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Moments $n = 2$ and $n = 3$ of the Wilson twist-two operators at three loops in the RI′/SMOM scheme

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Abstract

We study the renormalization of the matrix elements of the twist-two non-singlet bilinear quark operators, contributing to the $n = 2$ and $n = 3$ moments of the structure functions, at next-to-next-to-next-to-leading order in QCD perturbation theory at the symmetric subtraction point. This allows us to obtain conversion factors between the $\overline{\rm MS}$ scheme and the regularization-invariant symmetric MOM (RI/SMOM, RI′/SMOM) schemes. The obtained results can be used to reduce errors in determinations of moments of structure functions from lattice QCD simulations. The results are given in Landau gauge.

Keywords: Lattice QCD, Bilinear quark operators, $\overline{\rm MS}$ scheme, Regularization-invariant symmetric MOM scheme, Three-loop approximation

1. Introduction

The great success of QCD in the description of the structure of hadrons relies on the principle of factorization. Phenomenologically it is possible to access this problem only in some particular kinematical conditions, as provided, for instance, in experiments like deep-inelastic scattering, vector boson or heavy-meson production, Drell-Yan process and others.

In hard processes, QCD factorization and scaling violation manifest themselves in the well-known Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [1, 2, 3] and allow for nonperturbative information, on how the parton momenta are distributed inside the hadrons and how the hadron spins are

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generated, to be accumulated in parton distribution functions (PDFs). Besides PDFs, also other nonperturbative distributions and concepts like, e.g., light-cone distribution amplitudes (LCDAs) [4, 5, 6, 7, 8, 9] and generalized parton distributions (GPDs) [10, 11] have been introduced.

At the operator level, the most significant contributions in hard processes arise from operators of twist two. In particular, in case of the non-siglet distributions, bilinear quark operators play a crucial rôle. Such operators, contributing to the nth moment of a distribution, are given by symmetric traceless combinations, like

$$\mathcal{S}\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\dots D_{\mu_n}\psi\,, \tag{1}$$

where the symbol $\mathcal{S}$ denotes total symmetrization over indices $\mu_1,\dots,\mu_n$ (including the factor $1/n!$) and subtraction of all possible traces over pairs of indices.

Since the matrix elements of the operators in Eq. (1) are of nonpertubative nature, they can be accessed only by experiments, QCD sum rules, or lattice-QCD simulations. The most important examples of recent lattice studies include determinations of low moments of light-cone distribution amplitudes of mesons (see, e.g., Refs. [12, 13, 14, 15]) and low moments of the proton PDFs and GPDs (see, e.g., Refs. [16, 17, 18, 19, 20]).

To renormalize the matrix elements of the operators in Eq. (1) on the lattice, the regularization-invariant momentum-subtraction (RI/MOM) scheme and its modification, the RI′/MOM scheme, have been developed [21, 22]. Improved variants include the RI/SMOM and RI′/SMOM schemes [23, 24], which differ in the way three-point functions are treated. In the RI/MOM and RI′/MOM schemes, the subtraction is done at vanishing operator momenta, which potentially generates additional sensitivity to short-distance effects in this channel. On the other hand, in the RI/SMOM and RI′/SMOM schemes, the subtraction of three-point functions is performed at the symmetric Euclidean point, $-\mu^2$, by setting

$$p^2 = q^2 = (p+q)^2 = -\mu^2\,, \qquad p\cdot q = \frac{\mu^2}{2}\,, \tag{2}$$

where the four-momenta p and q are as depicted in Fig. 1. Thus, there is no channel with exceptional momenta in this scheme.

The next step after the nonpertubative renormalization is the perturbative convertion of the results from one of the above schemes into the modified minimal-subtraction ($\overline{\text{MS}}$) scheme of dimensional regularization, which serves as the worldwide standard in perturbative QCD calculations. Choosing the parameter $-\mu^2$ to be of the order of a few GeV, such a convertion can be done perturbatively order by order in the expansion in the strong-coupling constant $\alpha_s(-\mu^2)$. This matches the lattice simulations with the high-energy behavior determined by conventional perturbation theory in the continuum using the $\overline{\text{MS}}$ scheme.

