\
& \left. + \frac{64}{9}(-25 - z + 3z^2)\zeta_2 \right] + (-1 + 2z) \left(-\frac{4}{15}H_0^5 + \frac{8}{9}H_1^4 - \frac{128}{3}H_{0,1}^2 - \frac{64}{3}H_{0,0,0,1} \right. \\
& \left. - \frac{320}{3}H_{0,1,1,1} - \frac{976}{15}\zeta_2^2 \right) + \left[\frac{16}{3}(261 + 155z) + (-1 + 2z) \left(\frac{128}{3}H_{0,0,1} + \frac{256}{3}H_{0,1,1} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{32}{3} (2 - 5z + 4z^2) + (-1 + 2z) \left(-\frac{128H_{0,1}}{3} + \frac{64\zeta_2}{3} \right) \right] H_1 - \frac{64}{3} (-2 + 8z + z^2) H_{0,1} \\
& + \frac{4}{9} (769 + 316z) \zeta_2 - \frac{16}{9} (-47 + 4z) \zeta_3 \Big] H_0 + \left[\frac{8}{3} (142 - 11z + 8z^2) \right. \\
& + (-1 + 2z) \left(-\frac{64H_{0,1}}{3} - \frac{16\zeta_3}{3} \right) + \frac{16}{3} (-11 + 8z + 2z^2) H_1 + \frac{4}{3} (65 + 56z) \zeta_2 \Big] H_0^2 \\
& + \left[\frac{8}{9} (64 - 19z + 4z^2) + (-1 + 2z) \left(\frac{32H_1}{9} - 8\zeta_2 \right) \right] H_0^3 + \frac{2}{9} (25 + 16z) H_0^4 + \left[\frac{32}{3} (-5 + 7z) \right. \\
& + (-1 + 2z) \left(\frac{256}{3} H_{0,0,1} + \frac{256}{3} H_{0,1,1} - \frac{640}{9} \zeta_3 \right) + \frac{80}{9} (-5 + 4z) \zeta_2 \Big] H_1 \\
& + \left[-\frac{16}{3} (16 - 19z + 4z^2) + (-1 + 2z) \left(-\frac{64H_{0,1}}{3} + \frac{104\zeta_2}{3} \right) \right] H_1^2 + \frac{32}{9} (z - 1)(3 + z) H_1^3 \\
& + \left[-\frac{32}{3} (8 - 31z + 8z^2) + 48(-1 + 2z) \zeta_2 \right] H_{0,1} + \frac{32}{3} (3 + 24z + 2z^2) H_{0,0,1} \\
& + \frac{64}{3} (-9 + 2z + z^2) H_{0,1,1} + \frac{2}{3} (685 - 924z + 64z^2) \zeta_2 - \frac{16}{3} (-27 + 61z + 4z^2) \zeta_3 \Big\} \\
& + C_F N_F T_F^2 \left\{ -\frac{4}{3} (-827 + 914z) + L_M^3 \left[\frac{8}{3} (-41 + 42z) + (-1 + 2z) \left(\frac{16}{3} H_0^2 - \frac{32}{9} H_1 \right) \right. \right. \\
& - \frac{32}{9} (13 + 19z) H_0 \Big] + L_M^2 \left[-\frac{4}{3} (-285 + 296z) + (-1 + 2z) \left(-\frac{32}{3} H_0^3 - \frac{16}{3} H_1^2 - \frac{32}{3} H_{0,1} \right. \right. \\
& + \frac{32}{3} \zeta_2 \Big) + \frac{8}{9} (425 + 2z) H_0 + 16(7 + 6z) H_0^2 + \frac{32}{9} (1 + 4z) H_1 \Big] + L_M \left[\frac{2}{9} (-8933 + 9110z) \right. \\
& + (-1 + 2z) \left(\frac{20}{3} H_0^4 - \frac{32}{9} H_1^3 - \frac{64}{3} H_{0,1,1} + 192 H_{0,0,0,1} - \frac{384}{5} \zeta_2^2 \right) + \left[-\frac{8}{27} (3349 + 3562z) \right. \\
& + (-1 + 2z) \left(-64 H_{0,0,1} - 128 \zeta_3 \right) + \frac{16}{9} (323 - 322z + 6z^2) H_1 + \frac{32}{3} (31 + 28z) H_{0,1} \Big] H_0 \\
& + \left[\frac{8}{9} (-401 + 34z + 6z^2) + 16(-1 + 2z) H_1 \right] H_0^2 - 16(5 + 3z) H_0^3 - \frac{32}{27} (-58 + 77z) H_1 \\
& - \frac{16}{9} (-5 - 8z + 3z^2) H_1^2 - \frac{16}{9} (277 - 338z + 12z^2) H_{0,1} - \frac{32}{3} (59 + 62z) H_{0,0,1} \\
& + \frac{32}{9} (-23 - 8z + 3z^2) \zeta_2 + 192(3 + 4z) \zeta_3 \Big] + (-1 + 2z) \left(\frac{4}{15} H_0^5 + \frac{4}{9} H_1^4 - \frac{64}{3} H_{0,1}^2 \right. \\
& - \frac{160}{3} H_{0,1,1,1} - 128 H_{0,0,0,0,1} + 128 \zeta_5 \Big) + \left[\frac{8}{3} (-111 + 479z) + (-1 + 2z) \left(\frac{128}{3} H_{0,1,1} \right. \right. \\
& + 128 H_{0,0,0,1} \Big) + \left[\frac{16}{3} (-10 + 7z + 4z^2) + (-1 + 2z) \left(-\frac{64H_{0,1}}{3} + \frac{16\zeta_2}{3} \right) \right] H_1 \\
& - \frac{32}{3} (46 - 34z + z^2) H_{0,1} - \frac{32}{3} (41 + 44z) H_{0,0,1} + \frac{8}{9} (-259 + 86z) \zeta_2 + \frac{32}{9} (109 + 151z) \zeta_3 \Big] H_0 \\
& + \left[\frac{4}{3} (-116 - 437z + 8z^2) + (-1 + 2z) \left(-32 H_{0,0,1} - \frac{208\zeta_3}{3} \right) + \frac{8}{3} (115 - 118z + 2z^2) H_1 \right. \\
& + \frac{16}{3} (29 + 32z) H_{0,1} - \frac{232}{3} (1 + z) \zeta_2 \Big] H_0^2 + \left[\frac{4}{9} (-59 + 50z + 4z^2) \right. \\
& + (-1 + 2z) \left(\frac{16H_1}{9} + 8\zeta_2 \right) \Big] H_0^3 - \frac{4}{9} (8 + 11z) H_0^4 + \left[\frac{16}{3} (-5 + 7z) + (-1 + 2z) \left(\frac{128}{3} H_{0,0,1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{128}{3} H_{0,1,1} - \frac{352}{9} \zeta_3 \Big) + \frac{16}{9} (-7 + 2z) \zeta_2 \Big] H_1 + \left[-\frac{8}{3} (16 - 19z + 4z^2) \right. \\
& + (-1 + 2z) \left(-\frac{32 H_{0,1}}{3} + 16 \zeta_2 \right) \Big] H_1^2 + \frac{16}{9} (z - 1)(3 + z) H_1^3 + \left[-\frac{16}{3} (-4 - 19z + 8z^2) \right. \\
& + \frac{80}{3} (-1 + 2z) \zeta_2 \Big] H_{0,1} + \frac{16}{3} (69 - 18z + 2z^2) H_{0,0,1} + \frac{32}{3} (-9 + 2z + z^2) H_{0,1,1} \\
& + \frac{32}{3} (37 + 34z) H_{0,0,0,1} + \frac{2}{3} (-77 - 194z + 32z^2) \zeta_2 - \frac{64}{15} (29 + 50z) \zeta_2^2 \\
& - \frac{8}{3} (-169 + 250z + 4z^2) \zeta_3 \Big\} + C_F^2 T_F \left\{ 451 - 474z + L_M^3 \left[(-1 + 2z) \left(-6 - \frac{4}{3} H_0^2 - \frac{32}{3} H_1^2 \right. \right. \right. \\
& + \frac{32}{3} \zeta_2 \Big) + \left[-\frac{8}{3} (z - 1) - \frac{32}{3} (-1 + 2z) H_1 \right] H_0 + 16 H_1 \Big] + (1 + 2z) (16 H_{-1}^2 H_0 \zeta_2 \\
& + 16 H_{0,0,-1} \zeta_2 + 32 H_{0,-1,-1} \zeta_2) + L_M^2 \left[-439 + 472z + (-1 + 2z) (-16 H_1^3 + 48 H_{0,0,1} \right. \\
& + 16 H_{0,1,1}) + (1 + 2z) \left(-32 H_{-1}^2 H_0 - \frac{8}{3} H_0^3 - 32 H_{0,0,-1} - 64 H_{0,-1,-1} \right) + \left[-2(101 + 138z) \right. \\
& + (-1 + 2z) (-48 H_1^2 - 32 H_{0,1}) - 16(-7 + 4z) H_1 \\
& + 16(-1 + 6z) \zeta_2 \Big] H_0 + \left[-4(11 + 10z) - 16(-1 + 2z) H_1 \right] H_0^2 + \left[-8(-4 + 9z) \right. \\
& + 64(-1 + 2z) \zeta_2 \Big] H_1 - 4(-19 + 16z) H_1^2 + \left[(1 + 2z) (16 H_0^2 + 64 H_{0,-1} - 32 \zeta_2) \right. \\
& + 96(1 + z) H_0 \Big] H_{-1} - 16(3 + 7z) H_{0,1} - 96(1 + z) H_{0,-1} + 16(2 + 11z) \zeta_2 + 16(1 + 6z) \zeta_3 \Big] \\
& + L_M \left[\frac{1}{2} (3633 - 3782z) + (-1 + 2z) (-8 H_1^4 - 32 H_{0,1}^2 + 32 H_{0,-1}^2 - 80 H_{0,0,1,1} \right. \\
& - 32 H_{0,1,1,1} - 64 H_{0,-1,0,1}) + \left[2(209 + 350z) + (1 + 2z) (-96 H_{0,-1} - 96 H_{0,0,-1}) \right. \\
& + (-1 + 2z) \left(-\frac{32}{3} H_1^3 + 160 H_{0,1,1} - 64 H_{0,1,-1} - 64 H_{0,-1,1} \right) \\
& + \left[-12(-15 + 26z) + (-1 + 2z) (-96 H_{0,1} + 64 H_{0,-1} + 192 \zeta_2) \right] H_1 - 8(-5 + 6z) H_1^2 \\
& - 8(-21 + 22z + 4z^2) H_{0,1} - 16(1 + 14z) H_{0,0,1} - 256z H_{0,-1,-1} + 32(-7 + 12z) \zeta_2 \\
& + 16(-11 + 10z) \zeta_3 \Big] H_0 + \left[2(-21 - 220z + 24z^2) + (-1 + 2z) (-24 H_1^2 + 48 \zeta_2) \right. \\
& + 4(19 - 28z + 4z^2) H_1 + 8(3 + 2z) H_{0,1} + 16(1 + 6z) H_{0,-1} \Big] H_0^2 + \left[\frac{4}{3} (-17 - 23z + 4z^2) \right. \\
& - \frac{32}{3} (-1 + 2z) H_1 \Big] H_0^3 + \frac{1}{3} (-1 - 6z) H_0^4 + \left[-8(-83 + 81z) + (-1 + 2z) (96 H_{0,0,1} \right. \\
& - 128 H_{0,0,-1} + 96 H_{0,1,1} + 224 \zeta_3) - 192(z - 1) H_{0,1} + 16(-21 + 22z) \zeta_2 \Big] H_1 \\
& + \left[-8(-39 + 44z) + (-1 + 2z) (-64 H_{0,1} + 96 \zeta_2) \right] H_1^2 - 16(z - 1)(5 + z) H_1^3
\end{aligned}$$

$$\begin{aligned}
& + \left[(1+z)(64H_0^2 - 128H_{0,-1} + 64\zeta_2) + \left[-32(2+3z+3z^2) + 64(1+2z)H_{0,-1} \right] H_0 \right. \\
& + \left. \frac{16}{3}(1+2z)H_0^3 \right] H_{-1} + \left[64(1+z)H_0 - 32(1+2z)H_0^2 \right] H_{-1}^2 + \left[-4(-23+38z) \right. \\
& - 128z\zeta_2 \left. \right] H_{0,1} + \left[32(2+3z+3z^2) + 64(-1+2z)H_{0,1} - 128z\zeta_2 \right] H_{0,-1} \\
& + 8(-29+24z+4z^2)H_{0,0,1} + 64(1+4z)H_{0,0,-1} - 16(-19+3z+6z^2)H_{0,1,1} \\
& + 128(1+z)H_{0,-1,-1} + 64(-1+8z)H_{0,0,0,1} - 32(-5+2z)H_{0,0,0,-1} - 16(21-29z+6z^2)\zeta_2 \\
& - \frac{8}{5}(-47+126z)\zeta_2^2 + 16(-62+37z+6z^2)\zeta_3 \left. \right] + (-1+2z) \left(\frac{1}{15}H_0^5 + \frac{4}{3}H_1^5 - 112H_{0,0,0,0,1} \right. \\
& + 160H_{0,0,0,1,1} + 96H_{0,0,1,0,1} - 432H_{0,0,1,1,1} - 224H_{0,1,0,1,1} - 144H_{0,1,1,1,1} + 64\zeta_5 \left. \right) \\
& + \left[-2(91+33z) + (-1+2z) \left(\frac{4}{3}H_1^4 - 32H_{0,1}^2 + 48H_{0,0,0,1} - 160H_{0,1,1,1} \right) \right. \\
& + \left. \left[4(-214+215z+24z^2) + (-1+2z) \left(96H_{0,0,1} + 192H_{0,1,1} - \frac{448}{3}\zeta_3 \right) \right. \right. \\
& - 32(-7+4z+z^2)H_{0,1} - 4(-19+16z+8z^2)\zeta_2 \left. \right] H_1 + \left[112(z-1) \right. \\
& + (-1+2z)(-64H_{0,1} + 72\zeta_2) \left. \right] H_1^2 + \frac{16}{3}(z-1)(3+z)H_1^3 + \left[-8(3+2z)(-7+4z) \right. \\
& + 152(-1+2z)\zeta_2 \left. \right] H_{0,1} - 8(-19-18z+4z^2)H_{0,0,1} + 32(-21+2z+2z^2)H_{0,1,1} \\
& - 2(-143+12z)\zeta_2 - \frac{8}{5}(-37+94z)\zeta_2^2 - \frac{8}{3}(-20+95z+24z^2)\zeta_3 \left. \right] H_0 + \left[-128+263z \right. \\
& + 48z^2 + (-1+2z) \left(8H_{0,0,1} + 64H_{0,1,1} - \frac{68}{3}\zeta_3 \right) + \left[2(-27+42z+16z^2) \right. \\
& - 48(-1+2z)H_{0,1} \left. \right] H_1 + 4(-11+8z+2z^2)H_1^2 - 4(1+22z)H_{0,1} - 2(-21-3z \\
& + 8z^2)\zeta_2 \left. \right] H_0^2 + \left[\frac{1}{3}(-39+50z+32z^2) + (-1+2z) \left(\frac{8}{3}H_1^2 - 8H_{0,1} \right) \right. \\
& + \frac{4}{3}(-25+16z+4z^2)H_1 - \frac{2}{3}(-5+2z)\zeta_2 \left. \right] H_0^3 + \left[\frac{1}{3}(-2+5z+4z^2) + \frac{4}{3}(-1+2z)H_1 \right] H_0^4 \\
& + \left[-4(-159+169z) + (-1+2z) \left(-64H_{0,1}^2 - \frac{1056}{5}\zeta_2^2 \right) \right. \\
& + \left[-32(z-1) + 64(-1+2z)\zeta_2 \right] H_{0,1} + 64(-7+4z+z^2)H_{0,0,1} \\
& - 96H_{0,1,1} - 104(-3+4z)\zeta_2 - 32(-11+6z+2z^2)\zeta_3 \left. \right] H_1 \\
& + \left[-2(56-131z+24z^2) + (-1+2z) \left(96H_{0,0,1} + 64H_{0,1,1} - \frac{112}{3}\zeta_3 \right) + 24H_{0,1} \right. \\
& - 4(13-8z+4z^2)\zeta_2 \left. \right] H_1^2 + \left[-\frac{8}{3}(44-47z+12z^2) + (-1+2z) \left(-\frac{64H_{0,1}}{3} + 32\zeta_2 \right) \right] H_1^3
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3}(-51 + 32z + 16z^2)H_1^4 + \left[(1 + 2z)(-8H_0^2\zeta_2 - 32H_{0,-1}\zeta_2 + 16\zeta_2^2) - 48(1 + z)H_0\zeta_2 \right] H_{-1} \\
& + \left[-4(-364 + 97z + 48z^2) + (-1 + 2z)(-32H_{0,0,1} + 96H_{0,1,1} + 160\zeta_3) \right. \\
& + 4(73 + 74z)\zeta_2 \left. \right] H_{0,1} - 8(-13 + 8z + 2z^2)H_{0,1}^2 + \left[4(-85 - 40z + 16z^2) \right. \\
& - 200(-1 + 2z)\zeta_2 \left. \right] H_{0,0,1} + \left[-8(-1 - 85z + 24z^2) - 128(-1 + 2z)\zeta_2 \right] H_{0,1,1} \\
& + 16(-17 + 2z + 6z^2)H_{0,0,0,1} - 32(-30 - z + 2z^2)H_{0,0,1,1} + 8(-83 + 20z + 12z^2)H_{0,1,1,1} \\
& + \left[-384\ln(2)(-1 + 2z) + \frac{1}{2}(-1005 - 506z + 192z^2) + \frac{16}{3}(-7 + 2z)\zeta_3 \right] \zeta_2 \\
& + 48(1 + z)H_{0,-1}\zeta_2 - \frac{8}{5}(-46 + 251z + 42z^2)\zeta_2^2 + 2(141 - 154z + 96z^2)\zeta_3 \left. \right\} \\
& + C_A^2 T_F \left\{ \frac{8}{3}(-6427 + 6405z) + (1 + z)(-128H_{0,1,-1,-1} - 128H_{0,-1,1,-1} - 128H_{0,-1,-1,1}) \right. \\
& + (-7 + 10z)(-32H_{0,0,0,0,-1} + 32H_{0,0,-1,0,1}) + (-1 + 4z)(-64H_{0,0,-1,0,-1} - 512H_{0,0,-1,-1,-1} \\
& - 256H_{0,-1,0,-1,-1}) + (1 + 2z)(-16H_{-1}^4 H_0 - 128H_{0,0,1,1,-1} - 128H_{0,0,1,-1,1} - 192H_{0,0,1,-1,-1} \\
& - 128H_{0,0,-1,1,1} - 192H_{0,0,-1,1,-1} - 192H_{0,0,-1,-1,1} - 128H_{0,1,-1,-1,-1} - 64H_{0,-1,0,1,1} \\
& - 128H_{0,-1,0,1,-1} - 128H_{0,-1,0,-1,1} - 128H_{0,-1,1,-1,-1} - 128H_{0,-1,-1,0,1} - 128H_{0,-1,-1,1,-1} \\
& - 128H_{0,-1,-1,-1,1} - 384H_{0,-1,-1,-1,-1}) + L_M^3 \left[-\frac{1328}{3}(z - 1) + \left[\frac{16}{9}(107 + 59z) \right. \right. \\
& - \frac{32}{3}(-1 + 2z)H_1 \left. \right] H_0 - \frac{32}{3}(-2 + z)H_0^2 - \frac{8}{9}(-155 + 166z)H_1 - \frac{32}{3}(-1 + 2z)H_1^2 \\
& + 32(1 + 2z)H_{0,1} - \frac{128}{3}(1 + z)\zeta_2 \left. \right] + L_M^2 \left[-\frac{16}{3}(-301 + 298z) + (1 + 2z)(32H_{-1}^2 H_0 \right. \\
& + 16H_{0,0,1} + 32H_{0,1,-1} + 32H_{0,-1,1} + 64H_{0,-1,-1}) + \left[\frac{368}{9}(1 + 28z) + 128(z - 1)H_1 \right. \\
& - 48(1 + 2z)H_{0,1} + 16(-5 + 14z)H_{0,-1} + 32(-1 + 4z)\zeta_2 \left. \right] H_0 + \left[-\frac{16}{3}(25 + 33z) \right. \\
& + 8(-1 + 2z)H_1 \left. \right] H_0^2 + \frac{64}{3}(z - 1)H_0^3 + \left[-\frac{40}{9}(-115 + 98z) + 32(-1 + 2z)\zeta_2 \right] H_1 \\
& - \frac{4}{3}(-131 + 142z)H_1^2 - 16(-1 + 2z)H_1^3 + \left[(1 + 2z)(-24H_0^2 - 32H_{0,1} - 64H_{0,-1} \right. \\
& + 64\zeta_2) - \frac{8}{3}(155 + 166z)H_0 \left. \right] H_{-1} + 16(9 + 4z)H_{0,1} + \frac{8}{3}(155 + 166z)H_{0,-1} \\
& - 16(-13 + 22z)H_{0,0,-1} + 64(1 + z)H_{0,1,1} - \frac{8}{3}(161 + 72z)\zeta_2 + 16(-17 + 8z)\zeta_3 \left. \right] \\
& + L_M \left[-\frac{4}{9}(-33406 + 33719z) + (1 + z)(-64H_{0,1,-1} - 64H_{0,-1,1}) + (-1 + 2z)(-8H_1^4 \right.
\end{aligned}$$

