

Measurements of Partial Branching Fractions of Inclusive $B \rightarrow X_u \ell^+ \nu_\ell$ Decays with Hadronic Tagging

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We present measurements of partial branching fractions of inclusive semileptonic $B \rightarrow X_u \ell^+ \nu_\ell$ decays using the full Belle data set of 711 fb^{-1} of integrated luminosity at the $\Upsilon(4S)$ resonance and for $\ell = e, \mu$. Inclusive semileptonic $B \rightarrow X_u \ell^+ \nu_\ell$ decays are CKM suppressed and measurements are complicated by the large background from CKM-favored $B \rightarrow X_c \ell^+ \nu_\ell$ transitions, which have a similar signature. Using machine learning techniques, we reduce this and other backgrounds effectively, whilst retaining access to a large fraction of the $B \rightarrow X_u \ell^+ \nu_\ell$ phase space and high signal efficiency. We measure partial branching fractions in three phase-space regions covering about 31% to 86% of the accessible $B \rightarrow X_u \ell^+ \nu_\ell$ phase space. The most inclusive measurement corresponds to the phase space with lepton energies of $E_\ell^B > 1 \text{ GeV}$, and we obtain $\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (1.59 \pm 0.07 \pm 0.17) \times 10^{-3}$ from a two-dimensional fit of the hadronic mass spectrum and the four-momentum-transfer squared distribution, with the uncertainties denoting the statistical and systematic error. We find $|V_{ub}| = (4.10 \pm 0.09 \pm 0.22 \pm 0.15) \times 10^{-3}$ from an average of four calculations for the partial decay rate with the third uncertainty denoting the average theory error. This value is higher but compatible with the determination from exclusive semileptonic decays within 1.3 standard deviations. In addition, we report charmless inclusive partial branching fractions separately for B^+ and B^0 mesons as well as for electron and muon final states. No isospin breaking or lepton flavor universality violating effects are observed.

I. INTRODUCTION

Precision measurements of the absolute value of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{ub} are important to challenge the Standard Model of particle physics (SM) [1, 2]. In the SM, the CKM matrix is a 3×3 unitary matrix and responsible for the known charge-parity (CP) violating effects in the quark sector [3]. There are indications of CP violation in the neutrino sector [4], but it remains unclear if both sources of CP violation are sufficient to explain the matter dominance of today's universe. This motivates the search for new sources of CP-violating phenomena. If such exist in the form of heavy exotic particles that couple to quarks in some form, their presence might alter the properties of measurements constraining the unitarity of the CKM matrix [5]. Precise measurements of $|V_{ub}|$ and the CKM angle $\gamma = \phi_3$ are imperative to isolate such effects, as their measurements involve tree-level processes, which are expected to remain unaffected by new physics and thus provide an unbiased measure for the amount of CPV due to the Kobayashi-Maskawa (KM) mechanism [2] alone.

Charmless semileptonic decays of B mesons provide a clean avenue to measure $|V_{ub}|$, as their decay rate is theoretically better understood than purely hadronic transitions and their decay signature is more accessible than leptonic B meson decays. The existing measurements either focus on exclusive final states, with $B \rightarrow \pi \ell^+ \nu_\ell$ [6] and the ratio of $\Lambda_b \rightarrow p \mu^+ \nu_\mu$ and $\Lambda_b \rightarrow \Lambda_c \mu^+ \nu_\mu$ [7] providing the most precise measurements to date, and measurements reconstructing the $B \rightarrow X_u \ell^+ \nu_\ell$ decay fully inclusively¹. Central for both approaches are reliable predictions of the (partial) decay rates $\Delta\Gamma(B \rightarrow X_u \ell^+ \nu_\ell)$ (omitting the CKM factor) from theory to convert measured (partial or full) branching fractions, $\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell)$, into measurements of $|V_{ub}|$ via

$$|V_{ub}| = \sqrt{\frac{\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell)}{\tau_B \Delta\Gamma(B \rightarrow X_u \ell^+ \nu_\ell)}},$$

with τ_B denoting the B meson lifetime. For exclusive measurements, the non-perturbative parts of the decay rates can be reliably predicted by lattice QCD [8] or light-cone sum rules [9] and constrained by the measurements

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¹ Charge conjugation is implied throughout this paper. In addition, $B \rightarrow X_u \ell^+ \nu_\ell$ is defined as the average branching fraction of charged and neutral B meson decays and $\ell = e$ or μ .

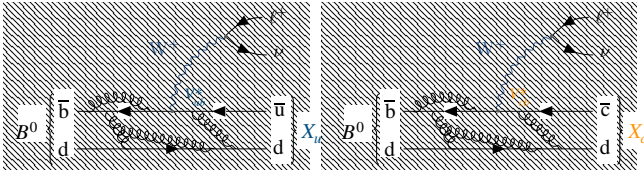


FIG. 1. The CKM suppressed and favored inclusive semileptonic processes $B \rightarrow X_u \ell^+ \nu_\ell$ (left) and $B \rightarrow X_c \ell^+ \nu_\ell$ (right) for a B^0 meson decay.

of the decay dynamics. The determination of $|V_{ub}|$ using inclusive decays is very challenging due to the large background from the CKM-favored $B \rightarrow X_c \ell^+ \nu_\ell$ process. Both processes have a very similar decay signature in the form of a high momentum lepton, a hadronic system, and missing energy from the neutrino that escapes detection. Figure 1 shows an illustration of both processes for a B^0 -meson decay. A clear separation of the processes is only possible in kinematic regions where $B \rightarrow X_c \ell^+ \nu_\ell$ is kinematically forbidden. In these regions, however, non-perturbative shape functions enter the description of the decay dynamics, making predictions for the decay rates dependent on the precise modeling. These functions parametrize at leading order the Fermi motion of the b quark inside the B meson. Properties of the leading-order Λ_{QCD}/m_b shape function can be determined using the photon energy spectrum of $B \rightarrow X_s \gamma$ decays and moments of the lepton energy or hadronic invariant mass in semileptonic B decays [10–12], but the modeling of both the leading and subleading shape functions introduces large theory uncertainties on the decay rate. In the future, more model-independent approaches aim to directly measure the leading-order shape function [13, 14].

As such methods are not yet realized, it is beneficial to extend the measurement region as much as possible into the $B \rightarrow X_c \ell^+ \nu_\ell$ dominated phase space. This was done, e.g., by Refs. [15, 16]. This reduces the theory uncertainties on the predicted partial rates [17–22], although making the measurement more prone to systematic uncertainties. This strategy is also adopted in the measurement described in this paper.

The corresponding world averages of $|V_{ub}|$ from both exclusive and inclusive determinations are [6]:

$$|V_{ub}^{\text{excl.}}| = (3.67 \pm 0.09 \pm 0.12) \times 10^{-3}, \quad (1)$$

$$|V_{ub}^{\text{incl.}}| = (4.32 \pm 0.12^{+0.12}_{-0.13}) \times 10^{-3}. \quad (2)$$

Here the uncertainties are experimental and from theory. Both world averages exhibit a disagreement of about 3 standard deviations between them. This disagreement is limiting the reach of present-day precision tests of the KM mechanism and searches for loop-level new physics, see e.g. Ref.[23] for a recent analysis.

One important experimental method to extend the probed $B \rightarrow X_u \ell^+ \nu_\ell$ phase space into regions dominated by $B \rightarrow X_c \ell^+ \nu_\ell$ transitions is the full reconstruction of the second B meson of the $e^+ e^- \rightarrow \Upsilon(4S) \rightarrow B\bar{B}$

process. This process is referred to as “tagging” and allows for the reconstruction of the hadronic X system of the semileptonic process. In addition, the neutrino four-momentum can be reconstructed. Properties of both are instrumental to distinguish $B \rightarrow X_u \ell^+ \nu_\ell$ and $B \rightarrow X_c \ell^+ \nu_\ell$ processes. In this manuscript the reconstruction of the second B meson and the separation of $B \rightarrow X_u \ell^+ \nu_\ell$ from $B \rightarrow X_c \ell^+ \nu_\ell$ processes were carried out using machine learning approaches. Several neural networks were trained to identify correctly reconstructed tag-side B mesons. The distinguishing variables of the classification algorithm were carefully selected in order not to introduce a bias in the measured partial branching fractions. In addition, the modeling of backgrounds was validated in $B \rightarrow X_c \ell^+ \nu_\ell$ enriched selections. We report the measurement of three partial branching fractions, covering 30% - 85% of the accessible $B \rightarrow X_u \ell^+ \nu_\ell$ phase space. The measurement of fully differential distributions, which allow one to determine the leading and subleading shape functions, is left for future work.

The main improvement over the previous Belle result of Ref. [16] lies in the adoption of a more efficient tagging algorithm for the reconstruction of the second B meson and the improvements of the $B \rightarrow X_u \ell^+ \nu_\ell$ signal and $B \rightarrow X_c \ell^+ \nu_\ell$ background descriptions. In addition, the full Belle data set of 711 fb^{-1} is analyzed and we avoid the direct use of kinematic properties of the candidate semileptonic decay in the background suppression. After the final selection we retain a factor of approximately 1.8 times more signal events than the previous analysis.

The remainder of this manuscript is organized as follows: Section II provides an overview of the data set and the simulated signal and background samples, that were used in the analysis. Section III details the analysis strategy and reconstruction of the hadronic X system of the semileptonic decay. Section IV introduces the fit procedure used to separate $B \rightarrow X_u \ell^+ \nu_\ell$ signal from background contributions. Section V lists the systematic uncertainties affecting the measurements and Section VI summarizes sideband studies central to validate the modeling of the crucial $B \rightarrow X_c \ell^+ \nu_\ell$ background processes. Finally, Section VII shows the selected signal events and compares them with the expectation from simulation. In Section VIII the measured partial branching fractions and subsequent values of $|V_{ub}|$ are discussed. Section IX presents our conclusions.

II. DATA SET AND SIMULATED SAMPLES

The analysis utilizes the full Belle data set of $(772 \pm 10) \times 10^6$ B meson pairs, which were produced at the KEKB accelerator complex [24] with a center-of-mass energy of $\sqrt{s} = 10.58 \text{ GeV}$ corresponding to the $\Upsilon(4S)$ resonance. In addition, 79 fb^{-1} of collision events recorded 60 MeV below the $\Upsilon(4S)$ resonance peak are used to derive corrections and for cross-checks.

The Belle detector is a large-solid-angle magnetic spec-

trometer that consists of a silicon vertex detector, a 50-layer central drift chamber (CDC), an array of aerogel threshold Cherenkov counters (ACC), a barrel-like arrangement of time-of-flight scintillation counters (TOF), and an electromagnetic calorimeter composed of CsI(Tl) crystals (ECL) located inside a superconducting solenoid coil that provides a 1.5 T magnetic field. An iron flux return located outside of the coil is instrumented to detect K_L^0 mesons and to identify muons (KLM). A more detailed description of the detector and its layout and performance can be found in Ref. [25] and in references therein.

Charged tracks are identified as electron or muon candidates by combining the information of multiple sub-detectors into a lepton identification likelihood ratio, $\mathcal{L}_{\text{LID}}$. For electrons, the most important identifying features are the ratio of the energy deposition in the ECL with respect to the reconstructed track momentum, the energy loss in the CDC, the shower shape in the ECL, the quality of the geometrical matching of the track to the shower position in the ECL, and the photon yield in the ACC [26]. Muon candidates can be identified from charged track trajectories extrapolated to the outer detector. The most important identifying features are the difference between expected and measured penetration depth as well as the transverse deviation of KLM hits from the extrapolated trajectory [27]. Charged tracks are identified as pions or kaons using a likelihood ratio $\mathcal{L}_{K/\pi \text{ ID}} = \mathcal{L}_{K \text{ ID}} / (\mathcal{L}_{K \text{ ID}} + \mathcal{L}_{\pi \text{ ID}})$. The most important identifying features of the kaon ($\mathcal{L}_{K \text{ ID}}$) and pion ($\mathcal{L}_{\pi \text{ ID}}$) likelihoods for low momentum particles with transverse momentum below 1 GeV in the laboratory frame are the recorded energy loss by ionization, dE/dx , in the CDC, and the time of flight information from the TOF. Higher-momentum kaon and pion classification relies on the Cherenkov light recorded in the ACC. In order to avoid the difficulties in understanding the efficiencies of reconstructing K_L^0 mesons, they are not explicitly reconstructed or used in this analysis.

Photons are identified as energy depositions in the ECL, vetoing clusters to which an associated track can be assigned. Only photons with an energy deposition of $E_\gamma > 100$ MeV, 150 MeV, and 50 MeV in the forward endcap, backward endcap and barrel part of the calorimeter, respectively, are considered. We reconstruct π^0 candidates from photon candidates. The invariant mass is required to fall inside a window² of $m_{\gamma\gamma} \in [0.12, 0.15]$ GeV, which corresponds to about 2.5 times the π^0 mass resolution.

Monte Carlo (MC) samples of B meson decays and continuum processes ($e^+e^- \rightarrow q\bar{q}$ with $q = u, d, s, c$) are simulated using the **EvtGen** generator [28]. These samples are used to evaluate reconstruction efficiencies and acceptance, and to estimate background contami-

TABLE I. Branching fractions for $B \rightarrow X_u \ell^+ \nu_\ell$ and $B \rightarrow X_c \ell^+ \nu_\ell$ background processes that were used are listed. More details on the applied corrections can be found in the text. We neglect the small contribution from $B^+ \rightarrow D_s^{(*)} K^+ \ell^+ \nu_\ell$ which has a branching fraction of similar size as $B \rightarrow D\pi\pi \ell^+ \nu_\ell$.

B	Value B^+	Value B^0
$B \rightarrow X_u \ell^+ \nu_\ell$		
$B \rightarrow \pi \ell^+ \nu_\ell$	$(7.8 \pm 0.3) \times 10^{-5}$	$(1.5 \pm 0.06) \times 10^{-4}$
$B \rightarrow \eta \ell^+ \nu_\ell$	$(3.9 \pm 0.5) \times 10^{-5}$	-
$B \rightarrow \eta' \ell^+ \nu_\ell$	$(2.3 \pm 0.8) \times 10^{-5}$	-
$B \rightarrow \omega \ell^+ \nu_\ell$	$(1.2 \pm 0.1) \times 10^{-4}$	-
$B \rightarrow \rho \ell^+ \nu_\ell$	$(1.6 \pm 0.1) \times 10^{-4}$	$(2.9 \pm 0.2) \times 10^{-4}$
$B \rightarrow X_u \ell^+ \nu_\ell$	$(2.2 \pm 0.3) \times 10^{-3}$	$(2.0 \pm 0.3) \times 10^{-3}$
$B \rightarrow X_c \ell^+ \nu_\ell$		
$B \rightarrow D \ell^+ \nu_\ell$	$(2.5 \pm 0.1) \times 10^{-2}$	$(2.3 \pm 0.1) \times 10^{-2}$
$B \rightarrow D^* \ell^+ \nu_\ell$	$(5.4 \pm 0.1) \times 10^{-2}$	$(5.1 \pm 0.1) \times 10^{-2}$
$B \rightarrow D_0^* \ell^+ \nu_\ell$	$(4.2 \pm 0.8) \times 10^{-3}$	$(3.9 \pm 0.7) \times 10^{-3}$
$(\hookrightarrow D\pi)$		
$B \rightarrow D_1^* \ell^+ \nu_\ell$	$(4.2 \pm 0.8) \times 10^{-3}$	$(3.9 \pm 0.8) \times 10^{-3}$
$(\hookrightarrow D^* \pi)$		
$B \rightarrow D_1 \ell^+ \nu_\ell$	$(4.2 \pm 0.3) \times 10^{-3}$	$(3.9 \pm 0.3) \times 10^{-3}$
$(\hookrightarrow D^* \pi)$		
$B \rightarrow D_2^* \ell^+ \nu_\ell$	$(1.2 \pm 0.1) \times 10^{-3}$	$(1.1 \pm 0.1) \times 10^{-3}$
$(\hookrightarrow D^* \pi)$		
$B \rightarrow D_2^* \ell^+ \nu_\ell$	$(1.8 \pm 0.2) \times 10^{-3}$	$(1.7 \pm 0.2) \times 10^{-3}$
$(\hookrightarrow D\pi)$		
$B \rightarrow D_1 \ell^+ \nu_\ell$	$(2.4 \pm 1.0) \times 10^{-3}$	$(2.3 \pm 0.9) \times 10^{-3}$
$(\hookrightarrow D\pi\pi)$		
$B \rightarrow D\pi\pi \ell^+ \nu_\ell$	$(0.6 \pm 0.6) \times 10^{-3}$	$(0.6 \pm 0.6) \times 10^{-3}$
$B \rightarrow D^* \pi\pi \ell^+ \nu_\ell$	$(2.2 \pm 1.0) \times 10^{-3}$	$(2.0 \pm 1.0) \times 10^{-3}$
$B \rightarrow D\eta \ell^+ \nu_\ell$	$(4.0 \pm 4.0) \times 10^{-3}$	$(4.0 \pm 4.0) \times 10^{-3}$
$B \rightarrow D^* \eta \ell^+ \nu_\ell$	$(4.0 \pm 4.0) \times 10^{-3}$	$(4.0 \pm 4.0) \times 10^{-3}$
$B \rightarrow X_c \ell^+ \nu_\ell$	$(10.8 \pm 0.4) \times 10^{-2}$	$(10.1 \pm 0.4) \times 10^{-2}$

nations. The sample sizes used correspond to approximately ten and five times, respectively, the Belle collision data for B meson and continuum decays. The interactions of particles traversing the detector are simulated using **Geant3** [29]. Electromagnetic final-state radiation is simulated using the **PHOTOS** [30] package for all charged final-state particles. The efficiencies in the MC are corrected using data-driven methods to account for, e.g., differences in identification and reconstruction efficiencies.

The most important background processes are semileptonic $B \rightarrow X_c \ell^+ \nu_\ell$ decays and continuum processes, which both can produce high-momentum leptons in a momentum range similar to the $B \rightarrow X_u \ell^+ \nu_\ell$ process. The semileptonic background from $B \rightarrow X_c \ell^+ \nu_\ell$ decays is dominated by $B \rightarrow D \ell^+ \nu_\ell$ and $B \rightarrow D^* \ell^+ \nu_\ell$ decays. The $B \rightarrow D \ell^+ \nu_\ell$ decays are modeled using the

² We use natural units: $\hbar = c = 1$.