The RI/SMOM to $\overline{\text{MS}}$ conversion functions of non-singlet bilinear quark operators without derivatives have been considered in Refs. [23, 25] at one loop

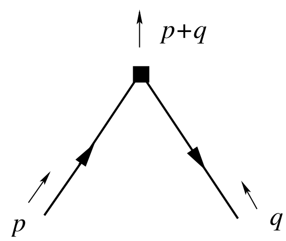

Figure 1: Matrix element $\langle\psi(q)\,O(-q-p)\,\bar{\psi}(p)\rangle$ of bilinear quark operator in momentum space. The black box denotes the operator and the solid lines the external quarks.

and in Refs. [24, 26, 27] at two loops. In our previous paper [28], we extended this analysis to the three-loop order numerically. Our three-loop result for the (pseudo)scalar current has been confirmed by an analytical calculation [29] in terms of constants constructed earlier in Ref. [30].

The corresponding conversions for the $n = 2,3$ moments of the bilinear quark operators of twist two with one or two covariant derivatives have been considered in Refs. [25, 27, 31, 32] at the one- and two-loop orders.

In this paper, we extend this analysis to the three-loop order. We concentrate on the cases of $n = 2$ and $n = 3$ and study the relevant operators at the symmetric kinematical point up to three loops. This paper is organized as follows. In Section 2, we introduce notations and the definitions. In Sections 3 and 4, we present our three-loop results for $n = 2$ and $n = 3$ moments, respectively. In Section 5, we conclude with a summary.

2. Setup

To fix the notation, we start from the following expression in Minkowsky coordinate space:

$$\int dx\,dy\,e^{-iq\cdot x-ip\cdot y}\langle\psi_{\xi,i}(x)O(0)\bar{\psi}_{\zeta,j}(y)\rangle = \delta_{ij}S_{\xi\xi'}(-q)\Lambda_{\xi'\zeta'}(p,q)S_{\zeta'\zeta}(p)\,, \quad (3)$$

where O stands for some bilinear quark operator, ξ,ζ are spinor indices, i,j are color indices in the fundamental representation of the $SU(N)$ group, $S(q)$ is the quark propagator, and $\Lambda(p,q)$ is the amputated Green's function, which is shown schematically in Fig. 1.

In the cases $n = 2$ and $n = 3$, we can write explicitly for any operators $O_{\mu\nu}$ and $O_{\mu\nu\sigma}$:

$$\mathcal{S}O_{\mu\nu} = \frac{1}{2}\big(O_{\mu\nu}+O_{\nu\mu}\big) - \frac{1}{d}\,g_{\mu\nu}O^{\alpha}_{\alpha}\,, \quad (4)$$

$$\mathcal{S}O_{\mu\nu\sigma} = \frac{1}{6}\big(O_{\mu\nu\sigma}+O_{\nu\mu\sigma}+O_{\nu\sigma\mu}+O_{\sigma\nu\mu}+O_{\sigma\mu\nu}+O_{\mu\sigma\nu}\big) - \frac{1}{3(d+2)}\big(g_{\mu\nu}g^{\rho}_{\sigma}+g_{\nu\sigma}g^{\rho}_{\mu}+g_{\sigma\mu}g^{\rho}_{\nu}\big)\big(O_{\rho\alpha\alpha'}+O_{\alpha\rho\alpha'}+O_{\alpha\alpha'\rho}\big)g^{\alpha\alpha'}\,, \quad (5)$$

where $g_{\mu\nu}$ is the metric tensor and $d = 4 - 2\varepsilon$ is the space-time dimension.

