Infinite Black Hole Entropies at Infinite Distances and Tower of States

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The aim of this paper is to elucidate a close connection between the black hole area law and the infinite distance conjecture in the context of the swampland. We will consider families of black hole geometries, being parametrized by their event horizon areas or by the values of their entropies, and show that the infinite entropy limit is always at infinite distance in the space of black hole geometries. We will argue that for any physical system with non-vanishing entropy, the infinite entropy limit can be mapped on an infinite distance direction in the corresponding field space. It then follows from the infinite distance conjecture that there must be a tower of states in the infinite entropy limit. We call this the black hole entropy distance conjecture. For extremal $\mathcal{N}=2$ black holes in string theory the tower of states in the large entropy limit can be nicely related to the internal Kaluza-Klein modes, via the use of the $\mathcal{N}=2$ attractor formalism. Furthermore, for non-extremal black holes we argue that the associated light tower of states is related to Goldstone-like modes that interpolate among the black hole microstates. Ignoring those towers on the horizon of the black hole would invalidate the effective theory when the entropy becomes large.

I. INTRODUCTION

In this paper we want to generalize the infinite distance conjecture of the swampland scenario [1–3] to black hole geometries with horizon. The famous area law of Bekenstein and Hawking [4, 5] is stating that the entropy $\mathcal{S}$ of a particular space-time geometry is proportional to the surface area A of its event horizon:

$$\mathcal{S} = \frac{1}{4}\frac{A}{L_p^2}\,, \tag{1}$$

where L_p is the Planck length. Viewing this relation from the perspective of statistical mechanics, it means that the number of microstates of a space-time geometry exponentially grows with the area of the event horizon. The agreement between the macroscopic and microscopic entropies was shown first for supersymmetric black holes in string theory [6], and subsequently investigated for several space-time geometries in string theory and in supergravity. For the Schwarzschild geometry the relation between the area law and black hole microstates is less understood, but progress towards its understanding was made by investigating the so-called soft hair of black holes [7–12]. Finally, for quantum field theories there is close relation between entropy and the number of quantum degrees of freedom via renormalization group flow and the c-theorem [13, 14].

A priori unrelated to the entropy/area law in quantum gravity, it is argued in the context of the swampland approach [1–3] that large distances in the field range of effective field theories lead to a tower of states when coupled to quantum gravity. This observation has led to the infinite distance conjecture stating that if, in quantum gravity, one tries to increase the range of some field ϕ beyond the Planck range some new light states emerge, invalidating the effective field theory. Specifically the masses of these additional states exponentially decrease as functions of the distance in field space,

$$m \sim M_p e^{-c|\Delta(\phi)|}\,, \tag{2}$$

where $c \sim \mathcal{O}(1)$. Hence for large field distances with

$|\Delta(\phi)| \to \infty$, we get in quantum gravity a massless tower of states, which invalidates the effective field theory description.

The infinite distance conjecture was discussed in string theory in many interesting instances [15–49]. In the context of string compactifications, ϕ is labelling a family of internal compact spaces, corresponding e.g. to a size modulus R of an internal Calabi-Yau manifold. Then in the limit where some internal cycles either become very large or shrink to zero size, generically two kind of towers emerge: (i) an infinite tower of Kaluza-Klein (or winding states), which scale as $m_{\rm KK} \sim n/R$ (or $m_{\rm wind.} \sim nR$ in string units), where n is some integer and R is the size of the internal cycle. The associated distance is given as

$$\Delta(R) \sim |\log R|\,. \tag{3}$$

(ii) Second, one can get also tensionless strings, and the associated tower becomes light in the limit of weak string coupling and scales as $m_{\rm string} \sim \sqrt{n} g_s$, where g_s is the string coupling constant. Here the distance in the string coupling constant is

$$\Delta(g_s) \sim |\log g_s|\,. \tag{4}$$

In fact, as advocated in [41], these two kinds of states are the only possible physical towers in the context of string compactifications.

The infinite distance conjecture can be also generalized to effective gravity theories, namely to the variation of the background metric itself. Here we consider gravity with a family of background metrics $g_{\mu\nu}(\tau)$, which are labeled by some parameter τ. Then the generalized distance conjecture [35] implies that, if the associated distance $\Delta(\tau)$ in the space of background metrics becomes infinite, always a light tower of states has to emerge. This generalized distance conjecture was applied in [35] to AdS space-times, which are labeled by the values of their (negative) cosmological constants Λ. Then the AdS distance conjecture (ADC) states that the corresponding infinite tower of states scales as $m_{\rm AdS} \sim |\Lambda|^\alpha$, with $\alpha = 1/2$ for the strong ADC.

When considering the variations of the background metric, it was also argued [40] that in gravity the Ricci-flow [50] and also generalized gradient flows [51] provide a refined criterion for the distance conjecture and its connection to the swampland: namely there exists an infinite tower of physical states which become massless when following the Ricci-flow towards a fixed point at infinite distance. The corresponding "Ricci distance" is given by the scalar curvature $\mathcal{R}$ of the background metric:

$$\Delta(\mathcal{R}) \sim |\log \mathcal{R}|\,. \tag{5}$$

This equation can be viewed as the generalization of the distance $\Delta(R)$ in (3). In addition, entropy functionals, which are related to the more general gradient flow equations, provide a sensible definition of the generalized distance in the space of background fields. Considering the combined dilaton-metric flow, the corresponding grading flow is derived from the entropy functional $\mathcal{F}$, which then provides a good definition for the distance in the combined metric-dilaton field space [40]:

$$\Delta(\mathcal{F}) \sim |\log \mathcal{F}|\,. \tag{6}$$

For example, if one considers the flow of the string coupling constant $g_s(\tau)$, one sees that the distance $\Delta(\mathcal{F})$ agrees with the distance in (4) and confirms that at least in the space of metrics and string coupling constants these two distances and their associated two towers of states are the only ones appearing in the context of string compactifications.