BGL parametrization [31] with form factor central values and uncertainties taken from the fit in Ref. [32]. For $B \rightarrow D^* \ell^+ \nu_\ell$ we use the BGL implementation proposed by Refs. [33, 34] with form factor central values and uncertainties from the fit to the measurement of Ref. [35]. Both backgrounds are normalized to the average branching fraction of Ref. [6] assuming isospin symmetry. Semileptonic $B \rightarrow D^{**} \ell^+ \nu_\ell$ decays with $D^{**} = \{D_0^*, D_1^*, D_1, D_2^*\}$ denoting the four orbitally excited charmed mesons are modeled using the heavy-quark-symmetry-based form factors proposed in Ref. [36]. We simulate all D^{**} decays using masses and widths from Ref. [37]. For the branching fractions we adopt the values of Ref. [6] and correct them to account for missing isospin-conjugated and other established decay modes, following the prescription given in Ref. [36]. To correct for the fact that the measurements were carried out in the $D^{**0} \rightarrow D^{(*)+} \pi^-$ decay modes, we account for the missing isospin modes with a factor of

$$f_\pi = \frac{\mathcal{B}(\overline{D}^{**0} \rightarrow D^{(*)-} \pi^+)}{\mathcal{B}(\overline{D}^{**0} \rightarrow \overline{D}^{(*)} \pi)} = \frac{2}{3}. \quad (3)$$

The measurements of the $B \rightarrow D_2^* \ell \bar{\nu}_\ell$ in Ref. [6] are converted to only account for the $\overline{D}_2^{*0} \rightarrow D^{*-} \pi^+$ decay. To also account for $\overline{D}_2^{*0} \rightarrow D^- \pi^+$ contributions, we apply a factor of [37]

$$f_{D_2^*} = \frac{\mathcal{B}(\overline{D}_2^{*0} \rightarrow D^- \pi^+)}{\mathcal{B}(\overline{D}_2^{*0} \rightarrow D^{*-} \pi^+)} = 1.54 \pm 0.15. \quad (4)$$

The world average of $B \rightarrow D_1^* \ell \bar{\nu}_\ell$ given in Ref. [6] combines measurements, which show poor agreement, and the resulting probability of the combination is below 0.01%. Notably, the measurement of Ref. [38] is in conflict with the measured branching fractions of Refs. [39, 40] and with the expectation of $\mathcal{B}(B \rightarrow D_1^* \ell \bar{\nu}_\ell)$ being of similar size than $\mathcal{B}(B \rightarrow D_0 \ell \bar{\nu}_\ell)$ [41, 42]. We perform our own average excluding the conflicting measurement and use

$$\mathcal{B}(B^+ \rightarrow \overline{D}_1^{*0} (\rightarrow D^{*-} \pi^+) \ell^+ \nu_\ell) = (0.28 \pm 0.06) \times 10^{-2}. \quad (5)$$

The world average of $B \rightarrow D_1 \ell \bar{\nu}_\ell$ does not include contributions from prompt three-body decays of $D_1 \rightarrow D \pi \pi$. We account for these using a factor [43]

$$f_{D_1} = \frac{\mathcal{B}(\overline{D}_1^0 \rightarrow D^{*-} \pi^+)}{\mathcal{B}(\overline{D}_1^0 \rightarrow \overline{D}^0 \pi^+ \pi^-)} = 2.32 \pm 0.54. \quad (6)$$

We subtract the contribution of $D_1 \rightarrow D \pi \pi$ from the measured non-resonant plus resonant $B \rightarrow D \pi \pi \ell \bar{\nu}_\ell$ branching fraction of Ref. [44]. To account for missing isospin-conjugated modes of the three-hadron final states we adopt the prescription from Ref. [44], which calculates an average isospin correction factor of

$$f_{\pi\pi} = \frac{\mathcal{B}(\overline{D}^{**0} \rightarrow \overline{D}^{(*)0} \pi^+ \pi^-)}{\mathcal{B}(\overline{D}^{**0} \rightarrow \overline{D}^{(*)} \pi \pi)} = \frac{1}{2} \pm \frac{1}{6}. \quad (7)$$

The uncertainty takes into account the full spread of final states ($f_0(500) \rightarrow \pi\pi$ or $\rho \rightarrow \pi\pi$ result in $f_{\pi\pi} = 2/3$ and $1/3$, respectively) and the non-resonant three-body decays ($f_{\pi\pi} = 3/7$). We further assume that

$$\begin{aligned} \mathcal{B}(\overline{D}_2^* \rightarrow \overline{D} \pi) + \mathcal{B}(\overline{D}_2^* \rightarrow \overline{D}^* \pi) &= 1, \\ \mathcal{B}(\overline{D}_1 \rightarrow \overline{D}^* \pi) + \mathcal{B}(\overline{D}_1 \rightarrow \overline{D} \pi \pi) &= 1, \\ \mathcal{B}(\overline{D}_1^* \rightarrow \overline{D}^* \pi) &= 1, \quad \text{and} \quad \mathcal{B}(\overline{D}_0 \rightarrow \overline{D} \pi) = 1. \end{aligned} \quad (8)$$

For the remaining $B \rightarrow D^{(*)} \pi \pi \ell^+ \nu_\ell$ contributions we use the measured value of Ref. [44]. The remaining “gap” between the sum of all considered exclusive modes and the inclusive $B \rightarrow X_c \ell^+ \nu_\ell$ branching fraction is filled in equal parts with $B \rightarrow D \eta \ell^+ \nu_\ell$ and $B \rightarrow D^* \eta \ell^+ \nu_\ell$ and for both we assume a 100% uncertainty. We simulate $B \rightarrow D^{(*)} \pi \pi \ell^+ \nu_\ell$ and $B \rightarrow D^{(*)} \eta \ell^+ \nu_\ell$ final states assuming that they are produced by the decay of two broad resonant states D_{gap}^{**} with masses and widths identical to D_1^* and D_0 . Although there is currently no experimental evidence for decays of charm $1P$ states into these final states or the existence of such an additional broad state (e.g. a $2S$) in semileptonic transitions, this description provides a better kinematic description of the initial three-body decay, $B \rightarrow D_{\text{gap}}^{**} \ell \bar{\nu}_\ell$, than e.g. a model based on the equidistribution of all final-state particles in phase space. For the form factors we adapt Ref. [36].

Semileptonic $B \rightarrow X_u \ell^+ \nu_\ell$ decays are modeled as a mixture of specific exclusive modes and non-resonant contributions. We normalize their corresponding branching fractions to the world averages from Ref. [37]: semileptonic $B \rightarrow \pi \ell^+ \nu_\ell$ decays are simulated using the BCL parametrization [45] with form factor central values and uncertainties from the global fit carried out by Ref. [46]. The processes of $B \rightarrow \rho \ell^+ \nu_\ell$ and $B \rightarrow \omega \ell^+ \nu_\ell$ are modeled using the BCL form factor parametrization. We fit the measurements of Refs. [47–49] in combination with the light-cone sum rule predictions of Ref. [9] to determine a set of form factor central values and uncertainties. The processes of $B \rightarrow \eta \ell^+ \nu_\ell$ and $B \rightarrow \eta' \ell^+ \nu_\ell$ are modeled using the LCSR calculation of Ref. [50]. For the uncertainties we assume for these states that the pole-parameters $\alpha^{+/0}$ and the form factor normalization $f_{B\eta}^+(0)$ at maximum recoil can be treated as uncorrelated. In addition to these narrow resonances, we simulate non-resonant $B \rightarrow X_u \ell^+ \nu_\ell$ decays with at least two pions in the final state following the DFN model [51]. The triple differential rate of this model is a function of the four-momentum-transfer squared (q^2), the lepton energy (E_ℓ^B) in the B rest-frame, and the hadronic invariant mass squared (M_X^2) of the X_u system at next-to-leading order precision in the strong coupling constant α_s . This triple differential rate is convolved with a non-perturbative shape function using an ad-hoc exponential model. The free parameters of the model are the b quark mass in the Kagan-Neubert scheme [52], $m_b^{\text{KN}} = (4.66 \pm 0.04) \text{ GeV}$ and a non-perturbative parameter $a^{\text{KN}} = 1.3 \pm 0.5$. The values of these parameters

were determined in Ref. [53] from a fit to $B \rightarrow X_c \ell^+ \nu_\ell$ and $B \rightarrow X_s \gamma$ decay properties. At leading order, the non-perturbative parameter a^{KN} is related to the average momentum squared of the b quark inside the B meson and determines the second moment of the shape function. It is defined as $a^{\text{KN}} = -3\bar{\Lambda}^2/\lambda_1 - 1$ with the binding energy $\bar{\Lambda} = m_B - m_b^{\text{KN}}$ and the kinetic energy parameter λ_1 . The hadronization of the parton-level $B \rightarrow X_u \ell^+ \nu_\ell$ DFN simulation is carried out using the JETSET algorithm [54], producing final states with two or more mesons. The inclusive and exclusive $B \rightarrow X_u \ell^+ \nu_\ell$ predictions are combined using a so-called ‘hybrid’ approach, which is a method originally suggested by Ref. [55], and our implementation closely follows Ref. [56] and uses the library of Ref. [57]. To this end, we combine both predictions such that the partial branching fractions in the triple differential rate of the inclusive ($\Delta\mathcal{B}_{ijk}^{\text{incl}}$) and combined exclusive ($\Delta\mathcal{B}_{ijk}^{\text{excl}}$) predictions reproduce the inclusive values. This is achieved by assigning weights to the inclusive contributions w_{ijk} such that

$$\Delta\mathcal{B}_{ijk}^{\text{incl}} = \Delta\mathcal{B}_{ijk}^{\text{excl}} + w_{ijk} \times \Delta\mathcal{B}_{ijk}^{\text{incl}}, \quad (9)$$

with i, j, k denoting the corresponding bin in the three dimensions of q^2 , E_ℓ^B , and M_X :

$$\begin{aligned} q^2 &= [0, 2.5, 5, 7.5, 10, 12.5, 15, 20, 25] \text{ GeV}^2, \\ E_\ell^B &= [0, 0.5, 1, 1.25, 1.5, 1.75, 2, 2.25, 3] \text{ GeV}, \\ M_X &= [0, 1.4, 1.6, 1.8, 2, 2.5, 3, 3.5] \text{ GeV}. \end{aligned}$$

To study the model dependence of the DFN shape function, we also determine weights using the BLNP model of Ref. [58] and treat the difference later as a systematic uncertainty. For the b quark mass in the shape-function scheme we use $m_b^{\text{SF}} = 4.61 \text{ GeV}$ and $\mu_\pi^{2\text{SF}} = 0.20 \text{ GeV}^2$. Figures detailing the hybrid model construction can be found in Appendix A.

Table I summarizes the branching fractions for the signal and the important $B \rightarrow X_c \ell^+ \nu_\ell$ background processes that were used. Figure 2 shows the generator-level distributions and yields of $B \rightarrow X_c \ell^+ \nu_\ell$ and $B \rightarrow X_u \ell^+ \nu_\ell$ after the tag-side reconstruction (cf. Section III). The $B \rightarrow X_u \ell^+ \nu_\ell$ yields were scaled up by a factor of 50 to make them visible. A clear separation can be obtained at low values of M_X and high values of E_ℓ^B .

III. ANALYSIS STRATEGY, HADRONIC TAGGING, AND X RECONSTRUCTION

A. Neutral Network Based Tag Side Reconstruction

We reconstruct collision events using the hadronic full reconstruction algorithm of Ref. [59]. The algorithm reconstructs one of the B mesons produced in the collision event using hadronic decay channels. We label

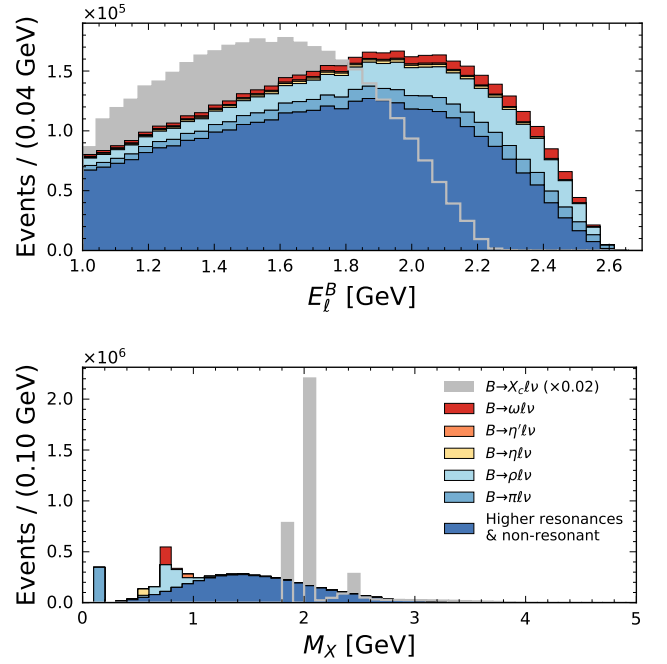


FIG. 2. The generator-level E_ℓ^B and M_X distributions of the CKM suppressed and favored inclusive semileptonic processes, $B \rightarrow X_u \ell^+ \nu_\ell$ (scaled up by a factor of 50) and $B \rightarrow X_c \ell^+ \nu_\ell$, respectively, are shown, using the models described in the text.

such B mesons in the following as B_{tag} . Instead of attempting to reconstruct as many B meson decay cascades as possible, the algorithm employs a hierarchical reconstruction ansatz in four stages: at the first stage, neural networks are trained to identify charged tracks and neutral energy depositions as detector stable particles ($e^+, \mu^+, K^+, \pi^+, \gamma$), neutral π^0 candidates, or K_S^0 candidates. At the second stage, these candidate particles are combined into heavier meson candidates ($J/\psi, D^0, D^+, D_s$) and for each target final state a neural network is trained to identify probable candidates. In addition to the classifier output from the first stage, vertex fit probabilities of the candidate combinations, and the full four-momentum of the combination are passed to the input layer. At the third stage, candidates for D^{*0}, D^{*+} , and D_s^* mesons are formed and separate neural networks are trained to identify viable combinations. The input layer aggregates the output classifiers from all previous reconstruction stages. The final stage combines the information from all previous stages to form B_{tag} candidates. The viability of such combinations is again assessed by a neural network that was trained to distinguish correctly reconstructed candidates from wrong combinations and whose output classifier score we denote by $\mathcal{O}_{\text{FR}}$. Over 1104 decay cascades are reconstructed in this manner, achieving an efficiency of 0.28% and 0.18% for charged and neutral B meson pairs [60], respectively. Finally, the output of this classifier is used as an input

and combined with a range of event shape variables to train a neural network to distinguish reconstructed B meson candidates from continuum processes. The output classifier score of this neural network is denoted as $\mathcal{O}_{\text{Cont}}$. Both classifier scores are mapped to a range of $[0, 1)$ signifying the reconstruction quality of poor to excellent candidates. We retain B_{tag} candidates that show at least moderate agreement based on these two outputs and require that $\mathcal{O}_{\text{FR}} > 10^{-4}$ and $\mathcal{O}_{\text{Cont}} > 10^{-4}$. Despite these relatively low values, knowledge of the charge and momentum of the decay constituents in combination with the known beam-energy allows one to infer the flavor and four-momentum of the B_{tag} candidate. We require the B_{tag} candidates to have at least a beam-constrained mass of

$$M_{\text{bc}} = \sqrt{E_{\text{beam}}^2 - |\mathbf{p}_{\text{tag}}|^2} > 5.27 \text{ GeV}, \quad (10)$$

with $\mathbf{p}_{\text{tag}}$ denoting the momentum of the B_{tag} candidate in the center-of-mass frame of the colliding e^+e^- -pair. Furthermore, $E_{\text{beam}} = \sqrt{s}/2$ denotes half the center-of-mass energy of the colliding e^+e^- -pair. The energy difference

$$\Delta E = E_{\text{tag}} - E_{\text{beam}}, \quad (11)$$

is already used in the input layer of the neural network trained in the final stage of the reconstruction. Here E_{tag} denotes the energy of the B_{tag} candidate in the center-of-mass frame of the colliding e^+e^- -pair. In each event a single B_{tag} candidate is then selected according to the highest $\mathcal{O}_{\text{FR}}$ score of the hierarchical full reconstruction algorithm. All tracks and clusters not used in the reconstruction of the B_{tag} candidate are used to define the signal side.

B. Signal Side Reconstruction

The signal side of the event is reconstructed by identifying a well-reconstructed lepton with $E_{\ell}^B = |\mathbf{p}_{\ell}^B| > 1 \text{ GeV}$ in the signal B rest frame³ using the likelihood mentioned in Section II. The signal B rest frame is calculated using the momentum of the B_{tag} candidate via

$$p_{\text{sig}} = p_{e^+e^-} - \left(\sqrt{m_B^2 + |\mathbf{p}_{\text{tag}}|^2}, \mathbf{p}_{\text{tag}} \right), \quad (12)$$

with $p_{e^+e^-}$ denoting the four-momentum of the colliding electron-positron pair. Leptons from J/ψ and photon conversions in detector material are rejected by combining the lepton candidate with oppositely charged tracks

(t) on the signal side and demanding that $m_{\ell t} > 0.14 \text{ GeV}$ and $m_{et} \notin [3.05, 3.15] \text{ GeV}$ or $m_{\mu t} \notin [3.06, 3.12] \text{ GeV}$. If multiple lepton candidates are present on the signal side, the event is discarded as multiple leptons are likely to originate from a double semileptonic $b \rightarrow c \rightarrow s$ cascade. For charged B_{tag} candidates, we demand that the charge assignment of the signal-side lepton be opposite that of the B_{tag} charge. The hadronic X system is reconstructed from the remaining unassigned charged particles and neutral energy depositions. Its four momentum is calculated as

$$p_X = \sum_i \left(\sqrt{m_{\pi}^2 + |\mathbf{p}_i|^2}, \mathbf{p}_i \right) + \sum_j (E_j, \mathbf{k}_j), \quad (13)$$

with $E_i = |\mathbf{k}_i|$ the energy of the neutral energy depositions and all charged particles with momentum $\mathbf{p}_i$ are assumed to be pions. With the X system reconstructed, we can also reconstruct the missing mass squared,

$$M_{\text{miss}}^2 = (p_{\text{sig}} - p_X - p_{\ell})^2, \quad (14)$$

which should peak at zero, $M_{\text{miss}}^2 \approx m_{\nu}^2 \approx 0 \text{ GeV}^2$, for correctly reconstructed semileptonic $B \rightarrow X_u \ell^+ \nu_{\ell}$ and $B \rightarrow X_c \ell^+ \nu_{\ell}$ decays. The hadronic mass of the X system is later used to discriminate $B \rightarrow X_u \ell^+ \nu_{\ell}$ signal decays from $B \rightarrow X_c \ell^+ \nu_{\ell}$ and other remaining backgrounds. It is reconstructed using

$$M_X = \sqrt{(p_X)^\mu (p_X)_\mu}. \quad (15)$$

In addition, we reconstruct the four-momentum-transfer squared, q^2 , as

$$q^2 = (p_{\text{sig}} - p_X)^2. \quad (16)$$

The resolution of both variables for $B \rightarrow X_u \ell^+ \nu_{\ell}$ is shown in Figure 3 as residuals with respect to the generated values of q^2 and M_X . The resolution for M_X has a root-mean-square (RMS) deviation of 0.47 GeV , but exhibits a large tail towards larger values. The distinct peak at 0 is from $B^0 \rightarrow \pi^- \ell^+ \nu_{\ell}$ and other low-multiplicity final states comprised of only charged pions. The four-momentum-transfer squared q^2 exhibits a large resolution, which is caused by a combination of the tag-side B and the X reconstruction. The RMS deviation for q^2 is 1.59 GeV^2 . The core resolution is dominated by the tagging resolution, whereas the large negative tail is dominated from the resolution of the reconstruction of the X system.