In the definition in Eq. (1), we still have the freedom to define in which directions the covariant derivatives act. Thus, in the case with one derivative, we can define two operators,

$$\mathcal{S}O^{L}_{\mu\nu} = \mathcal{S}\bar{\psi}\gamma_\mu \overleftarrow{D}_\nu \psi\,, \tag{6}$$

$$\mathcal{S}O^{R}_{\mu\nu} = \mathcal{S}\bar{\psi}\gamma_\mu \overrightarrow{D}_\nu \psi\,, \tag{7}$$

from which we can construct operators with either sign of charge conjugation (C),

$$O^{C=-1} = \mathcal{S}O^{L} + \mathcal{S}O^{R}\,, \tag{8}$$

$$O^{C=+1} = \mathcal{S}O^{L} - \mathcal{S}O^{R}\,, \tag{9}$$

where we have omitted the indices μ, ν for the ease of notation. Notice that the operators in Eqs. (8) and (9) do not mix under the renormalization, so that the operator renomalization matrix is diagonal in this basis. In Refs. [25, 27, 31], different operators, called W_2 and ∂W_2, have been introduced. These can be expressed in terms of the operators O^L and O^R with the help of a suitable 2×2 transformation matrix, as

$$\frac{1}{2}\begin{pmatrix} W_2 \\ \partial W_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} O^L \\ O^R \end{pmatrix}\,. \tag{10}$$

The factor 1/2 in Eq. (10) appears because it has been omitted in the definitions of W_2 and ∂W_2 in Refs. [25, 27, 31]. We should also note that, in these papers, W_2 corresponds to the operator where the covariant derivative acts to the right, while, according to our definitions, the derivative in W_2 acts to the left. Only with such conventions, we find agreement with Ref. [25, 27, 31].

For the operators with two derivatives, we introduce the following basis of three operators:

$$\mathcal{S}O^{LL}_{\mu\nu\sigma} = \mathcal{S}\bar{\psi}\gamma_\mu \overleftarrow{D}_\nu \overleftarrow{D}_\sigma \psi\,, \tag{11}$$

$$\mathcal{S}O^{LR}_{\mu\nu\sigma} = \mathcal{S}\bar{\psi}\gamma_\mu \overleftarrow{D}_\nu \overrightarrow{D}_\sigma \psi\,, \tag{12}$$

$$\mathcal{S}O^{RR}_{\mu\nu\sigma} = \mathcal{S}\bar{\psi}\gamma_\mu \overrightarrow{D}_\nu \overrightarrow{D}_\sigma \psi\,. \tag{13}$$

From these operators, we can define the following combinations with definite C parities:

$$O^{C=-1}_1 = O^{LL} - 2O^{LR} + O^{RR}\,, \tag{14}$$

$$O^{C=-1}_2 = O^{LL} + 2O^{LR} + O^{RR}\,, \tag{15}$$

$$O^{C=+1}_3 = O^{LL} - O^{RR}\,, \tag{16}$$

where we again omit the indices μ, ν, σ for simplicity. Operators O_1 and O_2 mix under renormalization, so that the 3×3 operator renormalization matrix takes a block diagonal form in this basis, with one block of size 2×2 and one of size 1×1.

In Refs. [25, 27, 32], a different triplet of operators, called W_3, ∂W_3, and $\partial\partial W_3$, has been introduced. We can express these in terms of the operators in Eqs. (11)–(13) as

$$\begin{pmatrix} W_3 \\ \partial W_3 \\ \partial\partial W_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} O^{LL} \\ O^{LR} \\ O^{RR} \end{pmatrix} . \tag{17}$$

Similarly to the previous case, we find that the directions in which the covariant derivatives act in the operator W_3 defined in Refs. [25, 27, 32] should be flipped. Upon this change, we find agreement with the previous one- and two-loop calculations.