In this paper we will consider effective gravity theories with a family of black hole metrics $g_{\mu\nu}(r_S)$ with horizons of size r_S. As we will discuss, the limit of infinite horizon $r_S \to \infty$ is at infinite distance in the space of metrics and the associated horizon distance can be always expressed as

$$\Delta(r_S) \sim \log r_S\,. \tag{7}$$

Using the Bekenstein-Hawking relation (1) between the size of the horizon and the entropy $\mathcal{S}$ of the horizon geometry, namely

$$\mathcal{S} \sim r_S^2\,, \tag{8}$$

it follows that the distance can be expressed in terms of the entropy as

$$\Delta(\mathcal{S}) \sim \log \mathcal{S}\,. \tag{9}$$

Now applying the infinite distance conjecture, there must be a corresponding mass scale with a tower of "states", whose masses in Planck units, according to (2), are given as

$$m_{\mathcal{S}} \sim \mathcal{S}^{-c}\,, \tag{10}$$

where c is a positive constant. We call this the *black hole entropy distance conjecture*, or in short BHEDC. For horizons larger than the Planck distance and correspondingly for large entropies this mass scale becomes very tiny. This means that the effective field theory of any macroscopic black hole must contain a very large number of almost massless modes. For $c = 1/2$, the black hole distance conjecture looks very similar to the strong ADC, and therefore this case is called strong BHEDC.

As we will discuss, examples of such light modes are best understood for extremal $\mathcal{N} = 2$ black holes from string compactifications. Here the attractor equations of supersymmetric extremal black holes [59, 60] imply that large horizons and large entropies follow from large values of internal moduli or dilaton fields on the horizon. The latter are accompanied by towers of Kaluza-Klein modes from the internal geometry, whose masses on the horizon go to zero in the infinite entropy limit. As we will discuss, under a certain assumption the strong BHEDC is satisfied.

The situation is different for large entropy limits for non-extremal black holes which are less understood from the point of view of string compactifications. For such cases, we address the problem directly on the black hole solution, without relying on higher dimensional constructions. In this setup, the infinite distance conjecture predicts a behaviour of the effective theory on the black hole horizon, i.e. it again infers a prediction in a lower dimensional effective theory. Indeed, as we will argue, for non-extremal black holes, like the Schwarzschild black hole, the mass scale $m_{\mathcal{S}}$ belongs to a new kind of tower of "states": these states are not given in terms of Kaluza-Klein or other real physical states in the effective theory, but this tower corresponds to the carriers of the information, namely they are closely related to the almost gapless modes that describe the degeneracies among the microstates of the black hole geometry. These modes were first discussed in the microscopic black hole N-portrait model of [52–57]. For finite horizons and finite entropies, the "masses", or better say the frequencies of these modes are finite. On the other hand, in the infinite horizon and infinite entropy limit the frequencies of the modes connecting the infinite microstates become zero. We will discuss this tower behaviour of non-extremal black holes in the context of the soft hair proposal [7–12] for the black hole entropy. The limit $r_S \to \infty$ corresponds to flat Minkowski space, where the horizon approaches null infinity of Minkowski space. In this limit the entropy of flat space is infinite [9, 58] and the corresponding microstates of Minkowski space can be viewed as the infinite number of soft gravitons, which are associated with the infinite number of BMS transformations at null infinity. From the swampland context, the emergence of this massless tower of modes in the infinite entropy limit of non-extremal black holes means that the effective horizon field theory must contain these states. If this is not the case, then the effective theory is in the swampland.

The paper is organized as follows. In section II we discuss the large entropy limit of $\mathcal{N}=2$ extremal black holes in string theory. As mentioned before, via the attractor equations, this limit is correlated to a limit of large moduli fields of the internal compact space. For large moduli at infinite distance, it follows that also the large entropy is at large distance in the space of $\mathcal{N}=2$ black hole string backgrounds and the associated tower of states is well understood in terms of the Kaluza-Klein states of the internal manifold. In the subsequent sections III - V, we show that also for non-extremal black holes, the limit of large horizon, or equivalently the limit of large entropy, is at infinite distance, when measured by a certain geometric distance functional in the space of black hole metrics. Since the geometric distance functional is not covariant under diffeomorphisms involving r_S, or a more general parameter τ, particular care should be taken in computing the distance. Consequently, we introduce a prescription to define the initial point in the τ flow. The tower of light modes for non-extremal black holes is then discussed in section VI. We present some conclusions in section VII and exile to appendix A some remarks on the properties of the geometric distance formula used here under diffeomorphisms that depend on the parameter τ along the path, and to appendix B a comparison between the formula used in this paper and the analogue without regulating volume factor, which appeared often in the literature.

II. $\mathcal{N}=2$ EXTREMAL BLACK HOLES, INFINITE DISTANCE AND KALUZA-KLEIN MODES AS INFINITE TOWER

In this section we consider $\mathcal{N}=2$ extremal black holes in string theory, and we show that the infinite entropy limit goes along with an infinite tower of light Kaluza-Klein modes or light string excitations. In the large entropy limit some of the black hole charges must become very large. Via the $\mathcal{N}=2$ attractor mechanism [59, 60] this implies that some of the internal moduli fields also have to take large field values and the associated Kaluza-Klein modes become very light. This means that the infinite entropy limit is at infinite distance with respect to the internal moduli space of $\mathcal{N}=2$ black hole solutions.

So let us consider four-dimensional extremal, $\mathcal{N}=2$ supersymmetric black holes in string theory. E.g. in type IIA string compactifications, four-dimensional, extremal black holes with non-vanishing entropy can be constructed by the intersection of N_V D4-branes, which are wrapped around four-cycles of the internal six-dimensional Calabi-Yau manifold. In addition to the D4-branes one also needs a certain number of D0-branes. The four-dimensional effective field theory is given in terms of $\mathcal{N}=2$ supergravity theory coupled to N_V $\mathcal{N}=2$ vector multiplets. The corresponding Abelian gauge group is given by

$$G = U(1)^{1+N_V} \,, \tag{11}$$

where the additional $U(1)$ factor comes from the graviphoton gauge symmetry. The four-dimensional metric of $\mathcal{N}=2$ extremal black hole solutions has the following form:

$$\mathrm{d}s^2 = -\left(1-\frac{\sqrt{A/4\pi}}{r}\right)^2 \mathrm{d}t^2 + \left(1-\frac{\sqrt{A/4\pi}}{r}\right)^{-2} \mathrm{d}r^2 + r^2 \mathrm{d}S_2^2 \,, \tag{12}$$

where A is the area of the event horizon. Via the so-called attractor mechanism [59, 60] the corresponding entropy is determined by the extremization of the central charge Z of the $\mathcal{N}=2$ supersymmetry algebra on the horizon. Specifically, the entropy takes the following form [61]:

$$\frac{\mathcal{S}_{\mathcal{N}=2}}{\pi} = \frac{A}{4\pi} = |Z_{\mathrm{hor}}|^2 = \sqrt{-4Q_0 d_{ijk} P^i P^j P^k} \,. \tag{13}$$