$$\begin{aligned}
& + 32H_{0,1,1,1}) + (1 + 2z) \left(-\frac{160}{3}H_{-1}^3H_0 + 368H_{0,0,0,-1} - 32H_{0,0,1,-1} - 32H_{0,0,-1,1} \right. \\
& + 160H_{0,0,-1,-1} + 64H_{0,1,1,-1} + 64H_{0,1,-1,1} + 64H_{0,-1,1,1} + 320H_{0,-1,-1,-1} \Big) \\
& + \left[\frac{8}{27}(13987 + 34141z) + (-1 + 2z) \left(\frac{80}{3}H_1^3 - 96H_{0,1,-1} - 96H_{0,-1,1} \right) \right. \\
& + \left. \left[\frac{4}{3}(z-1)(2363 + 11z) + (-1 + 2z)(64H_{0,1} + 96H_{0,-1} + 160\zeta_2) \right] H_1 + 480(z-1)H_1^2 \right. \\
& - \frac{8}{3}(574 + 181z + 6z^2)H_{0,1} + \frac{8}{3}(-275 + 26z)H_{0,-1} - 64(5 + 7z)H_{0,0,1} - 64(4 + 3z)H_{0,0,-1} \\
& - 64(2 + 11z)H_{0,1,1} - 128(-1 + 5z)H_{0,-1,-1} + 8(-29 + 15z)\zeta_2 - 96(11 + 6z)\zeta_3 \Big] H_0 \\
& + \left[\frac{2}{9}(2817 - 3380z + 141z^2) + \frac{4}{3}(62 - 79z + 6z^2)H_1 + 16(-1 + 2z)H_1^2 + 4(-1 + 46z)H_{0,1} \right. \\
& - 32(-2 + z)H_{0,-1} - 4(11 + 2z)\zeta_2 \Big] H_0^2 + \left[\frac{4}{9}(235 + 254z + 6z^2) - \frac{40}{3}(-1 + 2z)H_1 \right] H_0^3 \\
& - \frac{2}{3}(-19 + 20z)H_0^4 + \left[-\frac{4}{27}(-2629 + 3221z) + (-1 + 2z)(-256H_{0,0,1} - 192H_{0,0,-1} \right. \\
& + 544\zeta_3) - \frac{16}{3}(-61 + 50z)H_{0,1} - \frac{8}{3}(-17 + 28z)\zeta_2 \Big] H_1 + \left[-\frac{2}{9}(-1217 + 844z + 33z^2) \right. \\
& + (-1 + 2z)(-32H_{0,1} + 32\zeta_2) \Big] H_1^2 - \frac{8}{9}(-92 + 94z + 9z^2)H_1^3 + \left[(1 + 2z) \left(\frac{8}{3}H_0^3 \right. \right. \\
& + 32H_{0,0,1} - 160H_{0,0,-1} - 64H_{0,1,1} - 320H_{0,-1,-1} + 192\zeta_3 \Big) + \left[-\frac{16}{9}(1097 + 1030z + 27z^2) \right. \\
& + (1 + 2z)(64H_{0,-1} + 64\zeta_2) \Big] H_0 + \frac{4}{3}(101 + 112z)H_0^2 + 64(1 + z)H_{0,1} - \frac{16}{3}(221 + 232z)H_{0,-1} \\
& + \frac{8}{3}(197 + 208z)\zeta_2 \Big] H_{-1} + \left((1 + 2z)(8H_0^2 + 160H_{0,-1} - 80\zeta_2) + \frac{8}{3}(221 + 232z)H_0 \right) H_{-1}^2 \\
& + \left[-\frac{4}{3}(-2009 + 2064z + 22z^2) - 16(-13 + 18z)\zeta_2 \right] H_{0,1} + 32(-1 + 8z)H_{0,1}^2 \\
& + \left[\frac{16}{9}(1097 + 1030z + 27z^2) + 96(-1 + 2z)H_{0,1} - 16(-1 + 26z)\zeta_2 \right] H_{0,-1} \\
& + 32(-3 + 8z)H_{0,-1}^2 + 8(333 + 132z + 2z^2)H_{0,0,1} - \frac{8}{3}(-449 + 164z)H_{0,0,-1} \\
& - \frac{16}{3}(-46 + 44z + 9z^2)H_{0,1,1} + \frac{16}{3}(221 + 232z)H_{0,-1,-1} + 16(63 + 26z)H_{0,0,0,1} \\
& + 64(1 + 9z)H_{0,0,1,1} - 128(z-1)H_{0,-1,0,1} - \frac{4}{9}(3326 + 864z + 75z^2)\zeta_2 - \frac{4}{5}(879 + 490z)\zeta_2^2 \\
& + \frac{16}{3}(-946 + 100z + 9z^2)\zeta_3 \Big] + \left[-\frac{4}{3}(2971 + 8427z) + (1 + 2z)(-64H_{0,0,1,-1} \right. \\
& - 64H_{0,0,-1,1} - 96H_{0,0,-1,-1} + 64H_{0,1,1,-1} + 64H_{0,1,-1,1} + 64H_{0,-1,1,1} + 64H_{0,-1,1,1,1} \\
& + 64H_{0,-1,1,-1} + 64H_{0,-1,-1,1}) + \left[\frac{8}{3}(1388 - 1387z + 11z^2) + (-1 + 2z) \left(32H_{0,0,1} \right. \right. \\
& + 224H_{0,1,1} - \frac{160}{3}\zeta_3 \Big) - 8(-141 + 150z + 2z^2)H_{0,1} - 16(z-1)(27 + z)\zeta_2 \Big] H_1
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{8}{3}(-200 + 203z) + (-1 + 2z)(-96H_{0,1} - 8\zeta_2) \right] H_1^2 + \frac{8}{9}(92 - 106z + 3z^2)H_1^3 \\
& - \frac{20}{3}(-1 + 2z)H_1^4 + \left[-\frac{4}{3}(-2301 + 412z + 35z^2) + 8(19 + 22z)\zeta_2 \right] H_{0,1} \\
& + 48(3 + 4z)H_{0,1}^2 + \left[-\frac{16}{3}(71 + 131z) - 24(-1 + 10z)\zeta_2 \right] H_{0,-1} - 32(-1 + 4z)H_{0,-1}^2 \\
& - \frac{16}{3}(-330 - 118z + 3z^2)H_{0,0,1} - \frac{8}{3}(155 + 118z)H_{0,0,-1} + \frac{8}{3}(-577 + 782z \\
& + 12z^2)H_{0,1,1} + \frac{16}{3}(83 + 94z)H_{0,1,-1} + \frac{16}{3}(83 + 94z)H_{0,-1,1} - \frac{16}{3}(119 + 106z)H_{0,-1,-1} \\
& - 64(-7 + 4z)H_{0,0,0,1} + 32(-5 + 8z)H_{0,0,0,-1} - 32(7 + 18z)H_{0,0,1,1} - 32(-7 + 2z)H_{0,1,1,1} \\
& + 128(z - 1)H_{0,-1,0,1} - 768zH_{0,-1,-1,-1} - \frac{2}{9}(-667 + 5468z)\zeta_2 - \frac{32}{5}(-31 + 19z)\zeta_2^2 \\
& - \frac{16}{9}(344 + 695z + 18z^2)\zeta_3 \Big] H_0 + \left[\frac{2}{3}(-218 + 1781z + 22z^2) + (1 + 2z)(32H_{0,1,-1} \right. \\
& + 32H_{0,-1,1}) + \left. \frac{2}{3}(z - 1)(1559 + 35z) + (-1 + 2z)(-32H_{0,1} - 4\zeta_2) \right] H_1 \\
& + \frac{2}{3}(-209 + 214z + 6z^2)H_1^2 - \frac{8}{3}(139 + 154z)H_{0,1} + \frac{4}{3}(107 + 94z)H_{0,-1} \\
& + 32(-1 + 4z)H_{0,0,1} - 48(-1 + 2z)H_{0,0,-1} - 144H_{0,1,1} + 16(-1 + 10z)H_{0,-1,-1} \\
& - \frac{2}{3}(-87 - 226z + 12z^2)\zeta_2 + \frac{16}{3}(-31 + 20z)\zeta_3 \Big] H_0^2 + \left[\frac{2}{9}(201 + 324z + 35z^2) \right. \\
& + \frac{8}{3}(z - 1)(25 + z)H_1 + \frac{8}{3}(-1 + 2z)H_1^2 - \frac{64}{3}(1 + z)H_{0,1} + \frac{16}{3}(-1 + 4z)H_{0,-1} \\
& - \frac{16}{3}(-2 + 5z)\zeta_2 \Big] H_0^3 + \frac{1}{9}(71 + 94z + 6z^2)H_0^4 - \frac{8}{15}(z - 1)H_0^5 + \left[\frac{8}{3}(-752 + 739z) \right. \\
& + (-1 + 2z)(-80H_{0,1}^2 + 64H_{0,0,0,1} - 96H_{0,0,1,1} + 128H_{0,1}\zeta_2 - 184\zeta_2^2) \\
& + \frac{8}{3}(-763 + 806z + 12z^2)H_{0,0,1} + \frac{8}{3}(-83 + 94z)H_{0,1,1} + \frac{4}{9}(-2189 + 2122z)\zeta_2 \\
& - \frac{8}{9}(-919 + 938z + 36z^2)\zeta_3 \Big] H_1 + \left[-\frac{2}{3}(167 - 226z + 22z^2) + (-1 + 2z) \left(176H_{0,0,1} \right. \right. \\
& + 16H_{0,1,1} - \frac{208}{3}\zeta_3 \Big) - \frac{4}{3}(101 - 118z + 6z^2)\zeta_2 \Big] H_1^2 + \left[-\frac{2}{9}(415 - 440z + 61z^2) \right. \\
& + \frac{32}{3}(-1 + 2z)\zeta_2 \Big] H_1^3 + \frac{1}{9}(-203 + 190z + 24z^2)H_1^4 + \frac{4}{3}(-1 + 2z)H_1^5 \\
& + \left[(1 + z) \left(128H_{0,1,-1} + 128H_{0,-1,1} - \frac{2576}{3}\zeta_2 \right) + (1 + 2z) \left(-\frac{4}{3}H_0^4 - 32H_{0,0,0,1} \right. \right. \\
& - 160H_{0,0,0,-1} + 128H_{0,0,1,1} + 192H_{0,0,1,-1} + 192H_{0,0,-1,1} + 128H_{0,1,-1,-1} + 128H_{0,-1,0,1} \\
& + 128H_{0,-1,1,-1} + 128H_{0,-1,-1,1} + 384H_{0,-1,-1,-1} - \frac{376}{5}\zeta_2^2 \Big) + \left. \left[1712(1 + z) \right. \right. \\
& + (1 + 2z)(64H_{0,0,1} + 96H_{0,0,-1} - 64H_{0,1,1} - 64H_{0,1,-1} - 64H_{0,-1,1} + 128H_{0,-1,-1} - 64\zeta_3) \\
& - \frac{16}{3}(83 + 94z)H_{0,1} + \frac{16}{3}(95 + 106z)H_{0,-1} + \frac{8}{3}(71 + 82z)\zeta_2 \Big] H_0
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{1144(1+z)}{3} + (1+2z)(-32H_{0,1} - 16H_{0,-1} + 36\zeta_2) \right] H_0^2 - \frac{4}{9}(59+70z)H_0^3 + \left[192(1+z) \right. \\
& + 96(1+2z)\zeta_2 \left. \right] H_{0,1} + \left[\frac{4000(1+z)}{3} - 16(1+2z)\zeta_2 \right] H_{0,-1} + \frac{32}{3}(89+100z)H_{0,0,1} \\
& - \frac{16}{3}(71+82z)H_{0,0,-1} + \frac{32}{3}(107+118z)H_{0,-1,-1} - 8(111+122z)\zeta_3 \left. \right] H_{-1} \\
& + \left[(1+2z) \left(-\frac{8}{3}H_0^3 - 96H_{0,0,1} - 64H_{0,1,-1} - 64H_{0,-1,1} - 192H_{0,-1,-1} + 160\zeta_3 \right) \right. \\
& + \left[-\frac{2000}{3}(1+z) + (1+2z)(32H_{0,1} - 64H_{0,-1} + 40\zeta_2) \right] H_0 - \frac{4}{3}(119+130z)H_0^2 \\
& - 64(1+z)H_{0,1} - \frac{16}{3}(107+118z)H_{0,-1} + \frac{8}{3}(131+142z)\zeta_2 \left. \right] H_{-1}^2 + \left[(1+2z) \left(\frac{64}{3}H_0^2 \right. \right. \\
& + \frac{64}{3}H_{0,1} + 64H_{0,-1} - \frac{160}{3}\zeta_2 \left. \right) + \frac{16}{9}(107+118z)H_0 \left. \right] H_{-1}^3 + \left[-\frac{8}{3}(1442-779z+22z^2) \right. \\
& - 96(5+8z)H_{0,0,1} + 160(-1+2z)H_{0,1,1} - 32(17+2z)\zeta_2 + 160(1+2z)\zeta_3 \left. \right] H_{0,1} \\
& - \frac{8}{3}(-170+178z+3z^2)H_{0,1}^2 + \left[-1712(1+z) + (-1+4z)(64H_{0,0,-1} + 128H_{0,-1,-1}) \right. \\
& - 32(-7+10z)H_{0,0,1} - \frac{8}{3}(119+106z)\zeta_2 + 64(-2+5z)\zeta_3 \left. \right] H_{0,-1} + 64H_{0,-1}^2 \\
& + \left[\frac{4}{3}(-3019-472z+35z^2) - 8(23+14z)\zeta_2 \right] H_{0,0,1} + \left[\frac{16}{3}(-1+119z) \right. \\
& + 24(-5+14z)\zeta_2 \left. \right] H_{0,0,-1} + \left[-\frac{4}{3}(824-1048z+61z^2) - 16(-11+34z)\zeta_2 \right] H_{0,1,1} \\
& + \left[-192(1+z) - 96(1+2z)\zeta_2 \right] H_{0,1,-1} + \left[-192(1+z) - 96(1+2z)\zeta_2 \right] H_{0,-1,1} \\
& + \left[-\frac{4000}{3}(1+z) - 16(-5+14z)\zeta_2 \right] H_{0,-1,-1} + 16(-163+12z+3z^2)H_{0,0,0,1} \\
& + \frac{8}{3}(203+142z)H_{0,0,0,-1} - \frac{8}{3}(-517+926z+12z^2)H_{0,0,1,1} - \frac{32}{3}(89+100z)H_{0,0,1,-1} \\
& - \frac{32}{3}(89+100z)H_{0,0,-1,1} + \frac{16}{3}(71+82z)H_{0,0,-1,-1} + \frac{8}{3}(5+2z+18z^2)H_{0,1,1,1} \\
& - \frac{16}{3}(95+106z)H_{0,-1,0,1} - \frac{32}{3}(107+118z)H_{0,-1,-1,-1} + 32(-27+2z)H_{0,0,0,0,1} \\
& + 32(83+128z)H_{0,0,0,1,1} + 128(-5+8z)H_{0,0,0,1,-1} + 128(-5+8z)H_{0,0,0,-1,1} \\
& - 32(-11+14z)H_{0,0,0,-1,-1} + 32(29+44z)H_{0,0,1,0,1} + 32(-7+10z)H_{0,0,1,0,-1} \\
& - 256(-2+7z)H_{0,0,1,1,1} - 32(-7+20z)H_{0,1,0,1,1} - 32(1+4z)H_{0,1,1,1,1} \\
& + \left[\frac{2}{3}(344+4849z+44z^2) + \frac{8}{3}(139+166z)\zeta_3 \right] \zeta_2 - \frac{8}{15}(-3215-348z+63z^2)\zeta_2^2 \\
& + \frac{4}{3}(1156+2164z+61z^2)\zeta_3 - 544(-2+z)\zeta_5 \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + C_A C_F T_F \left\{ \frac{1}{3} (-1555 + 3178z) + (1+z)(128H_{0,1,-1,-1} + 128H_{0,-1,1,-1} + 128H_{0,-1,-1,1}) \right. \\
& + (1+2z)(16H_{-1}^4 H_0 - 32H_{0,0,0,0,-1} + 64H_{0,0,0,1,-1} + 64H_{0,0,0,-1,1} \\
& - 64H_{0,0,0,-1,-1} + 32H_{0,0,1,0,-1} + 128H_{0,0,1,1,-1} + 128H_{0,0,1,-1,1} + 192H_{0,0,1,-1,-1} \\
& + 32H_{0,0,-1,0,1} + 32H_{0,0,-1,0,-1} + 128H_{0,0,-1,1,1} + 192H_{0,0,-1,1,-1} \\
& + 192H_{0,0,-1,-1,1} + 256H_{0,0,-1,-1,-1} + 128H_{0,1,-1,-1,-1} + 64H_{0,-1,0,1,1} \\
& + 128H_{0,-1,0,1,-1} + 128H_{0,-1,0,-1,1} + 128H_{0,-1,0,-1,-1} + 128H_{0,-1,1,-1,-1} + 128H_{0,-1,-1,0,1} \\
& + 128H_{0,-1,-1,1,-1} + 128H_{0,-1,-1,-1,1} + 384H_{0,-1,-1,-1,-1}) + L_M^3 \left[\frac{2}{3} (-155 + 144z) \right. \\
& + \left[\frac{4}{9} (-131 + 214z) + \frac{64}{3} (-1 + 2z) H_1 \right] H_0 - \frac{32}{3} (1+z) H_0^2 + \frac{8}{9} (-173 + 166z) H_1 \\
& + \frac{64}{3} (-1 + 2z) H_1^2 - 32(1+2z) H_{0,1} + \frac{32}{3} (5+2z) \zeta_2 \left. \right] + L_M^2 \left[\frac{1}{3} (435 - 272z) \right. \\
& + (-1+2z) \left(32H_1^3 - \frac{128}{3} H_{0,1} \right) + (1+2z) \left[(-24H_0 + 8H_0^2 + 32H_{0,1} - 32\zeta_2) H_{-1} + 24H_{0,-1} \right. \\
& + 16H_{0,0,-1} - 48H_{0,1,1} - 32H_{0,1,-1} - 32H_{0,-1,1} \left. \right] + \left[\frac{2}{9} (155 + 3026z) + 320(z-1) H_1 \right. \\
& + 48(-1+2z) H_1^2 - 16(5+2z) H_{0,1} - 16(1+2z) H_{0,-1} - 16(-3+2z) \zeta_2 \left. \right] H_0 + \left[-8(1+6z) \right. \\
& + 8(-1+2z) H_1 \left. \right] H_0^2 + \left[\frac{8}{9} (-611 + 571z) - 96(-1+2z) \zeta_2 \right] H_1 + \frac{8}{3} (-94 + 95z) H_1^2 \\
& + 64(2+z) H_{0,0,1} - \frac{8}{3} (-95 + 88z) \zeta_2 - 16(-1+8z) \zeta_3 \left. \right] + L_M \left[\frac{1}{18} (-57083 + 54650z) \right. \\
& + (1+z) \left[-32(7+3z) H_{0,1,-1} - 32(7+3z) H_{0,-1,1} \right] + (1+2z) \left(\frac{160}{3} H_{-1}^3 H_0 - 96H_{0,0,1,-1} \right. \\
& - 96H_{0,0,-1,1} - 32H_{0,0,-1,-1} - 64H_{0,1,1,-1} - 64H_{0,1,-1,1} - 256H_{0,1,-1,-1} - 64H_{0,-1,1,1} \\
& - 256H_{0,-1,1,-1} - 256H_{0,-1,-1,1} - 320H_{0,-1,-1,-1} \left. \right) + \left[\frac{2}{27} (-15607 + 7346z) \right. \\
& + \left[-\frac{4}{9} (-1979 + 2068z + 66z^2) + (-1+2z)(32H_{0,1} - 96H_{0,-1} - 256\zeta_2) \right] H_1 \\
& - 8(-43 + 39z) H_1^2 - 16(-1+2z) H_1^3 + \frac{8}{3} (-65 + 34z + 18z^2) H_{0,1} + 8(-41 + 44z) H_{0,-1} \\
& + 16(-7 + 30z) H_{0,0,1} - 16(-1 + 22z) H_{0,0,-1} + 48(5 + 6z) H_{0,1,1} + 32(-5 + 2z) H_{0,1,-1} \\
& + 32(-5 + 2z) H_{0,-1,1} - 128(1+z) H_{0,-1,-1} + 24(17 + 28z + 8z^2) \zeta_2 + 16(45 + 32z) \zeta_3 \left. \right] H_0 \\
& + \left[-\frac{2}{9} (68 + 3197z + 390z^2) - 4(-59 + 56z + 6z^2) H_1 - 24(-1 + 2z) H_1^2 \right. \\
& - 16(-5 + 4z) H_{0,1} - 16(3 + 4z) H_{0,-1} - 8(5 + 18z) \zeta_2 \left. \right] H_0^2 + \left[-\frac{2}{3} (-39 + 46z + 36z^2) \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{8}{3}(-1 + 2z)H_1 \Big] H_0^3 + \frac{2}{3}(5 + 8z)H_0^4 + \left[\frac{8}{27}(-8663 + 8914z) + (-1 + 2z)(160H_{0,0,1} \right. \\
& + 192H_{0,0,-1} - 96H_{0,1,1} - 672\zeta_3) + 48(-11 + 10z)H_{0,1} + 8(13 - 18z + 6z^2)\zeta_2 \Big] H_1 \\
& + \left[\frac{4}{9}(-1435 + 1322z + 33z^2) + (-1 + 2z)(96H_{0,1} - 128\zeta_2) \right] H_1^2 + \frac{8}{9}(-182 + 166z \\
& + 27z^2)H_1^3 + 16(-1 + 2z)H_1^4 + \left[(1 + 2z) \left(\frac{104}{3}H_0^3 + 96H_{0,0,1} + 32H_{0,0,-1} + 64H_{0,1,1} \right. \right. \\
& + 256H_{0,1,-1} + 256H_{0,-1,1} + 320H_{0,-1,-1} - 384\zeta_3 \Big) + \left[16(1 + z)(1 + 9z) \right. \\
& + (1 + 2z)(64H_{0,1} + 128H_{0,-1} - 352\zeta_2) \Big] H_0 + 4(41 + 44z + 12z^2)H_0^2 \\
& + 32(1 + z)(7 + 3z)H_{0,1} + 48(11 + 12z + 2z^2)H_{0,-1} - 8(61 + 76z + 18z^2)\zeta_2 \Big] H_{-1} \\
& + \left[(1 + 2z)(-72H_0^2 - 128H_{0,1} - 160H_{0,-1} + 208\zeta_2) - 24(11 + 12z + 2z^2)H_0 \right] H_{-1}^2 \\
& + \left[\frac{4}{9}(-4939 + 1742z + 132z^2) + 64(-4 + 3z)\zeta_2 \right] H_{0,1} - 192zH_{0,1}^2 + \left[-16(1 + z)(1 + 9z) \right. \\
& - 96(-1 + 2z)H_{0,1} + 16(13 + 14z)\zeta_2 \Big] H_{0,-1} - 64zH_{0,-1}^2 - \frac{16}{3}(100 + 76z + 27z^2)H_{0,0,1} \\
& - 8(-41 + 132z + 12z^2)H_{0,0,-1} + \frac{8}{3}(-193 + 62z + 54z^2)H_{0,1,1} \\
& - 48(11 + 12z + 2z^2)H_{0,-1,-1} - 16(1 + 50z)H_{0,0,0,1} + 32(1 + 32z)H_{0,0,0,-1} \\
& - 32(3 + 10z)H_{0,0,1,1} + 64(-3 + 4z)H_{0,-1,0,1} + \frac{8}{9}(1498 + 163z + 129z^2)\zeta_2 + \frac{24}{5}(25 + 38z)\zeta_2^2 \\
& + 16(221 - 94z + 6z^2)\zeta_3 \Big] + \left[-\frac{2}{3}(-627 + 3853z) + (-1 + 2z) \left(\frac{16}{3}H_1^4 + 288H_{0,1,1,1} \right) \right. \\
& + (1 + 2z)(16H_{0,-1}^2 + 16H_{0,0,0,-1} + 64H_{0,0,1,-1} + 64H_{0,0,-1,1} + 96H_{0,0,-1,-1} - 64H_{0,1,1,-1} \\
& - 64H_{0,1,-1,1} - 64H_{0,1,-1,-1} + 32H_{0,-1,0,1} - 64H_{0,-1,1,1} - 64H_{0,-1,1,-1} - 64H_{0,-1,-1,1} \\
& + 192H_{0,-1,-1,-1}) + \left[-\frac{4}{3}(-872 + 851z + 80z^2) + (-1 + 2z) \left(-128H_{0,0,1} - 416H_{0,1,1} \right. \right. \\
& + \frac{608}{3}\zeta_3 \Big) + \frac{8}{3}(-433 + 404z + 18z^2)H_{0,1} + \frac{4}{3}(-181 + 122z + 36z^2)\zeta_2 \Big] H_1 \\
& + \left[8(-16 + 11z) + (-1 + 2z)(160H_{0,1} - 64\zeta_2) \right] H_1^2 - 8z^2H_1^3 + \left[\frac{8}{3}(-43 + 406z + 47z^2) \right. \\
& - 24(-3 + 14z)\zeta_2 \Big] H_{0,1} - 8(19 + 10z)H_{0,1}^2 + \left[-80(z - 1) + 48(1 + 2z)\zeta_2 \right] H_{0,-1} \\
& + \frac{16}{3}(-22 - 118z + 9z^2)H_{0,0,1} + 8(3 + 8z)H_{0,0,-1} - \frac{8}{3}(-791 + 712z + 36z^2)H_{0,1,1} \\
& - 48H_{0,1,-1} - 48H_{0,-1,1} + 16(3 + 4z)H_{0,-1,-1} + 64(5 + 4z)H_{0,0,0,1} + 32(7 + 18z)H_{0,0,1,1} \\
& - \frac{2}{9}(2590 + 5197z)\zeta_2 - \frac{8}{5}(131 + 28z)\zeta_2^2 + \frac{4}{9}(-631 - 58z + 216z^2)\zeta_3 \Big] H_0
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{1}{3} (556 + 1519z - 160z^2) + (1 + 2z)(-32H_{0,1,-1} - 32H_{0,-1,1} - 32H_{0,-1,-1}) \right. \\
& + \left[-\frac{2}{3} (-433 + 340z + 94z^2) + (-1 + 2z)(80H_{0,1} + 4\zeta_2) \right] H_1 - 2(-67 + 56z + 6z^2)H_1^2 \\
& + \frac{4}{3} (-109 + 266z)H_{0,1} + 4H_{0,-1} - 64(2 + 3z)H_{0,0,1} - 8(-23 + 22z)H_{0,1,1} \\
& + \frac{1}{3} (83 + 218z + 72z^2)\zeta_2 + \frac{8}{3} (-5 + 22z)\zeta_3 \Big] H_0^2 + \left[\frac{1}{9} (241 - 2020z - 188z^2) \right. \\
& - \frac{4}{9} (-299 + 286z + 18z^2)H_1 - \frac{16}{3} (-1 + 2z)H_1^2 + \frac{32}{3} (2 + 5z)H_{0,1} - \frac{8}{3} (1 + 2z)H_{0,-1} \\
& + 4(1 + 4z)\zeta_2 \Big] H_0^3 + \left[\frac{1}{18} (83 - 70z - 36z^2) - \frac{4}{3} (-1 + 2z)H_1 \right] H_0^4 + \frac{2}{15} H_0^5 \\
& + \left[-\frac{32}{3} (-215 + 208z) + (-1 + 2z) \left(144H_{0,1}^2 - 64H_{0,0,0,1} + 96H_{0,0,1,1} + \frac{1976}{5}\zeta_2^2 \right) \right. \\
& + \left[128(z - 1) - 192(-1 + 2z)\zeta_2 \right] H_{0,1} - \frac{8}{3} (-857 + 808z + 36z^2)H_{0,0,1} \\
& - \frac{8}{3} (-341 + 376z)H_{0,1,1} - \frac{4}{9} (-1649 + 1348z)\zeta_2 + \frac{8}{9} (-1759 + 1718z + 108z^2)\zeta_3 \Big] H_1 \\
& + \left[\frac{4}{3} (709 - 844z + 40z^2) + (-1 + 2z) \left(-272H_{0,0,1} - 80H_{0,1,1} + \frac{320}{3}\zeta_3 \right) \right. \\
& + \frac{8}{3} (-83 + 94z)H_{0,1} + 8(48 - 55z + 3z^2)\zeta_2 \Big] H_1^2 + \left[\frac{4}{9} (477 - 502z + 97z^2) \right. \\
& + (-1 + 2z) \left(\frac{64H_{0,1}}{3} - \frac{128\zeta_2}{3} \right) \Big] H_1^3 - \frac{2}{9} (-178 + 143z + 36z^2)H_1^4 - \frac{8}{3} (-1 + 2z)H_1^5 \\
& + \left[(1 + z)(-128H_{0,1,-1} - 128H_{0,-1,1}) + (1 + 2z) \left(\frac{4}{3} H_0^4 + 32H_{0,0,0,1} + 160H_{0,0,0,-1} \right. \right. \\
& - 128H_{0,0,1,1} - 192H_{0,0,1,-1} - 192H_{0,0,-1,1} - 128H_{0,1,-1,-1} - 128H_{0,-1,0,1} - 128H_{0,-1,1,-1} \\
& - 128H_{0,-1,-1,1} - 384H_{0,-1,-1,-1} + \frac{296}{5}\zeta_2^2 \Big) + \left[-16(11 + 19z) + (1 + 2z)(-64H_{0,0,1} \right. \\
& - 96H_{0,0,-1} + 64H_{0,1,1} + 64H_{0,1,-1} + 64H_{0,-1,1} - 128H_{0,-1,-1} + 64\zeta_3) + 48H_{0,1} \\
& - 16(7 + 4z)H_{0,-1} + 8(16 + 19z)\zeta_2 \Big] H_0 + \left[-8(11 + 5z) + (1 + 2z)(32H_{0,1} + 16H_{0,-1} \right. \\
& - 28\zeta_2) \Big] H_0^2 - \frac{4}{3} (5 + 8z)H_0^3 + \left[-192(1 + z) - 96(1 + 2z)\zeta_2 \right] H_{0,1} + \left[32(-5 + z) \right. \\
& + 48(1 + 2z)\zeta_2 \Big] H_{0,-1} - 32(5 + 2z)H_{0,0,1} - 16(1 + 4z)H_{0,0,-1} \\
& - 32(11 + 8z)H_{0,-1,-1} + 16(17 + 11z)\zeta_2 + 8(37 + 28z)\zeta_3 \Big] H_{-1} + \left[(1 + 2z) \left(\frac{8}{3} H_0^3 + 96H_{0,0,1} \right. \right. \\
& + 64H_{0,1,-1} + 64H_{0,-1,1} + 192H_{0,-1,-1} - 160\zeta_3 \Big) + \left[-16(-5 + z) + (1 + 2z)(-32H_{0,1} \right. \\
& + 64H_{0,-1} - 56\zeta_2) \Big] H_0 + 12(5 + 4z)H_0^2 + 64(1 + z)H_{0,1} + 16(11 + 8z)H_{0,-1} \\
& - 8(19 + 16z)\zeta_2 \Big] H_{-1}^2 + \left[(1 + 2z) \left(-\frac{64}{3} H_0^2 - \frac{64}{3} H_{0,1} - 64H_{0,-1} + \frac{160}{3}\zeta_2 \right) - \frac{16}{3} (11 + 8z)H_0 \right] H_{-1}^3
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{4}{3}(-272 + 421z + 160z^2) + 16(25 + 46z)H_{0,0,1} \right. \\
& - 32(-5 + 22z)H_{0,1,1} - \frac{4}{3}(-253 + 482z)\zeta_2 - 32(3 + 26z)\zeta_3 \left. \right] H_{0,1} \\
& + \frac{8}{3}(-247 + 296z + 9z^2)H_{0,1}^2 + (16(11 + 19z) + (1 + 2z)(-32H_{0,0,1} - 32H_{0,0,-1} \\
& - 64H_{0,-1,-1} - 16\zeta_3) - 8(12 + 11z)\zeta_2)H_{0,-1} + 32H_{0,-1}^2 + \left[-\frac{4}{3}(93 + 1146z + 94z^2) \right. \\
& + 8(1 + 70z)\zeta_2 \left. \right] H_{0,0,1} + \left[16(1 + 15z) - 40(1 + 2z)\zeta_2 \right] H_{0,0,-1} + \left[\frac{8}{3}(774 - 577z + 97z^2) \right. \\
& + 16(-13 + 62z)\zeta_2 \left. \right] H_{0,1,1} + \left[192(1 + z) + 96(1 + 2z)\zeta_2 \right] H_{0,1,-1} + \left[192(1 + z) \right. \\
& + 96(1 + 2z)\zeta_2 \left. \right] H_{0,-1,1} + \left[-32(-5 + z) - 16(1 + 2z)\zeta_2 \right] H_{0,-1,-1} \\
& - \frac{8}{3}(-157 - 172z + 54z^2)H_{0,0,0,1} - 8(7 + 16z)H_{0,0,0,-1} + 8(-187 + 142z + 12z^2)H_{0,0,1,1} \\
& + 32(5 + 2z)H_{0,0,1,-1} + 32(5 + 2z)H_{0,0,-1,1} + 16(1 + 4z)H_{0,0,-1,-1} \\
& - \frac{8}{3}(79 - 332z + 54z^2)H_{0,1,1,1} + 16(7 + 4z)H_{0,-1,0,1} + 32(11 + 8z)H_{0,-1,-1,-1} \\
& - 16(19 + 2z)H_{0,0,0,0,1} - 48(49 + 86z)H_{0,0,0,1,1} - 16(55 + 106z)H_{0,0,1,0,1} \\
& + 64(-8 + 55z)H_{0,0,1,1,1} + 16(-13 + 98z)H_{0,1,0,1,1} + 16(-7 + 26z)H_{0,1,1,1,1} \\
& + \left[192\ln(2)(-1 + 2z) + \frac{1}{6}(-11243 + 8530z - 640z^2) - \frac{4}{3}(151 + 178z)\zeta_3 \right] \zeta_2 \\
& + \frac{8}{15}(-1831 + 1325z + 189z^2)\zeta_2^2 - \frac{2}{3}(1289 - 2624z + 388z^2)\zeta_3 + 8(7 + 22z)\zeta_5 \left. \right\} + a_{Qg}^{(3)} \left. \right\}. \quad (\text{A10})
\end{aligned}$$