In order to renormalize the above operators, we use appropriate matrices Z of enormalization constants, a 2×2 matrix for $n = 2$ and a 3×3 matrix for $n = 3$. In the $\overline{\rm MS}$ scheme, we can write

$$Z = 1 + \frac{Z_1}{\varepsilon} + \frac{Z_2}{\varepsilon^2} + \frac{Z_3}{\varepsilon^3} + \cdots , \tag{18}$$

where Z_i are constant matrices depending on the QCD coupling constant,

$$a = \frac{\alpha_s}{4\pi} . \tag{19}$$

These matrices can be related to the matrix of anomalous dimensions γ by the following matrix equations:

$$a\,\partial_a Z_1 = -\gamma\,, \tag{20}$$

$$a\,\partial_a Z_2 = a\,\partial_a \left(\frac{1}{2} Z_1^2\right) + \beta\,\partial_a Z_1 - \xi\gamma_3\,\partial_\xi Z_1\,, \tag{21}$$

$$\begin{aligned} a\,\partial_a Z_3 &= a\,\partial_a \left(\frac{Z_1 Z_2 + Z_2 Z_1}{2} - \frac{1}{3} Z_1^3\right) + \beta\,\partial_a \left(Z_2 - \frac{1}{2} Z_1^2\right) \\ &\quad - \xi\gamma_3\,\partial_\xi \left(Z_2 - \frac{1}{2} Z_1^2\right) , \end{aligned} \tag{22}$$

where β is QCD β function, ξ is the gauge parameter, and γ_3 is the anomalous dimension associated with the latter [33].

The matrix γ for $n = 2$ has been evaluated analytically through $O(a^3)$ in Ref. [31]. The corresponding matrix for $n = 3$ can be found in Ref. [32].[1] Moreover, in Ref. [32], the nondiagonal matrix elements are only given through order $O(a^2)$. We evaluate the missing $O(a^3)$ contributions numerically for color

[1] Notice that the definition of γ given by Eq. (2.8) in Ref. [32] differs by a factor of 2 from the definition used for the results in Eq. (2.10) thereein.

group SU(3). In the basis $(W_3, \partial W_3, \partial\partial W_3)$, Eq. (2.10) in Ref. [32] should be extended by the following three-loop contributions

$$\gamma_{12}^{W_3,O(a^3)}(a) = a^3\big(- 385.466 + 66.199 n_f + 0.5329 n_f^2\big)\,, \tag{23}$$

$$\gamma_{13}^{W_3,O(a^3)}(a) = a^3\big(- 170.641 + 24.822 n_f + 0.3107 n_f^2\big)\,, \tag{24}$$

where n_f is the number of light quark flavors.

To represent our results, we adopt the tensor decompositions from Refs. [31, 32]. It is convenient to contract the open indices of the operators $O_{\mu\nu}$ and $O_{\mu\nu\sigma}$ with the light-cone vector Δ, with $\Delta^2 = 0$. This automatically takes into account the symmetry and the tracelessness of the operators. Specifically, we write

$$\begin{aligned}(-i)\Lambda_2(p,q) &= \not{\Delta}\left[2(p\cdot\Delta)\,F_1 + 2(q\cdot\Delta)\,F_2\right] \\ &+ \frac{1}{\mu^2}\not{p}\left[(p\cdot\Delta)^2\,F_3 + 2(p\cdot\Delta)(q\cdot\Delta)\,F_4 + (q\cdot\Delta)^2\,F_5\right] \\ &+ \frac{1}{\mu^2}\not{q}\left[(p\cdot\Delta)^2\,F_6 + 2(p\cdot\Delta)(q\cdot\Delta)\,F_7 + (q\cdot\Delta)^2\,F_8\right] \\ &+ \frac{1}{\mu^2}\Gamma_{3,\Delta pq}\left[2(p\cdot\Delta)\,F_9 + 2(q\cdot\Delta)\,F_{10}\right],\end{aligned} \tag{25}$$