Here Q_0 is the electric charge of the black hole with respect to the graviphoton $U(1)$ gauge group, and the P^i are magnetic charges with respect to $U(1)^{N_V}$. Moreover Q_0 corresponds to the number of D0-branes, and the magnetic charges P^i are the wrapping numbers of the D4-branes around the corresponding internal four-cycles. Finally d_{ijk} denotes the triple intersection numbers of the four-cycles. The entropy formula (13) is the $\mathcal{N}=2$ string generalization of the Reissner-Nordström entropy subsequently given in (63). $\mathcal{S}_{\mathcal{N}=2}$ becomes large in the limit of large electric charge Q_0 or large magnetic charges P_i, and, as we will show in section V B 2, the limit of large charges is at infinite distance in the space of extremal $\mathcal{N}=2$ metrics.

Let us consider the solutions of the attractor equations, which tell that the value of the scalar fields at the black hole horizon are entirely determined by the electric and magnetic charges in the following way. For the Kähler moduli T_i, which determine the sizes of the two-cycles perpendicular to the wrapped four-cycles of the D4-branes, one obtains

$$T^i = P^i \sqrt{-\frac{Q_0}{d_{jkl}P^jP^kP^l}}\,. \tag{14}$$

In addition the ten-dimensional IIA dilaton $e^{-2\phi^{(10)}}$ is determined by the attractor equations as

$$e^{-2\phi^{(10)}} = \sqrt{-\frac{d_{ijk}P^iP^jP^k}{Q_0^3}}\,. \tag{15}$$

In terms of the dilaton, the black hole entropy can be also written as

$$\mathcal{S}_{\mathcal{N}=2} = 2\pi Q_0^2 e^{-2\phi^{(10)}}\,. \tag{16}$$

The volume $\mathcal{V}$ of the internal space is given as

$$\mathcal{V} = c\, d_{ijk}T^iT^jT^k = c\sqrt{-\frac{Q_0^3}{d_{ijk}P^iP^jP^k}}\,. \tag{17}$$

Note that we have included in the expression for $\mathcal{V}$ a constant c, which must be determined from the full ten-dimensional background. In the following we will first assume that c is independent of the charges, and later we will comment on a possible charge dependence of c. The four-dimensional dilaton,

$$e^{-2\phi^{(4)}} = e^{-2\phi^{(10)}}\mathcal{V} = e^{-2\phi^{(10)}}c\, d_{ijk}T^iT^jT^k\,, \tag{18}$$

is, for constant c, independent of the charges Q_0 and P^i. In this case the four-dimensional Einstein frame is identical to the string frame, and the masses of the string states in IIA are also independent of the charges.[1] Specifically for the Kaluza-Klein mass scale one obtains:

$$m_{\rm KK} = \frac{M_s}{\mathcal{V}^{1/6}} = \frac{M_p}{c^{2/3}(d_{ijk}T^iT^jT^k)^{1/6}} = \frac{(d_{ijk}P^iP^jP^k)^{1/12}}{c^{2/3}(-Q_0)^{1/4}}\,. \tag{19}$$

Let us now compare the Kaluza-Klein mass scale (19) with the entropy (13) of the extremal $\mathcal{N}=2$ black hole:

$$m_{\rm KK} = \left(\frac{1}{\mathcal{S}_{\mathcal{N}=2}}\right)^{1/2}\frac{2(d_{ijk}P^iP^jP^k)^{1/3}}{c^{2/3}}\,. \tag{20}$$

We see that for fixed magnetic charges,[2] the black hole entropy becomes very large when the Kaluza-Klein masses are light. Specifically for fixed P^i, the mass scale of the tower of states is given in terms of a square root of the inverse entropy, i.e. one gets that $\alpha = 1/2$ for the power in (10). This relation between the relevant mass scale and the entropy is identical to the relation between the tower mass scale and the cosmological constant Λ in the strong AdS Distance Conjecture (ADC) [35]. On the other hand, for constant c there are also choices of electric and magnetic charges, for which $m_{\rm KK}$ can be kept constant, but the entropy becomes very large. This is in particular the case when sending all charges $-Q_0 = \text{const.}\times P_i \to \infty$. So the large entropy limit cannot always be mapped onto the limit where the Kaluza-Klein tower becomes light. It means that we are in a situation where the large distance limit in the internal six-dimensional moduli space always leads to a large entropy of the four-dimensional black hole. However the converse is apparently not true.

The comparison between the $\mathcal{N}=2$ black hole entropy and the Kaluza-Klein scale is quite analogous to the comparison between the AdS_4 cosmological constant Λ and the Kaluza-Klein mass scale in Calabi-Yau flux compactifications. As recently discussed in [48] one gets for a particular type IIA flux compactification for the ratio of $m_{\rm KK}$ and Λ an expression of the following form

$$\frac{m_{\rm KK}}{|\Lambda|^{1/2}} = \frac{1}{(ab^3)^{1/4}c^{1/6}}\,, \tag{21}$$

where a and b are certain IIA fluxes, related to $S^3\times S^3$ or twisted tori compactifications. As argued in [48] the "naive" four-dimensional effective field theory approach with $c=1$ would lead to a misleading result, namely to a flux-dependent ratio between the Kaluza-Klein and the cosmological constant scale. However the value of c, derived in [48] is precisely such that the flux dependence drops out from (21) and the strong ADC is fully

[1] In a dual heterotic compactification on $K3\times T^2$ one of the IIA moduli T_i corresponds to the heterotic dilaton field. Then the limit of large charges and large entropy corresponds to a very weakly coupled string, and the tower of states are given by light tensionless strings.

[2] For small magnetic charges P^i, the higher curvature corrections to the black hole entropy, which are proportional to the second Chern class c_2 of the Calabi-Yau manifold, are small.

satisfied for all values of the fluxes and not only for the fluxes a and b fixed. In the case of the black hole entropy versus the Kaluza-Klein mass scale one could try to use an analogous argument. Namely assuming that $c = \sqrt{d_{ijk}P^iP^jP^k}$, the magnetic charge dependence would drop out of (20) and the strong BHEDC for the $\mathcal{N} = 2$ entropy would be fully satisfied.