Finally, the single mass contributions to the OME $A_{gg,Q}^{(3)}$ are given by

$$\begin{aligned}
A_{gg,Q}^{(\delta)} = & \left\{ a_s \frac{4L_M}{3} T_F + a_s^2 \left[C_F T_F (-15 + 4L_M) + C_A T_F \frac{2}{9} (5 + 24L_M) + \frac{16L_M^2}{9} T_F^2 \right] \right. \\
& + a_s^3 \left\{ \frac{64L_M^3}{27} T_F^3 + C_A N_F T_F^2 \left[\frac{224}{27} - \frac{44L_M}{3} - \frac{4\zeta_2}{3} \right] + C_F N_F T_F^2 \left[\frac{118}{3} \right. \right. \\
& - \frac{268L_M}{9} + 28\zeta_2 \left. \right] + C_A T_F^2 \left[-\frac{8}{27} - 2L_M + \frac{56L_M^2}{3} - \frac{44}{3}\zeta_2 \right] + C_F T_F^2 \left[\frac{782}{9} \right. \\
& - \frac{584L_M}{9} + \frac{40L_M^2}{3} - \frac{40}{3}\zeta_2 \left. \right] + C_A^2 T_F \left[-\frac{616}{27} + L_M \left(\frac{277}{9} + \frac{16}{3}\zeta_2 + \frac{160}{9}\zeta_3 \right) \right. \\
& + L_M^2 \left(-\frac{2}{3} + \frac{16\zeta_3}{3} \right) + \left(4 - \frac{8\zeta_3}{3} \right) \zeta_2 \left. \right] + C_F^2 T_F [-39 - 2L_M + 16[-5 + 8\ln(2)]\zeta_2 \\
& - 32\zeta_3] + C_A C_F T_F \left[-\frac{1045}{6} + \frac{736L_M}{9} - \frac{22L_M^2}{3} - \frac{4}{3}[-5 + 48\ln(2)]\zeta_2 + 16\zeta_3 \right] \\
& \left. \left. - \frac{64}{27} T_F^3 \zeta_3 + a_{gg,Q}^{(3),\delta} \right\} \right\} \delta(1 - z) \quad (\text{A11})
\end{aligned}$$

$$\begin{aligned}
A_{gg,Q}^{(+)} = & -a_s^2 \frac{8(28 + 30L_M + 9L_M^2)}{27(z-1)} C_A T_F + a_s^3 \left\{ C_A N_F T_F^2 \left[\frac{1}{z-1} \left[-\frac{2176L_M}{81} - \frac{64L_M^3}{27} \right. \right. \right. \\
& - \frac{4}{243} (-2624 - 441\zeta_2 + 81z\zeta_2 - 144\zeta_3) + \frac{32}{9} H_0 \left. \right] + \frac{4}{3} \zeta_2 \left. \right] + C_A T_F^2 \left[\frac{1}{z-1} \left[-\frac{320L_M}{9} \right. \right. \\
& - \frac{640L_M^2}{27} - \frac{224L_M^3}{27} - \frac{4}{81} (-1312 - 717\zeta_2 + 297z\zeta_2 - 168\zeta_3) + \frac{16}{3} H_0 \left. \right] + \frac{44}{3} \zeta_2 \left. \right] \\
& + C_A^2 T_F \left\{ \frac{1}{z-1} \left[\frac{176L_M^3}{27} - \frac{4}{243} (5668 + 207\zeta_2 - 243z\zeta_2 + 324\zeta_2^2 + 396\zeta_3 - 162\zeta_2\zeta_3 \right. \right. \\
& + 162z\zeta_2\zeta_3) + \left(-\frac{88}{9} + \frac{32H_1\zeta_2}{3} \right) H_0 + \frac{8}{3} H_0^2 \zeta_2 \left. \right] + L_M \left[\frac{1}{z-1} \left[-\frac{16}{9} H_0^2 (10 + 3H_1) \right. \right. \\
& + \frac{8}{81} (-155 + 360\zeta_2 - 54\zeta_2^2 + 54z\zeta_2^2 - 1044\zeta_3 + 180z\zeta_3) + \left(-\frac{16}{3} - \frac{640}{9} H_1 \right. \\
& + \frac{32}{3} H_{0,1} - \frac{64}{3} H_{0,-1} \left. \right) H_0 - \frac{64}{3} H_{0,0,1} + \frac{128}{3} H_{0,0,-1} \left. \right] - \frac{16}{3} \zeta_2^2 - \frac{160}{9} \zeta_3 \left. \right] \\
& + L_M^2 \left[\frac{1}{z-1} \left[\frac{8}{9} (-23 + 12\zeta_2 - 6\zeta_3 + 6z\zeta_3) - \frac{16}{3} H_0^2 - \frac{64}{3} H_0 H_1 \right] - \frac{16}{3} \zeta_3 \right. \\
& + \left(-4 + \frac{8\zeta_3}{3} \right) \zeta_2 \left. \right\} + C_A C_F T_F \left\{ \frac{1}{z-1} \left[-\frac{2}{9} (233 + 210\zeta_2 - 288 \ln(2)\zeta_2 \right. \right. \\
& - 30z\zeta_2 + 288 \ln(2)z\zeta_2 + 72\zeta_3 - 72z\zeta_3) + \frac{8}{3} L_M (-5 + 24\zeta_3) - 8L_M^2 \left. \right] \\
& + \frac{4}{3} [-5 + 48 \ln(2)] \zeta_2 - 16\zeta_3 \left. \right\} + C_F^2 T_F \left\{ 16[-5\zeta_2 + 8 \ln(2)\zeta_2 - 2\zeta_3] \right. \\
& - 16[-5 + 8 \ln(2)] \zeta_2 + 32\zeta_3 \left. \right\} + a_{gg,Q}^{(3),(+)} \left. \right\} \quad (A12)
\end{aligned}$$

$$\begin{aligned}
A_{gg,Q}^{\text{reg}} = & a_s^2 \left\{ C_F T_F \left\{ -56(z-1) + L_M^2 [-20(z-1) + 8(1+z)H_0] + L_M [-40(z-1) \right. \right. \\
& - 8(-5+z)H_0 + 8(1+z)H_0^2 \left. \right] + 12(3+z)H_0 - 2(-5+z)H_0^2 + \frac{4}{3}(1+z)H_0^3 \left. \right\} \\
& + C_A T_F \left\{ -\frac{8}{3} L_M^2 (-1+2z) - \frac{2}{27} (-337 + 449z) + L_M \left[-\frac{16}{9} (-14 + 19z) + \frac{16}{3} (1+z)H_0 \right] \right. \\
& + \frac{4}{9} (22+z)H_0 + \frac{4}{3} (1+z)H_0^2 + \frac{4}{3} zH_1 \left. \right\} + a_s^3 \left\{ C_F T_F^2 \left\{ (z-1) \left(\frac{776\zeta_2}{9} + \frac{80H_1\zeta_2}{3} \right) \right. \right. \\
& + L_M^3 \left[-\frac{400}{9} (z-1) + \frac{160}{9} (1+z)H_0 \right] + L_M^2 \left[-\frac{32}{9} (z-1)(26 + 15H_1) + (1+z) \left(\frac{64}{3} H_0^2 \right. \right. \\
& + \frac{64}{3} H_{0,1} - \frac{64}{3} \zeta_2 \left. \right) - \frac{64}{9} (-13 + 8z)H_0 \left. \right] + L_M \left[-\frac{16}{9} (z-1)H_1(52 + 15H_1) + (1+z) \left(\frac{80}{9} H_0^3 \right. \right. \\
& - \frac{128}{3} H_{0,0,1} + \frac{64}{3} H_{0,1,1} \left. \right) + \frac{32}{27} (335 - 335z + 18\zeta_3 + 18z\zeta_3) + \left[\frac{16}{27} (373 + 55z) \right. \\
& + (1+z) \left(\frac{128H_{0,1}}{3} - \frac{128\zeta_2}{3} \right) - \frac{320}{3} (z-1)H_1 \left. \right] H_0 - \frac{8}{9} (-59 + 55z)H_0^2 + \frac{64}{9} (-2 + 7z)H_{0,1} \\
& + \frac{64}{9} (-13 + 8z)\zeta_2 \left. \right] + (1+z) \left(-\frac{4}{3} H_0^4 - \frac{32}{3} H_{0,1}\zeta_2 + \frac{32}{3} \zeta_2^2 \right) + \frac{16}{9} (-216 + 216z - 25\zeta_3)
\end{aligned}$$