$$\begin{aligned}(-i)^2\mu^2\Lambda_3(p,q) &= \frac{1}{\mu^2}\not{\Delta}\left[3(p\cdot\Delta)^2\,F_1 + 6(p\cdot\Delta)(q\cdot\Delta)\,F_2 + 3(q\cdot\Delta)^2\,F_3\right] \\ &+ \frac{1}{\mu^4}\not{p}\left[(p\cdot\Delta)^3\,F_4 + 3(p\cdot\Delta)^2(q\cdot\Delta)\,F_5 + 3(p\cdot\Delta)(q\cdot\Delta)^2\,F_6 + (q\cdot\Delta)^3\,F_7\right] \\ &+ \frac{1}{\mu^4}\not{q}\left[(p\cdot\Delta)^3\,F_8 + 3(p\cdot\Delta)^2(q\cdot\Delta)\,F_9 + 3(p\cdot\Delta)(q\cdot\Delta)^2\,F_{10} + (q\cdot\Delta)^3\,F_{11}\right] \\ &+ \frac{1}{\mu^4}\Gamma_{3,\Delta pq}\left[3(p\cdot\Delta)^2\,F_{12} + 6(p\cdot\Delta)(q\cdot\Delta)\,F_{13} + 3(q\cdot\Delta)^2\,F_{14}\right].\end{aligned} \tag{26}$$

Here

$$\Gamma_{3,\mu\nu\sigma} = \frac{1}{3!}\gamma_{[\mu}\gamma_\nu\gamma_{\sigma]} \tag{27}$$

is the fully antisymmetric combination of the Dirac γ matrices, and we use the short-hand notation $\Gamma_{3,\Delta pq} = \Gamma_{3,\mu\nu\sigma}\Delta^\mu p^\nu q^\sigma$. With the definitions in Eqs. (25) and (26), the definitions of the formactors $F_1, \ldots, F_{10}$ and $F_1, \ldots, F_{14}$ coincide with those in Ref. [31] and [32], respectively.

We refrain from describing our calculation because it is similar to the one in Ref. [28], where details may be found, and mere list our results, which we do for the $n = 2$ case in Section 3 and for the $n = 3$ case in Section 4.

3. Numerical results for $n = 2$ moment

Here, we present the numerical results for the formfactors F_j^L and F_j^R of the $n = 2$ moment at three loops in the $\overline{\text{MS}}$ scheme. For F_j^L, we have

$$
\begin{aligned}
F_1^L &= a\big(0.87497670537933942370\big) \\
&\quad + a^2\big(18.69246420435249196 - 2.5121840774766979282n_f\big) \\
&\quad + a^3\big(767.149(2) - 147.4921(2)n_f + 3.99834(1)n_f^2\big)\,, &(28)\\
F_2^L &= 0.5 + a\big(-1.6874417634483485593\big) \\
&\quad + a^2\big(-21.75488024301858658 + 2.394054755622383881n_f\big) \\
&\quad + a^3\big(-624.064(4) + 123.5347(3)n_f - 2.81311(1)n_f^2\big)\,, &(29)\\
F_3^L &= a\big(-0.62655873962365026074\big) \\
&\quad + a^2\big(-7.8618849118581104 + 0.4799770959727058987n_f\big) \\
&\quad + a^3\big(-254.42(1) + 48.002(1)n_f - 0.71738(1)n_f^2\big)\,, &(30)\\
F_4^L &= a\big(-0.82709837483873229037\big) \\
&\quad + a^2\big(-8.42794691505549605 + 0.8725648296451055402n_f\big) \\
&\quad + a^3\big(-313.419(4) + 64.3005(4)n_f - 1.38245(3)n_f^2\big)\,, &(31)\\
F_5^L &= a\big(-1.3116507463261931407\big) \\
&\quad + a^2\big(-30.32774293571586111 + 2.4134870684309379409n_f\big) \\
&\quad + a^3\big(-1281.20(1) + 213.0615(9)n_f - 5.33964(3)n_f^2\big)\,, &(32)\\
F_6^L &= a\big(-0.4661270314515846370\big) \\
&\quad + a^2\big(-14.05284249471245715 + 0.4507104624332595895n_f\big) \\
&\quad + a^3\big(-467.649(5) + 63.8280(5)n_f - 0.71048(3)n_f^2\big)\,, &(33)\\
F_7^L &= a\big(-0.79783174129928598074\big) \\
&\quad + a^2\big(-23.16109121325045065 + 1.2245098599795996977n_f\big) \\
&\quad + a^3\big(-890.390(6) + 134.2270(5)n_f - 2.708n_f^2\big)\,, &(34)\\
F_8^L &= a\big(-0.8455237148746085037\big) \\
&\quad + a^2\big(-10.93560591432546472 + 0.849974752412507047n_f\big) \\
&\quad + a^3\big(-404.35(2) + 72.164(2)n_f - 1.41340(7)n_f^2\big)\,, &(35)\\
F_9^L &= a\big(0.22222222222222222222\big) \\
&\quad + a^2\big(4.41926247296556700 - 0.22143896997362032903n_f\big) \\
&\quad + a^3\big(170.416(2) - 25.7049(1)n_f + 0.355738(1)n_f^2\big)\,, &(36)\\
F_{10}^L &= a\big(0.8195143283064261733\big) \\
&\quad + a^2\big(15.47442232012187938 - 1.2832916030122051316n_f\big) \\
&\quad + a^3\big(580.239(3) - 108.8446(4)n_f + 2.83040(3)n_f^2\big)\,. &(37)
\end{aligned}
$$