C. Background Suppression BDT

At this point in the reconstruction, the $B \rightarrow X_c \ell^+ \nu_{\ell}$ process completely dominates the selected events. To identify $B \rightarrow X_u \ell^+ \nu_{\ell}$, we combine several distinguishing features into a single discriminant. This is achieved

³ We neglect the small correction of the lepton mass term to the energy of the lepton.

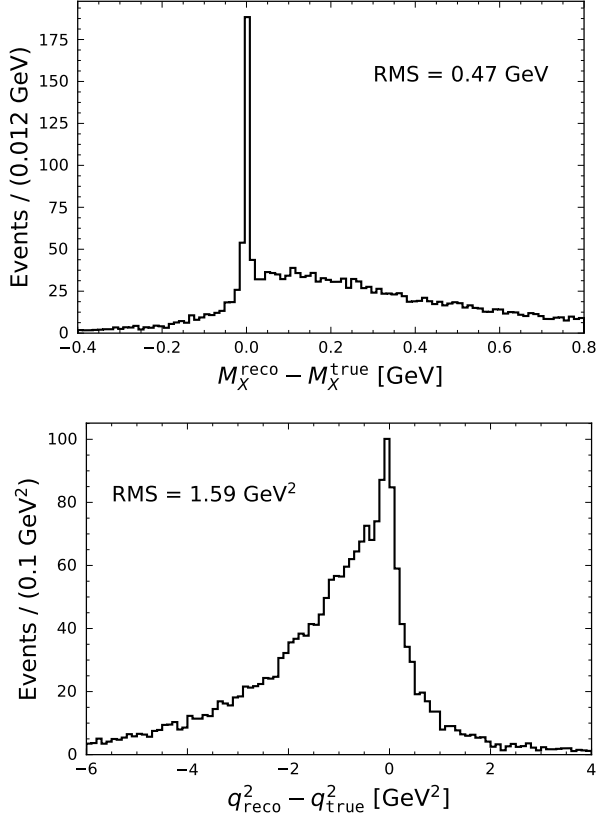


FIG. 3. The resolution of the reconstructed M_X and q^2 values for $B \rightarrow X_u \ell^+ \nu_\ell$ signal is shown as a residual with respect to the generated values.

by using a machine learning based classification with boosted decision trees (BDTs). Note that all momenta are in the center-of-mass frame of the colliding e^+e^- -pair. These features are:

1. M_{miss}^2 : The average $B \rightarrow X_c \ell^+ \nu_\ell$ multiplicity is higher than $B \rightarrow X_u \ell^+ \nu_\ell$, broadening the missing mass squared distribution.
2. D^* veto: We search for low momentum neutral and charged pions in the X system with $|\mathbf{p}_\pi| < 220$ MeV, compatible with a $D^* \rightarrow D\pi$ transition. The key idea of this is that due to the small available phase space from the small mass difference between the D^* and D mesons, the flight direction of the slow pion is strongly correlated with the D^* momentum direction. The energy and momentum of a D^* candidate can thus be approximated as

$$E_{D^*} = \frac{m_{D^*}}{m_{D^*} - m_D} \times E_\pi, \quad \mathbf{p}_{D^*} = \mathbf{p}_\pi \times \frac{\sqrt{E_{D^*}^2 - m_{D^*}^2}}{|\mathbf{p}_\pi|}, \quad (17)$$

with m_{D^*} and m_D denoting the D^* and D meson

masses, respectively, and $E_\pi = \sqrt{m_\pi^2 + |\mathbf{p}_\pi|^2}$ is the energy of the slow pion. Using the D^* candidate four momentum $p_{D^*} = (E_{D^*}, \mathbf{p}_{D^*})$ we can calculate

$$M_{\text{miss}, D^*}^2 = (p_{\text{sig}} - p_{D^*} - p_\ell)^2, \quad \cos \theta_{B, D^* \ell} = \frac{2E_{\text{beam}} E_{D^* \ell} - m_B^2 - m_{D^* \ell}^2}{2|\mathbf{p}_B||\mathbf{p}_{D^* \ell}|}, \quad \cos \theta^* = \frac{\mathbf{p}_\ell \cdot \mathbf{p}_{D^*}}{|\mathbf{p}_\ell||\mathbf{p}_{D^*}|}, \quad (18)$$

with $p_{D^* \ell} = p_{D^*} + p_\ell = (E_{D^* \ell}, \mathbf{p}_{D^* \ell})$ and $|\mathbf{p}_B| = \sqrt{E_B^2 - m_B^2}$. These three variables are used exclusively for events with charged and neutral slow pion candidates.

3. Kaons: We identify the number of K^+ candidates using the particle-identification likelihood, cf. Section II. In addition, we reconstruct K_S^0 candidates from displaced tracks found in the X system.
4. B_{sig} vertex fit: The charmed mesons produced in $B \rightarrow X_c \ell^+ \nu_\ell$ transitions exhibit a longer lifetime than their charmless counterparts produced in $B \rightarrow X_u \ell^+ \nu_\ell$ decays. This can be exploited by carrying out a vertex fit using the lepton and all charged constituents, not identified as kaons, of the X system and we use its χ^2 value as a discriminator.
5. Q_{tot} : The total event charge as calculated from the X system plus lepton on the signal and from the B_{tag} constituents. Due to the larger average multiplicity of $B \rightarrow X_c \ell^+ \nu_\ell$, the expected net zero event charge is more often violated in comparison to $B \rightarrow X_u \ell^+ \nu_\ell$ candidate events.

We use the BDT implementation of Ref. [61] and train a classifier $\mathcal{O}_{\text{BDT}}$ with simulated $B \rightarrow X_u \ell^+ \nu_\ell$ and $B \rightarrow X_c \ell^+ \nu_\ell$ events, which we discard in the later analysis. Ref. [61] uses optimized boosting and pruning procedures to maximize the classification performance. We choose a selection criteria on $\mathcal{O}_{\text{BDT}}$ that rejects 98.7% of $B \rightarrow X_c \ell^+ \nu_\ell$ and retains 18.5% of $B \rightarrow X_u \ell^+ \nu_\ell$ signal. This working point was chosen by maximizing the significance of the most inclusive partial branching fraction, taking into account the full set of systematic uncertainties and the full analysis procedure. The stability of the result as a function of the BDT selection is further discussed in Section VIII.

Table II lists the efficiencies for signal and $B \rightarrow X_c \ell^+ \nu_\ell$ background for the M_{bc} and the BDT selections. Figure 4 shows the output classifier of the background suppression BDT for MC and data. The classifier output shows good agreement between simulated and observed data over the full range. A comparison of the shape of all input variables for $B \rightarrow X_u \ell^+ \nu_\ell$ and $B \rightarrow X_c \ell^+ \nu_\ell$, and further MC and data comparisons can be found in Appendix B.

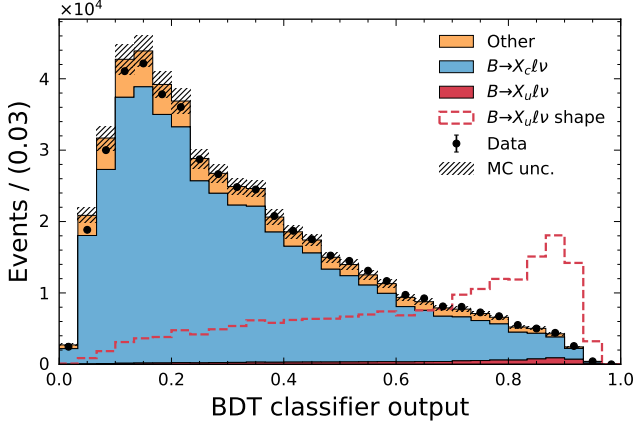


FIG. 4. The shape of the background suppression classifier $\mathcal{O}_{\text{BDT}}$ is shown. MC is divided into $B \rightarrow X_u \ell^+ \nu_\ell$ signal, the dominant $B \rightarrow X_c \ell^+ \nu_\ell$ background, and all other contributions. To increase visibility, the $B \rightarrow X_u \ell^+ \nu_\ell$ component is shown with a scaling factor (red dashed line). The uncertainties on the MC contain the full systematic errors and are further discussed in Section V.

TABLE II. The selection efficiencies for $B \rightarrow X_u \ell^+ \nu_\ell$ signal, $B \rightarrow X_c \ell^+ \nu_\ell$ and for data are listed after the reconstruction of the B_{tag} and lepton candidate. The nominal selection requirement on the BDT classifier $\mathcal{O}_{\text{BDT}}$ is 0.85. The other two requirements were introduced to test the stability of the result, cf. Section VIII.

Selection	$B \rightarrow X_u \ell^+ \nu_\ell$	$B \rightarrow X_c \ell^+ \nu_\ell$	Data
$M_{\text{bc}} > 5.27 \text{ GeV}$	84.8%	83.8%	80.2%
$\mathcal{O}_{\text{BDT}} > 0.85$	18.5%	1.3%	1.6%
$\mathcal{O}_{\text{BDT}} > 0.83$	21.9%	1.7%	2.1%
$\mathcal{O}_{\text{BDT}} > 0.87$	14.5%	0.9%	1.1%

D. Tagging Efficiency Calibration

The reconstruction efficiency of the hadronic full reconstruction algorithm of Ref. [59] differs between simulated samples and the reconstructed data. This difference mainly arises due to imperfections, e.g. in the simulation of detector responses, particle identification efficiencies, or incorrect branching fractions in the reconstructed decay cascades. To address this, the reconstruction efficiency is calibrated using a data-driven approach and we follow closely the procedure outlined in Ref. [32]. We reconstruct full reconstruction events by requiring exactly one lepton on the signal side, and apply the same B_{tag} and lepton selection criteria outlined in the previous section. This $B \rightarrow X \ell^+ \nu_\ell$ enriched sample is divided into groups of subsamples according to the B_{tag} decay channel and the multivariate classifier output $\mathcal{O}_{\text{FR}}$ used in the hierarchical reconstruction. Each of these groups of subsamples is studied individually to derive a calibration factor for the hadronic tagging efficiency: the calibra-

TABLE III. The binning choices of the four fits are given.

Fit variable	Bins
M_X	[0, 1.5, 1.9, 2.5, 3.1, 5.0] GeV
q^2	[0, 2, 4, 6, 8, 10, 12, 14, 26] GeV^2
E_ℓ^B	15 equidist. bins in [1, 2.5] GeV & [2.5, 2.7] GeV
$M_X : q^2$	[0, 1.5] GeV $\times$ [0, 2, 4, 6, 8, 10, 12, 14, 26] GeV^2
	[1.5, 1.9] GeV $\times$ [0, 2, 4, 6, 26] GeV^2
	[1.9, 2.5] GeV $\times$ [0, 2, 4, 26] GeV^2
	[2.5, 4.0] GeV $\times$ [0, 2, 26] GeV^2

tion factor is obtained by comparing the number of inclusive semileptonic B -meson decays, $N(B \rightarrow X \ell^+ \nu_\ell)$, in data with the expectation from the simulated samples, $N^{\text{MC}}(B \rightarrow X \ell^+ \nu_\ell)$. The semileptonic yield is determined via a binned maximum likelihood fit using the lepton energy spectrum. To reduce the modeling dependence of the $B \rightarrow X \ell^+ \nu_\ell$ sample this is done in a coarse granularity of five bins. The calibration factor of each these groups of subsamples is given by

$$C_{\text{tag}}(B_{\text{tag}} \text{ mode}, \mathcal{O}_{\text{FR}}) = \frac{N(B \rightarrow X \ell^+ \nu_\ell)}{N^{\text{MC}}(B \rightarrow X \ell^+ \nu_\ell)}. \quad (19)$$

The free parameters in the fit are the yield of the semileptonic $B \rightarrow X \ell^+ \nu_\ell$ decays, the yield of backgrounds from fake leptons and the yield of backgrounds from true leptons. Approximately 1200 calibration factors are determined this way. The leading uncertainty on the C_{tag} factors is from the assumed $B \rightarrow X \ell^+ \nu_\ell$ composition and the lepton PID performance, cf. Section V. We also apply corrections to the continuum efficiency. These are derived by using the off-resonance sample and comparing the number of reconstructed off-resonance events in data with the simulated on-resonance continuum events, correcting for differences in the selection.

IV. FITTING PROCEDURE

In order to determine the $B \rightarrow X_u \ell^+ \nu_\ell$ signal yield and constrain all backgrounds, we perform a binned likelihood fit in the discriminating variables. To reduce the dependence on the precise modeling of the $B \rightarrow X_u \ell^+ \nu_\ell$ signal, we use coarse bins over regions that are very sensitive to the admixture of resonant and non-resonant decays, cf. Section II. The total likelihood function is constructed as the product of individual Poisson distributions $\mathcal{P}$,

$$\mathcal{L} = \prod_i^{\text{bins}} \mathcal{P}(n_i; \nu_i) \times \prod_k \mathcal{G}_k, \quad (20)$$

with n_i denoting the number of observed data events and ν_i the total number of expected events in a given bin i .

Here, $\mathcal{G}_k$ are nuisance-parameter (NP) constraints, whose role is to incorporate systematic uncertainties of a source k into the fit. Their construction is further discussed in Section V. The number of expected events in a given bin, ν_i , is estimated using simulated collision events and is given by

$$\nu_i = \sum_k^{\text{processes}} f_{ik} \eta_k, \quad (21)$$

with η_k denoting the total number of events from a given process k , and f_{ik} denoting the fraction of such events being reconstructed in bin i as determined by the MC simulation.

We carry out four separate fits to measure three partial branching fractions, each using different discriminating variables to determine the $B \rightarrow X_u \ell^+ \nu_\ell$ yield. The fits and variables are:

1. The hadronic mass, M_X : Signal is expected to predominantly populate the low hadronic mass region, whereas remaining $B \rightarrow X_c \ell^+ \nu_\ell$ background will produce a sharp peak at around $M_X \approx 2 \text{ GeV}$. The sizeable resolution on the reconstruction of the X system will result in a non-negligible amount of these backgrounds to also be present in the low and high M_X region.
2. The four-momentum-transfer squared, q^2 : Signal will on average have a higher q^2 than $B \rightarrow X_c \ell^+ \nu_\ell$ background, whose kinematic endpoint is $q^2 = (m_B - m_D)^2 \approx 11.6 \text{ GeV}^2$. However, the reconstructed q^2 of $B \rightarrow X_c \ell^+ \nu_\ell$ events is smeared over the entire kinematic range due to the sizeable resolution in the reconstruction of the inclusive X system and the B_{tag} reconstruction.
3. The lepton energy in the B meson rest-frame, E_ℓ^B : Signal and $B \rightarrow X_c \ell^+ \nu_\ell$ can be separated beyond the kinematic endpoint of the $B \rightarrow X_c \ell^+ \nu_\ell$ background, which is $\frac{1}{2m_B} (m_B^2 - m_D^2 + m_\ell^2) \approx 2.3 \text{ GeV}$. The lepton energy is reconstructed using its momentum ($E_\ell^B = |\mathbf{p}_\ell^B|$), which has excellent resolution. This makes the measurement more sensitive to the exact composition of the $B \rightarrow X_c \ell^+ \nu_\ell$ background and $B \rightarrow X_u \ell^+ \nu_\ell$ signal. To minimize the dependence on the signal modeling, the endpoint of the lepton spectrum, ranging from $E_\ell^B \in [2.5, 2.7] \text{ GeV}$, is treated as a single coarse bin in the fit. To reduce the dependence on the exact modeling of $B \rightarrow X_c \ell^+ \nu_\ell$ we require $M_X < 1.7 \text{ GeV}$.
4. The final fit uses M_X and q^2 simultaneously in a two dimensional fit.

A summary of the binning choices is provided in Table III. The likelihood Eq. 20 is numerically maximized to fit the value of three different components, η_k , from the

observed events and by using the sequential least squares programming method implementation of Ref. [62]. The three components we determine are:

- a) Signal $B \rightarrow X_u \ell^+ \nu_\ell$ events that fall inside the phase-space region of the partial branching fraction we wish to determine.
- b) Signal $B \rightarrow X_u \ell^+ \nu_\ell$ events that fall outside said region. This component has a very similar shape as other backgrounds. We thus constrain this component in all fits to its expectation using the world average of $\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (2.13 \pm 0.30) \times 10^{-3}$ [37]. We also investigated different approaches: for instance linking this component with the component of a). This leads to small shifts of $\mathcal{O}(0.3 - 1\%)$ of the reported partial branching fractions using this component.
- c) Background events; dominated by $B \rightarrow X_c \ell^+ \nu_\ell$ and secondary $B \rightarrow h_1 h_2$ and $h_2 \rightarrow h_3 \ell^- \nu$ decays and simulated as described in Section III. Here h_1, h_2 , and h_3 denote hadronic final states.