In summary, denoting the Kaluza-Klein mass scale as $m_{\rm KK} = 1/R$, we have seen that

$$\Delta(R) \sim \log R \to \infty \implies \Delta(\mathcal{S}) \sim \log \mathcal{S} \to \infty \,. \qquad (22)$$

The emergence of an infinite tower of massless states in the infinite distance limit of large internal moduli is consistent with the swampland idea. Recall that the moduli fields can have arbitrary values at spatial infinity, but their horizon values must be expressed by the electric and magnetic black hole charges due to the attractor mechanism. Therefore in the limit of large internal moduli, the effective horizon field theory must contain an infinite tower of states, namely the light tower of internal Kaluza-Klein states. Without these light states, the effective field theory on the black hole horizon would be inconsistent, i.e. it would be lying in the swampland. In fact, the near horizon geometries of extremal $\mathcal{N} = 2$ extremal black holes are given by the space $\mathrm{AdS}_2 \times S^2$ with so-called Bertotti-Robinson metric. In the large moduli limit, the effective, four-dimensional $\mathrm{AdS}_2 \times S^2$ theory must contain an infinite tower of states, which are just the light Kaluza-Klein modes from the internal Calabi-Yau manifold. In the following sections we will see that the large entropy limit of non-extremal (and extremal) black holes - although a priori not being linked to large internal moduli fields - is also at infinite distance in the space of four-dimensional black hole metrics.

III. GEOMETRIC DISTANCE FORMULA

In order to analyze the behaviour of towers of states when we scan over physically different space-times, we first need to define a distance which measures the length of the geodesic path corresponding to this scan. In this paper, we use the geometric distance formula [62],

$$\begin{aligned}\Delta_g &= c\int_{\tau_i}^{\tau_f}\left(\frac{1}{V_M}\int_M \sqrt{g}g^{MN}g^{OP}\frac{\partial g_{MO}}{\partial\tau}\frac{\partial g_{NP}}{\partial\tau}\right)^{\frac{1}{2}}\mathrm{d}\tau\\ &= c\int_{\tau_i}^{\tau_f}\left(\frac{1}{V_M}\int_M\sqrt{g}\,\mathrm{tr}\left[\left(g^{-1}\frac{\partial g}{\partial\tau}\right)^2\right]\right)^{\frac{1}{2}}\mathrm{d}\tau\,,\end{aligned} \qquad (23)$$

where g is the metric on the manifold M, $V_M = \int_M \sqrt{g}$ is the volume of the latter, τ is a parameter along the path between $g(\tau_i)$ and $g(\tau_f)$ and $c \sim \mathcal{O}(1)$. This distance was used in [35] to formulate the AdS distance conjecture.

A. Geodesic equation

The shortest path between two space-time metrics can be identified by extremizing Δ_g. This leads to the geodesic equation[3]

$$\begin{aligned}\ddot{g} = \dot{g}g^{-1}\dot{g} + \tfrac{1}{4}\,\mathrm{tr}\big[(g^{-1}\dot{g})^2\big]g - \tfrac{1}{2}\,\mathrm{tr}\big[g^{-1}\dot{g}\big]\dot{g}\\ - \tfrac{1}{4}\langle\mathrm{tr}\big[(g^{-1}\dot{g})^2\big]\rangle g + \tfrac{1}{2}\langle\mathrm{tr}\big[g^{-1}\dot{g}\big]\rangle\dot{g}\,,\end{aligned} \qquad (24)$$

where

$$\langle X\rangle = \frac{\int_M \sqrt{g}\;X}{V_M} \qquad (25)$$

and a dot indicates differentiation with respect to the proper time[4] λ, defined as

$$\mathrm{d}\lambda = \mathrm{d}\tau\left(\frac{1}{V_M}\int_M\sqrt{g}\,\mathrm{tr}\left[\left(g^{-1}\frac{\partial g}{\partial\tau}\right)^2\right]\right)^{1/2}\,. \qquad (26)$$

Defining

$$f = g^{-1}\dot{g}\,, \qquad (27)$$

the geodesic equation can be concisely written as

$$\dot{f} = \tfrac{1}{4}\,\mathrm{tr}\big(f^2\big)\mathrm{Id} - \tfrac{1}{2}\,\mathrm{tr}(f)f - \tfrac{1}{4}\langle\mathrm{tr}\big(f^2\big)\rangle\mathrm{Id} + \tfrac{1}{2}\langle\mathrm{tr}(f)\rangle f\,. \qquad (28)$$

B. First examples

We now want to solve (28) to define geodesic paths between physically different space-times. Then, using (26), the distance is given in terms of the proper time variation:

$$\Delta_g = c\int_{\tau_i}^{\tau_f}\mathrm{d}\lambda = c(\lambda_f - \lambda_i)\,. \qquad (29)$$

Important examples of solutions to the geodesic equation (28) are the metrics with a constant parameter α entering as an overall factor:

$$\mathrm{d}s^2 = g_{\mu\nu}(x,\alpha)\mathrm{d}x^\mu\mathrm{d}x^\nu \text{ with } g_{\mu\nu}(x,\alpha) = \alpha\tilde{g}_{\mu\nu}(x)\,. \qquad (30)$$

Upgrading $\alpha \to \alpha(\lambda)$, and using (30) as an ansatz, it is straightforward to see that the geodesic equation is solved for

$$\alpha(\lambda) = \alpha_0\, e^{\lambda/\sqrt{d}}\,, \qquad (31)$$

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[3] Note that the geodesic equation is modified with respect to the case of a metric distance without the volume normalization, as in e.g. [82]. See also appendix B for details.

[4] The geodesic equation holds for more general affine parameters λ, but in order to limit the number of unfixed constants in the solutions to be discussed later, we always express them in terms of the proper time. For instance, in (31), the power $\sqrt{d}^{-1}$ in the exponential is fixed when one imposes that the affine parameter λ is the proper time as defined in (26).

where d is the dimension of M. Therefore, the geodesic distance between two space-times of the form (30) with conformal factors α_i and α_f is given by

$$\Delta_g = c\sqrt{d}\log\left(\frac{\alpha_f}{\alpha_i}\right) . \tag{32}$$

IV. ONE-PARAMETER SPACE-TIMES

Let us first discuss metrics which depend only on one dimension-full parameter, i.e. a single mass scale. This parameter defines a family of metrics and we are interested in measuring the distance between two representatives of such a family.