Via crossing symmetry in the decomposition in Eq. (25), we obtain for F_j^R

$$F^R_{1,2} = F^L_{2,1}\,, \qquad F^R_{3,4,5} = F^L_{8,7,6}\,, \qquad F^R_{6,7,8} = F^L_{5,4,3}\,, \qquad F^R_{9,10} = F^L_{10,9}\,. \quad (38)$$

Comparing with the previous calculations by Gracey [25, 27, 31], we find agreement by verifying the relations $F_j^L = -\frac{1}{2}\Sigma^{W_2}_{(j)}$ and $F_j^L + F_j^R = -\frac{1}{2}\Sigma^{\partial W_2}_{(j)}$ for $j = 1, \ldots, 10$ through the two-loop order.

4. Numerical results for $n = 3$ moment

Here, we present the numerical results for the formfactors F_j^{LL}, F_j^{RR}, and F_j^{LR} of the $n = 3$ moment at three loops in the $\overline{\mathrm{MS}}$ scheme. For F_j^{LL}, we have

$$\begin{aligned}
F_1^{LL} &= a\big(0.12809418462663994519\big) \\
&\quad + a^2\big(3.57396324725023741 - 0.39276632576413 07273 n_f\big) \\
&\quad + a^3\big(142.934(4) - 23.3744(5)n_f + 0.38322(1)n_f^2\big)\,, && (39)\\
F_2^{LL} &= a\big(0.64814814814814814815\big) \\
&\quad + a^2\big(11.92146760129898963 - 1.66859762023073406 04n_f\big) \\
&\quad + a^3\big(470.434(6) - 94.2894(8)n_f + 2.69132(1)n_f^2\big)\,, && (40)\\
F_3^{LL} &= 0.33333333333333333333 + a\big(-1.5801847945918187101\big) \\
&\quad + a^2\big(-23.39093305099714828 + 2.878059562968590479n_f\big) \\
&\quad + a^3\big(-784.543(3) + 157.3102(3)n_f - 4.15775(1)n_f^2\big)\,, && (41)\\
F_4^{LL} &= a\big(-0.34254600335127144000\big) \\
&\quad + a^2\big(-4.53423910035660995 + 0.2361911259199855009n_f\big) \\
&\quad + a^3\big(-138.88(5) + 25.985(4)n_f - 0.34336(2)n_f^2\big)\,, && (42)\\
F_5^{LL} &= a\big(-0.42384760642772557827\big) \\
&\quad + a^2\big(-4.59354266387785339 + 0.3660534795383755215n_f\big) \\
&\quad + a^3\big(-152.53(5) + 31.366(5)n_f - 0.528634(1)n_f^2\big)\,, && (43)\\
F_6^{LL} &= a\big(-0.5842793145997912020\big) \\
&\quad + a^2\big(-4.85747463745270533 + 0.6841872158771917737n_f\big) \\
&\quad + a^3\big(-193.46(4) + 43.965(3)n_f - 1.08542(4)n_f^2\big)\,, && (44)\\
F_7^{LL} &= a\big(-1.0677459370968307259\big) \\
&\quad + a^2\big(-23.37760835366161187 + 2.126160364999081326n_f\big) \\
&\quad + a^3\big(-1037.27(2) + 177.931(2)n_f - 4.87027(7)n_f^2\big)\,, && (45)\\
F_8^{LL} &= a\big(-0.22222222222222222222\big) \\
&\quad + a^2\big(-7.10270791265820791 + 0.1633837590014029747n_f\big) \\
&\quad + a^3\big(-223.72(4) + 28.698(4)n_f - 0.241101(1)n_f^2\big)\,, && (46)
\end{aligned}$$