$$\begin{aligned}
& + 25z\zeta_3) + \left[-\frac{32}{9}(81 + 9z + 5\zeta_3 + 5z\zeta_3) + \frac{8}{9}(-97 + 41z)\zeta_2 \right] H_0 + \left[-24(3 + z) \right. \\
& - \frac{56}{3}(1 + z)\zeta_2 \left. \right] H_0^2 + \frac{8}{3}(-5 + z)H_0^3 \left\} + C_F N_F T_F^2 \left\{ L_M^3 \left[-\frac{320}{9}(z - 1) + \frac{128}{9}(1 + z)H_0 \right] \right. \right. \\
& + (z - 1) \left[\left(-\frac{3712}{27} - \frac{80\zeta_2}{3} \right) H_1 + \frac{64}{9}H_1^2 - \frac{80}{9}H_1^3 \right] + L_M \left[-\frac{16}{3}(z - 1)H_1(32 + 15H_1) \right. \\
& + (1 + z) \left(\frac{32}{3}H_0^3 - \frac{256}{3}H_{0,0,1} + 64H_{0,1,1} \right) + \frac{64}{9}(89 - 89z + 3\zeta_3 + 3z\zeta_3) + \left[\frac{32}{9}(85 + 19z) \right. \\
& + (1 + z) \left(\frac{256H_{0,1}}{3} - \frac{256\zeta_2}{3} \right) - \frac{640}{3}(z - 1)H_1 \left. \right] H_0 - \frac{16}{9}(-29 + 49z)H_0^2 + \frac{64}{3}(-2 + 3z)H_{0,1} \\
& + \frac{64}{3}(-8 + 7z)\zeta_2 \left. \right] + (1 + z) \left(-\frac{8}{9}H_0^4 + \frac{64}{3}H_{0,1,1,1} - \frac{96}{5}\zeta_2^2 \right) + \frac{64}{81}(-494 + 494z - 27\zeta_3 \\
& + 90z\zeta_3) + \left[-\frac{64}{81}(341 + 191z + 18\zeta_3 + 18z\zeta_3) - \frac{16}{9}(19 + 7z)\zeta_2 \right] H_0 + \left[-16(3 + z) \right. \\
& - \frac{16}{3}(1 + z)\zeta_2 \left. \right] H_0^2 + \frac{16}{9}(-5 + z)H_0^3 + \left[\frac{128}{27}(11 + 14z) + \frac{32}{3}(1 + z)\zeta_2 \right] H_{0,1} \\
& - \frac{64}{9}(2 + 5z)H_{0,1,1} - \frac{80}{27}(29 + 11z)\zeta_2 \left\} + C_A N_F T_F^2 \left\{ -\frac{64}{27}L_M^3(-1 + 2z) \right. \right. \\
& + L_M \left[-\frac{32}{81}(-329 + 397z) + (1 + z) \left(\frac{32}{3}H_0^2 + \frac{256}{9}H_{0,1} - \frac{256}{9}\zeta_2 \right) - \frac{80}{27}(-20 + 13z)H_0 \right. \\
& - \frac{16}{9}(-41 + 39z)H_1 \left. \right] + \frac{32}{243}(-1147 + 1475z - 18\zeta_3 + 36z\zeta_3) + \left[-\frac{16}{81}(295 + 31z) \right. \\
& - \frac{32}{9}(1 + z)\zeta_2 \left. \right] H_0 - \frac{16}{27}(22 + z)H_0^2 - \frac{32}{27}(1 + z)H_0^3 - \frac{256}{27}zH_1 + \frac{16}{9}zH_1^2 \\
& + \frac{32}{27}(-14 + 19z)\zeta_2 \left. \right\} + C_A T_F^2 \left\{ -\frac{224}{27}L_M^3(-1 + 2z) + L_M^2 \left[-\frac{128}{27}(-14 + 19z) \right. \right. \\
& + \frac{128}{9}(1 + z)H_0 \left. \right] + L_M \left[-\frac{32}{9}(-37 + 47z) + (1 + z) \left(\frac{80}{9}H_0^2 + \frac{128}{9}H_{0,1} - \frac{128}{9}\zeta_2 \right) \right. \\
& - \frac{8}{27}(-188 + 61z)H_0 - \frac{8}{9}(-41 + 35z)H_1 \left. \right] + \frac{16}{81}(-1147 + 1475z - 42\zeta_3 + 84z\zeta_3) \\
& + \left[-\frac{8}{27}(295 + 31z) - \frac{112}{9}(1 + z)\zeta_2 \right] H_0 - \frac{8}{9}(22 + z)H_0^2 - \frac{16}{9}(1 + z)H_0^3 - \frac{128}{9}zH_1 + \frac{8}{3}zH_1^2 \\
& + \frac{112}{27}(-14 + 19z)\zeta_2 \left. \right\} + C_A^2 T_F^2 \left\{ \frac{176}{27}L_M^3(-1 + 2z) + L_M \left[(1 + z) \left[\frac{16(2 - 35z + 2z^2)H_{-1}^2 H_0}{3z} \right. \right. \right. \\
& + \left(-\frac{1832}{9} + 96\zeta_2 \right) H_{0,1} + \frac{32(2 - 35z + 2z^2)H_{0,-1,-1}}{3z} - 192H_{0,0,0,1} + 192H_{0,0,0,-1} \left. \right] \left. \right. \\
& - \frac{4}{81}(39029 - 38719z - 10512\zeta_3 + 6696z\zeta_3 + 1080z^2\zeta_3) + \left[-\frac{4}{27}(2870 + 7703z - 1296\zeta_3) \right. \\
& - \frac{64}{9}(-19 + 29z)H_1 - \frac{8}{3}(-37 + 17z)H_{0,1} + \frac{64(1 + 2z^2)H_{0,-1}}{3z} + 96(1 + z)H_{0,0,1} \\
& + 32(-5 + z)H_{0,0,-1} + 192(z - 1)H_{0,-1,-1} - \frac{8(-59 - 2z + 69z^2 + 16z^3)}{3(1 + z)}\zeta_2 \left. \right] H_0
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{4(84 + 169z + 81z^2 + 36z^3)}{9(1+z)} + \frac{4}{3}(-29 + 25z)H_1 - 24(1+z)H_{0,1} - 48(z-1)H_{0,-1} \right. \\
& + 8(5 + 3z)\zeta_2 \left. \right] H_0^2 + \frac{8}{9}(1 - 12z + 4z^2)H_0^3 - \frac{4}{3}H_0^4 + \left[\frac{4}{9}(-1087 + 1065z) \right. \\
& - \frac{16(z-1)(2 + 35z + 2z^2)}{3z} \zeta_2 \left. \right] H_1 + \left[(1+z) \left[-\frac{8(4 - 31z + 4z^2)H_0^2}{3z} \right. \right. \\
& - \frac{32(2 - 35z + 2z^2)H_{0,-1}}{3z} + \frac{16(2 - 9z + 2z^2)\zeta_2}{z} \left. \right] - \frac{16}{9z(1+z)}(18 - 233z - 462z^2 \\
& - 233z^3 + 18z^4)H_0 - \frac{64(1+z)^3H_{0,1}}{3z} \left. \right] H_{-1} + \left[\frac{16}{9z(1+z)}(18 - 233z - 462z^2 \right. \\
& - 233z^3 + 18z^4) + 96(z-1)\zeta_2 \left. \right] H_{0,-1} - 96(z-1)H_{0,-1}^2 + \frac{16}{3}(-23 + 31z + 4z^2)H_{0,0,1} \\
& + \frac{16(1+4z)(-4 - 11z + z^2)}{3z}H_{0,0,-1} + \frac{64(1+z)^3H_{0,1,-1}}{3z} + \frac{64(1+z)^3H_{0,-1,1}}{3z} \\
& - \frac{8(-539 - 960z - 425z^2 + 36z^3)}{9(1+z)}\zeta_2 - \frac{24}{5}(13 + 23z)\zeta_2^2 - \frac{160}{9}\zeta_3 \left. \right] \\
& + L_M^2 \left[\frac{8}{9}(-100 + 77z + 6\zeta_3) + \left[-\frac{32}{3}(-3 + 13z) - \frac{64}{3}(-1 + 2z)H_1 \right] H_0 \right. \\
& + \frac{64(2 + 3z + 2z^2)}{3(1+z)}H_{-1}H_0 + \frac{16(3 + 4z)H_0^2}{3(1+z)} - \frac{64(2 + 3z + 2z^2)}{3(1+z)}H_{0,-1} + \frac{32(1+2z)^2\zeta_2}{3(1+z)} \\
& - \frac{16}{3}\zeta_3 \left. \right] - \frac{8}{243}(-11843 + 14677z - 198\zeta_3 + 396z\zeta_3) + \left[\frac{44}{81}(295 + 31z) + \frac{8}{9}(-7 + 89z)\zeta_2 \right. \\
& + \frac{32}{3}(-1 + 2z)H_1\zeta_2 \left. \right] H_0 + \left[\frac{44(22 + z)}{27} - \frac{8(3 + 4z)\zeta_2}{3(1+z)} \right] H_0^2 + \frac{88}{27}(1+z)H_0^3 + \frac{704}{27}zH_1 \\
& - \frac{44}{9}zH_1^2 + \left[-\frac{4}{27}(-563 + 559z + 18\zeta_3) + \frac{8}{3}\zeta_3 \right] \zeta_2 - \frac{32(2 + 3z + 2z^2)}{3(1+z)}H_{-1}H_0\zeta_2 \\
& + \frac{32(2 + 3z + 2z^2)}{3(1+z)}H_{0,-1}\zeta_2 - \frac{16(1+2z)^2\zeta_2^2}{3(1+z)} \left. \right\} + C_A C_F T_F \left\{ L_M^3 \left[\frac{8}{9}(z-1)(223 + 30H_1) \right. \right. \\
& + (1+z) \left(-\frac{32H_{0,1}}{3} + \frac{32\zeta_2}{3} \right) - \frac{16}{9}(53 + 38z)H_0 + \frac{16}{3}(-2 + z)H_0^2 \left. \right] + (1+z) \left[-\frac{160}{3}H_{-1}^3H_0 \right. \\
& + (224H_0 + 40H_0^2 + 160H_{0,-1} - 80\zeta_2)H_{-1}^2 + 320H_{0,0,1,-1} + 320H_{0,0,-1,1} - 160H_{0,0,-1,-1} \\
& + 160H_{0,-1,0,1} + 320H_{0,-1,-1,-1} - 864H_{0,0,0,1,1} - 288H_{0,0,1,0,1} + 128H_{0,0,1,1,1} + 32H_{0,1,0,1,1} \\
& + 32H_{0,1,1,1,1} \left. \right] + (z-1) \left[\frac{4}{15}H_0^5 + (-80 - 20\zeta_2)H_1^2 - \frac{88}{3}H_1^3 - \frac{10}{3}H_1^4 + 80H_{0,1}^2 \right. \\
& + 128H_{0,0,0,0,-1} - 384H_{0,0,0,1,-1} - 384H_{0,0,0,-1,1} + 192H_{0,0,0,-1,-1} - 128H_{0,0,1,0,-1} \\
& - 128H_{0,0,-1,0,1} + 64H_{0,0,-1,0,-1} + 512H_{0,0,-1,-1,-1} + 256H_{0,-1,0,-1,-1} \left. \right] \\
& + L_M^2 \left[(1+z)(160H_{-1}H_0 - 160H_{0,-1} - 32H_{0,0,1}) + (z-1) \left(-\frac{32}{3}H_0^3 + \frac{440}{3}H_1 + 128H_{0,0,-1} \right) \right. \\
& - \frac{8}{9}(862 - 853z - 144\zeta_3 + 72z\zeta_3) + \left[-\frac{8}{9}(187 + 571z) + (z-1)(-80H_1 - 64H_{0,-1} - 32\zeta_2) \right. \\
& + 32(1+z)H_{0,1} \left. \right] H_0 + \frac{56}{3}(1+4z)H_0^2 - \frac{32}{3}(10+z)H_{0,1} + \frac{16}{3}(35+17z)\zeta_2 \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + L_M \left[(z-1) \left[\left[-\frac{7720}{9} + 160H_{0,1} + \frac{32(2+47z+2z^2)\zeta_2}{3z} \right] H_1 + \frac{4}{3}H_1^2 - \frac{40}{3}H_1^3 + 224H_{0,-1}^2 \right] \right. \\
& + (1+z) \left[-\frac{16(4-79z+4z^2)H_{-1}^2H_0}{3z} - 32H_{0,1}^2 - \frac{32(4-79z+4z^2)H_{0,-1,-1}}{3z} - 32H_{0,0,1,1} \right. \\
& + 32H_{0,1,1,1} \left. \right] + \frac{8}{27}(3845 - 3890z + 990\zeta_3 + 2016z\zeta_3 + 360z^2\zeta_3) + \left[(z-1) \left(-\frac{2960}{3}H_1 \right. \right. \\
& - 120H_1^2 - 448H_{0,-1,-1} \left. \right) - \frac{8}{27}(-4205 + 2956z + 216\zeta_3 + 324z\zeta_3) + \frac{16}{3}(38 + 107z)H_{0,1} \\
& - \frac{16(8+9z+93z^2)H_{0,-1}}{3z} - 192(1+2z)H_{0,0,1} + 64(3+z)H_{0,0,-1} + 96(1+z)H_{0,1,1} \\
& + \frac{64}{3}(-11 + 13z + 4z^2)\zeta_2 \left. \right] H_0 + \left[-\frac{4}{9}(-1261 + 659z + 108z^2) \right. \\
& + (z-1)(-232H_1 + 48H_{0,-1}) + (1+z)(112H_{0,1} - 96\zeta_2) \left. \right] H_0^2 - \frac{16}{9}(-79 - 31z + 4z^2)H_0^3 \\
& - \frac{4}{3}(-11 + 7z)H_0^4 + \left[(1+z) \left[\frac{16(18-37z+18z^2)H_0}{3z} + \frac{8(8-11z+8z^2)H_0^2}{3z} \right. \right. \\
& + \frac{32(4-79z+4z^2)H_{0,-1}}{3z} - \frac{16(4-21z+4z^2)\zeta_2}{z} \left. \right] + \frac{128(1+z)^3H_{0,1}}{3z} \left. \right] H_{-1} \\
& + \left[\frac{272}{9}(-19 + 44z) - 256(1+z)\zeta_2 \right] H_{0,1} + \left[-\frac{16(1+z)(18-37z+18z^2)}{3z} \right. \\
& - 224(z-1)\zeta_2 \left. \right] H_{0,-1} - \frac{16}{3}(131 + 179z + 8z^2)H_{0,0,1} - \frac{16(-8-21z-189z^2+8z^3)}{3z}H_{0,0,-1} \\
& - \frac{16}{3}(-19 + 29z)H_{0,1,1} - \frac{128(1+z)^3H_{0,1,-1}}{3z} - \frac{128(1+z)^3H_{0,-1,1}}{3z} + 96(1+7z)H_{0,0,0,1} \\
& - 96(3+5z)H_{0,0,0,-1} + \frac{16}{9}(-289 - 193z + 54z^2)\zeta_2 + \frac{16}{5}(89 + 73z)\zeta_2^2 \left. \right] \\
& - \frac{2}{9}(-34613 + 34846z + 3860\zeta_3 + 6472z\zeta_3 + 2520\zeta_5 - 1080z\zeta_5) + \left[(1+z)(-48H_{0,1}^2 - 160H_{0,1,-1} \right. \\
& - 160H_{0,-1,1} + 128H_{0,0,1,1} - 64H_{0,1,1,1}) + (z-1) \left[(1344 + 240H_{0,1} + 40\zeta_2)H_1 + 112H_1^2 \right. \\
& + \frac{80}{3}H_1^3 + 32H_{0,-1}^2 - 96H_{0,0,0,-1} - 64H_{0,-1,0,1} + 128H_{0,-1,-1,-1} \left. \right] + \frac{8}{9}(2676 + 5541z \\
& + 178\zeta_3 + 598z\zeta_3) + [224(-6+z) - 16(1+z)\zeta_2]H_{0,1} + [32(1+8z) + 48(z-1)\zeta_2]H_{0,-1} \\
& - 32(29+11z)H_{0,0,1} + 16(13+5z)H_{0,0,-1} - 16(-21+25z)H_{0,1,1} + 32(9+5z)H_{0,-1,-1} \\
& + 64(-5+2z)H_{0,0,0,1} + \frac{4}{9}(127 + 745z)\zeta_2 + \frac{16}{5}(-38 + 5z)\zeta_2^2 \left. \right] H_0 + \left[(z-1)(-424H_1 \right. \\
& - 20H_1^2 + 32H_{0,0,-1} - 32H_{0,-1,-1}) - \frac{4}{3}(-192 + 441z - 56\zeta_3 + 52z\zeta_3) + 8(29+19z)H_{0,1} \\
& - 8(9+5z)H_{0,-1} - 32(-2+z)H_{0,0,1} + 16(1+z)H_{0,1,1} - \frac{8}{3}(-2+19z)\zeta_2 \left. \right] H_0^2 \\
& + \left[-\frac{4}{9}(-28+5z) + (z-1) \left(-\frac{16}{3}H_{0,-1} + 8\zeta_2 \right) \right] H_0^3 - \frac{2}{9}(1+4z)H_0^4 \\
& + \left[(z-1) \left(-400H_{0,0,1} - 80H_{0,1,1} - \frac{460}{3}\zeta_2 \right) + \frac{8}{3}(81 - 75z - 50\zeta_3 + 50z\zeta_3) \right] H_1
\end{aligned}$$

$$\begin{aligned}
& + \left[(1+z) \left[(-672 + 160H_{0,1} - 160H_{0,-1} - 120\zeta_2)H_0 - 112H_0^2 + \frac{40}{3}H_0^3 - 448H_{0,-1} \right. \right. \\
& - 320H_{0,0,1} + 160H_{0,0,-1} - 320H_{0,-1,-1} + 224\zeta_2 \left. \right] + 240(\zeta_3 + z\zeta_3) \left. \right] H_{-1} + \left[-\frac{40}{3}(-96 + 93z \right. \\
& + 4\zeta_3 + 4z\zeta_3) + 160(1+z)H_{0,0,1} + \frac{8}{3}(32 + 5z)\zeta_2 \left. \right] H_{0,1} + [-96(-7 - 7z - \zeta_3 + z\zeta_3) \\
& + (z-1)(128H_{0,0,1} - 64H_{0,0,-1} - 128H_{0,-1,-1}) + 8(23 + 15z)\zeta_2]H_{0,-1} - 64H_{0,-1}^2 \\
& + [16(115 + 29z) + 16(1+z)\zeta_2]H_{0,0,1} + [-32(-5 + 9z) - 96(z-1)\zeta_2]H_{0,0,-1} \\
& + [-8(-30 + 31z) + 16(1+z)\zeta_2]H_{0,1,1} + [448(1+z) + 64(z-1)\zeta_2]H_{0,-1,-1} \\
& + 48(29 + 3z)H_{0,0,0,1} - 16(17 + 5z)H_{0,0,0,-1} + 16(-17 + 28z)H_{0,0,1,1} + 96zH_{0,1,1,1} \\
& - 128(-4 + z)H_{0,0,0,0,1} - \frac{4}{9}(-487 + 2179z + 258\zeta_3 + 222z\zeta_3)\zeta_2 - \frac{8}{15}(1402 + 229z)\zeta_2^2 \\
& - 16\zeta_3 \left. \right\} + C_F^2 T_F \left\{ L_M^3 \left[-\frac{4}{3}(z-1)(13 + 20H_1) + (1+z) \left(\frac{8}{3}H_0^2 + \frac{32}{3}H_{0,1} - \frac{32}{3}\zeta_2 \right) \right. \right. \\
& - \frac{16}{3}(-2 + 3z)H_0 \left. \right] + (z-1) \left[(280 - 80H_{0,1} + 100\zeta_2)H_1^2 + 24H_1^3 + \frac{10}{3}H_1^4 \right] \\
& + (1+z) \left(-\frac{2}{15}H_0^5 - 32H_{0,0,1,1} + 192H_{0,0,0,0,1} + 768H_{0,0,0,1,1} + 320H_{0,0,1,0,1} - 416H_{0,0,1,1,1} \right. \\
& - 192H_{0,1,0,1,1} - 32H_{0,1,1,1,1} \left. \right) + L_M^2 \left[(1+z) \left(\frac{16}{3}H_0^3 - 32H_{0,0,1} \right) + 8(25 - 25z + 4\zeta_3 + 4z\zeta_3) \right. \\
& + [-8(-19 + 7z) + (1+z)(32H_{0,1} - 32\zeta_2) - 80(z-1)H_1]H_0 - 16(-3 + 2z)H_0^2 \\
& - 120(z-1)H_1 + 16(1 + 5z)H_{0,1} - 96\zeta_2 \left. \right] + L_M \left[(z-1) \left[(-200 - 160H_{0,1} - 256\zeta_2)H_1 \right. \right. \\
& - 20H_1^2 + \frac{40}{3}H_1^3 - 128H_{0,-1}^2 \left. \right] + (1+z) \left[-256H_{-1}^2H_0 + 2H_0^4 + \left[-\frac{32(2 + 3z + 2z^2)H_0}{z} \right. \right. \\
& + 128H_0^2 + 512H_{0,-1} - 256\zeta_2 \left. \right] H_{-1} + 32H_{0,1}^2 - 512H_{0,-1,-1} - 416H_{0,0,0,1} + 384H_{0,0,0,-1} \\
& - 32H_{0,1,1,1} \left. \right] + 8(-31 + 31z - 70\zeta_3 + 54z\zeta_3) + [(1+z)(288H_{0,0,1} - 64H_{0,1,1}) \\
& + (z-1)(168H_1 + 80H_1^2 + 256H_{0,-1} + 256H_{0,-1,-1}) - 4(-13 + 133z - 16\zeta_3 + 48z\zeta_3) \\
& - 64(-3 + 4z)H_{0,1} - 256H_{0,0,-1} - 416\zeta_2]H_0 + [2(69 + 73z + 16z^2) \\
& + (z-1)(128H_1 - 64H_{0,-1}) + (1+z)(-64H_{0,1} - 32\zeta_2)]H_0^2 - \frac{8}{3}(-11 + 5z)H_0^3 \\
& + [-8(-67 + 41z) + 128(1+z)\zeta_2]H_{0,1} + \left[\frac{32(1+z)(2 + 3z + 2z^2)}{z} + 128(z-1)\zeta_2 \right] H_{0,-1} \\
& + 32(9 + 8z)H_{0,0,1} - 256(-1 + 3z)H_{0,0,-1} + 16(-7 + 13z)H_{0,1,1} - 16(33 - 10z + 4z^2)\zeta_2 \\
& - \frac{32}{5}(13 + 23z)\zeta_2^2 \left. \right] - \frac{4}{3}(-660 + 660z - 323\zeta_3 + 17z\zeta_3 + 120\zeta_5 + 120z\zeta_5) \\
& + \left[(z-1)[(168 - 160H_{0,1} + 80\zeta_2)H_1 + 320H_{0,1,1}] + (1+z) \left(32H_{0,1}^2 - 160H_{0,0,0,1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -128H_{0,0,1,1} + \frac{448}{5}\zeta_2^2 \Big) - \frac{8}{3}(-51 - 282z - 32\zeta_3 + 6z\zeta_3) + [-16(-17 + 27z) \\
& - 32(1 + z)\zeta_2]H_{0,1} + 16(9 + 19z)H_{0,0,1} + 2(83 + 255z)\zeta_2 \Big] H_0 + \left[\frac{2}{3}(-51 - 201z + 44\zeta_3 \right. \\
& + 44z\zeta_3) + 140(z - 1)H_1 - 8(5 + 13z)H_{0,1} + 48(1 + z)H_{0,0,1} + 4(-7 + 3z)\zeta_2 \Big] H_0^2 \\
& + \left[\frac{2}{3}(1 + 83z) + (1 + z) \left(-\frac{16H_{0,1}}{3} - 4\zeta_2 \right) + \frac{40}{3}(z - 1)H_1 \right] H_0^3 + \frac{2}{3}(-3 + z)H_0^4 \\
& + \left[(z - 1)(-32H_{0,1} + 320H_{0,0,1} + 320H_{0,1,1} + 148\zeta_2) - \frac{8}{3}(27 - 27z - 110\zeta_3 + 110z\zeta_3) \right] H_1 \\
& + \left[(1 + z)(-128H_{0,0,1} + 64H_{0,1,1}) + \frac{8}{3}(15 + 174z + 44\zeta_3 + 44z\zeta_3) + 8(-5 + 13z)\zeta_2 \right] H_{0,1} \\
& - 32(-3 + 5z)H_{0,1}^2 + [8(-45 + 43z) - 32(1 + z)\zeta_2]H_{0,0,1} \\
& + [8(-58 + 5z) - 80(1 + z)\zeta_2]H_{0,1,1} - 16(7 + 23z)H_{0,0,0,1} - 16(-23 + 27z)H_{0,1,1,1} \\
& + \frac{2}{3}(633 - 1389z + 40\zeta_3 + 40z\zeta_3)\zeta_2 - \frac{8}{5}(-163 + 23z)\zeta_2^2 + 32\zeta_3 \Big\} + a_{gg,Q}^{(3),\text{reg}} \Big\}. \tag{A13}
\end{aligned}$$

APPENDIX B: THE z -SPACE EXPRESSIONS FOR THE ASYMPTOTIC MASSIVE WILSON COEFFICIENTS

Here we list the z -space contributions to the massive Wilson coefficient $L_q^{\text{PS}}, L_g^{\text{S}}$ and H_g^{S} . For $L_q^{\text{PS}}(z)$ we obtain

$$\begin{aligned}
L_q^{\text{PS}}(z) = & a_s^3 C_F N_F T_F^2 \left\{ L_M^2 \left[\frac{32}{3}(z - 1)(12 + 5H_1) + (1 + z) \left(-\frac{32}{3}H_0^2 - \frac{64}{3}H_{0,1} + \frac{64}{3}\zeta_2 \right) \right. \right. \\
& + \frac{32}{3}(-7 + z)H_0 \Big] + (z - 1) \left[\frac{3296}{9} - \frac{32}{27}H_1(-125 + 12H_0) + \frac{16}{9}H_1^2(28 + 15H_0) + \frac{80}{3}H_1^3 \right] \\
& + L_Q \left[(1 + z) \left(\frac{64}{3}H_{0,1,1} - \frac{64\zeta_3}{3} \right) + L_M^2 \left[-\frac{160}{3}(z - 1) + \frac{64}{3}(1 + z)H_0 \right] \right. \\
& + L_M \left[\frac{64}{9}(z - 1)(-4 + 15H_1) + (1 + z) \left(-\frac{128H_{0,1}}{3} + \frac{128\zeta_2}{3} \right) + \frac{128}{9}(2 + 5z)H_0 \right] \\
& + \frac{128}{27}(11 + 14z)H_0 + (z - 1) \left(-\frac{3712}{27} + \frac{128}{9}H_1 - \frac{80}{3}H_1^2 \right) - \frac{64}{9}(2 + 5z)H_{0,1} \\
& + \frac{64}{9}(2 + 5z)\zeta_2 \Big] + L_M \left[(z - 1) \left[-\frac{64}{3} - \frac{64}{9}H_1(32 + 15H_0) - \frac{320}{3}H_1^2 \right] \right. \\
& + (1 + z) \left[\frac{128}{9}H_{0,1}(1 + 3H_0) - \frac{128}{3}H_{0,0,1} + \frac{256}{3}H_{0,1,1} - \frac{128}{3}\zeta_3 \right] - \frac{512}{9}(1 + 2z)H_0 \\
& - \frac{64}{9}(2 + 5z)H_0^2 + \left[\frac{64}{9}(-17 + 13z) - \frac{128}{3}(1 + z)H_0 \right] \zeta_2 \Big] + (1 + z) \left(\frac{64}{3}H_{0,0,1,1} - 64H_{0,1,1,1} + \frac{32}{15}\zeta_2^2 \right) \\
& + \frac{128}{27}(-40 + z)H_0 - \frac{64}{27}(11 + 14z)H_0^2 + \left(\frac{128}{27}(-8 + z) + \frac{64}{9}(2 + 5z)H_0 \right) H_{0,1} - \frac{64}{9}(2 + 5z)H_{0,0,1} \\
& + \left[\frac{64}{9}(1 + 4z) - \frac{64}{3}(1 + z)H_0 \right] H_{0,1,1} + \left[\frac{128}{27}(5 + 2z) - \frac{64}{9}(2 + 5z)H_0 - \frac{160}{3}(z - 1)H_1 + \frac{64}{3}(1 + z)H_{0,1} \right] \zeta_2 \\
& + \left[-\frac{32}{9}(-17 + 13z) + \frac{64}{3}(1 + z)H_0 \right] \zeta_3 + A_{qq,Q}^{\text{PS}(3)}(z) + N_F \hat{C}_q^{\text{PS}(3)}(L_Q, N_F, z) \Big\}. \tag{B1}
\end{aligned}$$