The discussion in Section III B can be immediately applied to this case. Indeed, for dimensional reasons, the metric can be put in the form

$$g_{\mu\nu}(x, M) = M^{-2}\tilde{g}_{\mu\nu}(x) , \tag{33}$$

by using dimensionless coordinates x^μ. It then follows from (30)-(32) that

$$\Delta_g = 2c\sqrt{d}\log\left(\frac{M_i}{M_f}\right) . \tag{34}$$

For AdS spaces, this result agrees with calculations made in [35] regarding the AdS distance conjecture. We will now recover it for specific one parameter space-times.

A. de Sitter space distance

Let us first consider briefly the case of de Sitter space. The details are very close to the ones encountered in [35], so that this section can be seen as a warm-up for the later use of the metric distance (23).

As it is well know, de Sitter space possesses a finite cosmological horizon and the d-dimensional de Sitter metric in static coordinates can be written as

$$\mathrm{d}s^2 = -\left(1-\frac{r^2}{r_S^2}\right)\mathrm{d}t^2+\left(1-\frac{r^2}{r_S^2}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}S_{d-2}^2 . \tag{35}$$

The horizon radius r_S is related to the positive de Sitter cosmological constant Λ_{dS} by

$$\Lambda_{\mathrm{dS}} = \frac{(d-1)(d-2)}{2r_S^2} . \tag{36}$$

We then measure the distance between two de Sitter space-times of cosmological constants Λ_i and Λ_f by solving (28). We choose for instance a solution ansatz for which the initial metric $g(\lambda_i)$ in the flow reproduces (35) with $r_S = r_S(\lambda_i)$, and similarly for $g(\lambda_f)$. However with this line element the space-time integrations present in (28) complicate the analysis. This comes about because this set of coordinates is singular at the horizon. Instead, we use global coordinates,

$$\mathrm{d}s^2 = -\frac{r_S^2}{\cos^2(\eta)}\left(-\mathrm{d}\eta^2+\mathrm{d}\psi^2+\sin^2(\psi)\mathrm{d}S_{d-2}^2\right) . \tag{37}$$

The computation of the distance for this kind of space-times has already been carried out in Section III B, with (37) being an explicit realization of (30). In addition, since de Sitter spaces are fully determined by their cosmological constants, the discussion around (33) also applies. Thus, (37) represents a realization of those previous considerations in terms of well-behaved coordinates for which the computation is well defined. Eventually, we obtain the de Sitter distance

$$\Delta_{\mathrm{dS}} \sim \log r_S \sim -\log\Lambda_{\mathrm{dS}} , \tag{38}$$

as already determined in [35].

Alternatively for de Sitter space, being an Einstein space of constant curvature

$$R_{\mathrm{dS}} = \frac{d(d-2)}{r_S^2} , \tag{39}$$

one can also follow the Ricci-flow along the space of metrics $g_{\mu\nu}(\Lambda_{\mathrm{dS}})$. Then it turns out that $r_S \to \infty$ corresponds to an infinite distance fixed point of the Ricci flow and the Ricci distance (5) agrees with the distance given in (38).

Let us recall that the Gibbons-Hawking entropy [63] is proportional to the area of the event horizon, namely

$$\mathcal{S}_{\mathrm{dS}} = 1/\Lambda_{\mathrm{dS}} \tag{40}$$

Therefore we can easily express the distance in terms of the de Sitter entropy as

$$\Delta_{\mathrm{dS}} \sim \log\mathcal{S}_{\mathrm{dS}} , \tag{41}$$

and the limit of infinite entropy is at infinite distance in the space of de Sitter geometries.[5] This conclusion also holds for black hole geometries, which are the main focus of this paper and to which we now turn.

[5] The infinite distance at vanishing cosmological constant for AdS spaces was argued in [35] to announce the presence of Kaluza-Klein modes from compact dimensions, as clearly seen in string constructions. The existence of such constructions for dS being unclear, similar arguments are not available to explain the infinite distances encountered in this section. However, the latter being driven by infinite entropies, it would be interested, although beyond the scope of this paper, to generalize the arguments to be developed in what follows for black hole geometries to the case of dS space.

B. Schwarzschild black hole

We now consider asymptotically flat geometries with finite horizon. These are nothing else than black hole geometries. Let us first consider the well-known 4-dimensional Schwarzschild black hole of mass M. The metric reads

$$\mathrm{d}s^2 = -\left(1-\frac{r_S}{r}\right)\mathrm{d}t^2 + \left(1-\frac{r_S}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}S_2^2\ , \quad (42)$$

with horizon size $r_S = 2M$. Since the Schwarzschild metric is Ricci flat, we cannot use the Ricci flow distance in order to compute Δ_{bh}. As in the case of de Sitter space, the choice of coordinates (42) is not the one for which the distance ought to be computed. Instead, we can use Kruskal-Szekeres coordinates, in which the metric reads

$$\mathrm{d}s^2 = -4\frac{r_S^3}{r}e^{-\frac{r}{r_S}}\mathrm{d}T^2 + 4\frac{r_S^3}{r}e^{-\frac{r}{r_S}}\mathrm{d}R^2 + r^2\mathrm{d}S_2^2\ , \quad (43)$$

with $\left(1-\frac{r}{r_S}\right)e^{\frac{r}{r_S}} = T^2 - R^2$. This again leads to

$$\Delta_{bh} \sim \log r_S\ . \quad (44)$$

The result again agrees with the analysis of (33), as it should since Schwarzschild geometry is fully specified in terms of the black hole mass. Indeed, (43) is a realization of (30) once we redefined $r \to r_S \times r$, or equivalently r/r_S is fully specified in terms of the dimensionless coordinates T and R. As for the dS global coordinates, (43) offers a realization of well-behaved coordinates for which the computation is well defined and agrees with our general considerations for one-parameter space-times.

Expressed in terms of the black hole entropy the result reads

$$\Delta_{bh} \sim \log \mathcal{S}_{bh}\ . \quad (45)$$

Instead of the full Schwarzschild geometry we can also consider the near horizon geometry. For that we introduce the coordinate $\epsilon = r - r_S$ and for small ϵ we obtain the metric in the near horizon limit

$$\mathrm{d}s^2 = -\frac{\epsilon}{r_S}\mathrm{d}t^2 + \frac{r_S}{\epsilon}\mathrm{d}\epsilon^2 + r_S^2\mathrm{d}S_2^2\ . \quad (46)$$

The distance when varying the horizon size is again

$$\Delta_{bh} \sim \log r_S\ . \quad (47)$$

At large distances, meaning at large black hole horizon, the near-horizon metric approaches Minkowski space[6], whose entropy is infinite [9, 58].