$$
\begin{aligned}
F_9^{LL} &= a\big(-0.28292698728195749432\big) \\
&\quad + a^2\big(-9.25460402258154041 + 0.26455836743429509 50 n_f\big) \\
&\quad + a^3\big(-306.93(4) + 40.838(5)n_f - 0.426065 n_f^2\big)\,, \qquad (47)\\
F_{10}^{LL} &= a\big(-0.47732484248508078617\big) \\
&\quad + a^2\big(-15.44021253016360790 + 0.72401605190797200 91 n_f\big) \\
&\quad + a^3\big(-587.16(3) + 86.038(3)n_f - 1.64434(2)n_f^2\big)\,, \qquad (48)\\
F_{11}^{LL} &= a\big(-0.5615109786022296830\big) \\
&\quad + a^2\big(-7.60796010282396424 + 0.606188782359786649 n_f\big) \\
&\quad + a^3\big(-288.83(1) + 50.148(2)n_f - 1.03938(10)n_f^2\big)\,, \qquad (49)\\
F_{12}^{LL} &= a\big(0.060704765059735272099\big) \\
&\quad + a^2\big(1.27591955027728723 - 0.0499048842151173 1147 n_f\big) \\
&\quad + a^3\big(49.32(1) - 6.8348(9)n_f + 0.07301(1)n_f^2\big)\,, \qquad (50)\\
F_{13}^{LL} &= a\big(0.12140953011947054420\big) \\
&\quad + a^2\big(2.55370383484854647 - 0.127070554624646822 28 n_f\big) \\
&\quad + a^3\big(98.40(1) - 14.3840(9)n_f + 0.206262(1)n_f^2\big)\,, \qquad (51)\\
F_{14}^{LL} &= a\big(0.45889950244920457284\big) \\
&\quad + a^2\big(8.64602611504816215 - 0.757806639574173 8465 n_f\big) \\
&\quad + a^3\big(322.542(7) - 62.2614(7)n_f + 1.72278(2)n_f^2\big)\,. \qquad (52)
\end{aligned}
$$

Via crossing symmetry in the decomposition in Eq. (26), we obtain for F_j^{RR}

$$
F_{1,2,3}^{RR} = F_{3,2,1}^{LL}\,, \quad F_{4,5,6,7}^{RR} = F_{11,10,9,8}^{LL}\,, \quad F_{8,9,10,11}^{RR} = F_{7,6,5,4}^{LL}\,, \quad F_{12,13,14}^{RR} = F_{14,13,12}^{LL}\,. \qquad (53)
$$