For $L_g^S(z)$ we obtain

$$\begin{aligned}
L_g^S(z) = & a_s^2 N_F T_F^2 \left\{ \frac{16}{3} L_M L_Q (2z-1) + L_M \left[-\frac{16}{3} (-3+4z) + (2z-1) \left(-\frac{16}{3} H_0 - \frac{16}{3} H_1 \right) \right] \right\} \\
& + a_s^3 \left\{ N_F T_F^3 \left\{ \frac{64}{9} L_M^2 L_Q (2z-1) + L_M^2 \left[-\frac{64}{9} (-3+4z) + (2z-1) \left(-\frac{64}{9} H_0 - \frac{64}{9} H_1 \right) \right] \right\} \right. \\
& + C_A N_F T_F^2 \left\{ \frac{8}{27} (-3943 + 3928z) + L_M L_Q^2 \left[-64(z-1) + \frac{64}{3} (1+z) H_0 - \frac{32}{3} (2z-1) H_1 \right] \right. \\
& + L_M^2 \left[\frac{512}{3} (z-1) + (1+z) \left(-\frac{32}{3} H_0^2 - \frac{64}{3} H_{0,1} \right) + \left[\frac{64}{3} (-4+z) + \frac{32}{3} (2z-1) H_1 \right] H_0 \right. \\
& + \frac{32}{3} (-9+10z) H_1 + \frac{32}{3} (2z-1) H_1^2 + 32\zeta_2 \left. \right] + L_Q \left[-\frac{40}{9} (-108+107z) \right. \\
& + L_M^2 \left[-64(z-1) + \frac{64}{3} (1+z) H_0 - \frac{32}{3} (2z-1) H_1 \right] + L_M \left[-\frac{32}{3} (-22+19z) \right. \\
& + \left. \left. \left[\frac{32}{9} (5+98z) + \frac{64}{3} (2z-1) H_1 \right] H_0 - \frac{64}{3} (1+2z) H_{-1} H_0 - \frac{64}{3} (1+3z) H_0^2 \right. \right. \\
& + \frac{128}{9} (-8+7z) H_1 + \frac{32}{3} (2z-1) H_1^2 - \frac{128}{3} (1+z) H_{0,1} + \frac{64}{3} (1+2z) H_{0,-1} + \frac{128}{3} \zeta_2 \left. \right] \\
& + \frac{8}{27} (785+644z) H_0 + \frac{16}{9} (17+2z) H_0^2 - \frac{16}{9} (2z-1) H_0^3 - \frac{16}{27} (-47+103z) H_1 - \frac{32}{3} z H_{0,1} \\
& + \frac{32}{3} z \zeta_2 \left. \right\} + (2z-1) \left(\frac{4}{9} H_0^4 + \frac{32}{3} H_{0,0,1} - \frac{64}{15} \zeta_2^2 \right) + L_M \left[\frac{16}{9} (-325+301z) \right. \\
& + \left. \left. \left[\frac{32}{9} (-58-42z+3z^2) - \frac{128}{9} (-8+7z) H_1 - 16(2z-1) H_1^2 - \frac{32}{3} (-3-6z+2z^2) H_{0,1} \right. \right. \right. \\
& + \frac{64}{3} (-1-2z+z^2) H_{0,-1} \left. \right] H_0 + \frac{32}{3} (1+2z+2z^2) H_{-1}^2 H_0 + \left[\frac{8}{9} (23-124z+44z^2) \right. \\
& + \frac{16}{3} (3-6z+2z^2) H_1 \left. \right] H_0^2 + \frac{16}{9} (5+14z) H_0^3 + \left[\frac{16}{9} (-180+161z+6z^2) \right. \\
& + \frac{64}{3} (2z-1) H_{0,1} \left. \right] H_1 - \frac{32}{9} (-8+z) H_1^2 - \frac{16}{9} (2z-1) H_1^3 \\
& + \left[-\frac{64(2+6z+12z^2+11z^3)}{9z} H_0 - \frac{32}{3} (-1-2z+z^2) H_0^2 + \frac{64}{3} (1+2z) H_{0,1} \right. \\
& - \frac{64}{3} (1+2z+2z^2) H_{0,-1} \left. \right] H_{-1} - \frac{32}{9} (41+62z) H_{0,1} + \frac{64(2+6z+12z^2+11z^3)}{9z} H_{0,-1} \\
& + \frac{32}{3} (-3+2z+2z^2) H_{0,0,1} - \frac{64}{3} (-1-2z+z^2) H_{0,0,-1} - \frac{32}{3} (-7+2z) H_{0,1,1} \\
& - \frac{64}{3} (1+2z) H_{0,1,-1} - \frac{64}{3} (1+2z) H_{0,-1,1} + \frac{64}{3} (1+2z+2z^2) H_{0,-1,-1} \\
& + \left[-\frac{32}{9} (3-90z+22z^2) - \frac{64}{3} (3+4z) H_0 - \frac{64}{3} (z-1)^2 H_1 + \frac{32}{3} (-1-2z+2z^2) H_{-1} \right] \zeta_2 \\
& - \frac{64}{3} (2+z+2z^2) \zeta_3 \left. \right] + \left[\frac{8}{27} (-2405+413z) + \frac{16}{27} (-47+103z) H_1 + \frac{32}{3} z H_{0,1} \right] H_0 \\
& - \frac{4}{27} (989+764z) H_0^2 + \frac{160}{27} (-2+z) H_0^3 + \frac{8}{27} (-1884+1981z) H_1 + \frac{16}{27} (-47+103z) H_1^2 \\
& - \frac{8}{27} (785+572z) H_{0,1} - \frac{32}{9} (17+5z) H_{0,0,1} + \frac{64}{3} z H_{0,1,1} + \left[\frac{8}{9} (293+122z) - \frac{32}{9} (-17+z) H_0 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16}{3}(2z-1)H_0^2\Big]\zeta_2 + \left[-\frac{32}{9}(-17+z) - \frac{32}{3}(2z-1)H_0\right]\zeta_3\Big\} + C_F N_F T_F^2 \Big\{ 4(-889+904z) \\
& + L_M L_Q^2 \left[8(-41+42z) + (2z-1) \left(16H_0^2 - \frac{32}{3}H_1 \right) - \frac{32}{3}(13+19z)H_0 \right] \\
& + L_Q \left[-20(-67+70z) + L_M^2 \left[-336(z-1) + 48(3+4z)H_0 - 16(2z-1)H_0^2 \right] \right. \\
& + L_M \left[-\frac{16}{3}(-472+473z) + (2z-1) \left(-32H_0^3 + \frac{64}{3}H_1^2 - 64H_{0,0,1} + 64\zeta_3 \right) \right. \\
& + \left. \left[\frac{32}{3}(156+25z) + \frac{128}{3}(2z-1)H_1 \right] H_0 + \frac{16}{3}(73+52z)H_0^2 - \frac{16}{3}(-109+106z)H_1 \right. \\
& + \left. \frac{64}{3}(14+17z)H_{0,1} + \left[-64(4+7z) + 64(2z-1)H_0 \right] \zeta_2 \right] + 32(23+17z)H_0 \\
& - 8(-23+9z)H_0^2 + \frac{8}{3}(9+4z)H_0^3 - \frac{4}{3}(2z-1)H_0^4 \Big] + L_M^2 \left[928(z-1) \right. \\
& + (2z-1) \left(\frac{16}{3}H_0^3 + 32H_{0,0,1} - 32\zeta_3 \right) - 16(30+7z)H_0 - 8(11+4z)H_0^2 \\
& + 336(z-1)H_1 - 48(3+4z)H_{0,1} + \left. \left[48(3+4z) - 32(2z-1)H_0 \right] \zeta_2 \right] \\
& + L_M \left[\frac{8}{9}(-8533+8509z) + (1+z)^2 \left(-\frac{128}{3}H_{-1}^2 H_0 - \frac{256}{3}H_{0,-1,-1} \right) + (2z-1) \left(\frac{28}{3}H_0^4 \right. \right. \\
& - \frac{80}{9}H_1^3 + 64H_{0,0,1,1} - \frac{32}{5}\zeta_2^2 \Big) + \left. \left[-\frac{8}{9}(5606+209z+24z^2) + (2z-1) \left(-\frac{64}{3}H_1^2 \right. \right. \right. \\
& + \left. \left. 64H_{0,0,1} \right) \right] + 48(-11+10z)H_1 + \frac{32}{3}(-25-40z+4z^2)H_{0,1} \\
& - \frac{128}{3}(-2+4z+z^2)H_{0,-1} \Big] H_0 + \left[-\frac{4}{9}(3063+462z+32z^2) - \frac{16}{3}(-1+2z+4z^2)H_1 \right] H_0^2 \\
& - \frac{16}{9}(104+17z)H_0^3 + \left[-\frac{32}{3}(229-230z+2z^2) - \frac{64}{3}(2z-1)H_{0,1} \right] H_1 + 24(-11+10z)H_1^2 \\
& + \left[(1+z)^2 \left(\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1} \right) + \frac{64(4+45z+48z^2+4z^3)}{9z}H_0 \right] H_{-1} - \frac{16}{3}(203+156z)H_{0,1} \\
& - \frac{64(4+45z+48z^2+4z^3)}{9z}H_{0,-1} - \frac{64}{3}(z-1)(-11+2z)H_{0,0,1} \\
& + \frac{128}{3}(-5+6z+z^2)H_{0,0,-1} - \frac{32}{3}(31+28z)H_{0,1,1} + \left[\frac{32}{9}(543+99z+8z^2) \right. \\
& + \frac{32}{3}(83+56z)H_0 - 96(2z-1)H_0^2 + \frac{64}{3}(-1+2z+2z^2)H_1 - \frac{128}{3}(1+z)^2H_{-1} \Big] \zeta_2 \\
& + \left[\frac{64}{3}(33-4z+4z^2) - 128(2z-1)H_0 \right] \zeta_3 \Big] + (2z-1) \left(\frac{4}{15}H_0^5 + 32H_{0,0,0,0,1} - 32\zeta_5 \right) \\
& + 4(-519+10z)H_0 - 8(69+8z)H_0^2 - \frac{8}{3}(32+3z)H_0^3 + \frac{2}{3}(-11+4z)H_0^4 + 20(-67+70z)H_1 \\
& - 32(23+17z)H_{0,1} + 16(-23+9z)H_{0,0,1} - 16(9+4z)H_{0,0,0,1} + \left[32(23+17z) \right. \\
& - 16(-23+9z)H_0 + 8(9+4z)H_0^2 - \frac{16}{3}(2z-1)H_0^3 \Big] \zeta_2 + \left[\frac{32}{5}(9+4z) - \frac{64}{5}(2z-1)H_0 \right] \zeta_2^2
\end{aligned}$$

$$+ \left[-16(-23 + 9z) + 16(9 + 4z)H_0 - 16(2z - 1)H_0^2 \right] \zeta_3 \Big\} + A_{qg,Q}^{(3)} + N_F \hat{C}_g^{S(3)}(L_Q, N_F, z) \Big\}. \quad (\text{B2})$$

For $H_q^{\text{PS}}(z)$ we obtain

$$\begin{aligned} H_q^{\text{PS}} = & a_s^2 C_F T_F \left\{ -\frac{4}{3}(z-1)(148 + 66H_1 + 15H_1^2) + L_Q[8(z-1)(11 + 5H_1) \right. \\ & + (1+z)(-16H_0^2 - 16H_{0,1} + 16\zeta_2) + 32(-2+z)H_0] + L_M^2[20(z-1) - 8(1+z)H_0] \\ & + L_Q^2[-20(z-1) + 8(1+z)H_0] + L_M[8(z-1) - 8(-1+3z)H_0 + 8(1+z)H_0^2] \\ & + (1+z) \left(\frac{16}{3}H_0^3 - 32H_{0,0,1} + 16H_{0,1,1} + 16\zeta_3 \right) + \left[-\frac{256}{3}(-2+z) - 80(z-1)H_1 \right. \\ & - \frac{32(1+z)^3 H_{-1}}{3z} + 32(1+z)H_{0,1} \Big] H_0 + \frac{8}{3}(21 + 2z^2)H_0^2 + 16(-1+3z)H_{0,1} \\ & + \frac{32(1+z)^3 H_{0,-1}}{3z} + \left. \left[-\frac{32}{3}(9-3z+z^2) - 32(1+z)H_0 \right] \zeta_2 \right\} \\ & + a_s^3 \left\{ C_F T_F^2 \left\{ L_M^2 \left[\frac{32}{3}(z-1)(12 + 5H_1) + (1+z) \left(-\frac{32}{3}H_0^2 - \frac{64}{3}H_{0,1} + \frac{64}{3}\zeta_2 \right) \right. \right. \right. \\ & + \frac{32}{3}(-7+z)H_0 \Big] + L_Q \left[(1+z) \left(\frac{64}{3}H_{0,1,1} - \frac{64\zeta_3}{3} \right) + L_M^2 \left[-\frac{160}{3}(z-1) + \frac{64}{3}(1+z)H_0 \right] \right. \\ & + L_M \left[\frac{64}{9}(z-1)(-4 + 15H_1) + (1+z) \left(-\frac{128H_{0,1}}{3} + \frac{128\zeta_2}{3} \right) + \frac{128}{9}(2+5z)H_0 \right] \\ & + \frac{128}{27}(11 + 14z)H_0 + (z-1) \left(-\frac{3712}{27} + \frac{128}{9}H_1 - \frac{80}{3}H_1^2 \right) - \frac{64}{9}(2+5z)H_{0,1} \\ & + \frac{64}{9}(2+5z)\zeta_2 \Big] + L_M \left[(1+z) \left(\frac{128}{9}H_{0,1} - \frac{128}{3}H_{0,0,1} + \frac{256}{3}H_{0,1,1} - \frac{128}{3}\zeta_3 \right) \right. \\ & + \left. \left[-\frac{512}{9}(1+2z) - \frac{320}{3}(z-1)H_1 + \frac{128}{3}(1+z)H_{0,1} \right] H_0 - \frac{64}{9}(2+5z)H_0^2 \right. \\ & + (z-1) \left(-\frac{64}{3} - \frac{2048}{9}H_1 - \frac{320}{3}H_1^2 \right) + \left. \left[\frac{64}{9}(-17+13z) - \frac{128}{3}(1+z)H_0 \right] \zeta_2 \right] \\ & + (1+z) \left(\frac{64}{3}H_{0,0,1,1} - 64H_{0,1,1,1} + \frac{32}{15}\zeta_2^2 \right) + \left[\frac{128}{27}(-40+z) + \frac{16}{9}(z-1)H_1(-8+15H_1) \right. \\ & + \frac{64}{9}(2+5z)H_{0,1} - \frac{64}{3}(1+z)H_{0,1,1} \Big] H_0 - \frac{64}{27}(11+14z)H_0^2 + (z-1) \left(\frac{3296}{9} + \frac{4000}{27}H_1 \right. \\ & + \frac{448}{9}H_1^2 + \frac{80}{3}H_1^3 \Big) + \frac{128}{27}(-8+z)H_{0,1} - \frac{64}{9}(2+5z)H_{0,0,1} + \frac{64}{9}(1+4z)H_{0,1,1} \\ & + \left[\frac{128}{27}(5+2z) - \frac{64}{9}(2+5z)H_0 - \frac{160}{3}(z-1)H_1 + \frac{64}{3}(1+z)H_{0,1} \right] \zeta_2 \\ & + \left. \left[-\frac{32}{9}(-17+13z) + \frac{64}{3}(1+z)H_0 \right] \zeta_3 \right\} + C_F^2 T_F \left\{ (z-1) \left[2656 + 8H_1(103 + 40H_{0,0,1}) \right. \right. \\ & + \frac{8}{3}H_0^4 + 144H_1^2 - 80H_{0,1}^2 + 160H_{0,1,1,1} \Big] + L_M^2 \left[4(z-1)(92 + 43H_1 + 10H_1^2) \right. \\ & + (1+z) \left(-\frac{8}{3}H_0^3 + 48H_{0,0,1} - 32H_{0,1,1} - 16\zeta_3 \right) + \left. \left[-4(51+7z) + 80(z-1)H_1 \right. \right. \end{aligned}$$

$$\begin{aligned}
& -32(1+z)H_{0,1}]H_0 + 8(-6+5z)H_0^2 - 80zH_{0,1} + [80+16(1+z)H_0]\zeta_2 \Big] \\
& + L_Q[L_M^2[-4(z-1)(13+20H_1) + (1+z)(8H_0^2+32H_{0,1}-32\zeta_2) - 16(-2+3z)H_0] \\
& + (z-1)(-392-288H_1-160H_{0,1,1}) + L_M \Big[-8(z-1)(5+4H_1) \\
& + (1+z) \Big(-8H_0 - \frac{16}{3}H_0^3 - 64H_{0,0,1} + 64\zeta_3 \Big) + 32zH_0^2 + 32(-1+3z)H_{0,1} \\
& + [-32(-1+3z) + 64(1+z)H_0]\zeta_2 \Big] + (1+z) \Big(\frac{2}{3}H_0^4 - 64H_{0,0,0,1} + 128H_{0,0,1,1} - \frac{256}{5}\zeta_2^2 \Big) \\
& + [-52(-3+z) + 8(z-1)H_1(13+10H_1) + (1+z)(96H_{0,0,1} - 64H_{0,1,1}) \\
& - 32(-3+2z)H_{0,1}]H_0 + [6(3+19z) + 80(z-1)H_1 - 32(1+z)H_{0,1}]H_0^2 - \frac{16}{3}zH_0^3 \\
& + 8(19+17z)H_{0,1} - 16(1+5z)H_{0,0,1} + [-48(1+5z) + (1+z)(-16H_0^2 + 64H_{0,1}) \\
& + 16(-1+3z)H_0 - 160(z-1)H_1]\zeta_2 + [48(-3+5z) - 96(1+z)H_0]\zeta_3 \Big] \\
& + L_M \Big[(z-1) \Big(312 - \frac{64}{3}H_0^3 + 88H_1 + 16H_1^2 \Big) + (1+z) \Big(\frac{4}{3}H_0^4 - 160H_{0,0,0,1} + 64H_{0,0,1,1} \\
& + \frac{288}{5}\zeta_2^2 \Big) + [-8(-2+21z) + 32(z-1)H_1 - 32(-1+3z)H_{0,1} + 64(1+z)H_{0,0,1}]H_0 \\
& + 4(7+13z)H_0^2 - 8(-11+21z)H_{0,1} + 32(1+7z)H_{0,0,1} - 32(-1+3z)H_{0,1,1} \\
& + [8(-7+17z) - 32(3+z)H_0 - 16(1+z)H_0^2]\zeta_2 + [-64(1+2z) + 32(1+z)H_0]\zeta_3 \Big] \\
& + (1+z) \Big(-\frac{2}{15}H_0^5 - 80H_{0,0,0,0,1} + 832H_{0,0,0,1,1} + 256H_{0,0,1,0,1} - 128H_{0,0,1,1,1} + 80\zeta_5 \Big) \\
& + \Big[-4(263+195z) + (z-1) \Big[-32H_1(14+5H_{0,1}) - 172H_1^2 - \frac{80}{3}H_1^3 \Big] \\
& + (1+z)(32H_{0,1}^2 + 96H_{0,0,0,1} - 224H_{0,0,1,1} + 64H_{0,1,1,1}) + 16(1+10z)H_{0,1} \\
& - 16(-1+17z)H_{0,0,1} + 160(-2+3z)H_{0,1,1} \Big] H_0 + [-6(26+15z) - 4(z-1)H_1(43+10H_1) \\
& + (1+z)(-48H_{0,0,1} + 32H_{0,1,1}) + 80zH_{0,1}]H_0^2 + \Big[-2(5+23z) - \frac{80}{3}(z-1)H_1 \\
& + \frac{32}{3}(1+z)H_{0,1} \Big] H_0^3 + [4(-169+35z) - 128(1+z)H_{0,0,1}]H_{0,1} - 4(109+33z)H_{0,0,1} \\
& + 8(-49+13z)H_{0,1,1} + 32(-8+15z)H_{0,0,0,1} - 144(-1+3z)H_{0,0,1,1} + \Big[4(57+77z) \\
& + 8(z-1)H_1(43+10H_1) + (1+z) \Big(\frac{8}{3}H_0^3 + 96H_{0,0,1} - 64H_{0,1,1} - 96\zeta_3 \Big) + [12(5+13z) \\
& + 160(z-1)H_1 - 64(1+z)H_{0,1}]H_0 + 8(3+z)H_0^2 - 160zH_{0,1} \Big] \zeta_2 + \Big[-\frac{8}{5}(-145+83z) \\
& + \frac{48}{5}(1+z)H_0 \Big] \zeta_2^2 + [4(207+7z) + (1+z)(24H_0^2 + 128H_{0,1}) - 32(-13+8z)H_0 \\
& - 320(z-1)H_1]\zeta_3 \Big\} + A_{Qq}^{\text{PS}(3)} + \tilde{C}^{\text{PS}(3)}(L_Q, N_F + 1) \Big\}. \tag{B3}
\end{aligned}$$