[6] Defining new coordinates $\rho = \sqrt{r_S\epsilon}$ and $\omega = t/r_S$, (46) is the metric of $M^{1,1}\times S^2$, where $M^{1,1}$ is the 2-dimensional Minkowski space in Rindler coordinates. It flows at large r_S to the metric of 4D Minkowski space in Rindler coordinates.

V. TWO-PARAMETERS SPACE-TIMES

We now turn to the study of space-times defined by two dimensionful parameters.

A. A prescription to compute the distance

For two-parameters space-times the discussion of section III B does not straightforwardly apply. Indeed, for a metric g defined by two mass parameters M_1 and M_2, dimensional-analysis arguments only teach us that the metric can be put in the following form

$$g_{\mu\nu}(x, M_1, M_2) = M_1^{-2}\tilde{g}_{\mu\nu}\left(x, \frac{M_1}{M_2}\right)\ , \quad (48)$$

for dimensionless coordinates x. Thus, the geodesic equation and the distance depend on the precise functional form of g.

Nonetheless, in this paper we will only study 4-dimensional static space-times with two Killing vectors, for which the metric can always be brought in the form

$$\mathrm{d}s^2 = -V(r,\alpha_i)\ \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{V(r,\alpha_i)} + r^2\mathrm{d}S_2^2\ , \quad (49)$$

where V is a given function and the α_i dependence indicates the possible presence of parameters defining the solution. For such space-times, we compute the distance as follows: first, we reach the Eddington-Finkelstein gauge performing the transformation

$$\mathrm{d}t = \mathrm{d}v - \frac{\mathrm{d}r}{V(r,\alpha_i)}\ , \quad (50)$$

which brings the metric to the form

$$\mathrm{d}s^2 = -V(r,\alpha_i)\ \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}S_2^2\ . \quad (51)$$

One can then rescale the r and v coordinates, $r \to \alpha\tilde{r}, v \to \alpha\tilde{v}$, where α is a function of the parameters[7] α_i. The metric becomes

$$\mathrm{d}s^2 = -\alpha^2 V(\alpha\tilde{r},\alpha_i)\ \mathrm{d}\tilde{v}^2 + 2\alpha^2\mathrm{d}\tilde{v}\mathrm{d}\tilde{r} + \alpha^2\tilde{r}^2\mathrm{d}S_2^2 \quad (52)$$

Using this ansatz with $\alpha = \alpha(\lambda)$, the distance can be easily computed and yields

$$\Delta_g = 4\ c\log\left(\frac{\alpha(\lambda_f)}{\alpha(\lambda_i)}\right). \quad (53)$$

This holds for every metric of the form (49) and any α since $V(\alpha\tilde{r},\alpha_i)$ does not contribute when using (23).

[7] For metrics defined in terms of at least one mass scale M, $M = \alpha$ can be used to make the coordinates dimensionless, as we discussed in section III B.

Our prescription to compute the distance is thus to bring the metric in the form (52), then to use it to solve the geodesic equation (28) and to compute the distance (23). Note that the geodesic equation for the ansatz (52) is not trivial, as is shown later on for specific cases.

The reader may wonder why we need to define such a precise prescription. The answer lies in the fact that (23) is not compatible with diffeomorphisms which vary along the geodesic flow, so that the initial frame from which the flow is computed has to be specified, as explained in greater details in appendix A. At this point a legitimate question about the universality of our results can be raised. To answer this question we notice however that in all the examples studied, if we change the coordinates system, the geometric distance is either not computable – due to problems in the coordinates integration range – or again logarithmic in the parameter. Therefore, although not a proof, this is a confirmation that our prescription is sensible.

B. Reissner-Nordström black hole

1. Non-extremal case

The first example of a two-parameter family of solutions is the non-extremal Reissner-Nordström metric, which describes a charged black hole. It can be written as in (49) with

$$V(r,\alpha_i) = \frac{(r-r_S^+)(r-r_S^-)}{r^2}\,. \tag{54}$$

This geometry possesses two horizons $r_S^\pm$. In terms of the black hole mass M and charge Q, the horizons positions are given by

$$r_S^\pm = M \pm \sqrt{\delta}\ ,\ \text{with } \delta = M^2 - Q^2. \tag{55}$$

To avoid the appearance of a naked singularity we must impose $|Q| \leq M$, such that r_S^+ and r_S^- are the outer and inner horizon, respectively.

We now apply the prescription of section V A, i.e. we rescale the coordinates using a dimensionful parameter α of the metric, so that we obtain a distance which scales logarithmically with respect to α. In the current setup, α can be built combining M and Q in an arbitrary fashion: $\alpha = \alpha(M,Q)$. Thus, the infinite distance in α corresponds to some geodesic path in (M,Q) space, which we now study. It is easy to see that, for the metric (52), the right hand side of (28) vanishes, so that we are left with

$$\dot{f} = 0 \tag{56}$$

as a geodesic equation. For instance, the solutions for the choice $\alpha = M$ are

$$M(\lambda) = c_1 e^{\frac{\lambda}{4}}\ ,\quad Q(\lambda) = c_3\sqrt{2\lambda - c_2}\, e^{\frac{2\lambda - c_2}{8}} \tag{57}$$

(remember that we specialized to $d = 4$). Imposing $|Q| \leq M$, we must choose $c_3 = 0$ (and $c_1 > 0$) and we actually describe the flow of Schwarzschild black holes, with infinite distance points which are either zero or infinite mass black holes, consistently with the results of section IV B. If $\alpha = Q$, the solutions are

$$M(\lambda) = (c_2\lambda + c_3)e^{\frac{\lambda}{4}}\ ,\quad Q(\lambda) = c_1 e^{\frac{\lambda}{4}}\ , \tag{58}$$

and the only (asymptotic) bound is $c_3 > |c_1|$ if $c_2 = 0$ (and $\mathrm{sgn}(c_2) = \mathrm{sgn}(\lambda)$ always). This case allows to describe the infinite distance limits of a zero ($\lambda \to -\infty$) or infinite ($\lambda \to +\infty$) mass and charge black hole, either with $\frac{Q}{M}$ fixed ($c_2 = 0$) or going to zero ($c_2 \neq 0$)[8].