Finally, for F_j^{LR} we have

$$
\begin{aligned}
F_1^{LR} = F_3^{LR} &= a\big(0.45522361895958633728\big) \\
&\quad + a^2\big(8.88767955565142389 - 1.282023059220334 5582 n_f\big) \\
&\quad + a^3\big(368.500(5) - 74.9538(4)n_f + 2.28234(1)n_f^2\big)\,, \qquad (54)\\
F_2^{LR} &= 0.16666666666666666667 + a\big(-0.91896983417115119333\big) \\
&\quad + a^2\big(-12.94227294752102117 + 1.629221179612629 3778 n_f\big) \\
&\quad + a^3\big(-422.739(7) + 86.3035(6)n_f - 2.29625(1)n_f^2\big)\,, \qquad (55)\\
F_4^{LR} = F_{11}^{LR} &= a\big(-0.2840127362723788 2074\big) \\
&\quad + a^2\big(-3.32764581150150047 + 0.243785970052720 3978 n_f\big) \\
&\quad + a^3\big(-115.52(3) + 22.016(3)n_f - 0.37402(3)n_f^2\big)\,, \qquad (56)\\
F_5^{LR} = F_{10}^{LR} &= a\big(-0.33640422333931270222\big) \\
&\quad + a^2\big(-3.64571691677851411 + 0.375648772215930 1382 n_f\big)
\end{aligned}
$$

$$+ a^3\big(- 141.21(5) + 27.501(4)n_f - 0.632127(1)n_f^2\big)\,, \tag{57}$$

$$\begin{aligned} F_6^{LR} = F_9^{LR} &= a\big(- 0.40433651740142803852\big) \\ &+ a^2\big(- 10.87040428448957907 + 0.70201836002985790003n_f\big) \\ &+ a^3\big(- 442.55(5) + 69.922(4)n_f - 1.61609(1)n_f^2\big)\,, \end{aligned} \tag{58}$$

$$\begin{aligned} F_7^{LR} = F_8^{LR} &= a\big(- 0.2439048092293624148\big) \\ &+ a^2\big(- 6.95013458205424924 + 0.2873267034318566148n_f\big) \\ &+ a^3\big(- 243.93(3) + 35.130(3)n_f - 0.46937(3)n_f^2\big)\,, \end{aligned} \tag{59}$$

$$\begin{aligned} F_{12}^{LR} = F_{14}^{LR} &= a\big(0.087443383088412876049\big) \\ &+ a^2\big(1.67025543169975743 - 0.09772109576729624122n_f\big) \\ &+ a^3\big(64.285(8) - 10.3017(10)n_f + 0.164148(1)n_f^2\big)\,, \end{aligned} \tag{60}$$

$$\begin{aligned} F_{13}^{LR} &= a\big(0.22583598672341225432\big) \\ &+ a^2\big(4.07752442951393566 - 0.37450630303729499795n_f\big) \\ &+ a^3\big(151.82(1) - 30.4658(9)n_f + 0.855783(1)n_f^2\big)\,. \end{aligned} \tag{61}$$

Comparing with the previous calculations by Gracey [25, 27, 32], we find agreement by verifying the relations $F_j^{LL} = -\Sigma_{(j)}^{W_3}$, $F_j^{LL} + F_j^{LR} = -\Sigma_{(j)}^{\partial W_3}$, and $F_j^{LL} + 2F_j^{LR} + F_j^{RR} = -\Sigma_{(j)}^{\partial\partial W_3}$ for $j = 1, \ldots, 14$ through the two-loop order.

5. Conclusion

In this paper, we have calculated the $n = 2$ and $n = 3$ moments of the twist-two non-singlet bilinear quark operators in SMOM kinematics at three loops in QCD. This allows us to match, with unprecedented precision, lattice QCD simulations of these quantities to their high-energy behaviors in the continuum limit as determined from perturbative QCD calculations in the $\overline{\text{MS}}$ scheme. We have presented the relevant conversion factors between the RI/SMOM and $\overline{\text{MS}}$ schemes in numerical form, ready to use by the lattice community. The three-loop corrections are comparable in size to the two-loop contributions available from Refs. [25, 27, 31, 32], which we were able to reproduce after clarifying some issues with the definitions.

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