Finally, $H^S(z)$ reads

$$\begin{aligned}
H_g^S = & a_s T_F \{ -4(-3 + 4z) + (2z - 1)(-4L_M + 4L_Q - 4H_0 - 4H_1) \} \\
& + a_s^2 \left\{ C_A T_F \left[-\frac{8}{3}(-101 + 104z) + L_Q^2[-48(z - 1) \right. \right. \\
& + 16(1 + z)H_0 - 8(2z - 1)H_1] + L_M^2[48(z - 1) - 16(1 + z)H_0 + 8(2z - 1)H_1] \\
& + (1 + 2z)(-16H_{0,1,-1} - 16H_{0,-1,1}) + L_M[8(-12 + 11z) + (1 + 2z)(16H_{-1}H_0 + 8H_0^2 \\
& - 16H_{0,-1}) - 8(1 + 8z)H_0 + 32(z - 1)H_1 + 8(2z - 1)H_1^2 + 16\zeta_2] + L_Q[8(-20 + 21z) \\
& + (1 + 2z)(-16H_{-1}H_0 + 16H_{0,-1}) + [24(-5 + 4z) + 16(2z - 1)H_1]H_0 - 8(3 + 4z)H_0^2 \\
& + 16(-7 + 8z)H_1 + 8(2z - 1)H_1^2 - 32(1 + z)H_{0,1} + 32\zeta_2] + \left. \left[\frac{4}{3}(194 - 163z + 6z^2) \right. \right. \\
& - 4(-53 + 56z + z^2)H_1 - 16(2z - 1)H_1^2 - 8(-7 - 10z + 2z^2)H_{0,1} \\
& + 8(-3 - 6z + 2z^2)H_{0,-1} \left. \right] H_0 + 16z^2H_{-1}^2H_0 + \left[\frac{2}{3}(126 - 48z + 41z^2) \right. \\
& + 4(3 - 6z + 2z^2)H_1 \left. \right] H_0^2 + \frac{8}{3}(3 + 4z)H_0^3 + [4(43 - 53z + 2z^2) + 16(2z - 1)H_{0,1}]H_1 \\
& + 2(19 - 24z + z^2)H_1^2 + \left[-\frac{16(2 + 3z + 9z^2 + 11z^3)}{3z}H_0 - 4(-3 - 6z + 2z^2)H_0^2 \right. \\
& + 16(1 + 2z)H_{0,1} - 32z^2H_{0,-1} \left. \right] H_{-1} + 4(-19 + 28z + 2z^2)H_{0,1} \\
& + \frac{16(2 + 3z + 9z^2 + 11z^3)}{3z}H_{0,-1} + 8(-9 - 10z + 2z^2)H_{0,0,1} - 8(-3 - 6z + 2z^2)H_{0,0,-1} \\
& + 48H_{0,1,1} + 32z^2H_{0,-1,-1} + \left[-\frac{4}{3}(114 - 84z + 47z^2) - 32(2 + z)H_0 \right. \\
& - 16(z - 1)^2H_1 + 16(-1 - 2z + z^2)H_{-1} \left. \right] \zeta_2 - 8(-1 - 10z + 4z^2)\zeta_3 \left. \right\} \\
& + C_F T_F \left\{ -\frac{20}{3}(-20 + 17z) + L_M^2(6 + (2z - 1)(-4H_0 - 8H_1)) \right. \\
& + L_Q^2[6 + (2z - 1)(-4H_0 - 8H_1)] + L_M[-4(-17 + 13z) + (2z - 1)(-8H_0^2 - 16H_1^2 \\
& + 8H_{0,1} + 24\zeta_2) + [-16(-3 + 2z) - 32(2z - 1)H_1]H_0 - 4(-17 + 20z)H_1] \\
& + L_Q[4(-17 + 13z) + L_M[-12 + (2z - 1)(8H_0 + 16H_1)] + (2z - 1)(8H_0^2 + 16H_1^2 \\
& - 8H_{0,1} - 24\zeta_2) + [16(-3 + 2z) + 32(2z - 1)H_1]H_0 + 4(-17 + 20z)H_1] \\
& + (2z - 1) \left(-\frac{8}{3}H_0^3 - 8H_1^3 + 24H_{0,1,1} \right) + (1 + z)^2(-32H_{-1}^2H_0 - 64H_{0,-1,-1}) \\
& + \left[-\frac{8}{3}(-46 + 53z + 6z^2) + 8(8 - 14z + z^2)H_1 - 16(2z - 1)H_1^2 + 32(z - 1)^2H_{0,1} \right. \\
& - 32(z - 1)^2H_{0,-1} \left. \right] H_0 + \left[-\frac{4}{3}(-27 + 6z + 23z^2) - 16z^2H_1 \right] H_0^2 + [-4(-47 + 41z + 4z^2) \\
& - 16(2z - 1)H_{0,1}]H_1 - 2(-33 + 40z + 2z^2)H_1^2 + \left[(1 + z)^2(16H_0^2 + 64H_{0,-1}) \right. \\
& + \frac{16(4 + 12z^2 + 13z^3)}{3z}H_0 \left. \right] H_{-1} - 16(-6 + z^2)H_{0,1} - \frac{16(4 + 12z^2 + 13z^3)}{3z}H_{0,-1} \\
& - 32(z - 1)^2H_{0,0,1} + 32(1 - 6z + z^2)H_{0,0,-1} + \left. \left[\frac{8}{3}(-60 + 42z + 29z^2) + 32(2z - 1)H_0 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + 16(-1 + 2z + 2z^2)H_1 - 32(1 + z)^2H_{-1} \Big] \zeta_2 + 8(1 + 14z + 8z^2)\zeta_3 \Big\} \\
& + T_F^2 \Big\{ (2z - 1) \Big(-\frac{16L_M^2}{3} + \frac{16L_M L_Q}{3} \Big) + L_M \Big[-\frac{16}{3}(-3 + 4z) + (2z - 1) \Big(-\frac{16}{3}H_0 - \frac{16}{3}H_1 \Big) \Big] \Big\} \\
& + a_s^3 \Big\{ C_F N_F T_F^2 \Big\{ L_M L_Q^2 \Big[8(-43 + 42z) + (2z - 1) \Big(16H_0^2 + \frac{32}{3}H_1 \Big) - \frac{32}{3}(14 + 17z)H_0 \Big] \\
& + (L_M L_Q) \Big[-\frac{8}{3}(-709 + 690z) + (2z - 1) \Big(-\frac{64}{3}H_0^3 - \frac{32}{3}H_1^2 - 64H_{0,0,1} + 64\zeta_3 \Big) \\
& + \Big[-\frac{16}{9}(-673 + 26z) - \frac{128}{3}(2z - 1)H_1 \Big] H_0 + \frac{32}{3}(25 + 16z)H_0^2 - \frac{16}{9}(-425 + 454z)H_1 \\
& + 64(4 + 7z)H_{0,1} + \Big[-\frac{64}{3}(14 + 17z) + 64(2z - 1)H_0 \Big] \zeta_2 \Big] + L_M \Big[\frac{2}{9}(-29415 + 28958z) \\
& + (1 + z)[-64(-5 + z)H_{-1}H_0 + 64(-5 + z)H_{0,-1}] + (2z - 1) \Big(\frac{20}{3}H_0^4 + \frac{32}{9}H_1^3 \\
& + 256H_{0,0,-1} - 64H_{0,0,0,1} + 64H_{0,0,1,1} + \frac{96}{5}\zeta_2^2 \Big) + \Big[-\frac{8}{27}(13571 + 1958z) \\
& + (2z - 1) \Big(\frac{32}{3}H_1^2 - 128H_{0,-1} + 64H_{0,0,1} \Big) + \frac{16}{3}(-151 + 170z)H_1 - \frac{32}{3}(23 + 44z)H_{0,1} \Big] H_0 \\
& + \Big[8(-129 + 18z + 4z^2) + \frac{80}{3}(2z - 1)H_1 \Big] H_0^2 - \frac{304}{9}(4 + z)H_0^3 + \Big[\frac{104}{27}(-523 + 542z) \\
& + \frac{64}{3}(2z - 1)H_{0,1} \Big] H_1 + \frac{8}{9}(-425 + 454z)H_1^2 - \frac{704}{9}(5 + 11z)H_{0,1} + \frac{32}{3}(1 + 46z)H_{0,0,1} \\
& - \frac{64}{3}(11 + 23z)H_{0,1,1} + \Big[-\frac{16}{9}(-853 + 26z + 36z^2) + (2z - 1) \Big(-64H_0^2 - \frac{128}{3}H_1 \Big) \\
& + \frac{64}{3}(31 + 16z)H_0 \Big] \zeta_2 + \Big[-\frac{64}{3}(-20 + 19z) - 64(2z - 1)H_0 \Big] \zeta_3 \Big\} + C_A T_F^2 \Big\{ \frac{8}{27}(-3943 \\
& + 3928z) + L_M L_Q^2 \Big[-64(z - 1) + \frac{64}{3}(1 + z)H_0 - \frac{32}{3}(2z - 1)H_1 \Big] + L_M^2 \Big[\frac{512}{3}(z - 1) \\
& + (1 + z) \Big(-\frac{32}{3}H_0^2 - \frac{64}{3}H_{0,1} \Big) + \Big[\frac{64}{3}(-4 + z) + \frac{32}{3}(2z - 1)H_1 \Big] H_0 + \frac{32}{3}(-9 + 10z)H_1 \\
& + \frac{32}{3}(2z - 1)H_1^2 + 32\zeta_2 \Big] + L_Q \Big[-\frac{40}{9}(-108 + 107z) + L_M^2 \Big[-64(z - 1) + \frac{64}{3}(1 + z)H_0 \\
& - \frac{32}{3}(2z - 1)H_1 \Big] + L_M \Big[-\frac{32}{3}(-22 + 19z) + (1 + 2z) \Big(-\frac{64}{3}H_{-1}H_0 + \frac{64}{3}H_{0,-1} \Big) \\
& + \Big[\frac{32}{9}(5 + 98z) + \frac{64}{3}(2z - 1)H_1 \Big] H_0 - \frac{64}{3}(1 + 3z)H_0^2 + \frac{128}{9}(-8 + 7z)H_1 + \frac{32}{3}(2z - 1)H_1^2 \\
& - \frac{128}{3}(1 + z)H_{0,1} + \frac{128}{3}\zeta_2 \Big] + \frac{8}{27}(785 + 644z)H_0 + \frac{16}{9}(17 + 2z)H_0^2 - \frac{16}{9}(2z - 1)H_0^3 - \frac{16}{27}(-47 + 103z)H_1 \\
& - \frac{32}{3}zH_{0,1} + \frac{32}{3}z\zeta_2 \Big] + (2z - 1) \Big(\frac{4}{9}H_0^4 + \frac{32}{3}H_{0,0,0,1} - \frac{64}{15}\zeta_2^2 \Big) \\
& + L_M \Big[\frac{16}{9}(-325 + 301z) + (1 + 2z) \Big(-\frac{64}{3}H_{0,1,-1} - \frac{64}{3}H_{0,-1,1} \Big) + \Big[\frac{32}{9}(-58 - 42z + 3z^2)
\end{aligned}$$

$$\begin{aligned}
& -\frac{128}{9}(-8+7z)H_1 - 16(2z-1)H_1^2 - \frac{32}{3}(-3-6z+2z^2)H_{0,1} \\
& + \frac{64}{3}(-1-2z+z^2)H_{0,-1} \Big] H_0 + \frac{32}{3}(1+2z+2z^2)H_{-1}^2 H_0 + \left[\frac{8}{9}(23-124z+44z^2) \right. \\
& + \frac{16}{3}(3-6z+2z^2)H_1 \Big] H_0^2 + \frac{16}{9}(5+14z)H_0^3 + \left[\frac{16}{9}(-180+161z+6z^2) \right. \\
& + \frac{64}{3}(2z-1)H_{0,1} \Big] H_1 - \frac{32}{9}(-8+z)H_1^2 - \frac{16}{9}(2z-1)H_1^3 + \left[-\frac{64(2+6z+12z^2+11z^3)}{9z} H_0 \right. \\
& - \frac{32}{3}(-1-2z+z^2)H_0^2 + \frac{64}{3}(1+2z)H_{0,1} - \frac{64}{3}(1+2z+2z^2)H_{0,-1} \Big] H_{-1} \\
& - \frac{32}{9}(41+62z)H_{0,1} + \frac{64(2+6z+12z^2+11z^3)}{9z} H_{0,-1} + \frac{32}{3}(-3+2z+2z^2)H_{0,0,1} \\
& - \frac{64}{3}(-1-2z+z^2)H_{0,0,-1} - \frac{32}{3}(-7+2z)H_{0,1,1} + \frac{64}{3}(1+2z+2z^2)H_{0,-1,-1} \\
& + \left[-\frac{32}{9}(3-90z+22z^2) - \frac{64}{3}(3+4z)H_0 - \frac{64}{3}(z-1)^2 H_1 + \frac{32}{3}(-1-2z+2z^2)H_{-1} \right] \zeta_2 \\
& - \frac{64}{3}(2+z+2z^2)\zeta_3 \Big] + \left[\frac{8}{27}(-2405+413z) + \frac{16}{27}(-47+103z)H_1 + \frac{32}{3}zH_{0,1} \right] H_0 \\
& - \frac{4}{27}(989+764z)H_0^2 + \frac{160}{27}(-2+z)H_0^3 + \frac{8}{27}(-1884+1981z)H_1 + \frac{16}{27}(-47+103z)H_1^2 \\
& - \frac{8}{27}(785+572z)H_{0,1} - \frac{32}{9}(17+5z)H_{0,0,1} + \frac{64}{3}zH_{0,1,1} + \left[\frac{8}{9}(293+122z) - \frac{32}{9}(-17+z)H_0 \right. \\
& - \frac{16}{3}(2z-1)H_0^2 \Big] \zeta_2 + \left[-\frac{32}{9}(-17+z) - \frac{32}{3}(2z-1)H_0 \right] \zeta_3 \Big\} + C_F T_F^2 \left\{ 4(-889+904z) \right. \\
& + L_M L_Q^2 \left[8+(2z-1) \left(-\frac{16}{3}H_0 - \frac{32}{3}H_1 \right) \right] + L_Q \left[-20(-67+70z) + L_M^2 \left[-16(-20+21z) \right. \right. \\
& + (2z-1) \left(-16H_0^2 + \frac{64}{3}H_1 \right) + \frac{80}{3}(5+8z)H_0 \Big] + L_M \left[-\frac{16}{3}(-136+137z) \right. \\
& + (2z-1) \left(-\frac{32}{3}H_0^3 + \frac{64}{3}H_1^2 - \frac{32}{3}H_{0,1} - 32\zeta_2 \right) + \left[\frac{32}{3}(51+25z) + \frac{128}{3}(2z-1)H_1 \right] H_0 \\
& + \frac{16}{3}(25+16z)H_0^2 + \frac{16}{3}(-17+20z)H_1 \Big] + 32(23+17z)H_0 - 8(-23+9z)H_0^2 \\
& + \frac{8}{3}(9+4z)H_0^3 - \frac{4}{3}(2z-1)H_0^4 \Big] + L_M^2 \left[\frac{16}{3}(-155+164z) + (2z-1) \left(\frac{16}{3}H_0^3 - \frac{32}{3}H_1^2 \right. \right. \\
& + 32H_{0,0,1} - 32\zeta_3 \Big) + \left[-\frac{16}{3}(79+13z) - \frac{64}{3}(2z-1)H_1 \right] H_0 - \frac{8}{3}(31+16z)H_0^2 \\
& + 16(-18+17z)H_1 - \frac{16}{3}(29+32z)H_{0,1} + \left(\frac{80}{3}(5+8z) - 32(2z-1)H_0 \right) \zeta_2 \Big] \\
& + L_M \left[\frac{8}{9}(-1549+1525z) + (1+z)^2 \left(-\frac{128}{3}H_{-1}^2 H_0 - \frac{256}{3}H_{0,-1,-1} \right) + (2z-1) \left(\frac{8}{3}H_0^4 \right. \right. \\
& - \frac{80}{9}H_1^3 + \frac{128}{3}H_{0,1,1} + 64H_{0,0,0,1} - \frac{128}{5}\zeta_2^2 \Big) + \left[-\frac{8}{9}(1394-547z+24z^2) - 48(-3+4z)H_1 \right. \\
& - \frac{64}{3}(2z-1)H_1^2 + \frac{64}{3}(1-2z+2z^2)H_{0,1} - \frac{128}{3}(z-1)^2 H_{0,-1} \Big] H_0
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{4}{9}(885 + 732z + 104z^2) - \frac{16}{3}(-1 + 2z + 4z^2)H_1 \right] H_0^2 + \frac{8}{9}(-61 + 14z)H_0^3 \\
& + \left[-\frac{32}{3}(61 - 62z + 2z^2) - \frac{64}{3}(2z - 1)H_{0,1} \right] H_1 - 24(-3 + 4z)H_1^2 \\
& + \left[(1 + z)^2 \left(\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1} \right) + \frac{64(4 + 12z^2 + 13z^3)}{9z}H_0 \right] H_{-1} - \frac{16}{3}(119 + 30z)H_{0,1} \\
& - \frac{64(4 + 12z^2 + 13z^3)}{9z}H_{0,-1} - \frac{64}{3}(14 + 5z + 2z^2)H_{0,0,1} + \frac{128}{3}(1 - 6z + z^2)H_{0,0,-1} \\
& + \left[\frac{32}{9}(138 + 99z + 26z^2) + \frac{32}{3}(23 + 20z)H_0 - 32(2z - 1)H_0^2 + \frac{64}{3}(-1 + 2z + 2z^2)H_1 \right. \\
& \left. - \frac{128}{3}(1 + z)^2H_{-1} \right] \zeta_2 + \left[\frac{32}{3}(27 + 28z + 8z^2) - 64(2z - 1)H_0 \right] \zeta_3 \Big] \\
& + (2z - 1) \left(\frac{4}{15}H_0^5 + 32H_{0,0,0,0,1} - 32\zeta_5 \right) + 4(-519 + 10z)H_0 - 8(69 + 8z)H_0^2 \\
& - \frac{8}{3}(32 + 3z)H_0^3 + \frac{2}{3}(-11 + 4z)H_0^4 + 20(-67 + 70z)H_1 - 32(23 + 17z)H_{0,1} \\
& + 16(-23 + 9z)H_{0,0,1} - 16(9 + 4z)H_{0,0,0,1} + \left[32(23 + 17z) - 16(-23 + 9z)H_0 \right. \\
& \left. + 8(9 + 4z)H_0^2 - \frac{16}{3}(2z - 1)H_0^3 \right] \zeta_2 + \left[\frac{32}{5}(9 + 4z) - \frac{64}{5}(2z - 1)H_0 \right] \zeta_2^2 \\
& + [-16(-23 + 9z) + 16(9 + 4z)H_0 - 16(2z - 1)H_0^2] \zeta_3 \Big\} + C_F^2 T_F \left\{ -2(101 + 7z) \right. \\
& + L_M L_Q^2 [(2z - 1)(-18 - 4H_0^2 - 32H_1^2 + 32\zeta_2) + [-8(z - 1) - 32(2z - 1)H_1]H_0 + 48H_1] \\
& + L_M^2 \left[-2(-67 + 94z) + (2z - 1) \left(-\frac{4}{3}H_0^3 - 16H_1^3 + 8H_{0,0,1} + 16H_{0,1,1} + 8\zeta_3 \right) \right. \\
& + [-2(-31 + 58z) - 96(z - 1)H_1 - 32(2z - 1)H_1^2]H_0 + [4(5 + 3z) - 16(2z - 1)H_1]H_0^2 \\
& - 2(-103 + 58z)H_1 - 12(-7 + 8z)H_1^2 + 8(3 + 13z)H_{0,1} + [-8(15 + z) \\
& + (2z - 1)(24H_0 + 64H_1)]\zeta_2 \Big] + L_Q \left[2(23 + 13z) + L_M^2 [(2z - 1)(18 + 4H_0^2 + 32H_1^2 - 32\zeta_2) \right. \\
& + [8(z - 1) + 32(2z - 1)H_1]H_0 - 48H_1] + (2z - 1) \left(\frac{16}{3}H_1^4 + 16H_{0,1}^2 + 16H_{0,0,0,1} - 80H_{0,0,1,1} \right. \\
& \left. + 80H_{0,1,1,1} + \frac{184}{5}\zeta_2^2 \right) + L_M [-2(-305 + 284z) + (2z - 1)(64H_1^3 - 112H_{0,0,1} - 64H_{0,1,1}) \\
& + (1 + 2z)(64H_{-1}^2 H_0 + 64H_{0,0,-1} + 128H_{0,-1,-1}) + [56(5 + 14z) + (2z - 1)(160H_1^2 + 64H_{0,1}) \\
& + 32(-13 + 10z)H_1]H_0 + [8(6 + 7z) + 64(2z - 1)H_1]H_0^2 + \frac{8}{3}(1 + 6z)H_0^3 \\
& + 4(-119 + 94z)H_1 + 320(z - 1)H_1^2 + [(1 + 2z)(-32H_0^2 - 128H_{0,-1}) - 192(1 + z)H_0]H_{-1} \\
& + 16(3 + z)H_{0,1} + 192(1 + z)H_{0,-1} + [-16(-11 + 21z) - 16(-5 + 18z)H_0 - 256(2z - 1)H_1 \\
& + 64(1 + 2z)H_{-1}]\zeta_2 - 16(1 + 14z)\zeta_3 \Big] + \left[2(28 + 37z) + (2z - 1) \left(\frac{16}{3}H_1^3 - 48H_{0,0,1} \right. \right. \\
& \left. \left. + 32H_{0,1,1} \right) \right] + [4(-53 + 46z) + 32(2z - 1)H_{0,1}]H_1 - 8(-5 + 4z)H_1^2 + 8[-49 + 14z \\
& + 4z^2]H_{0,1}]H_0 + [15 + 14z + (2z - 1)(-8H_1^2 + 24H_{0,1}) - 4(-25 + 16z + 4z^2)H_1]H_0^2
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{4}{3}(-2 + 5z + 4z^2) - \frac{16}{3}(2z - 1)H_1 \right] H_0^3 + \frac{1}{3}(1 - 2z)H_0^4 + [12(-10 + 3z) \\
& - 64(2z - 1)H_{0,0,1}]H_1 + 16(-17 + 16z)H_1^2 + 4(-13 + 8z + 4z^2)H_1^3 + 4(55 + 46z)H_{0,1} \\
& - 32(-15 + 4z + z^2)H_{0,0,1} + 8(-87 + 28z + 12z^2)H_{0,1,1} + [-8(1 + 46z) \\
& + (2z - 1)(8H_0^2 - 16H_1^2 - 96H_{0,1}) + (8(13 + 4z) + 32(2z - 1)H_1)H_0 + 16(-5 + 4z)H_1]\zeta_2 \\
& + [-16(-21 + 10z + 6z^2) + 32(2z - 1)H_0]\zeta_3 \Big] + L_M \left[\frac{1}{2}(-2757 + 2786z) \right. \\
& + (1 + z)(-192H_{0,1,-1} - 192H_{0,-1,1} - 192H_{0,-1,-1}) + (2z - 1)(-24H_1^4 - 16H_{0,1}^2 \\
& - 288H_{0,0,0,-1} - 192H_{0,0,1,1} + 224H_{0,1,1,1} - 192H_{0,-1,0,1}) + (1 + 2z) \left(\frac{128}{3}H_{-1}^3H_0 + 80H_{0,1,1} \right. \\
& - 64H_{0,0,1,-1} - 64H_{0,0,-1,1} - 64H_{0,0,-1,-1} - 128H_{0,1,-1,-1} - 128H_{0,-1,1,-1} - 128H_{0,-1,-1,1} \\
& \left. \left. - 256H_{0,-1,-1,-1} \right) + \left[-6(149 + 196z) + (2z - 1) \left(-\frac{256}{3}H_1^3 + 224H_{0,1,1} - 192H_{0,1,-1} \right. \right. \right. \\
& \left. \left. - 192H_{0,-1,1} \right) + [-4(-307 + 230z) + (2z - 1)(-160H_{0,1} + 192H_{0,-1})]H_1 \right. \\
& - 8(-67 + 82z)H_1^2 + 8(25 + 18z)H_{0,1} - 224(1 + 2z)H_{0,-1} - 16(3 + 10z)H_{0,0,1} \\
& \left. \left. - 32(5 + 2z)H_{0,0,-1} - 256zH_{0,-1,-1} \right] H_0 + (4(-64 - 125z + 12z^2) - 4(-67 + 76z)H_1 \right. \\
& - 56(2z - 1)H_1^2 + 8(3 + 2z)H_{0,1} + 16(1 + 6z)H_{0,-1})H_0^2 + \left[\frac{4}{3}(-27 + z) - \frac{64}{3}(2z - 1)H_1 \right] H_0^3 \\
& + \frac{1}{3}(-5 - 14z)H_0^4 + [-2(-353 + 332z) + (2z - 1)(224H_{0,0,1} - 384H_{0,0,-1} - 96H_{0,1,1}) \\
& + 48(-3 + 4z)H_{0,1}]H_1 + [-2(-365 + 322z) + 32(2z - 1)H_{0,1}]H_1^2 - 16(-13 + 16z)H_1^3 \\
& + [(1 + z)(192H_0^2 + 192H_{0,1} + 192H_{0,-1}) + (1 + 2z)(16H_0^3 + 64H_{0,0,1} + 64H_{0,0,-1} \\
& + 128H_{0,1,-1} + 128H_{0,-1,1} + 256H_{0,-1,-1}) + [-32(-12 - 11z + 3z^2) \\
& + 128(1 + 2z)H_{0,-1}]H_0]H_{-1} + [(1 + 2z)(-80H_0^2 - 64H_{0,1} - 128H_{0,-1}) - 96(1 + z)H_0]H_{-1}^2 \\
& - 12(49 + 6z)H_{0,1} + [32(-12 - 11z + 3z^2) + 192(2z - 1)H_{0,1}]H_{0,-1} - 64H_{0,-1}^2 \\
& + 8(-71 + 18z)H_{0,0,1} + 64(1 + 8z)H_{0,0,-1} + 16(-3 + 22z)H_{0,0,0,1} + [-32(8 - 31z + 3z^2) \\
& + [16(-13 + 11z) + 320(2z - 1)H_1]H_0 + 8(-5 + 18z)H_0^2 + 64(-13 + 16z)H_1 \\
& + 160(2z - 1)H_1^2 + [-288(1 + z) - 64(1 + 2z)H_0]H_{-1} + 128(1 + 2z)H_{-1}^2 \\
& - 64(-1 + 4z)H_{0,1} - 32(1 + 6z)H_{0,-1}]\zeta_2 + \frac{48}{5}(13 + 2z)\zeta_2^2 + [16(-6 + 31z) \\
& + 16(-3 + 26z)H_0 + 256(2z - 1)H_1 - 224(1 + 2z)H_{-1}]\zeta_3 + (2z - 1) \left(\frac{1}{15}H_0^5 - \frac{8}{3}H_1^5 \right. \\
& \left. + 104H_{0,0,0,0,1} - 656H_{0,0,0,1,1} - 192H_{0,0,1,0,1} + 32H_{0,0,1,1,1} - 64H_{0,1,0,1,1} - 128H_{0,1,1,1,1} - 72\zeta_5 \right) \\
& + \left[-2(159 + 625z) + (2z - 1) \left(-\frac{16}{3}H_1^4 - 24H_{0,1}^2 - 80H_{0,0,0,1} + 192H_{0,0,1,1} - 144H_{0,1,1,1} \right) \right. \\
& + (-4(-11 - 43z + 18z^2) + 8(-13 + 4z + 4z^2)H_{0,1} + 32(2z - 1)H_{0,1,1})H_1 \\
& + [-10(-19 + 18z) - 16(2z - 1)H_{0,1}]H_1^2 - \frac{4}{3}(-23 + 20z + 8z^2)H_1^3 \\
& \left. - 16(-14 + 39z + 3z^2)H_{0,1} + 16(-30 + 43z + 4z^2)H_{0,0,1} \right]
\end{aligned}$$