The most natural choice for α is provided by the use of the entropy of the RN black bole, which is proportional to the area of the outer event horizon,

$$\mathcal{S}_{RN} = \pi {r_S^+}^2\,. \tag{59}$$

Therefore, in order to express the distance in terms of the black hole entropy, we take $\alpha = r_S^+$ as the relevant parameter and follow the prescription explained before. This leads to the distance with respect to r_S^+, where the ratio r_S^+/r_S^- is kept fixed,[9]

$$\Delta_{RN} \sim \log r_S^+ \sim \log \mathcal{S}_{RN}\,. \tag{61}$$

This behaviour is also reproduced by (57) and (58), up to possible logarithmic corrections, the first one scaling for instance as $\log\log S_{RN}$ if $\alpha = Q$ and $c_2 \neq 0$.

2. Extremal case

In the extremal limit ($Q^2 = M^2$) we obtain

$$V(r,\alpha_i) = \left(1 - \frac{|Q|}{r}\right)^2 \tag{62}$$

Here the horizon is at $r_S = |Q|$ and the entropy simply becomes

$$\mathcal{S}_{RN} = \pi Q^2\,. \tag{63}$$

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[8] $M(\lambda) = c_2\lambda + c_3$, $Q(\lambda) = c_1$ is also a solution, which corresponds to asymptotically large M for fixed Q, with a zero associated distance. This describes a null geodesic, for which our notion of proper time and distance does not apply. We disregard this possibility and similar ones in what follows.

[9] The fact that the ratio r_S^+/r_S^- is kept fixed is found by solving the geodesic equation, which leads to

$$r_S^+ = c_1 e^{c_2\lambda}\ ,\quad r_S^- = (c_3 + c_4\lambda)e^{c_2\lambda}\ , \tag{60}$$

for an affine parameter λ. Demanding $r_S^- < r_S^+$ imposes $c_4 = 0$ (and $c_1 > c_3$), so that eventually $r_S^+/r_S^- = c_1/c_3$. c_2 is fixed when demanding that λ is the proper time.

Since the extremal RN black hole is given in terms of a single scale Q, there is no possible subtlety in the prescription and we are back to the analysis from section IV. The distance is thus:

$$\Delta_{RN} \sim \log Q \sim \log \mathcal{S} \,. \tag{64}$$

This result is in good agreement with the distance formula (22), which was obtained from the moduli field distance via the attractor mechanism in section II.

C. AdS Schwarzschild black hole

The AdS Schwarzschild black hole has

$$V(r, \alpha_i) = 1 - \frac{2M}{r} + \frac{r^2}{\ell_S^2} \tag{65}$$

Choosing $\alpha = M$ we obtain the distance

$$\Delta_g \sim \log M \tag{66}$$

and

$$M(\lambda) = c_1 e^{\frac{\lambda}{4}} \tag{67}$$

$$l_S(\lambda) = \frac{c_3 e^{\frac{\lambda}{4}}}{\sqrt{c_2 + 2\lambda}}, \tag{68}$$

whereas if $\alpha = \ell_S$,

$$\Delta_g \sim \log \ell_S \tag{69}$$

and

$$M(\lambda) = (c_2 \lambda + c_3)\, e^{\frac{\lambda}{4}} \tag{70}$$

$$l_S(\lambda) = c_1 e^{\frac{\lambda}{4}}. \tag{71}$$

Depending on the values of the integration constants, different scenarios are depicted. For instance, it is possible that $M \to 0$ faster than $\ell_S \to 0$.

We can again rewrite the distance in terms of the entropy

$$S_{bh} = \pi r_h^2 \tag{72}$$

with r_h the horizon radius implicitly defined by

$$M = \frac{r_h}{2} \left(1 + \frac{r_h^2}{l_S^2} \right) \,. \tag{73}$$

This implies that asymptotically

$$r_h \sim (M l_S^2)^{\frac{1}{3}} \text{ and } \Delta_g \sim \log S_{bh} \,. \tag{74}$$

Again, there might be logarithmic corrections to the distance written in terms of the entropy, for instance if $c_2 \neq 0$ when $\alpha = l_S$.

VI. NON-EXTREMAL BLACK HOLES AND MICROSTATES AS INFINITE TOWER

In the last section we have seen that for Schwarzschild and for non-extremal Reissner-Nordström black holes the limit of infinite horizon $r_S \to \infty$, or equivalently the limit of infinite entropy $\mathcal{S} \to \infty$, is at infinite distance in the space of black hole metrics. Via the infinite distance conjecture we now would get a prediction, namely the BHEDC stating that there is an infinite tower of modes, whose masses scale with a certain power of entropy as

$$m_{\mathcal{S}} \sim \mathcal{S}^{-\alpha} \,, \tag{75}$$

where α is a positive constant. For large horizons and correspondingly for large entropies this mass scale becomes very tiny and goes to zero in the infinite entropy limit. This is the same result we have found for extremal black holes. However, while for the latter one could rely on higher dimensional string-theoretical constructions, for non-extremal black holes we are bound to an intrinsic analysis in four dimensions in terms of properties of the black hole.

In this section we thus try to provide an interpretation of the infinite tower of modes in terms of black hole horizon pseudo Goldstone modes, which are closely related to the microstates of the black hole. Concretely, besides the black hole microstates, all with masses of the order M, there are additional light modes that describe the transition between the different black hole microstates, i.e. the excitations from one microstate into another one. They are these light modes, which we like to identify with the light tower of modes from the infinite distance conjecture. It is of course known that any system in the limit of infinite entropy must possess modes, whose number goes to infinity together with the entropy. However a rather nontrivial question is what happens to the mass differences (frequencies) between these modes? Here the infinite distance limit for the horizon suggests that the frequencies of these modes must vanish with a certain power of the entropy. That is, the energy gap of each mode must scale as $\mathcal{S}^{-\alpha}$. In the following we like to argue that this behaviour is true when considering the Goldstone modes of black holes in terms of soft hair and related BMS-like diffeomorphisms on the horizon of a black hole. In other words, we claim that the infinite tower of massless modes, which follows from the infinite horizon distance conjecture, corresponds to the Goldstone modes of the BMS-like transformations on the black hole horizon. We stress again that these modes connect a microstate to another. If the microstates are completely degenerate and infinite, then the energy of a mode that brings from a state to another is zero, and there are infinitely many modes.