Confidence intervals for the three components are constructed using the profile likelihood ratio method. For a given component η_k the ratio is

$$\Lambda(\eta_k) = -2 \ln \frac{\mathcal{L}(\eta_k, \hat{\boldsymbol{\eta}}_{\eta_k}, \hat{\boldsymbol{\theta}}_{\eta_k})}{\mathcal{L}(\hat{\eta}_k, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\theta}})}, \quad (22)$$

where $\hat{\eta}_k, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\theta}}$ are the values of the component of interest, the remaining components, and a vector of nuisance parameters (NPs), respectively, that maximize the likelihood function, whereas the remaining components $\hat{\boldsymbol{\eta}}_{\eta_k}$ and nuisance parameters $\hat{\boldsymbol{\theta}}_{\eta_k}$ maximize the likelihood for the specific value η_k . In the asymptotic limit, the test statistic Eq. 22 can be used to construct approximate confidence intervals through

$$1 - \text{CL} = \int_{\Lambda(\eta_k)}^{\infty} f_{\chi^2}(x; 1 \text{ dof}) dx, \quad (23)$$

with $f_{\chi^2}(x; 1 \text{ dof})$ denoting the χ^2 distribution of the variable x with a single degree of freedom. Further, CL denotes the desired confidence level. The determined signal yields $\hat{\eta}_k = \hat{\eta}_{\text{sig}}$ are translated into partial branching fractions via

$$\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell; \text{Reg.}) = \frac{\hat{\eta}_{\text{sig}} \cdot \epsilon_{\Delta\mathcal{B}(\text{Reg.})}}{4 (\epsilon_{\text{tag}} \cdot \epsilon_{\text{sel}}) \cdot N_{BB}}. \quad (24)$$

Here ϵ_{tag} denotes the tagging efficiency, as determined after applying the calibration factor introduced in Section IIID. Further, ϵ_{sel} and $\epsilon_{\Delta\mathcal{B}(\text{Reg.})}$ denote the signal side selection efficiency and a correction to the efficiency to account for the fraction of $B \rightarrow X_u \ell^+ \nu_\ell$ phase-space region that is measured. The factor of 4 in the denominator is due to the factor $N_{BB} = (771.58 \pm 9.78) \times 10^6$

B meson pairs and our averaging over electron and muon final states.

To validate the fit procedure we generated ensembles of pseudoexperiments for different input branching fractions for $B \rightarrow X_u \ell^+ \nu_\ell$ signal and $B \rightarrow X_c \ell^+ \nu_\ell$ background. Fits to these ensembles show no biases in central values and no under- or overcoverage of CI. Using the current world average of $\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (2.13 \pm 0.30) \times 10^{-3}$, we expect approximately between 930 - 2070 $B \rightarrow X_u \ell^+ \nu_\ell$ signal events with significances $s = \hat{\eta}_{\text{sig}}/\epsilon$ ranging from about 9 to 15 standard deviations, depending on the signal region under study, and with ϵ being the expected fit error determined from Asimov data sets [63].

V. SYSTEMATIC UNCERTAINTIES

Several systematic uncertainties affect the determination of the reported partial branching fractions. The most important uncertainties arise from the modeling of the $B \rightarrow X_u \ell^+ \nu_\ell$ signal component and from the tagging calibration correction. This is followed by uncertainties on particle identification of kaons and leptons, the uncertainty on the number of B -meson pairs, the statistical uncertainty on the used MC samples, and uncertainties related to the efficiency of the track reconstructions. Table IV summarizes the systematic uncertainties for the five measured partial branching fractions probing three phase-space regions. The table separates uncertainties that originate from the background subtraction (‘Additive uncertainties’) and uncertainties related to the translation of the fitted signal yields into partial branching fractions (‘Multiplicative uncertainties’).

The tagging calibration uncertainties are evaluated by producing different sets of calibration factors. These sets take into account the correlation structure from common systematic uncertainties (cf. Section IIID) and that individual channels and ranges of the output classifier are statistically independent. When applying the different sets of calibration factors, we notice only negligible shape changes on the signal and background template shapes, but the overall tagging efficiency is affected. The associated uncertainty on the calibration factors is found to be 3.6% and is identical for the five measured partial branching fractions. The $B \rightarrow X_u \ell^+ \nu_\ell$ and $B \rightarrow X_c \ell^+ \nu_\ell$ modeling uncertainties do directly affect the shapes of M_X , q^2 , and E_ℓ^B signal and background distributions. Further, the $B \rightarrow X_u \ell^+ \nu_\ell$ modeling affects the overall reconstruction efficiencies and migrations of events inside and outside of the phase-space regions we measure. We evaluate the uncertainties on the composition of the hybrid $B \rightarrow X_u \ell^+ \nu_\ell$ MC by variations of the $B \rightarrow \pi \ell^+ \nu_\ell$, $B \rightarrow \rho \ell^+ \nu_\ell$, $B \rightarrow \omega \ell^+ \nu_\ell$, $B \rightarrow \eta \ell^+ \nu_\ell$, $B \rightarrow \eta' \ell^+ \nu_\ell$ branching fractions and form factors. The uncertainty on non-resonant $B \rightarrow X_u \ell^+ \nu_\ell$ contributions in the hybrid model is estimated by changing the underlying model from that of DFN [51] to that of BLNP [17]. In addition,

the uncertainty on the used DFN parameters m_b^{1S} and a (cf. Section II) are incorporated. For each of these variations, new hybrid weights are calculated to propagate the uncertainties into shapes and efficiencies. We estimate the uncertainties of X_u fragmentation into $s\bar{s}$ quark pairs by variations of the corresponding JETSET parameter γ_s (cf. Ref. [54]). As our BDT is trained to reject final states with kaon candidates, a change in this fraction will directly impact the signal efficiency. The $s\bar{s}$ production probability has been measured by Refs. [64, 65] at center-of-mass energies of 12 and 36 GeV with values of $\gamma_s = 0.35 \pm 0.05$ and $\gamma_s = 0.27 \pm 0.06$, respectively. We adopt the value and error of $\gamma_s = 0.30 \pm 0.09$, which spans the range of both measurements including their uncertainties. The X_u system of the non-resonant signal component is hadronized by JETSET into final states with two or more pions. We test the impact on the signal efficiency by changing the post-fit charged pion multiplicity of non-resonant $B \rightarrow X_u \ell^+ \nu_\ell$ to the distribution observed in data in the signal enriched region of $M_X < 1.7$ GeV (cf. Section VIIE and Appendix C). The $B \rightarrow X_c \ell^+ \nu_\ell$ background after the BDT selection is dominated by $B \rightarrow D \ell^+ \nu_\ell$ and $B \rightarrow D^* \ell^+ \nu_\ell$ decays. We evaluate the uncertainties on the modeling of $B \rightarrow D \ell^+ \nu_\ell$, $B \rightarrow D^* \ell^+ \nu_\ell$ and $B \rightarrow D^{**} \ell^+ \nu_\ell$ by variations of the BGL parameters and heavy quark form factors within their uncertainties. In addition, we propagate the branching fraction uncertainties. The uncertainties on the $B \rightarrow X_c \ell^+ \nu_\ell$ gap branching fractions are taken to be large enough to account for the difference between the sum of all exclusive branching fractions measured and the inclusive branching fraction measured. We also evaluate the impact on the efficiency of the lepton- and hadron-identification uncertainties, and the overall tracking efficiency uncertainty. The statistical uncertainty on all generated MC samples is also evaluated and propagated into the systematic errors.

We incorporate the effect of additive systematic uncertainties directly into the likelihood function. This can be done by introducing a vector of NPs, θ_k , for each fit template of a process k (e.g. signal or background). Each element of this vector represents one bin of the fitted observables of interest (e.g. M_X, q^2, E_ℓ^B or a 2D bin of $M_X : q^2$). These NPs are constrained parameters in the likelihood Eq. 20 using multivariate Gaussian distributions, $\mathcal{G}_k = \mathcal{G}_k(\mathbf{0}; \theta_k, \Sigma_k)$. Here Σ_k denotes the systematic covariance matrix for a given template k and θ_k is a vector of NPs. The covariance Σ_k is the sum over all possible uncertainty sources for a given template k ,

$$\Sigma_k = \sum_s^{\text{error sources}} \Sigma_{ks}, \quad (25)$$

with Σ_{ks} denoting the covariance matrix of error source s . The covariance matrices Σ_{ks} depend on uncertainty vectors σ_{ks} , which represent the absolute error in bins of the fit variable of template k . Uncertainties from the same error source are either fully correlated, or for the case of MC

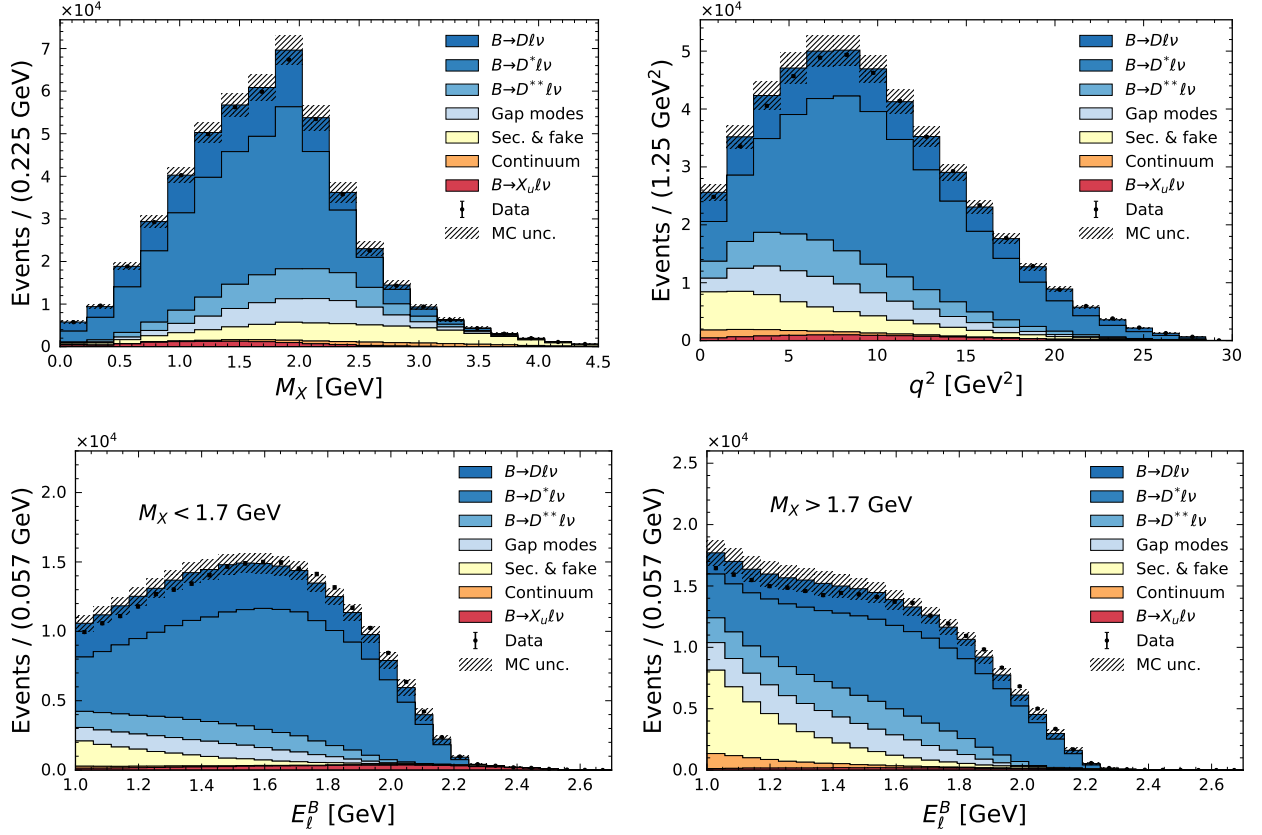


FIG. 5. (Top) The M_X and q^2 spectra of the selected candidates prior to applying the background BDT are shown. (Bottom) The E_ℓ^B spectrum of the selected candidates prior to applying the background BDT are shown for events with $M_X < 1.7$ GeV and $M_X > 1.7$ GeV.

or other statistical uncertainties, are treated as uncorrelated. Both cases can be expressed as $\Sigma_{ks} = \sigma_{ks} \otimes \sigma_{ks}$ or $\Sigma_{ks} = \text{Diag}(\sigma_{ks}^2)$, respectively. For particle identification uncertainties, we estimate Σ_{ks} using sets of correction tables, sampled according to their statistical and systematic uncertainties. The systematic NPs are incorporated in Eq. 21 by rewriting the fractions f_{ik} for all templates as

$$f_{ik} = \frac{\eta_{ik}^{\text{MC}}}{\sum_j \eta_{jk}^{\text{MC}}} \rightarrow \frac{\eta_{ik}^{\text{MC}} (1 + \theta_{ik})}{\sum_j \eta_{jk}^{\text{MC}} (1 + \theta_{jk})}, \quad (26)$$

to take into account changes in the signal or background shape. Here η_{ik}^{MC} denotes the predicted number of MC events of a given bin i and a process k , and θ_{ik} is the associated nuisance parameter constrained by $\mathcal{G}_k$.

VI. $B \rightarrow X_c \ell \bar{\nu}_\ell$ CONTROL REGION

Figure 5 compares the reconstructed M_X , q^2 , and E_ℓ^B distributions with the expectation from MC before applying the background suppression BDT. All corrections

are applied and the MC uncertainty contains all systematic uncertainties discussed in Section V. The agreement of M_X and q^2 is excellent, but some differences in the shape of the lepton momentum spectrum are seen. This is likely due to imperfections of the modeling of the inclusive $B \rightarrow X_c \ell^+ \nu_\ell$ background. The discrepancy reduces in the $M_X < 1.7$ GeV region. The main results of this paper will be produced by fitting q^2 and M_X in two dimensions. We use the lepton spectrum to measure the same regions of phase space, to validate the obtained results.

VII. $B \rightarrow X_u \ell^+ \nu_\ell$ SIGNAL REGION

Figure 6 shows the reconstructed M_X , q^2 , and E_ℓ^B distributions after the BDT selection is applied. The $B \rightarrow X_u \ell^+ \nu_\ell$ contribution is now clearly visible at low M_X and high E_ℓ^B , while the reconstructed events and the MC expectation show good agreement. The $B \rightarrow X_c \ell^+ \nu_\ell$ background is dominated by contributions from $B \rightarrow D \ell^+ \nu_\ell$ and $B \rightarrow D^* \ell^+ \nu_\ell$ decays, and the remaining background is predominantly from secondary leptons, and misidentified lepton candidates.

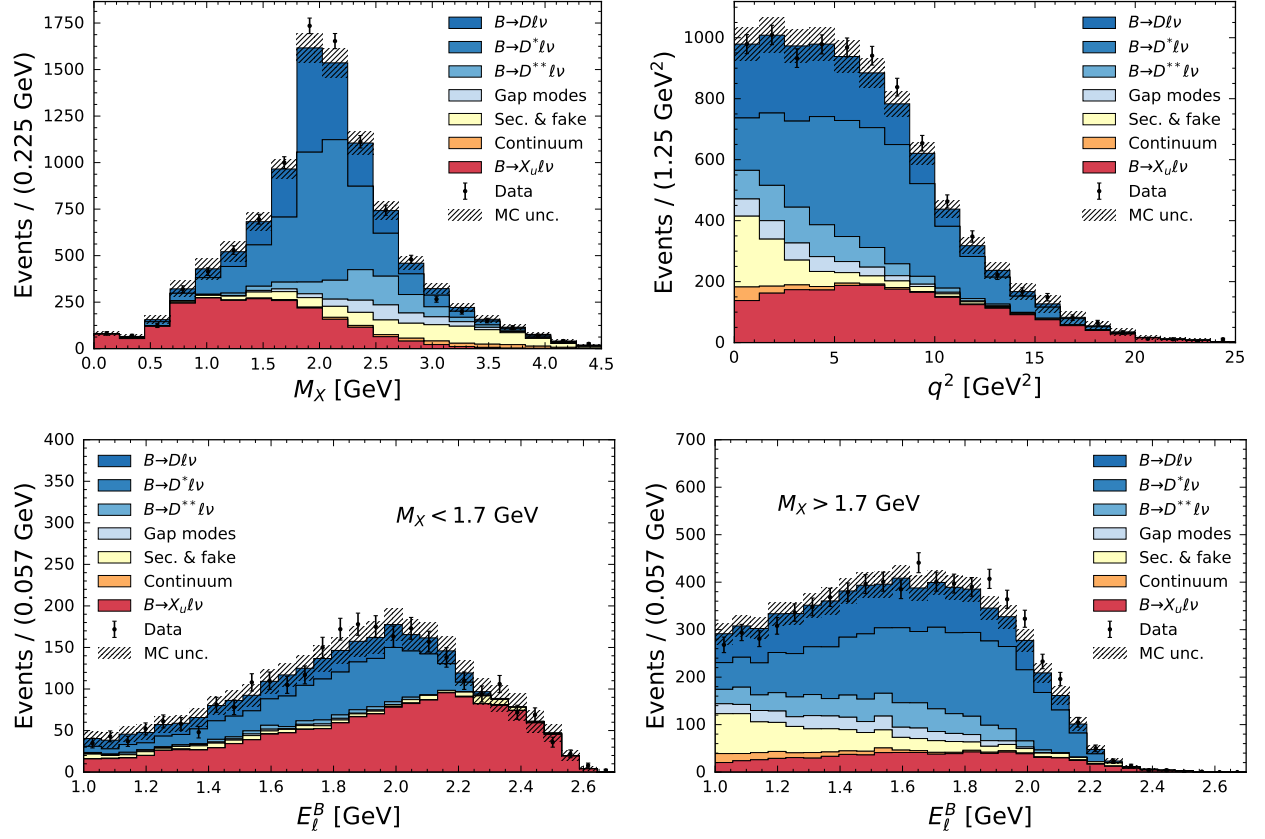


FIG. 6. The M_X , q^2 and E_ℓ^B spectra after applying the background BDT but before the fit are shown. The $B \rightarrow X_u \ell^+ \nu_\ell$ contribution is shown in red and scaled to the world average of $\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (2.13 \pm 0.30) \times 10^{-3}$. The data and MC agreement is reasonable in all variables. The E_ℓ^B spectra is shown with selections of $M_X < 1.7$ GeV and $M_X > 1.7$ GeV. The cut of $M_X < 1.7$ GeV is later used in the fit to reduce the dependence on the $B \rightarrow X_c \ell^+ \nu_\ell$ modeling of higher charmed states.

TABLE IV. The fractional uncertainty on the extracted $B \rightarrow X_u \ell^+ \nu_\ell$ partial branching fractions are shown. For definitions of additive and multiplicative errors, see text.