$$\begin{aligned}
& -16(-61 + 31z + 8z^2)H_{0,1,1}]H_0 + [-103 + 287z - 36z^2 + (2z - 1)(32H_{0,0,1} - 40H_{0,1,1}) \\
& + 2(-51 + 22z + 12z^2)H_1 + 2(-19 + 20z)H_1^2 - 8(z - 1)(25 + 2z)H_{0,1}]H_0^2 \\
& + \left[\frac{1}{3}(-53 - 6z + 24z^2) + (2z - 1)\left(\frac{8}{3}H_1^2 - 8H_{0,1}\right) + \frac{4}{3}(-31 + 28z + 4z^2)H_1 \right] H_0^3 \\
& + \left[\frac{1}{3}(-6 + z + 4z^2) + \frac{4}{3}(2z - 1)H_1 \right] H_0^4 + [-2(-593 + 575z) - 16(2z - 1)H_{0,1}^2 \\
& - 16(-13 + 4z + 4z^2)H_{0,0,1}]H_1 + [4(172 - 166z + 9z^2) + 32(2z - 1)H_{0,0,1}]H_1^2 \\
& - \frac{4}{3}(-181 + 148z + 18z^2)H_1^3 - 4(-9 + 8z + 2z^2)H_1^4 + [2(226 - 423z + 72z^2) \\
& + (2z - 1)(96H_{0,0,1} + 32H_{0,1,1})]H_{0,1} + 4(-13 + 4z + 4z^2)H_{0,1}^2 + 2(149 + 562z + 24z^2)H_{0,0,1} \\
& - 4(-187 + 104z + 36z^2)H_{0,1,1} - 8(-74 + 141z + 16z^2)H_{0,0,0,1} \\
& + 8(-155 + 100z + 16z^2)H_{0,0,1,1} - 8(-129 + 44z + 16z^2)H_{0,1,1,1} \\
& + \left[-2(248 - 337z + 36z^2) + (2z - 1)\left(-\frac{4}{3}H_0^3 + \frac{64}{3}H_1^3 - 96H_{0,0,1} + 80H_{0,1,1}\right) \right. \\
& + [2(-271 + 18z) - 8(-19 + 20z)H_1 + 80(2z - 1)H_{0,1}]H_0 + [-4(13 + 7z) \\
& - 16(2z - 1)H_1]H_0^2 + [20(-19 + 18z) + 32(2z - 1)H_{0,1}]H_1 + 4(-23 + 20z + 8z^2)H_1^2 \\
& + 16(-48 + 27z + 4z^2)H_{0,1}]\zeta_2 + \left. \left[\frac{16}{5}(25 + 74z + 14z^2) - 16(2z - 1)H_0 \right] \zeta_2^2 \right. \\
& + [2(-815 - 130z + 72z^2) + (2z - 1)(-4H_0^2 + 32H_1^2 - 32H_{0,1}) \\
& + [8(-39 + 33z + 16z^2) + 32(2z - 1)H_1]H_0 + 32(-17 + 12z + 4z^2)H_1]\zeta_3 \Big\} \\
& + C_A C_F T_F \left\{ 4(-1921 + 1903z) + L_M L_Q^2 \left[22 + (2z - 1)\left(-\frac{44}{3}H_0 - \frac{88}{3}H_1\right) \right] \right. \\
& + (1 + z)(-176H_{0,1,-1} - 176H_{0,-1,1} - 64H_{0,1,1,-1} - 64H_{0,1,-1,1} - 64H_{0,-1,1,1} + 512H_{0,0,1,0,1}) \\
& + (2z - 1)\left(\frac{8}{3}H_1^5 - 128H_{0,1,0,1,1} + 32H_{0,1,1,1,1}\right) + (1 + 2z)\left(-8H_{-1}^4 H_0 - \frac{2}{15}H_0^5 + 64H_{0,0,0,0,-1} \right. \\
& - 80H_{0,0,0,1,-1} - 80H_{0,0,0,-1,1} + 144H_{0,0,0,-1,-1} - 48H_{0,0,1,0,-1} - 32H_{0,0,1,1,-1} - 32H_{0,0,1,-1,1} \\
& + 32H_{0,0,1,-1,-1} - 32H_{0,0,-1,0,1} + 48H_{0,0,-1,0,-1} - 32H_{0,0,-1,1,1} + 32H_{0,0,-1,1,-1} + 32H_{0,0,-1,-1,1} \\
& - 352H_{0,0,-1,-1,-1} - 64H_{0,1,1,-1,-1} - 64H_{0,1,-1,1,-1} - 64H_{0,1,-1,-1,1} - 128H_{0,1,-1,-1,-1} \\
& - 32H_{0,-1,0,1,1} + 32H_{0,-1,0,1,-1} + 32H_{0,-1,0,-1,1} - 128H_{0,-1,0,-1,-1} - 64H_{0,-1,1,1,-1} \\
& - 64H_{0,-1,1,-1,1} - 128H_{0,-1,1,-1,-1} + 64H_{0,-1,-1,0,1} - 64H_{0,-1,-1,1,1} - 128H_{0,-1,-1,1,-1} \\
& \left. - 128H_{0,-1,-1,-1,1} - 192H_{0,-1,-1,-1,-1}\right) + L_M^2[992(z - 1) + [32(-16 + z) \\
& + 24(-11 + 12z)H_1 + 32(2z - 1)H_1^2 - 48(1 + 2z)H_{0,1}]H_0 + [8(-13 + 12z) \\
& + 16(2z - 1)H_1]H_0^2 - \frac{16}{3}(1 + z)H_0^3 + 8(-73 + 64z)H_1 + 24(-7 + 8z)H_1^2 + 16(2z - 1)H_1^3 \\
& - 320zH_{0,1} + 16(5 + 8z)H_{0,0,1} - 32(1 + 4z)H_{0,1,1} + [8(33 + 4z) + 48H_0 - 64(2z - 1)H_1]\zeta_2 \\
& \left. - 16(1 + 4z)\zeta_3] + L_M \left[\frac{1}{18}(-15897 + 22658z) + (2z - 1)(16H_1^4 - 16H_{0,-1}^2 - 224H_{0,1,1,1}) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + (1 + 2z) \left[-\frac{32}{3} H_{-1}^3 H_0 + (-24H_0 + 24H_0^2 + 32H_{0,-1}) H_{-1}^2 - 48H_{0,1,-1} \right. \\
& - 48H_{0,-1,1} - 48H_{0,-1,-1} + 96H_{0,0,1,-1} + 96H_{0,0,-1,1} - 32H_{0,0,-1,-1} - 64H_{0,1,1,-1} \\
& - 64H_{0,1,-1,1} - 64H_{0,-1,1,1} + 64H_{0,-1,-1,-1} \left. \right] + \left[-\frac{2}{27} (-7111 + 4982z) \right. \\
& + (2z - 1) \left(\frac{112}{3} H_1^3 - 80H_{0,1,1} \right) + \left[-\frac{4}{3} (211 + 52z) + (2z - 1) (64H_{0,1} - 96H_{0,-1}) \right] H_1 \\
& + \frac{8}{3} (-73 + 71z) H_1^2 + \frac{80}{3} (-5 + z) H_{0,1} - 48(-3 + 2z) H_{0,-1} + 16(5 + 14z) H_{0,0,1} \\
& + 16(7 + 6z) H_{0,0,-1} + 32(-5 + 2z) H_{0,1,-1} + 32(-5 + 2z) H_{0,-1,1} + 32(1 + 6z) H_{0,-1,-1} \left. \right] H_0 \\
& + \left[-2(-146 - 91z + 12z^2) - \frac{20}{3} (-5 + 16z) H_1 - 16H_{0,1} - 8(3 + 10z) H_{0,-1} \right] H_0^2 \\
& + \left[-\frac{10}{9} (-59 + 70z) - 8(2z - 1) H_1 \right] H_0^3 + \frac{2}{3} (5 + 4z) H_0^4 + \left[\frac{2}{27} (-10201 + 6722z) \right. \\
& + (2z - 1) (-96H_{0,0,1} + 192H_{0,0,-1} + 96H_{0,1,1}) - \frac{16}{3} (-29 + 40z) H_{0,1} \left. \right] H_1 \\
& + \left[\frac{2}{9} (-2279 + 1516z) - 32(2z - 1) H_{0,1} \right] H_1^2 + \frac{16}{9} (-71 + 79z) H_1^3 + \left[(1 + 2z) \left(\frac{8}{3} H_0^3 + 48H_{0,1} \right. \right. \\
& + 48H_{0,-1} - 96H_{0,0,1} + 32H_{0,0,-1} + 64H_{0,1,1} - 64H_{0,-1,-1} \left. \right) + [16(7 - z + 3z^2) \\
& + (1 + 2z) (64H_{0,1} - 64H_{0,-1})] H_0 - 24H_0^2 \left. \right] H_{-1} - \frac{8}{9} (-124 + 71z) H_{0,1} + [-16(7 - z + 3z^2) \\
& - 96(2z - 1) H_{0,1}] H_{0,-1} + \frac{8}{3} (133 + 40z) H_{0,0,1} + 48(-5 + 4z) H_{0,0,-1} - \frac{8}{3} (85 + 76z) H_{0,1,1} \\
& - 128(1 + 4z) H_{0,0,0,1} + 16(-13 + 10z) H_{0,0,0,-1} + 64(-1 + 6z) H_{0,0,1,1} + 64(-1 + 4z) H_{0,-1,0,1} \\
& + \left[\frac{4}{9} (637 + 298z + 108z^2) + (1 + 2z) (-8H_{-1} (9 + 8H_0) - 16H_0^2 - 16H_{-1}^2) \right. \\
& + \left[\frac{16}{3} (-11 + 10z) - 64(2z - 1) H_1 \right] H_0 - \frac{8}{3} (-97 + 80z) H_1 - 64(2z - 1) H_1^2 \\
& + 16(-3 + 10z) H_{0,1} + 64(1 + 3z) H_{0,-1} \left. \right] \zeta_2 + \frac{16}{5} (15 + 22z) \zeta_2^2 + \left[-\frac{16}{3} (-71 + 94z) + 80H_0 \right. \\
& - 112(2z - 1) H_1 + 16(1 + 2z) H_{-1} \left. \right] \zeta_3 \left. \right] + L_Q \left[-4(-281 + 275z) + (1 + z) (80H_{0,-1} \right. \\
& + 64H_{0,1,-1} + 64H_{0,-1,1}) + (2z - 1) \left(-\frac{16}{3} H_1^4 + 16H_{0,1}^2 - 48H_{0,1,1,1} \right) + (1 + 2z) \left(-\frac{64}{3} H_{-1}^3 H_0 \right. \\
& + \frac{2}{3} H_0^4 + 16H_{0,-1}^2 - 16H_{0,0,0,-1} + 192H_{0,0,1,1} + 32H_{0,0,1,-1} + 32H_{0,0,-1,1} + 32H_{0,0,-1,-1} \\
& + 64H_{0,1,-1,-1} + 32H_{0,-1,0,1} + 64H_{0,-1,1,-1} + 64H_{0,-1,-1,1} + 128H_{0,-1,-1,-1} \left. \right) \\
& + L_M^2 [-144(z - 1) + [-16(-5 + 8z) - 32(2z - 1) H_1] H_0 + 16(1 + z) H_0^2 \\
& - 24(-9 + 8z) H_1 - 32(2z - 1) H_1^2 + 48(1 + 2z) H_{0,1} - 16(5 + 2z) \zeta_2]
\end{aligned}$$

$$\begin{aligned}
& + L_M \left[\frac{2}{3} (-425 + 174z) + (2z - 1)(-32H_1^3 + 32H_{0,1,1}) + (1 + 2z)[(48H_0 - 16H_0^2 \right. \\
& - 64H_{0,1})H_{-1} + 56H_{0,1} - 48H_{0,-1} - 32H_{0,0,-1} + 64H_{0,1,-1} + 64H_{0,-1,1}] \\
& + \left[-\frac{4}{9}(623 + 566z) + (2z - 1)(-32H_1^2 - 32H_{0,1}) + \frac{32}{3}(1 + 10z)H_1 + 32(1 + 2z)H_{0,-1} \right] H_0 \\
& + \left[\frac{8}{3}(-23 + 58z) + 16(2z - 1)H_1 \right] H_0^2 - \frac{32}{3}(1 + z)H_0^3 - \frac{4}{9}(-515 + 76z)H_1 \\
& - \frac{16}{3}(-23 + 13z)H_1^2 - 32(3 + 2z)H_{0,0,1} + \left[-\frac{8}{3}(7 + 82z) + (1 + 2z)(64H_0 + 64H_{-1}) \right. \\
& + 64(2z - 1)H_1 \left. \right] \zeta_2 + 192z\zeta_3 \left. \right] + \left[-8(-49 + 24z) + [16(-15 + 16z) + 32(2z - 1)H_{0,1}]H_1 \right. \\
& + 4(-43 + 40z)H_1^2 + \frac{16}{3}(2z - 1)H_1^3 - 8(-29 + 26z + 2z^2)H_{0,1} + 56H_{0,-1} + 192(1 + z)H_{0,0,1} \\
& - 48(1 + 6z)H_{0,1,1} \left. \right] H_0 + [2(29 + 148z) + 8(z - 1)(25 + z)H_1 + 8(2z - 1)H_1^2 - 64(1 + z)H_{0,1} \\
& + 8(1 + 2z)H_{0,-1}]H_0^2 + \frac{8}{3}(1 + z^2)H_0^3 + [-44(-20 + 21z) - 64(2z - 1)H_{0,0,1}]H_1 \\
& - 8(-5 + 3z)H_1^2 - 4(-11 + 8z + 2z^2)H_1^3 + \left[(1 + 2z) \left(-\frac{16}{3}H_0^3 - 32H_{0,0,1} - 32H_{0,0,-1} \right. \right. \\
& - 64H_{0,1,-1} - 64H_{0,-1,1} - 128H_{0,-1,-1} \left. \right) + [-80(1 + z) - 32(1 + 2z)H_{0,-1}]H_0 \\
& - 4(11 + 8z)H_0^2 - 64(1 + z)H_{0,1} - 16(7 + 4z)H_{0,-1} \left. \right] H_{-1} + [(1 + 2z)(24H_0^2 + 32H_{0,1} \\
& + 64H_{0,-1}) + 8(7 + 4z)H_0]H_{-1}^2 + 8(59 + 43z)H_{0,1} + 16(-1 + z + z^2)H_{0,0,1} \\
& + 8(-3 + 8z)H_{0,0,-1} - 16(-24 + 31z + 3z^2)H_{0,1,1} + 16(7 + 4z)H_{0,-1,-1} \\
& - 32(5 + 4z)H_{0,0,0,1} + [-24(13 + 25z) + (1 + 2z)(-16H_0^2 - 64H_{-1}^2 - 48H_{0,-1}) \\
& + 16(-5 + z)H_0 - 288(z - 1)H_1 + 16(2z - 1)H_1^2 + [24(5 + 4z) + 64(1 + 2z)H_0]H_{-1} \\
& + 128(1 + z)H_{0,1}]\zeta_2 - \frac{144}{5}(3 + 2z)\zeta_2^2 + [16(-28 + 28z + 3z^2) - 48(3 + 4z)H_0 \\
& + 48(2z - 1)H_1 + 112(1 + 2z)H_{-1}]\zeta_3 \left. \right] + [-20(143 + 100z) + (1 + z)(-64H_{0,1,-1} \\
& - 64H_{0,-1,1}) + (1 + 2z)(-8H_{0,-1}^2 - 32H_{0,0,0,-1} - 32H_{0,0,1,-1} - 32H_{0,0,-1,1} - 64H_{0,0,-1,-1} \\
& - 64H_{0,1,-1,-1} - 32H_{0,-1,0,1} - 64H_{0,-1,1,-1} - 64H_{0,-1,-1,1} - 96H_{0,-1,-1,-1}) \\
& + [4(231 - 235z + 9z^2) + (2z - 1)(32H_{0,0,1} + 64H_{0,1,1}) - 8(-59 + 60z + 2z^2)H_{0,1}]H_1 \\
& + [-4(-101 + 96z) - 32(2z - 1)H_{0,1}]H_1^2 + \frac{4}{3}(29 - 36z + 4z^2)H_1^3 + \frac{4}{3}(2z - 1)H_1^4 \\
& + 8(-6 + 47z + 3z^2)H_{0,1} + 24(3 + 2z)H_{0,1}^2 - 40(3 + 2z)H_{0,-1} - 8(-7 + 96z + 4z^2)H_{0,0,1} \\
& + 8(-9 + 28z)H_{0,0,-1} + 32(-29 + 49z + 2z^2)H_{0,1,1} + 8(-9 + 28z)H_{0,-1,-1} \\
& + 32(7 + 6z)H_{0,0,0,1} - 16(25 + 34z)H_{0,0,1,1} + 16(7 + 10z)H_{0,1,1,1}]H_0 \\
& + \left[2(-223 - 174z + 9z^2) + (1 + 2z)(8H_{0,0,-1} + 24H_{0,-1,-1}) + [-4(-105 + 104z + 3z^2) \right. \\
& - 16(2z - 1)H_{0,1}]H_1 - 2(-49 + 52z)H_1^2 - \frac{8}{3}(2z - 1)H_1^3 + 4(-5 + 50z + 2z^2)H_{0,1}
\end{aligned}$$

$$\begin{aligned}
& -8(1+10z)H_{0,-1} - 96(1+z)H_{0,0,1} + 24(1+6z)H_{0,1,1} \Big] H_0^2 + \Big[-2(17+56z+2z^2) \\
& -\frac{8}{3}(z-1)(25+z)H_1 - \frac{8}{3}(2z-1)H_1^2 + \frac{64}{3}(1+z)H_{0,1} - \frac{8}{3}(1+2z)H_{0,-1} \Big] H_0^3 \\
& -\frac{2}{3}(5-4z+z^2)H_0^4 + [4(-697+712z) - 32(2z-1)H_{0,1}^2 + 16(-59+60z+2z^2)H_{0,0,1}]H_1 \\
& + [-2(337-348z+9z^2) + 64(2z-1)H_{0,0,1}]H_1^2 + \frac{4}{3}(-88+64z+9z^2)H_1^3 \\
& + 4(-8+8z+z^2)H_1^4 + \Big[(1+z)(176H_{0,1} + 64H_{0,1,1}) + (1+2z)\Big(\frac{4}{3}H_0^4 - 64H_{0,0,0,1} \\
& + 32H_{0,0,1,1} - 32H_{0,0,1,-1} - 32H_{0,0,-1,1} + 96H_{0,0,-1,-1} + 64H_{0,1,1,-1} + 64H_{0,1,-1,1} \\
& + 128H_{0,1,-1,-1} - 32H_{0,-1,0,1} + 64H_{0,-1,1,1} + 128H_{0,-1,1,-1} + 128H_{0,-1,-1,1} + 192H_{0,-1,-1,-1}\Big) \\
& + [64(1+z)(8+H_{0,1}) + (1+2z)(32H_{0,0,1} + 32H_{0,0,-1} + 64H_{0,1,-1} + 64H_{0,-1,1} \\
& + 32H_{0,-1,-1}) + 24(3+4z)H_{0,-1}]H_0 + 4(41+32z)H_0^2 + \frac{4}{3}(17+20z)H_0^3 \\
& + 48(10+7z)H_{0,-1} + 8(1+4z)H_{0,0,1} + 8(17+20z)H_{0,0,-1} + 16(13+16z)H_{0,1,-1} \\
& + 16(13+16z)H_{0,-1,1} + 32(11+14z)H_{0,-1,-1} \Big] H_{-1} + [(1+2z)(-8H_0^3 + 16H_{0,0,1} \\
& - 48H_{0,0,-1} - 32H_{0,1,1} - 64H_{0,1,-1} - 64H_{0,-1,1} - 96H_{0,-1,-1}) + [-24(10+7z) \\
& + (1+2z)(-32H_{0,1} - 16H_{0,-1})]H_0 - 2(35+44z)H_0^2 - 8(13+16z)H_{0,1} \\
& - 16(11+14z)H_{0,-1}]H_{-1}^2 + \Big[(1+2z)\Big(\frac{40}{3}H_0^2 + \frac{64}{3}H_{0,1} + 32H_{0,-1}\Big) + \frac{16}{3}(11+14z)H_0 \Big] H_{-1}^3 \\
& + [-4(437-86z+18z^2) - 256(1+z)H_{0,0,1} + 64(2z-1)H_{0,1,1}]H_{0,1} \\
& - 4(-59+60z+2z^2)H_{0,1}^2 + [-512(1+z) + (1+2z)(48H_{0,0,1} - 16H_{0,0,-1} \\
& + 64H_{0,-1,-1})]H_{0,-1} - 160zH_{0,-1}^2 - 4(235+166z+6z^2)H_{0,0,1} - 8(11+12z)H_{0,0,-1} \\
& + 8(-157+52z+9z^2)H_{0,1,1} - 48(10+7z)H_{0,-1,-1} + 64(-8+22z+z^2)H_{0,0,0,1} \\
& - 32(-4+11z)H_{0,0,0,-1} - 16(-27+105z+4z^2)H_{0,0,1,1} - 8(1+4z)H_{0,0,1,-1} \\
& - 8(1+4z)H_{0,0,-1,1} - 8(17+20z)H_{0,0,-1,-1} + 8(-55+70z+8z^2)H_{0,1,1,1} \\
& - 16(13+16z)H_{0,1,-1,-1} - 8(5+4z)H_{0,-1,0,1} - 16(13+16z)H_{0,-1,1,-1} \\
& - 16(13+16z)H_{0,-1,-1,1} - 32(11+14z)H_{0,-1,-1,-1} - 16(17+10z)H_{0,0,0,0,1} \\
& + 64(26+27z)H_{0,0,0,1,1} - 16(-7+62z)H_{0,0,1,1,1} + \Big[4(334+149z+9z^2) \\
& + (1+2z)\Big(\frac{8}{3}H_0^3 - \frac{112}{3}H_{-1}^3 - 16H_{0,0,-1} - 32H_{0,-1,-1}\Big) + [4(101+186z) + 288(z-1)H_1 \\
& - 16(2z-1)H_1^2 - 128(1+z)H_{0,1} + 32(1+2z)H_{0,-1}]H_0 + 8(6+5z)H_0^2 + [8(-71+75z) \\
& + 64(2z-1)H_{0,1}]H_1 - 4(7-8z+4z^2)H_1^2 - \frac{64}{3}(2z-1)H_1^3 + [-8(52+43z) \\
& + (1+2z)(-32H_0^2 - 32H_{0,-1}) - 48(5+6z)H_0]H_{-1} + [48(4+5z) + 80(1+2z)H_0]H_{-1}^2 \\
& - 16(1+28z+2z^2)H_{0,1} + 8(13+32z)H_{0,-1} + 192(1+z)H_{0,0,1} - 16(1+22z)H_{0,1,1} \\
& - 32(7+3z)\zeta_3 \Big] \zeta_2 + \Big[-\frac{4}{5}(-587+302z+28z^2) - \frac{32}{5}(-7+4z)H_0 - 48(2z-1)H_1
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{5}(1+2z)H_{-1} \Big] \zeta_2^2 + \Big[-4(-597-112z+18z^2) + (1+2z)(104H_{-1}^2 + 24H_{0,-1}) \\
& + [-32(-27+19z+2z^2) - 48(2z-1)H_1]H_0 + 48(1+z)H_0^2 \\
& - 4(-267+260z+16z^2)H_1 - 88(2z-1)H_1^2 + [-12(29+36z) - 80(1+2z)H_0]H_{-1} \\
& + 8(29+38z)H_{0,1}\zeta_3 - 4(-73+46z)\zeta_5 \Big\} + T_F^3 \Big\{ \frac{64}{9}L_M^2L_Q(2z-1) + L_M^2 \Big[-\frac{64}{9}(-3+4z) \\
& + (2z-1) \Big(-\frac{64}{9}H_0 - \frac{64}{9}H_1 \Big) \Big] \Big\} + A_{Qg}^{(3)} + \tilde{C}_g^{(3)}(N_F+1) \Big\}. \tag{B4}
\end{aligned}$$

APPENDIX C: THE TRANSFORMATION BETWEEN $\overline{\text{MS}}$ AND THE LARIN SCHEME IN THE NONSINGLET CASE

In Refs. [22,25] the massive OME $A_{qq,Q}^{(3),\text{NS}}$ has been given in the $\overline{\text{MS}}$ scheme. This also applies to the three-loop heavy flavor massive nonsinglet Wilson coefficient in [28], related to the associated three-loop massless Wilson coefficient [79]. As we present here all OMEs in the Larin scheme, we also provide $A_{qq,Q}^{(3),\text{NS}}$ in this scheme. We first decompose $A_{qq,Q}^{(3),\text{NS}}$ into its single and double mass pieces

$$A_{qq,Q}^{(3),\text{NS}} = A_{qq,Q}^{(3),\text{NS},s} + \tilde{A}_{qq,Q}^{(3),\text{NS}}. \tag{C1}$$

The renormalized single mass OME [22] is given by

$$A_{qq,Q}^{(2),\text{NS},L} = A_{qq,Q}^{(2),\text{NS},M} + \frac{1}{2}[\hat{\gamma}_{qq}^{(1),\text{NS},L} - \hat{\gamma}_{qq}^{(1),\text{NS},M}]L + a_{qq,Q}^{(2),\text{NS},L} - a_{qq,Q}^{(2),\text{NS},M}, \tag{C2}$$

$$\begin{aligned}
A_{qq,Q}^{(3),\text{NS},L,s} &= A_{qq,Q}^{(3),\text{NS},M,s} + \frac{1}{2}[(\gamma_{qq}^{(1),\text{NS},L} - \gamma_{qq}^{(1),\text{NS},M})\beta_{0,Q} - (\hat{\gamma}_{qq}^{(1),\text{NS},L} - \hat{\gamma}_{qq}^{(1),\text{NS},M})(\beta_0 + \beta_{0,Q})]L^2 \\
&+ \frac{1}{2}[\hat{\gamma}_{qq}^{(2),\text{NS},L} - \hat{\gamma}_{qq}^{(2),\text{NS},M} - 4(a_{qq,Q}^{(2),\text{NS},L} - a_{qq,Q}^{(2),\text{NS},M})(\beta_0 + \beta_{0,Q})]L + 4(\bar{a}_{qq,Q}^{(2),\text{NS},L} - \bar{a}_{qq,Q}^{(2),\text{NS},M}) \\
&- \frac{\beta_{0,Q}\zeta_2}{4}[\gamma_{qq}^{(1),\text{NS},L} - \gamma_{qq}^{(1),\text{NS},M}] + \delta m_1^{(0)}[\hat{\gamma}_{qq}^{(1),\text{NS},L} - \hat{\gamma}_{qq}^{(1),\text{NS},M}] + 2\delta m_1^{(-1)}(a_{qq,Q}^{(2),\text{NS},L} - a_{qq,Q}^{(2),\text{NS},M}) \\
&+ a_{qq,Q}^{(3),\text{NS},L} - a_{qq,Q}^{(3),\text{NS},M}, \tag{C3}
\end{aligned}$$

where the formula applies structurally in N and z space. In z space products have to be understood as convolutions. The constant part of the unrenormalized OME $A_{qq,Q}^{(3),\text{NS},L}$, $a_{qq,Q}^{(3),\text{NS},L}$, is related to the corresponding expression in the M scheme by

$$\begin{aligned}
a_{qq,Q}^{(3),\text{NS},L} &= a_{qq,Q}^{(3),\text{NS},M} + C_F \Big\{ T_F^2 \Big[-\frac{128W_2}{81N^3(1+N)^3} - N_F \Big(\frac{128W_3}{81N^3(1+N)^3} + \frac{64}{9N(1+N)}\zeta_2 \Big) - \frac{128}{9N(1+N)}\zeta_2 \Big] \\
&+ C_A T_F \Big[\frac{32W_1}{9N^3(1+N)^3}S_1 - \frac{16W_4}{81N^4(1+N)^4} + \frac{64}{3N(1+N)}S_3 - \frac{128(-3+4N+10N^2)}{27N^2(1+N)^2}S_{-2} \\
&+ \frac{128}{9N(1+N)}S_{-3} + \frac{256}{9N(1+N)}S_{-2,1} + \frac{176\zeta_2}{9N(1+N)} - \frac{128\zeta_3}{3N(1+N)} \Big] \Big\} \\
&+ C_F^2 T_F \Big[-\frac{128(12+17N-14N^3+3N^4)}{27N^3(1+N)^3}S_1 + \frac{32W_5}{27N^4(1+N)^4} + \frac{256(-3-N+5N^2)}{27N^2(1+N)^2}S_2 \\
&+ \frac{256(-3+4N+10N^2)}{27N^2(1+N)^2}S_{-2} - \frac{512}{9N(1+N)}S_3 - \frac{256}{9N(1+N)}S_{-3} - \frac{512}{9N(1+N)}S_{-2,1} + \frac{128}{3N(1+N)}\zeta_3 \Big], \tag{C4}
\end{aligned}$$

with the polynomials

$$W_1 = 3N^4 + 6N^3 + 5N^2 + 2N + 2, \quad (C5)$$

$$W_2 = 29N^4 + 38N^3 + 17N^2 + 8N + 6, \quad (C6)$$

$$W_3 = 30N^4 + 50N^3 + 24N^2 + 4N + 3, \quad (C7)$$

$$W_4 = 13N^6 - 113N^5 - 107N^4 - 255N^3 - 136N^2 - 294N - 144, \quad (C8)$$

$$W_5 = 115N^6 + 198N^5 + 199N^4 + 28N^3 + 72N^2 + 16N - 12. \quad (C9)$$

Here the expansion coefficients of the QCD- β function [41,80] are

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F N_F \quad (C10)$$

$$\beta_1 = \frac{34}{3}C_A^2 - 4\left(\frac{5}{3}C_A + C_F\right)T_F N_F \quad (C11)$$

and the expansion parameters of the unrenormalized heavy quark mass [41,81] are

$$\delta m_1^{(-1)} = 6C_F \quad (C12)$$

$$\delta m_1^{(0)} = -4C_F. \quad (C13)$$

Furthermore, the transformation relation of the asymptotic massive Wilson coefficient is

$$L_{qq,Q}^{(2),NS,L} = L_{qq,Q}^{(2),NS,M} + A_{qq,Q}^{(2),NS,L} - A_{qq,Q}^{(2),NS,M} + \hat{C}_q^{(2),NS,L} - \hat{C}_q^{(2),NS,M} \quad (C14)$$

$$L_{qq,Q}^{(3),NS,L} = L_{qq,Q}^{(3),NS,M} + A_{qq,Q}^{(3),NS,L} - A_{qq,Q}^{(3),NS,M} + A_{qq,Q}^{(2),NS,L} C_q^{(1),NS,L} - A_{qq,Q}^{(2),NS,M} C_q^{(1),NS,M} + C_q^{(3),NS,L} - C_q^{(3),NS,M}, \quad (C15)$$

which structurally again applies for Mellin N and z space. Here, the following functions contribute

$$C_q^{(1),NS,L} = C_q^{(1),NS,M} - C_F \frac{8}{N(N+1)} \quad (C16)$$

$$\hat{C}_q^{(2),NS,L} = \hat{C}_q^{(2),NS,M} - C_F T_F \frac{16(3+N-5N^2)}{9N^2(1+N)^2} \quad (C17)$$

$$\gamma_{qq}^{(1),NS,L} = \gamma_{qq}^{(1),NS,M} - 2\beta_0 z_{qq}^{(1)} \quad (C18)$$

$$a_{qq,Q}^{(2),NS,L} = a_{qq,Q}^{(2),NS,M} - 16C_F T_F \frac{5N^2 - N - 3}{9N^2(N+1)^2} \quad (C19)$$

$$\bar{a}_{qq,Q}^{(2),NS,L} = \bar{a}_{qq,Q}^{(2),NS,M} - C_F T_F \left[\frac{8(9+12N+10N^2+26N^3+28N^4)}{27N^3(N+1)^3} + \frac{8\zeta_2}{3N(N+1)} \right] \quad (C20)$$

$$\gamma_{qq}^{(2),NS,L} = \gamma_{qq}^{(2),NS,M} + 2\beta_0 [(z_{qq}^{(1)})^2 - 2z_{qq}^{(2),NS}] - 2\beta_1 z_{qq}^{(1)}. \quad (C21)$$

The functions $z_{ij}^{(k)}$ are given in Refs. [64,67,68].

$$z_{qq}^{(1)} = -\frac{8C_F}{N(N+1)}, \quad (C22)$$

$$z_{qq}^{(2),\text{NS}} = C_F T_F N_F \frac{16(-3 - N + 5N^2)}{9N^2(1 + N)^2} + C_A C_F \left\{ -\frac{4W_6}{9N^3(1 + N)^3} - \frac{16}{N(1 + N)} S_{-2} \right\} \\ + C_F^2 \left\{ \frac{8(2 + 5N + 8N^2 + N^3 + 2N^4)}{N^3(1 + N)^3} + \frac{16(1 + 2N)}{N^2(1 + N)^2} S_1 + \frac{16}{N(1 + N)} S_2 + \frac{32}{N(1 + N)} S_{-2} \right\}, \quad (\text{C23})$$

with

$$W_6 = 103N^4 + 140N^3 + 58N^2 + 21N + 36. \quad (\text{C24})$$

The corresponding expressions in z space read

$$a_{qq,Q}^{(3),\text{NS,L}}(z) = a_{qq,Q}^{(3),\text{NS,M}}(z) + C_F \left\{ T_F^2 \left[N_F \left((1 - z) \left(-\frac{1280}{27} - \frac{640}{81} H_0 - \frac{64}{27} H_0(z)^2 \right) - \frac{64(1 - z)}{9} \zeta_2 \right) \right. \right. \\ + (1 - z) \left(-\frac{3712}{81} - \frac{1280}{81} H_0 - \frac{128}{27} H_0^2 \right) - \frac{128}{9} (1 - z) \zeta_2 \Big] \\ + C_A T_F \left[(1 - z) \left(-\frac{3664}{81} + \frac{352}{9} H_1 + \frac{32}{3} H_0^2 H_1 - \frac{64}{3} H_0 H_{0,1} \right) \right. \\ + (1 + z) \left(\frac{1280}{27} H_{-1} H_0 + \frac{64}{9} H_{-1} H_0^2 - \frac{1280}{27} H_{0,-1} - \frac{128}{9} H_{0,0,-1} + \frac{256}{9} H_{0,1,-1} \right. \\ + \frac{256}{9} H_{0,-1,1} - \frac{128}{9} H_{0,1}(-1 + 2H_{-1}) \Big) - \frac{32}{81} (200 + 7z) H_0 - \frac{16}{27} (47 + 53z) H_0^2 \\ - \frac{128}{27} H_0^3 + \frac{256}{9} H_{0,0,1} + \left(-\frac{16}{27} (-89 + 57z) - \frac{64}{9} (-1 + 3z) H_0 \right. \\ + \frac{256}{9} (1 + z) H_{-1} \Big) \zeta_2 + \frac{128}{3} (-2 + z) \zeta_3 \Big] \Big\} + C_F^2 T_F \left[(1 - z) \left(\frac{4480}{27} \right. \right. \\ - \frac{896}{9} H_1 - \frac{1280}{27} H_0 H_1 - \frac{256}{9} H_0^2 H_1 + \frac{256}{9} H_0 H_{0,1} \Big) + (1 + z) \left(-\frac{2560}{27} H_{-1} H_0 \right. \\ - \frac{128}{9} H_{-1} H_0^2 + \frac{2560}{27} H_{0,-1} + \frac{256}{9} H_{0,0,-1} - \frac{512}{9} H_{0,1,-1} - \frac{512}{9} H_{0,-1,1} \\ + \frac{128}{9} H_{0,1}(-3 + 4H_{-1}) \Big) - \frac{32}{27} (-112 + 3z) H_0 + \frac{256}{27} (4 + 7z) H_0^2 \\ + \frac{64}{27} (1 + 3z) H_0^3 - \frac{512}{9} H_{0,0,1} + \left(\frac{256}{9} H_0 + (1 + z) \left(-\frac{128}{27} - \frac{512}{9} H_{-1} \right) \right) \zeta_2 - \frac{256}{9} (-4 + z) \zeta_3 \Big] \quad (\text{C25})$$

and

$$C_q^{(1),\text{NS,L}} = C_q^{(2),\text{NS,M}} - 8C_F(1 - z), \quad (\text{C26})$$

$$\hat{C}_q^{(2),\text{NS,L}} = \hat{C}_q^{(2),\text{NS,M}} + \frac{16}{9} C_F T_F (1 - z) (5 + 3H_0), \quad (\text{C27})$$

$$\gamma_{qq}^{(1),\text{NS,L}} = \gamma_{qq}^{(1),\text{NS,M}} + 16\beta_0 C_F (1 - z), \quad (\text{C28})$$

$$a_{qq,Q}^{(2),\text{NS,L}} = a_{qq,Q}^{(2),\text{NS,M}} - \frac{16}{9} C_F T_F (1 - z) (5 + 3H_0), \quad (\text{C29})$$

$$\bar{a}_{qq,Q}^{(2),\text{NS,L}} = \bar{a}_{qq,Q}^{(2),\text{NS,M}} - (1 - z) C_F T_F \left(\frac{224}{27} + \frac{8}{3} \zeta_2 + \frac{40}{9} H_0 + \frac{4}{3} H_0^2 \right), \quad (\text{C30})$$

$$\begin{aligned}
\hat{\gamma}_{qq}^{(2),\text{NS,L}} = & \hat{\gamma}_{qq}^{(2),\text{NS,M}} + C_A C_F T_F \left[-\frac{19904}{27} - \left(\frac{256}{9}(10-z) - \frac{256}{3}(1+z)H_{-1} \right) H_0 - \frac{128}{3}H_0^2 - \frac{256}{3}(1+z)H_{0,-1} + \frac{256}{3}\zeta_2 \right] \\
& + C_F^2 T_F \left[\frac{1472}{3} + \left(\frac{128}{3}(5+6z) - \frac{512}{3}(1+z)H_{-1} \right) H_0 + \frac{128}{3}(1+z)H_0^2 - \frac{256}{3}(1-z)H_0H_1 + \frac{512}{3}(1+z)H_{0,-1} \right. \\
& \left. - \frac{256}{3}(1+z)\zeta_2 \right] + C_F T_F^2 (1-2N_F)(1-z) \left[\frac{1280}{27} + \frac{256}{9}H_0 \right]. \quad (\text{C31})
\end{aligned}$$

For the double mass contributions [25] we obtain

$$\begin{aligned}
\tilde{A}_{qq,Q}^{(3),\text{NS,L}} = & \tilde{A}_{qq,Q}^{(3),\text{NS,M}} + \frac{1}{2}\beta_{0,Q}(\hat{\gamma}_{qq}^{(1),\text{NS,L}} - \hat{\gamma}_{qq}^{(1),\text{NS,M}})(L_1^2 + L_2^2) + 4(a_{qq,Q}^{(2),\text{NS,L}} - a_{qq,Q}^{(2),\text{NS,M}})(L_1 + L_2) \\
& + 8\beta_{0,Q}(\tilde{a}_{qq,Q}^{(2),\text{NS,L}} - \tilde{a}_{qq,Q}^{(2),\text{NS,M}}) + \tilde{a}_{qq,Q}^{(3),\text{NS,L}} - \tilde{a}_{qq,Q}^{(3),\text{NS,M}}, \quad (\text{C32})
\end{aligned}$$

with

$$\begin{aligned}
\tilde{a}_{qq,Q}^{(3),\text{NS,L}}(N) = & \tilde{a}_{qq,Q}^{(3),\text{NS,M}}(N) + C_F T_F^2 \left\{ -\frac{256}{9N(N+1)}(L_1^2 + L_1L_2 + L_2^2 + \zeta_2) + \frac{256(3+N+5N^2)}{27N^2(N+1)^2}(L_1 + L_2) \right. \\
& \left. - \frac{256(6+8N+17N^2+38N^3+29N^4)}{81N^3(N+1)^3} \right\}, \quad (\text{C33})
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_{qq,Q}^{(3),\text{NS,L}}(z) = & \tilde{a}_{qq,Q}^{(3),\text{NS,M}}(z) + C_F T_F^2 (1-z) \left\{ -\frac{256}{9}(L_1^2 + L_1L_2 + L_2^2 + \zeta_2) \right. \\
& \left. - \frac{256}{27}(5+3H_0)(L_1 + L_2) - \frac{256}{27}H_0^2 - \frac{2560}{81}H_0 - \frac{7424}{81} \right\}. \quad (\text{C34})
\end{aligned}$$

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