Phase-space region	Fractional uncertainties [%]				
	$M_X < 1.7 \text{ GeV},$ $E_\ell^B > 1 \text{ GeV}$	$M_X < 1.7 \text{ GeV},$ $E_\ell^B > 1 \text{ GeV}$	$M_X < 1.7 \text{ GeV},$ $q^2 > 8 \text{ GeV}^2,$ $E_\ell^B > 1 \text{ GeV}$	$E_\ell^B > 1 \text{ GeV}$	$E_\ell^B > 1 \text{ GeV}$
Fit variable(s)	$(M_X \text{ fit})$	$(E_\ell^B \text{ fit})$	$(q^2 \text{ fit})$	$(E_\ell^B \text{ fit})$	$(M_X : q^2 \text{ fit})$
Additive uncertainties					
$B \rightarrow X_u \ell^+ \nu_\ell$ modeling					
$B \rightarrow \pi \ell^+ \nu_\ell$ FFs	0.1	0.7	1.4	0.6	0.4
$B \rightarrow \rho \ell^+ \nu_\ell$ FFs	0.2	1.9	4.3	1.9	0.7
$B \rightarrow \omega \ell^+ \nu_\ell$ FFs	0.5	3.2	5.2	3.1	0.8
$B \rightarrow \eta \ell^+ \nu_\ell$ FFs	0.1	0.7	1.4	0.8	0.3
$B \rightarrow \eta' \ell^+ \nu_\ell$ FFs	0.1	0.7	1.4	0.8	1.2
$\mathcal{B}(B \rightarrow \pi \ell^+ \nu_\ell)$	0.2	0.1	0.2	0.1	0.2
$\mathcal{B}(B \rightarrow \rho \ell^+ \nu_\ell)$	0.3	0.7	0.8	0.5	0.4
$\mathcal{B}(B \rightarrow \omega \ell^+ \nu_\ell)$	<0.1	0.1	0.8	0.1	0.1
$\mathcal{B}(B \rightarrow \eta \ell^+ \nu_\ell)$	<0.1	0.1	<0.1	0.1	<0.1
$\mathcal{B}(B \rightarrow \eta' \ell^+ \nu_\ell)$	<0.1	<0.1	0.1	0.1	<0.1
$\mathcal{B}(B \rightarrow X_u \ell^+ \nu)$	0.7	2.0	2.1	2.1	2.1
DFN parameters	2.3	3.5	1.1	3.5	5.0
Hybrid model	2.7	8.7	4.6	8.7	3.1
$B \rightarrow X_c \ell^+ \nu_\ell$ modeling					
$B \rightarrow D \ell^+ \nu_\ell$ FFs	0.1	0.1	0.9	0.1	<0.1
$B \rightarrow D^* \ell^+ \nu_\ell$ FFs	1.4	1.2	3.0	1.3	1.1
$B \rightarrow D^{**} \ell^+ \nu_\ell$ FFs	0.4	0.5	0.3	0.5	0.4
$\mathcal{B}(B \rightarrow D \ell^+ \nu_\ell)$	0.1	<0.1	0.2	<0.1	0.2
$\mathcal{B}(B \rightarrow D^* \ell^+ \nu_\ell)$	<0.1	<0.1	0.3	<0.1	0.2
$\mathcal{B}(B \rightarrow D^{**} \ell^+ \nu_\ell)$	0.6	0.1	0.3	0.1	0.5
Gap modeling	1.1	0.1	0.3	0.1	1.0
MC statistics	1.3	1.6	3.8	1.7	1.6
Tracking efficiency	0.3	-	0.8	-	0.4
$\mathcal{L}_{\text{ID}}$ shape	1.0	0.5	1.3	0.6	1.2
$\mathcal{L}_{K/\pi \text{ ID}}$ shape	1.2	-	1.3	-	1.0
$D \rightarrow X \ell \nu_\ell$	0.1	0.1	0.1	0.1	0.1
π_s efficiency	<0.1	-	0.1	-	0.1
Multiplicative uncertainties					
$B \rightarrow X_u \ell^+ \nu_\ell$ modeling					
$B \rightarrow \pi \ell^+ \nu_\ell$ FFs	0.2	0.2	1.9	0.2	0.2
$B \rightarrow \rho \ell^+ \nu_\ell$ FFs	0.7	0.8	3.7	0.8	0.6
$B \rightarrow \omega \ell^+ \nu_\ell$ FFs	1.3	1.6	6.1	1.6	1.1
$B \rightarrow \eta \ell^+ \nu_\ell$ FFs	0.3	0.3	1.7	0.3	0.2
$B \rightarrow \eta' \ell^+ \nu_\ell$ FFs	0.2	0.3	1.7	0.3	0.2
$\mathcal{B}(B \rightarrow \pi \ell^+ \nu_\ell)$	0.3	0.4	0.4	0.4	0.3
$\mathcal{B}(B \rightarrow \rho \ell^+ \nu_\ell)$	0.4	0.6	0.6	0.6	0.4
$\mathcal{B}(B \rightarrow \omega \ell^+ \nu_\ell)$	<0.1	<0.1	0.1	<0.1	<0.1
$\mathcal{B}(B \rightarrow \eta \ell^+ \nu_\ell)$	0.1	0.1	<0.1	0.1	<0.1
$\mathcal{B}(B \rightarrow \eta' \ell^+ \nu_\ell)$	0.1	0.1	0.1	0.1	0.1
$\mathcal{B}(B \rightarrow X_u \ell^+ \nu)$	3.0	3.2	2.9	4.8	3.8
DFN parameters	2.5	2.5	2.7	6.8	3.6
Hybrid model	0.2	0.8	1.4	4.7	2.8
π^+ multiplicity	1.7	2.5	2.3	3.1	1.7
γ_s ($s\bar{s}$ fragmentation)	0.5	0.8	1.1	1.1	0.8
$\mathcal{L}_{\text{ID}}$ efficiency	1.5	1.6	1.6	1.6	1.5
$\mathcal{L}_{K/\pi \text{ ID}}$ efficiency	0.7	0.6	0.6	0.6	0.7
$N_{B\bar{B}}$	1.3	1.3	1.3	1.3	1.3
Tracking efficiency	0.8	0.3	0.8	0.3	0.9
Tagging calibration	3.6	3.6	3.6	3.6	3.6
Total syst. uncertainty	7.8	12.6	14.6	15.4	10.4

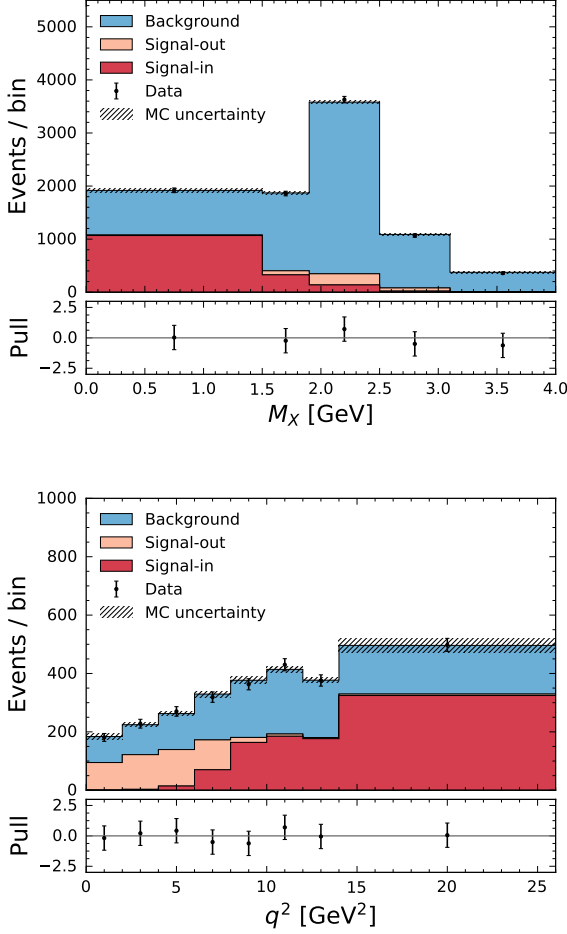


FIG. 7. The post-fit distributions of the one-dimensional fits to M_X and q^2 are shown, corresponding to the measured partial branching fractions for $E_\ell^B > 1$ GeV with additional requirements of $M_X < 1.7$ GeV, and $M_X < 1.7$ GeV and $q^2 > 8$ GeV², respectively.

VIII. RESULTS

We report partial branching fractions for three phase-space regions from five fits to the reconstructed variables introduced in Section IV. All partial branching fractions correspond to a selection with $E_\ell^B > 1$ GeV, also reverting the effect of final state radiation photons, and possible additional phase-space restrictions. The resulting fit yields are listed in Table V.

A. Partial Branching Fraction Results

For the partial branching fraction with $M_X < 1.7$ GeV from the fit to M_X we find

$$\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (1.09 \pm 0.05 \pm 0.08) \times 10^{-3}, \quad (27)$$

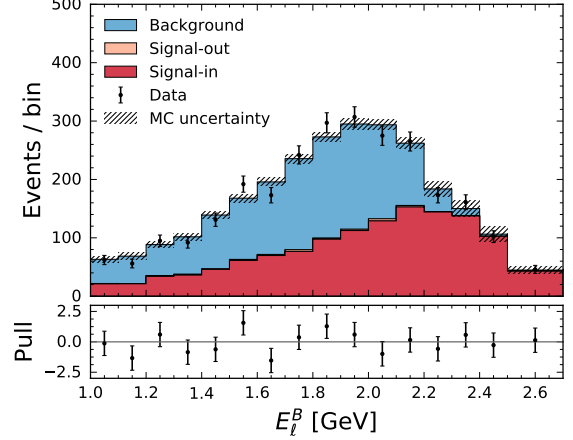


FIG. 8. The post-fit distributions of the fit to E_ℓ^B with $M_X < 1.7$ GeV is shown. The resulting yields were corrected to correspond to the partial branching fraction with $E_\ell^B > 1$ GeV with and without an additional requirement of $M_X < 1.7$ GeV, respectively.

with the first and second error denoting the statistical and systematic uncertainty, respectively. The resulting post-fit distribution is shown in the top panel of Figure 7. With this selection about 56% of the available $B \rightarrow X_u \ell^+ \nu_\ell$ phase space is probed. The partial branching fraction is in good agreement with the value obtained by fitting E_ℓ^B and corrected to the same phase space. The fit is shown in Figure 8 and we measure

$$\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (1.12 \pm 0.06 \pm 0.14) \times 10^{-3}, \quad (28)$$

with a larger systematic and statistical uncertainty than Eq. 27. To further probe the $B \rightarrow X_u \ell^+ \nu_\ell$ enriched region, we carry out a measurement for $M_X < 1.7$ GeV and $q^2 > 8$ GeV² from a fit to the q^2 spectrum. This selection only probes about 30% of the available $B \rightarrow X_u \ell^+ \nu_\ell$ phase space. We find

$$\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (0.67 \pm 0.07 \pm 0.10) \times 10^{-3}. \quad (29)$$

The corresponding post-fit distribution of q^2 is shown in the bottom panel of Figure 7. The most precise determinations of $B \rightarrow X_u \ell^+ \nu_\ell$ are obtained from a two-dimensional fit, exploiting the full combined discriminatory power of M_X and q^2 . The resulting partial branching fraction probes about 85% of the available $B \rightarrow X_u \ell^+ \nu_\ell$ phase space. We measure

$$\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (1.59 \pm 0.07 \pm 0.17) \times 10^{-3}. \quad (30)$$

The projection of the 2D fit onto M_X and the q^2 distribution for the signal enriched region of $M_X < 1.5$ GeV are shown in Figure 9. The remaining q^2 distributions are given in Appendix D. The partial branching fraction is also in good agreement from the measurement obtained

TABLE V. The fitted signal yields in ($\hat{\eta}_{\text{sig}}$) and outside ($\hat{\eta}_{\text{sig-out}}$) the measured phase-space regions, the background yields ($\hat{\eta}_{\text{bkg}}$) and the product of tagging and selection efficiency are listed.

Phase-space region	Additional Selection	Fit variable(s)	$\hat{\eta}_{\text{sig}}$	$\hat{\eta}_{\text{sig-out}}$	$\hat{\eta}_{\text{bkg}}$	$10^3 (\epsilon_{\text{tag}} \cdot \epsilon_{\text{sel}})$
$M_X < 1.7 \text{ GeV}$, $E_\ell^B > 1 \text{ GeV}$	-	M_X fit	$1558 \pm 66 \pm 72$	364 ± 51	6912 ± 138	0.26 ± 0.07
$M_X < 1.7 \text{ GeV}$, $E_\ell^B > 1 \text{ GeV}$	$M_X < 1.7 \text{ GeV}$	E_ℓ^B fit	$1285 \pm 68 \pm 136$	22 ± 3	1362 ± 153	0.21 ± 0.07
$M_X < 1.7 \text{ GeV}$, $q^2 > 8 \text{ GeV}^2$, $E_\ell^B > 1 \text{ GeV}$	$M_X < 1.7 \text{ GeV}$	q^2 fit	$938 \pm 101 \pm 98$	474 ± 58	1253 ± 194	0.14 ± 0.07
$E_\ell^B > 1 \text{ GeV}$	$M_X < 1.7 \text{ GeV}$	E_ℓ^B fit	$1303 \pm 69 \pm 138$	-	1366 ± 154	0.21 ± 0.19
$E_\ell^B > 1 \text{ GeV}$		$M_X : q^2$ fit	$1801 \pm 81 \pm 127$	-	7032 ± 167	0.31 ± 0.12

by fitting E_ℓ^B , covering the same phase space (c.f. Figure 8):

$$\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (1.69 \pm 0.09 \pm 0.26) \times 10^{-3}. \quad (31)$$

The uncertainties are larger, but both results are compatible. The nuisance parameter pulls of all fits are provided in Appendix D. The result of Eq. 30 can be further compared with the most precise measurement to date of this region of Ref. [66], where $\Delta\mathcal{B}(B \rightarrow X_u \ell \nu_\ell) = (1.55 \pm 0.12) \times 10^{-3}$, and shows good agreement. The measurement can also be compared to Ref. [15] using a similar experimental approach. The measured partial branching fraction of $E_\ell^B > 1 \text{ GeV}$ is $\Delta\mathcal{B}(B \rightarrow X_u \ell \nu_\ell) = (1.82 \pm 0.19) \times 10^{-3}$, which is compatible with Eq. 30 within 0.9 standard deviations. Belle previously reported in Ref. [16] using also a similar approach for the same phase space a higher value of $\Delta\mathcal{B}(B \rightarrow X_u \ell \nu_\ell) = (1.96 \pm 0.19) \times 10^{-3}$. We cannot quantify the statistical overlap between both results, but by comparing the number of determined signal events one can estimate it to be below 55%. The dominant systematic uncertainties of Ref. [16] were evaluated using different approaches, but fully correlating the dominant systematic uncertainties and assuming a statistical correlation of 55% we obtain a compatibility of 1.7 standard deviations. The main difference of this analysis with Ref. [16] lies in the modeling of signal and background processes: since its publication our understanding improved and more precise measurements of branching fractions and form factors were made available. Further, for the $B \rightarrow X_u \ell^+ \nu_\ell$ signal process in this paper a hybrid approach was adopted (see Section II and Appendix A), whereas Ref. [16] used an alternative approach to model signal as a mix of inclusive and exclusive decay modes. Note that this work supersedes Ref. [16].

B. $|V_{ub}|$ Determination

We determine $|V_{ub}|$ from the measured partial branching fractions using a range of theoretical rate predictions. In principle, the total $B \rightarrow X_u \ell^+ \nu_\ell$ decay rate can be calculated using the same approach as $B \rightarrow X_c \ell^+ \nu_\ell$ using the heavy quark expansion (HQE) in inverse powers of m_b . Unfortunately, the measurement requirements necessary to separate $B \rightarrow X_u \ell^+ \nu_\ell$ from the dominant $B \rightarrow X_c \ell^+ \nu_\ell$ background spoil the convergence of this approach. In the predictions for the partial rates corresponding to our measurements, perturbative and non-perturbative uncertainties are largely enhanced and as outlined in the introduction the predictions are sensitive to the shape function modeling.

The relationship between measured partial branching fractions, predictions of the rate (omitting CKM factors) $\Delta\Gamma(B \rightarrow X_u \ell^+ \nu_\ell)$, and $|V_{ub}|$ is

$$|V_{ub}| = \sqrt{\frac{\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell)}{\tau_B \cdot \Delta\Gamma(B \rightarrow X_u \ell^+ \nu_\ell)}}. \quad (32)$$

with $\tau_B = (1.579 \pm 0.004) \text{ ps}$ denoting the average of the charged and neutral B meson lifetime [37]. We use four predictions for the theoretical partial rates. All predictions use the same input values as Ref. [6] chooses for their world averages. The four predictions are:

- **BLNP**: The prediction of Bosch, Lange, Neubert, and Paz (short BLNP) of Ref. [17] provides a prediction at next-to-leading-order accuracy in terms of the strong coupling constant α_s and incorporates all known corrections. Predictions are interpolated between the shape-function dominated region (end-point of the lepton spectrum, small hadronic mass) to the region of phase space, that can be described via the operator product expansion (OPE). As input we use $m_b^{\text{SF}} = 4.58 \pm 0.03 \text{ GeV}$ and $\mu_\pi^{2\text{SF}} = 0.20_{-0.10}^{+0.09} \text{ GeV}^2$.

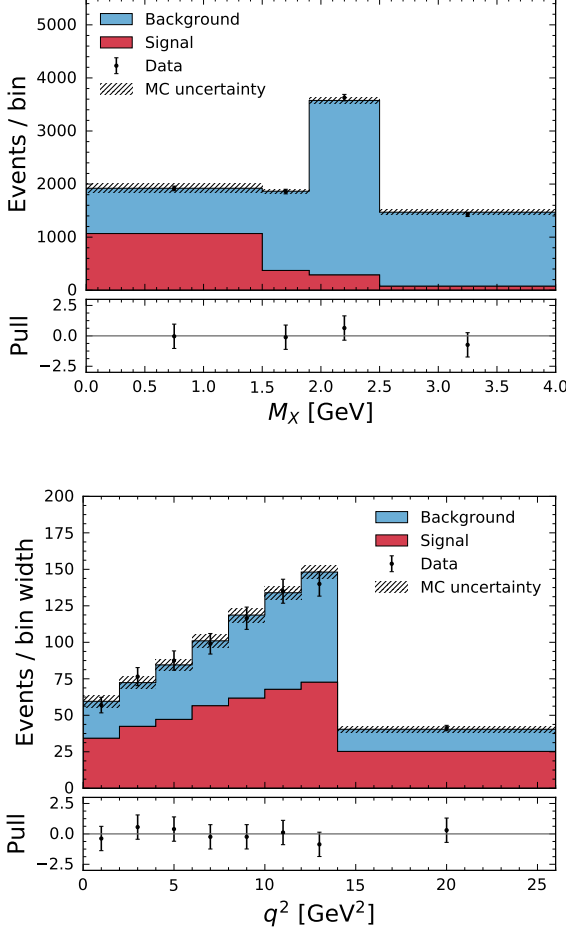


FIG. 9. The post-fit projection of M_X of the two-dimensional fit to $M_X : q^2$ on M_X and the q^2 distribution in the range of $M_X \in [0, 1.5]$ GeV are shown. The resulting yields are corrected to correspond to a partial branching fraction with $E_\ell^B > 1$ GeV. The remaining q^2 distributions are given in Figure 21 (Appendix D).

- **DGE**: The Dressed Gluon Approximation (short DGE) from Andersen and Gardi [19, 20] makes predictions by avoiding the direct use of shape functions, but produces predictions for hadronic observables using the on-shell b -quark mass. The calculation is carried out in the $\overline{\text{MS}}$ scheme and we use $m_b(\overline{\text{MS}}) = 4.19 \pm 0.04$ GeV.
- **GGOU**: The prediction from Gambino, Giordano, Ossola, and Uraltsev [18] (short GGOU) incorporates all known perturbative and non-perturbative effects up to the order $\mathcal{O}(\alpha_s^2 \beta_0)$ and $\mathcal{O}(1/m_b^3)$, respectively. The shape function dependence is incorporated by parametrizing its effects in each structure function with a single light-cone function. The calculation is carried out in the kinetic scheme and we use as inputs $m_b^{\text{kin}} = 4.55 \pm 0.02$ GeV and

$$\mu_\pi^{2\text{kin}} = 0.46 \pm 0.08 \text{ GeV}^2.$$

- **ADFR**: The calculation of Aglietti, Di Lodovico, Ferrera, and Ricciardi [21, 22] makes use of the ratio of $B \rightarrow X_u \ell^+ \nu_\ell$ to $B \rightarrow X_c \ell^+ \nu_\ell$ rates and soft-gluon resummation at next-to-next-to-leading-order and an effective QCD coupling approach. The calculation uses the $\overline{\text{MS}}$ scheme and we use $m_b(\overline{\text{MS}}) = 4.19 \pm 0.04$ GeV.

Table VI lists the decay rates and their associated uncertainties for the probed regions of phase space, which we use to extract $|V_{ub}|$ from the measured partial branching fractions with Eq. 32.

C. $|V_{ub}|$ Results

From the partial branching fractions with $E_\ell^B > 1$ GeV and $M_X < 1.7$ GeV determined from fitting M_X we find

$$\begin{aligned} |V_{ub}| \text{ (BLNP)} &= (3.90 \pm 0.08 \pm 0.15 \pm 0.21) \times 10^{-3}, \\ |V_{ub}| \text{ (DGE)} &= (4.08 \pm 0.09 \pm 0.16_{-0.26}^{+0.20}) \times 10^{-3}, \\ |V_{ub}| \text{ (GGOU)} &= (3.97 \pm 0.08_{-0.16}^{+0.15+0.15}) \times 10^{-3}, \\ |V_{ub}| \text{ (ADFR)} &= (3.63 \pm 0.08 \pm 0.14 \pm 0.17) \times 10^{-3}. \end{aligned} \quad (33)$$

The uncertainties denote the statistical uncertainty, the systematic uncertainty and the theory error from the partial rate prediction. For the partial branching fraction with $E_\ell^B > 1$ GeV, $M_X < 1.7$ GeV, and $q^2 > 8 \text{ GeV}^2$ we find

$$\begin{aligned} |V_{ub}| \text{ (BLNP)} &= (4.24_{-0.23-0.32-0.28}^{+0.22+0.30+0.26}) \times 10^{-3}, \\ |V_{ub}| \text{ (DGE)} &= (4.16_{-0.23-0.31-0.21}^{+0.22+0.29+0.18}) \times 10^{-3}, \\ |V_{ub}| \text{ (GGOU)} &= (4.25_{-0.24-0.32-0.26}^{+0.22+0.30+0.24}) \times 10^{-3}, \\ |V_{ub}| \text{ (ADFR)} &= (3.68_{-0.20-0.28}^{+0.19+0.26} \pm 0.17) \times 10^{-3}. \end{aligned} \quad (34)$$

Finally, the most inclusive determination with $E_\ell^B > 1$ GeV from the two-dimensional fit of M_X and q^2 results in

$$\begin{aligned} |V_{ub}| \text{ (BLNP)} &= (4.05 \pm 0.09_{-0.22-0.20}^{+0.21+0.18}) \times 10^{-3}, \\ |V_{ub}| \text{ (DGE)} &= (4.16 \pm 0.09_{-0.22-0.12}^{+0.21+0.11}) \times 10^{-3}, \\ |V_{ub}| \text{ (GGOU)} &= (4.15 \pm 0.09_{-0.22-0.09}^{+0.21+0.08}) \times 10^{-3}, \\ |V_{ub}| \text{ (ADFR)} &= (4.05 \pm 0.09_{-0.22}^{+0.21} \pm 0.18) \times 10^{-3}. \end{aligned} \quad (35)$$

In order to quote a single value for $|V_{ub}|$ we adapt the procedure of Ref. [67] and calculate a simple arithmetic

TABLE VI. The theory rates $\Delta\Gamma(B \rightarrow X_u \ell^+ \nu_\ell)$ from various theory calculations are listed. The rates are given in units of ps^{-1} .

Phase-space region	BLNP [17]	DGE [19, 20]	GGOU [18]	ADFR [21, 22]
$M_X < 1.7 \text{ GeV}$	$45.2^{+5.4}_{-4.6}$	$42.3^{+5.8}_{-3.8}$	$43.7^{+3.9}_{-3.2}$	$52.3^{+5.4}_{-4.7}$
$M_X < 1.7 \text{ GeV}, q^2 > 8 \text{ GeV}^2$	$23.4^{+3.4}_{-2.6}$	$24.3^{+2.6}_{-1.9}$	$23.3^{+3.2}_{-2.4}$	$31.1^{+3.0}_{-2.6}$
$E_\ell^B > 1 \text{ GeV}$	$61.5^{+6.4}_{-5.1}$	$58.2^{+3.6}_{-3.0}$	$58.5^{+2.7}_{-2.3}$	$61.5^{+5.8}_{-5.1}$

average of the most precise determinations in Eq. 35 to obtain

$$|V_{ub}| = (4.10 \pm 0.09 \pm 0.22 \pm 0.15) \times 10^{-3}. \quad (36)$$

This value is larger, but compatible with the exclusive measurement of $|V_{ub}|$ from $B \rightarrow \pi \ell^+ \nu_\ell$ of $|V_{ub}| = (3.67 \pm 0.09 \pm 0.12) \times 10^{-3}$ within 1.3 standard deviations.

D. Stability Checks

To check the stability of the result we redetermine the partial branching fractions using two additional working points. We change the BDT selection to increase and decrease the amount of $B \rightarrow X_c \ell^+ \nu_\ell$ and other backgrounds, and repeat the full analysis procedure. The resulting values of $\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell)$ are determined using the two-dimensional fit of $M_X : q^2$ and are shown in Figure 10. The background contamination changes by

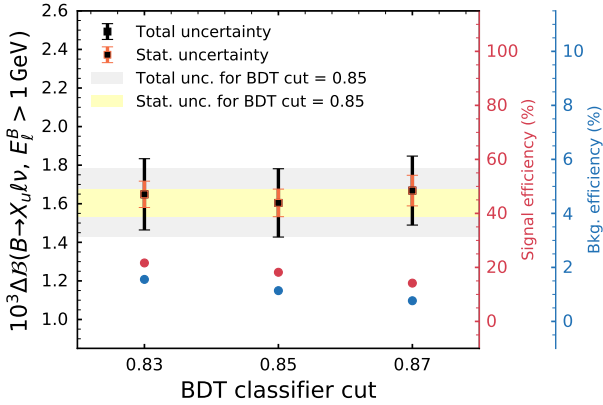


FIG. 10. The stability of the determined partial branching fraction $\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell)$ using the $M_X : q^2$ fit is studied as a function of the BDT selection requirement. The classifier output selection of 0.83 and 0.87 correspond to signal efficiencies after the pre-selection of 22% and 15%, respectively. These selections increase, or decrease the background from $B \rightarrow X_c \ell^+ \nu_\ell$ and other processes by 37% and 33%, respectively. The grey and yellow bands show the total and statistical error, respectively, with the nominal BDT working point of 0.85.

+37% and -33%, respectively. The small shifts in central value are well contained within the quoted systematic uncertainties. To further estimate the compatibility of the result we determine the full statistical and systematic correlations of the results and recover that the partial branching fraction with looser and tighter BDT selection are in agreement with the nominal result within 1.1 and 1.4 standard deviations, respectively.

E. $B \rightarrow X_u \ell^+ \nu_\ell$ Charged Pion Multiplicity

The modeling the $B \rightarrow X_u \ell^+ \nu_\ell$ signal composition is crucial to all presented measurements. One aspect difficult to assess is the X_u fragmentation simulation: the charmless X_u state can decay via many different channels producing a number of charged or neutral pions or kaons. In Section V we discussed how we assess the uncertainty on the number of $s\bar{s}$ quark pairs produced in the X_u fragmentation. Due to the BDT removing such events to suppress the dominant $B \rightarrow X_c \ell^+ \nu_\ell$ background, no signal-enriched region can be easily obtained. The accuracy of the fragmentation into the number of charged pions can be tested in the signal enriched region of $M_X < 1.7 \text{ GeV}$. Figure 11 compares the charged pion multiplicity between simulated signal and background processes and data. The signal and background predictions are scaled to their respective normalizations obtained from the two-

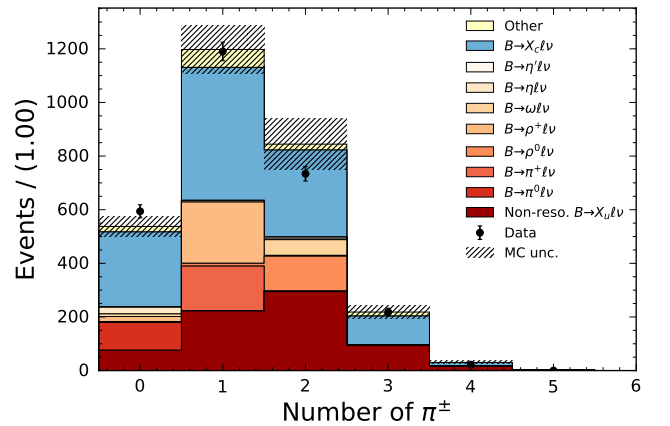


FIG. 11. The post-fit charged pion multiplicity is shown for events with $M_X < 1.7 \text{ GeV}$. The uncertainties on the MC stack include all systematic uncertainties.

dimensional fit in $M_X : q^2$. The uncertainty band shown on the MC includes the full systematic uncertainties discussed in Section V. The agreement overall lies within the assigned uncertainties, with the data having more events in the zero multiplicity bin and less in the two charged pion multiplicity bin. We use this distribution to correct our simulation to assign an additional uncertainty from the charged pion fragmentation. More details can be found in Section V and Appendix C.

F. Lepton Flavor Universality and Weak Annihilation Contributions

To test the lepton flavor universality in $B \rightarrow X_u \ell^+ \nu_\ell$ we also carry out fits to determine the partial branching fraction for electron and muon final states. For this we categorize the selected events accordingly and carry out a fit to the $M_X : q^2$ distributions using the same granularity as the fit described in Section VIII A. We carry out a simultaneous analysis of both samples, such that shared NPs for the modeling of the signal or background components can be correctly correlated afterwards. The resulting yields are corrected to a partial branching fraction with $E_\ell^B > 1$ GeV and we obtain

$$\Delta\mathcal{B}(B \rightarrow X_u e^+ \nu_e) = (1.56 \pm 0.10 \pm 0.17) \times 10^{-3}, \quad (37)$$

$$\Delta\mathcal{B}(B \rightarrow X_u \mu^+ \nu_\mu) = (1.63 \pm 0.10 \pm 0.18) \times 10^{-3}, \quad (38)$$

with a total correlation of $\rho = 0.57$. The ratio of the electron to the muon final state is

$$R_{e\mu} = \frac{\Delta\mathcal{B}(B \rightarrow X_u e^+ \nu_e)}{\Delta\mathcal{B}(B \rightarrow X_u \mu^+ \nu_\mu)} = 0.96 \pm 0.09 \pm 0.04, \quad (39)$$

with the first error denoting the statistical uncertainty and the second the systematic uncertainty. We observe no significant deviation from lepton flavor universality. More details on the fit can be found in Appendix E.

Isospin breaking effects can be studied by separately measuring the partial branching fraction for charged and neutral B meson final states. We determine the ratio

$$R_{\text{iso}} = \frac{\tau_{B^0}}{\tau_{B^+}} \times \frac{\Delta\mathcal{B}(B^+ \rightarrow X_u \ell^+ \nu_\ell)}{\Delta\mathcal{B}(B^0 \rightarrow X_u \ell^+ \nu_\ell)}, \quad (40)$$

by using the information from the composition of the fully reconstructed tag-side B -meson decays to separate charged and neutral B candidates. The partial branching fraction is then determined by a simultaneous fit of both samples in $M_X : q^2$ to correctly correlate common systematic uncertainties. To account for the small contamination of wrongly assigned B tag flavors, we use the wrong-tag fractions from our simulation. The measured number of signal events in the reconstructed neutral and charged B candidate categories (denoted in the following

as N_{reco}^0 and N_{reco}^+) are related to the number of neutral and charged B mesons (N_{true}^0 and N_{true}^+) via

$$N_{\text{reco}}^0 = \mathcal{P}_{B_{\text{true}}^0 \rightarrow B_{\text{reco}}^0} N_{\text{true}}^0 + \mathcal{P}_{B_{\text{true}}^+ \rightarrow B_{\text{reco}}^0} N_{\text{true}}^+, \quad (41)$$

$$N_{\text{reco}}^+ = \mathcal{P}_{B_{\text{true}}^0 \rightarrow B_{\text{reco}}^+} N_{\text{true}}^0 + \mathcal{P}_{B_{\text{true}}^+ \rightarrow B_{\text{reco}}^+} N_{\text{true}}^+. \quad (42)$$

Here e.g. $\mathcal{P}_{B_{\text{true}}^0 \rightarrow B_{\text{reco}}^+}$ denotes the probability to identify in the reconstruction of the tag-side B -meson a true B^0 as a B^+ candidate. In the simulation we find

$$\mathcal{P}_{B_{\text{true}}^0 \rightarrow B_{\text{reco}}^0} = 0.985 \quad \mathcal{P}_{B_{\text{true}}^0 \rightarrow B_{\text{reco}}^+} = 0.015, \quad (43)$$

$$\mathcal{P}_{B_{\text{true}}^+ \rightarrow B_{\text{reco}}^0} = 0.977 \quad \mathcal{P}_{B_{\text{true}}^+ \rightarrow B_{\text{reco}}^+} = 0.023. \quad (44)$$

Using this procedure we determine for the individual partial branching fractions with $E_\ell^B > 1$ GeV

$$\Delta\mathcal{B}(B^+ \rightarrow X_u \ell^+ \nu_\ell) = (1.65 \pm 0.10 \pm 0.18) \times 10^{-3}, \quad (45)$$

$$\Delta\mathcal{B}(B^0 \rightarrow X_u \ell^+ \nu_\ell) = (1.51 \pm 0.09 \pm 0.16) \times 10^{-3}, \quad (46)$$

with a total correlation of $\rho = 0.52$ and for the ratio Eq. 40

$$R_{\text{iso}} = 1.02 \pm 0.09 \pm 0.05, \quad (47)$$

compatible with the expectation of equal semileptonic rates for both isospin states. Isospin breaking effects would for instance arise from weak annihilation contributions, which only can contribute to charged B meson final states. Using Eq. 47 the relative contribution from weak annihilation processes to the total semileptonic $B \rightarrow X_u \ell^+ \nu_\ell$ rate can be constrained via

$$\frac{\Gamma_{\text{wa}}}{\Gamma(B \rightarrow X_u \ell^+ \nu_\ell)} = \frac{f_u}{f_{\text{wa}}} \times (R_{\text{iso}} - 1). \quad (48)$$

Here f_u is a factor that corrects the measured partial branching fraction to the full inclusive phase space. We estimate it using the DFN model [51] (cf. Section II for details) and find $f_u = 0.86$. We further assume that $f_{\text{wa}} = 1$, as such processes would produce a high momentum lepton. We recover

$$\frac{\Gamma_{\text{wa}}}{\Gamma(B \rightarrow X_u \ell^+ \nu_\ell)} = 0.02 \pm 0.09, \quad (49)$$

which translates into a limit of $[-0.13, 0.17]$ at 90% CL. This result is more stringent than the limit of Ref. [15], but weaker than the result of Ref. [68], that directly used the shape of the q^2 distribution to constrain weak annihilation processes. Our result is also weaker than the estimates of Refs. [69–72] that constrain weak annihilation contributions to be of the order 2-3%.

IX. SUMMARY AND CONCLUSIONS

We report measurements of partial branching fractions with different requirements on the properties of the hadronic system of the $B \rightarrow X_u \ell^+ \nu_\ell$ decay and with a lepton energy of $E_\ell^B > 1 \text{ GeV}$ in the B rest-frame, covering 31-86% of the available phase space. The sizeable background from semileptonic $B \rightarrow X_c \ell^+ \nu_\ell$ decays is suppressed using multivariate methods in the form of a BDT. This approach allows us to reduce such backgrounds to an acceptable level, whilst retaining a high signal efficiency. Signal yields are obtained using a binned likelihood fit in either the reconstructed hadronic mass M_X , the four-momentum-transfer squared q^2 , or the lepton energy E_ℓ^B . The most precise result is obtained from a two-dimensional fit of M_X and q^2 . Translated to a partial branching fraction for $E_\ell^B > 1 \text{ GeV}$ we obtain

$$\Delta\mathcal{B}(B \rightarrow X_u \ell^+ \nu_\ell) = (1.59 \pm 0.07 \pm 0.17) \times 10^{-3}, \quad (50)$$

with the errors denoting statistical and systematic uncertainties. The partial branching fraction is compatible with the value obtained by a fit of the lepton energy spectrum E_ℓ^B and with the most precise determination of Ref. [66]. In addition, it is stable under variations of the background suppression BDT. From this partial branching fraction we obtain a value of

$$|V_{ub}| = (4.10 \pm 0.09 \pm 0.22 \pm 0.15) \times 10^{-3} \quad (51)$$

from an average over four theoretical calculations. This value is higher than, but compatible with, the value of $|V_{ub}|$ from exclusive determinations by 1.3 standard deviations. The compatibility with the value expected from CKM unitarity from a fit of Ref. [73] of $|V_{ub}| = (3.62^{+0.11}_{-0.08}) \times 10^{-3}$ is 1.6 standard deviations. Figure 12 summarizes the situation. The result presented here supersedes Ref. [16]: this paper uses a more efficient tagging algorithm, incorporates improvements of the $B \rightarrow X_u \ell^+ \nu_\ell$ signal and $B \rightarrow X_c \ell^+ \nu_\ell$ background descriptions, and analyzes the full Belle data set of 711 fb^{-1} . The measurement of kinematic differential shapes of M_X , q^2 , and other properties are left for future work. These results will be crucial for future direct measurements with Belle II that will attempt to use data-driven methods to directly constrain the shape function using $B \rightarrow X_u \ell^+ \nu_\ell$ information.

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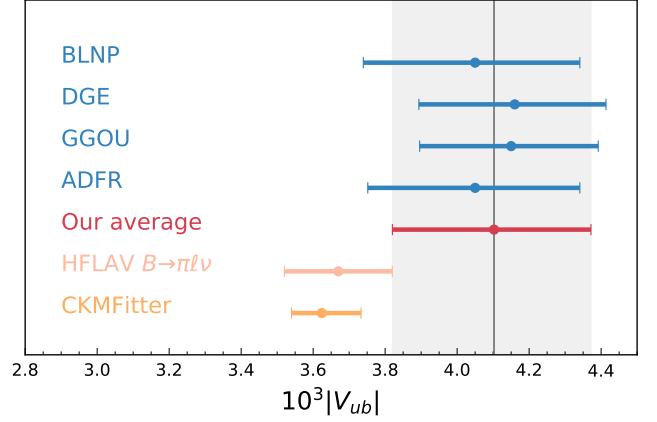


FIG. 12. The obtained values of $|V_{ub}|$ from the four calculations and the arithmetic average is compared to the determination from exclusive $B \rightarrow \pi \ell^+ \nu_\ell$ and the expectation from CKM unitarity [73] without the direct constraints from semileptonic and leptonic decays.

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A. $B \rightarrow X_u \ell^+ \nu_\ell$ HYBRID MC DETAILS

Figure 13 shows the generator level hybrid $B \rightarrow X_u \ell^+ \nu_\ell$ signal sample for E_ℓ^B , M_X , and q^2 described in Section II.

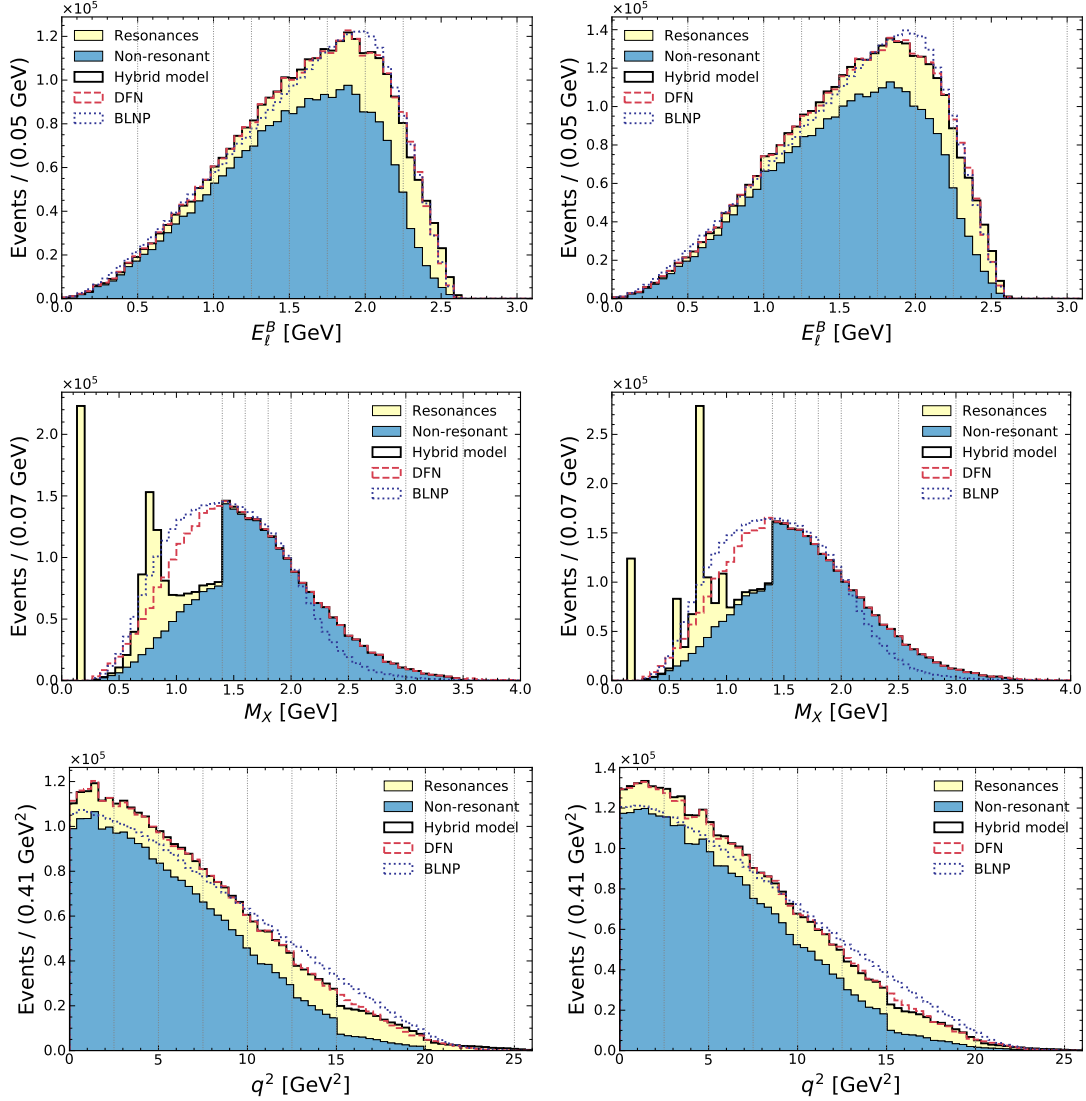


FIG. 13. The generator level $B \rightarrow X_u \ell^+ \nu_\ell$ distributions E_ℓ^B , M_X , and q^2 for neutral (left) and charged (right) B mesons are shown. The black histogram shows the merged hybrid model, composed of resonant and non-resonant contributions. For more details on the used models and how the hybrid $B \rightarrow X_u \ell^+ \nu_\ell$ signal sample is constructed, see Section II.

B. INPUT VARIABLES OF $B \rightarrow X_c \ell \bar{\nu}_\ell$ SUPPRESSION BDT

The shapes of the variables used in the $B \rightarrow X_c \ell^+ \nu_\ell$ background suppression BDT are shown in Figures 14 and 16. The most discriminating variables are M_{miss}^2 , the B_{sig} vertex fit probability, and M_{miss, D^*}^2 . Figures 15 and 17 show the agreement between recorded and simulated events, taking into account the full uncertainties detailed in Section V. More details about the BDT can be found in Section III C.

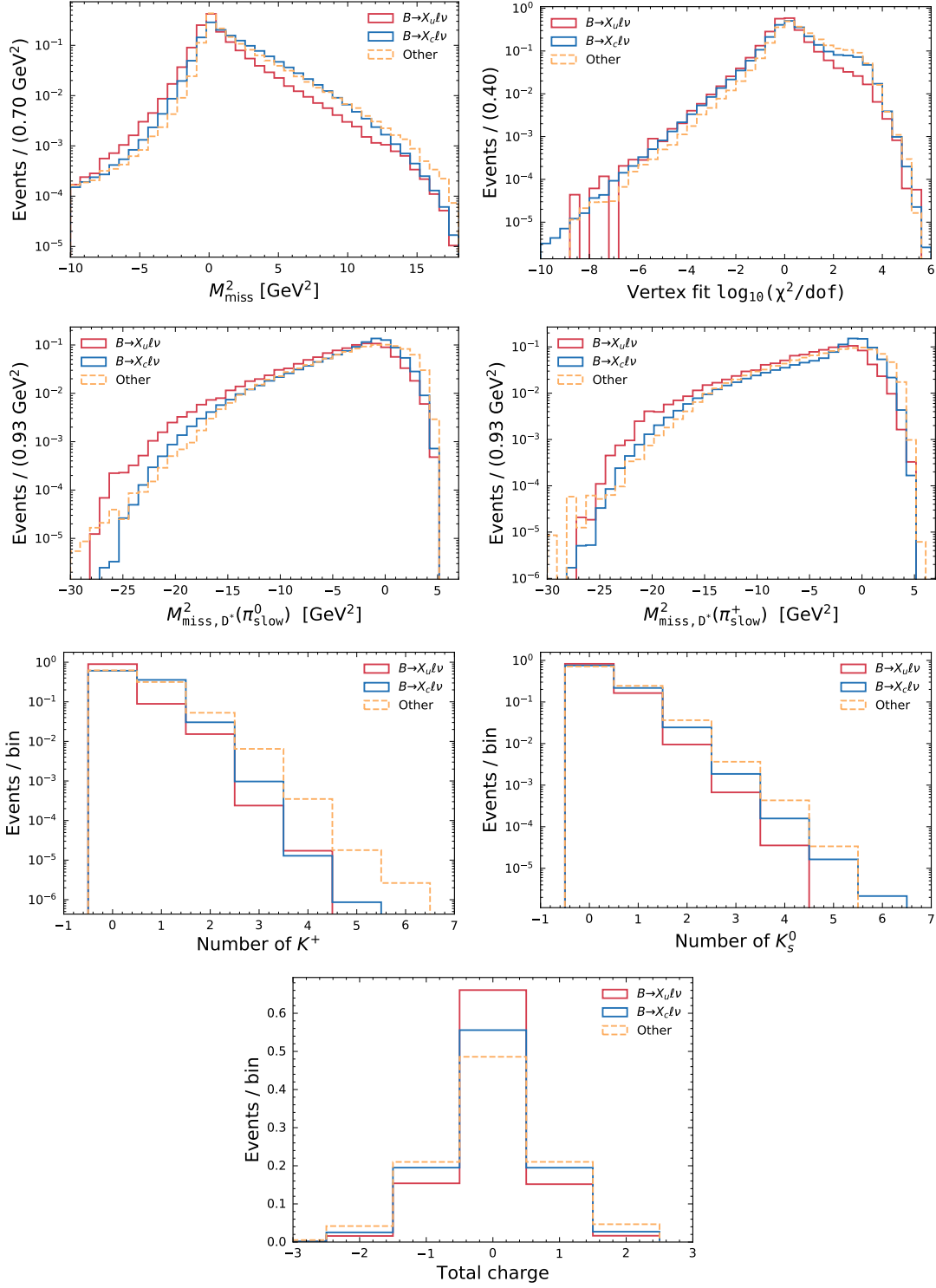


FIG. 14. The shape of the input variables for the $B \rightarrow X_c \ell^+ \nu_\ell$ background suppression BDT are shown. For details and definitions see Section III C.

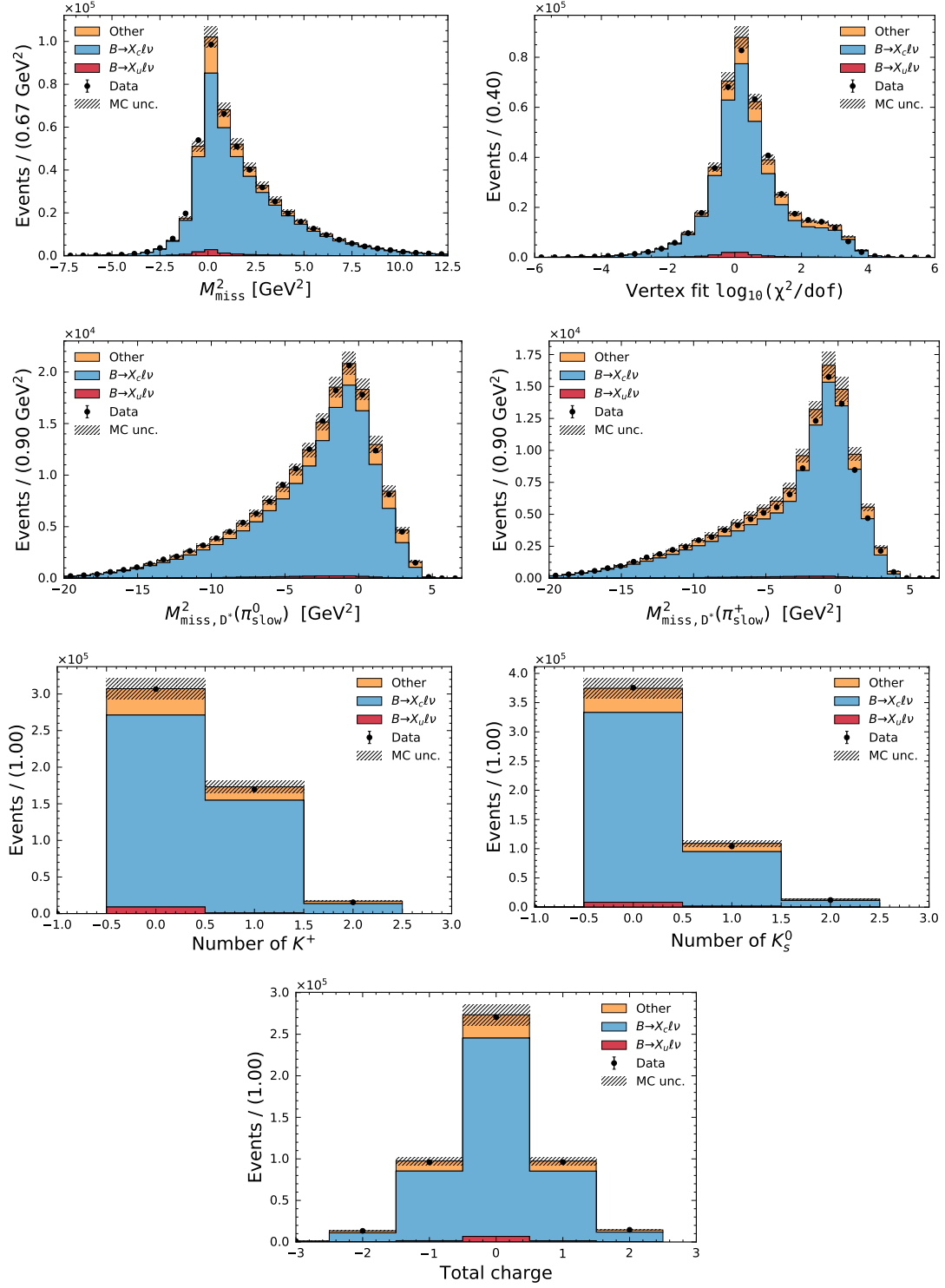


FIG. 15. The input variables for the $B \rightarrow X_c \ell^+ \nu_\ell$ background suppression BDT for recorded and simulated events are shown. The uncertainty on the simulated events incorporate the full systematic uncertainties detailed in Section V.

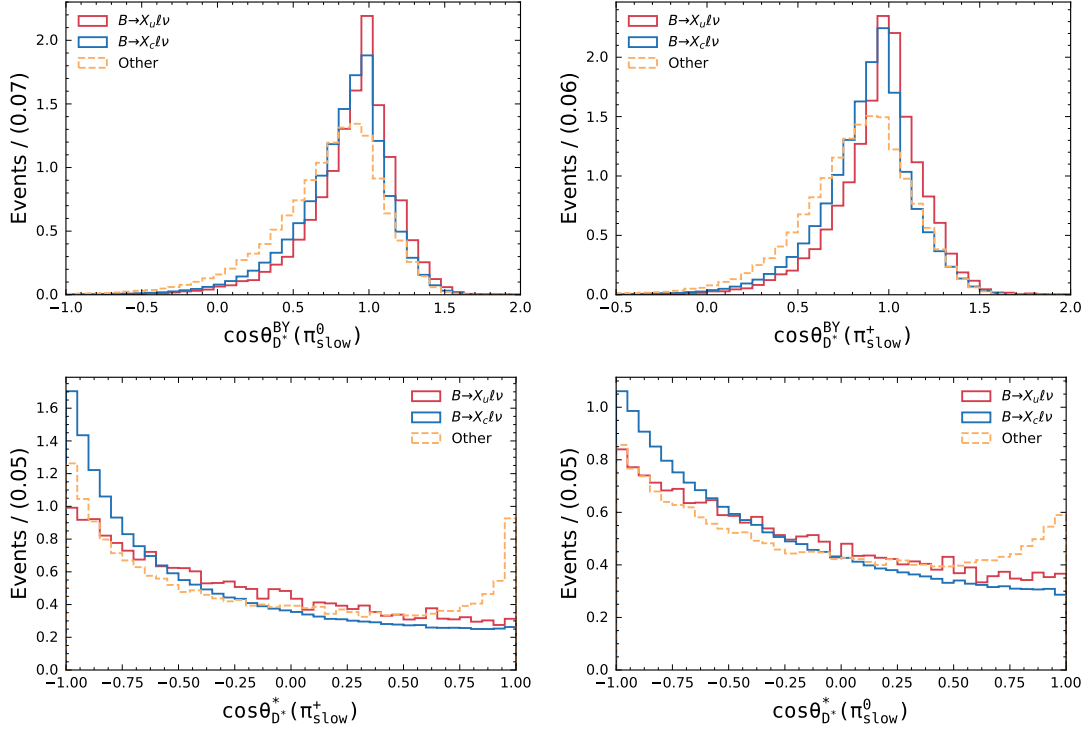


FIG. 16. The shape of the input variables for the $B \rightarrow X_c \ell^+ \nu_\ell$ background suppression BDT are shown. For details and definitions see Section III C.

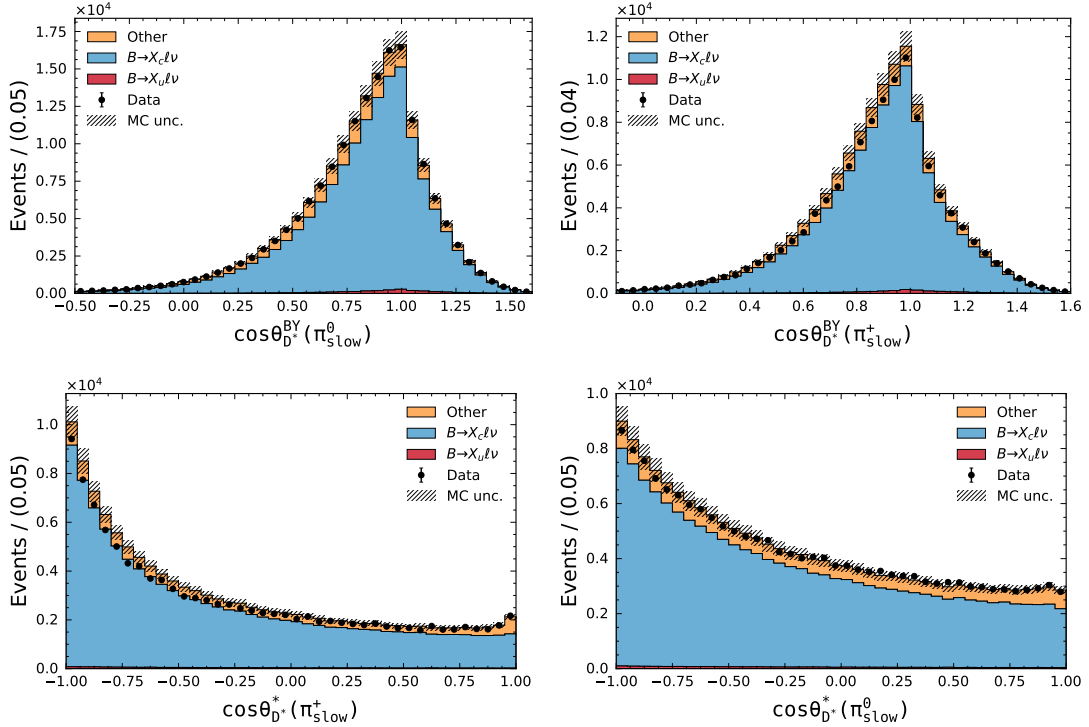


FIG. 17. The input variables for the $B \rightarrow X_c \ell^+ \nu_\ell$ background suppression BDT for recorded and simulated events are shown. The uncertainty on the simulated events incorporate the full systematic uncertainties detailed in Section V.

C. $B \rightarrow X_u \ell^+ \nu_\ell$ CHARGED PION FRAGMENTATION MODELING

Figure 18 compares the charged pion multiplicity at different stages in the selection. This variable is not used in the signal extraction, but its modeling is tested to make sure that the $B \rightarrow X_u \ell^+ \nu_\ell$ fragmentation probabilities cannot bias the final result. The agreement in the signal enriched region with $M_X < 1.7$ GeV after the BDT selection is fair, but shows some deviations. We correct the generator level charged pion multiplicity to match the $n_{\pi^\pm}$ observed in this selection by assigning the non-resonant $B \rightarrow X_u \ell^+ \nu_\ell$ events a correction weight as a function of the true charged pion multiplicity. After this procedure the agreement is perfect and we use the difference in the reconstruction efficiency as an uncertainty on the pion fragmentation on the partial branching fractions and $|V_{ub}|$ (cf. Section V).

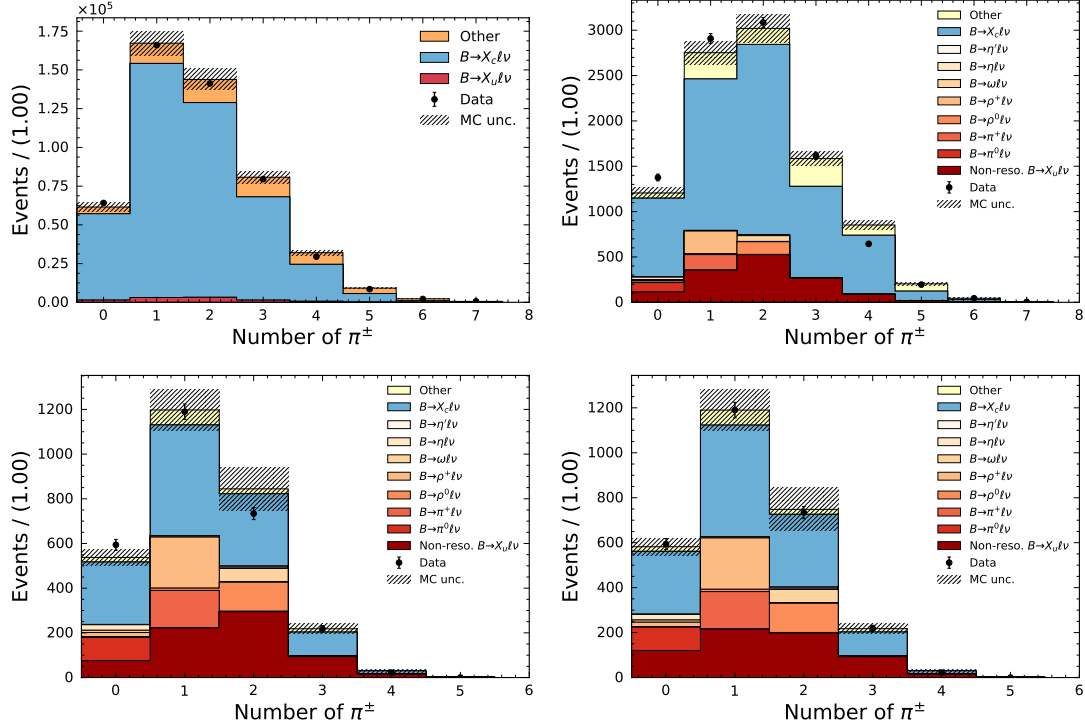


FIG. 18. The charged pion multiplicity ($n_{\pi^\pm}$) are compared between data and the simulation: (top left) for all events prior the BDT selection; (top right) for all events after the BDT selection; (bottom left): for the signal enriched region of $M_X < 1.7$ GeV; (bottom right) for the same region but after rescaling the non-resonant contributions such that the $n_{\pi^\pm}$ fragmentation probability to match the one observed in data.

D. NUISANCE PARAMETER PULLS AND ADDITIONAL FIT PLOTS

Figures 19 and 20 show the nuisance parameter pulls for each fit category k and bin i defined as

$$\left(\hat{\theta}_{ik} - \theta_{ik} \right) / \sqrt{\Sigma_{k,ii}}, \quad (52)$$

of the partial branching fraction fits, with $\hat{\theta}$ (θ) corresponding to the post-fit (pre-fit) value of the nuisance parameter. Note that uncertainties of each pull shows the post-fit error

$$\sqrt{\hat{\Sigma}_{k,ii}} \quad (53)$$

normalized to the pre-fit constraint

$$\sqrt{\Sigma_{k,ii}}. \quad (54)$$

Figure 21 shows the post-fit q^2 distributions of the two-dimensional fit to $M_X : q^2$ on M_X .

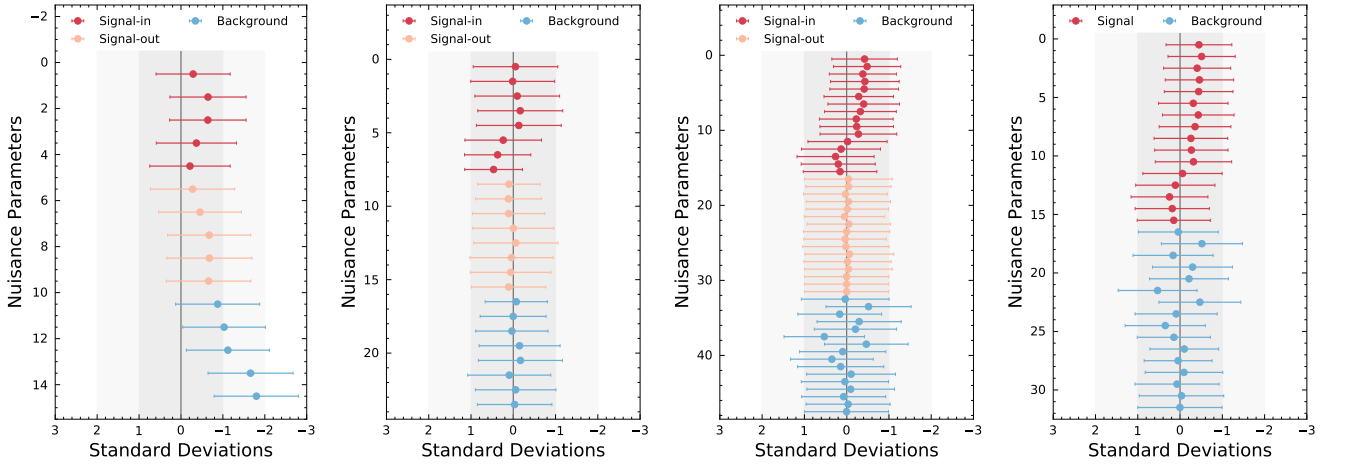


FIG. 19. The nuisance parameter pulls on the 1D fits of M_X , q^2 , and E_ℓ^B with and without $M_X < 1.7$ GeV events separated out, are shown from left to right.

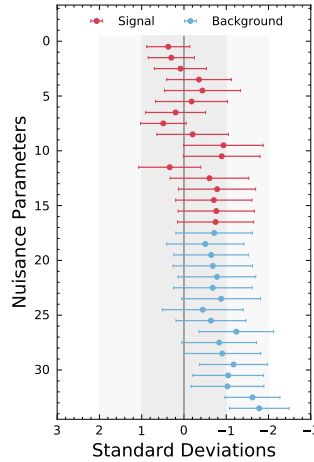


FIG. 20. The nuisance parameter pulls on the 2D fit of $M_X : q^2$ is shown.

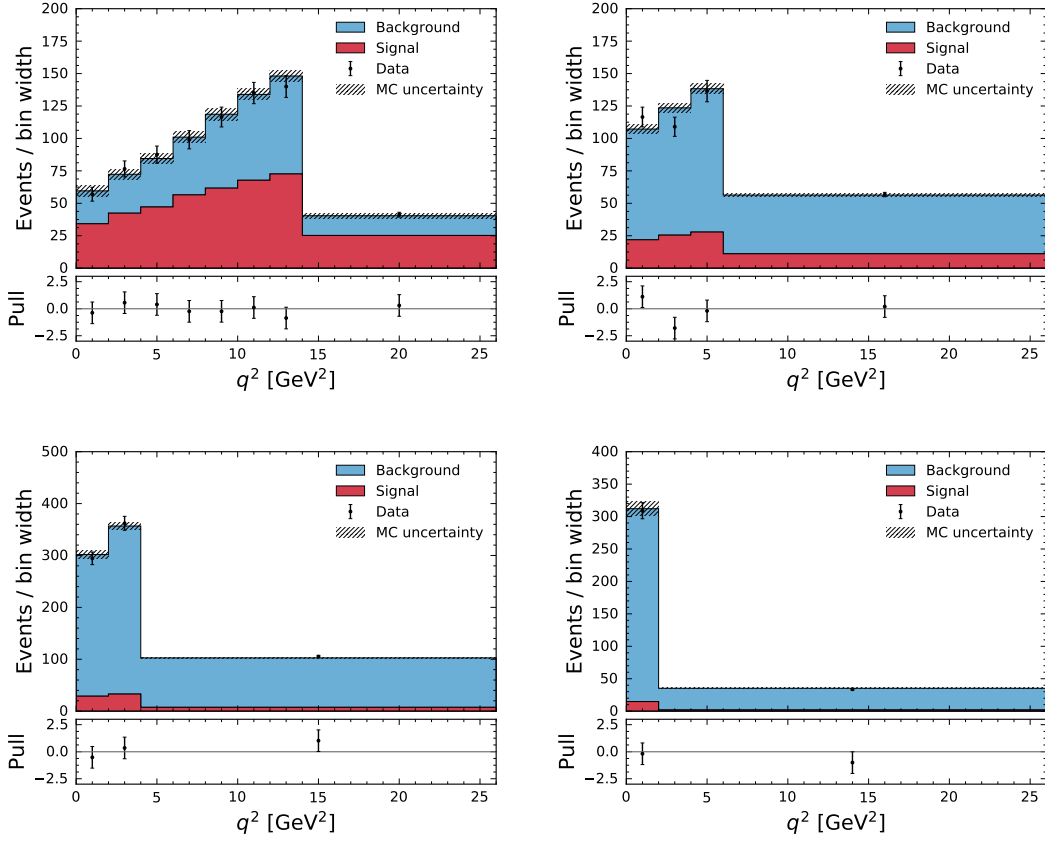


FIG. 21. The post-fit q^2 distributions of the two-dimensional fit to $M_X : q^2$ on M_X are shown. The panels correspond to: $M_X \in [0, 1.5]$ GeV (top left), $M_X \in [1.5, 1.9]$ GeV (top right), $M_X \in [1.9, 2.4]$ GeV (bottom left) and $M_X \in [2.4, 4]$ GeV (bottom right). The resulting yields are corrected to correspond to a partial branching fraction with $E_\ell^B > 1$ GeV.

TABLE VII. The fitted yields separated in electron and muon candidates, as well as in charged or neutral B mesons.

Decay mode	$\hat{\eta}_{\text{sig}}$	$\hat{\eta}_{\text{bkg}}$	$10^3 (\epsilon_{\text{tag}} \cdot \epsilon_{\text{sel}})$	$10^3 \Delta \mathcal{B}$
$B^+ \rightarrow X_u \ell^+ \nu$	$915 \pm 56 \pm 65$	$3667 \pm 77 \pm 64$	0.30 ± 0.13	$1.65 \pm 0.10 \pm 0.18$
$B^0 \rightarrow X_u \ell^+ \nu$	$876 \pm 58 \pm 65$	$3375 \pm 76 \pm 64$	0.33 ± 0.11	$1.51 \pm 0.09 \pm 0.16$
$B \rightarrow X_u e^+ \nu$	$866 \pm 56 \pm 64$	$3315 \pm 75 \pm 65$	0.31 ± 0.12	$1.56 \pm 0.10 \pm 0.17$
$B \rightarrow X_u \mu^+ \nu$	$940 \pm 58 \pm 74$	$3712 \pm 78 \pm 73$	0.32 ± 0.13	$1.63 \pm 0.10 \pm 0.18$

E. ADDITIONAL FIT DETAILS TO THE LEPTON FLAVOR UNIVERSALITY AND WEAK ANNIHILATION TESTS

The fitted yields of the two-dimensional fit to $M_X : q^2$ separated in electron and muon candidates, as well as in charged or neutral B mesons are listed in Table VII.

F. BDT EFFICIENCIES

Figure 22 shows the efficiency of the BDT selection as a function of the reconstructed variables q^2 , M_X , and the lepton energy E_ℓ^B for simulated $B \rightarrow X_u \ell^+ \nu_\ell$ events. Although we avoided using these variables in the boosted decision tree, a residual dependence on the kinematic variables is seen. For instance the efficiency increases with an increase in E_ℓ^B and a decrease with respect to high q^2 . The efficiency on the hadronic mass M_X is relatively flat. This efficiency dependence is linked to the used variables in the BDT. Although we carefully avoided kinematic variables that would allow the BDT to learn these kinematic properties, there are indirect connections: e.g. high E_ℓ^B final states have a lower multiplicity as they are dominated by $B \rightarrow \pi \ell \bar{\nu}_\ell$ decays. Further, their corresponding hadronic system carries little momentum and on average such decays retain a better resolution in discriminating variables of the background suppression BDT. A concrete example is M_{miss}^2 (cf. Figure 15): high multiplicity $B \rightarrow X_u \ell^+ \nu_\ell$ decays will retain a larger tail in this variable and will be selected with a lower efficiency by the BDT.

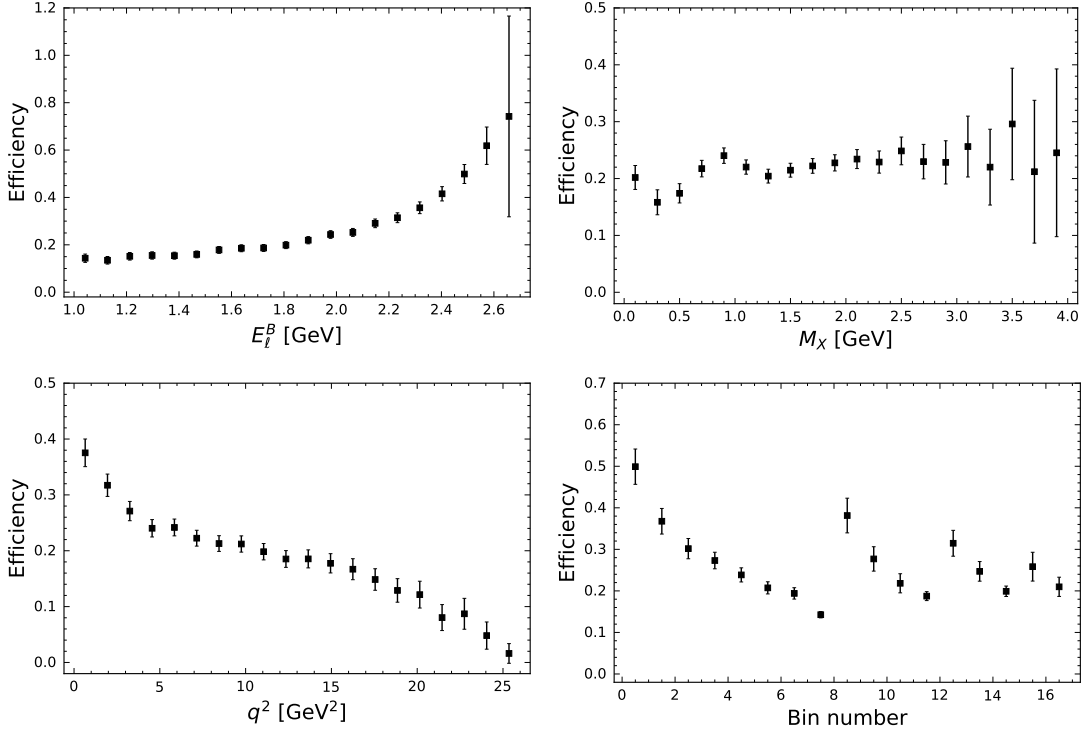


FIG. 22. The $B \rightarrow X_u \ell^+ \nu_\ell$ efficiency after the BDT selection is shown as a function of the reconstructed kinematic variables (E_ℓ^B , M_X , q^2) used in the signal extraction. The bottom right plot shows the efficiencies in the bins of $M_X : q^2$ and the binning can be found in the text. The uncertainties are statistical only.