As by now advocated in many papers [64–71], in the presence of an event horizon, the classical symmetry algebra of a black hole geometry is enhanced, and it is this enhancement that is responsible for microstates of the black hole. So let us recall the classical BMS-like transformations on the black hole horizon. For this it is useful to choose Eddington-Finkelstein coordinates, so that the Schwarzschild metric gets the form

$$\mathrm{d}s^2 = -\left(1 - \frac{r_S}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2. \tag{76}$$

Performing the change of variable

$$r = r_S(1 + 2\kappa\rho), \tag{77}$$

where $r_S = 1/(2\kappa)$ for Schwarzschild, and expanding in powers of ρ, one finds that (76) becomes

$$\mathrm{d}s^2 = -2\kappa\rho\,\mathrm{d}v^2 + 2\mathrm{d}\rho\mathrm{d}v + \left(\frac{1}{4\kappa^2} + \frac{\rho}{\kappa}\right)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,, \tag{78}$$

up to $O(\rho^2)$ terms. This metric belongs to a larger class of near horizon metrics. Using this metric one can compute the asymptotic Killing vectors ζ_f, i.e. the set of vectors preserving the falloffs of the metric on the horizon. Using (77) one can then rewrite this result in terms of r and r_S to get

$$\zeta_f^\mu = \left(f(\theta,\phi), 0, \frac{\partial f}{\partial\theta}\left(\frac{1}{r} - \frac{1}{r_S}\right), \frac{1}{\sin^2\theta}\frac{\partial f}{\partial\phi}\left(\frac{1}{r} - \frac{1}{r_S}\right)\right)\,, \tag{79}$$

where $f(\theta,\phi)$ is an arbitrary function on the two-sphere. As shown in [9, 10], these are precisely the diffeomorphisms acting on the Schwarzschild metric that leave the horizon and the black hole mass M invariant. The algebra of the asymptotic Killing vectors of the Schwarzschild black hole becomes an infinite-dimensional, commutative supertranslation BMS algebra.[10]

In the classical limit $\hbar \to 0$ the entropy and hence number of black hole microstates becomes infinite. This may sound a bit paradoxical, but it means that the classical ground state of a black hole is infinitely degenerate. All the infinite number of classical microstate geometries have mass M, and they are completely degenerate. They are all connected to each other by the BMS-like transformations (79) on the black hole horizon. Choosing one of the infinitely many classical black hole "ground states" with mass M, the event horizon symmetries are spontaneously broken. The associated Goldstone modes are massless and describe the transitions among the infinitely many microstate geometries. In fact one can compare the different vacua of the black hole geometry with the infinitely many Higgs vacua for the case of spontaneous symmetry breaking. Each value of the Higgs field corresponds to a black hole microstate of mass M, and the classical, massless Goldstone modes correspond to the transitions from one Higgs vacuum to another one.

As we will mention at the end, there are also Higgs systems with non-vanishing entropy, where the distance to some critical Higgs field value in the scalar field space and the associated entropy become infinite. The black hole entropy becomes infinite in the semiclassical limit of large horizon, i.e. $r_S \to \infty$. Performing this takes the future horizon at $r = r_S$ to past null infinity $\mathcal{I}^-$. In this limit the vector fields (79) reduce to

$$\zeta_f^\mu = \left(f(\theta,\phi), 0, \frac{\partial f}{\partial\theta}\frac{1}{r}, \frac{1}{\sin^2\theta}\frac{\partial f}{\partial\phi}\frac{1}{r}\right)\,, \tag{80}$$

which are the standard BMS supertranslation vector fields at null infinity. In this limit the associated Goldstones modes at the horizon will coincide with the standard BMS modes, which are massless and are closely related to the soft graviton modes at $\mathcal{I}^-$. So for infinite horizon, the massless tower of states from the infinite distance conjecture should coincide with the infinitely many true BMS modes at $\mathcal{I}^-$.

Now we like to argue that in the quantum case the different black hole microstates and their associated entropy carriers will exhibit the right tower behaviour of the infinite distance conjecture.[11] In the quantum theory, namely in case of finite $\hbar$ and finite r_S, the BMS-like symmetries on the horizon will be explicitly broken, and the Goldstone modes will acquire masses, i.e. they will become pseudo Goldstone modes. In the N-portrait model, this means that at the quantum level the black hole microstates are not any more completely degenerate, but they possess masses of the order $M + m_l$, where the mass differences m_l agree with the masses of the pseudo Goldstone bosons and go to zero in the limit of large r_S or $\hbar \to 0$. Hence the transition among the non-degenerate black hole microstates are given in terms of the pseudo Goldstone excitation modes with finite masses m_l. It therefore costs a finite amount of energy to go from one black hole quantum microstate to another one.

What is now the quantum spectrum of these pseudo Goldstone modes? Without having a microscopic quantum theory that could tells us what is the energy difference of going from one microstate to another one, one can only make some guesses. For example, the modes in the Hawking radiation can be viewed as entropy carriers. These have typical frequencies of the order of the Hawking temperature $T_H \sim 1/M$, and one could conclude that the mass gap is of order

$$m \sim \frac{\hbar}{r_S} \sim \left(\frac{1}{\mathcal{S}}\right)^{1/2}. \tag{81}$$

However, based on the role of quantum criticality and entropy scaling of an underlying microscopic black hole

[10] One can derive another set of supertranslations and superrotations but we will focus only on (79) here.

[11] Related arguments for the case of de Sitter potential via towers of particles and species as entropy carriers were provided in [72].

N-portrait model [52–57], there must be an additional entropy suppression by a factor $1/\mathcal{S}$, which takes into account the relative deviations from the thermal spectrum in the Hawking radiation. It follows that the mass gap is bounded to be at most

$$m_{\max} \sim \frac{1}{\mathcal{S}} \frac{\hbar}{r_S} \sim \left(\frac{1}{\mathcal{S}}\right)^{3/2} . \tag{82}$$

Therefore in terms of the black hole mass M, the energy gap has to satisfy $m_{\max} \leq 1/M^3$. Finally, in order to get the right number of microstates, it was argued in [9] (see also [73]) that the masses of the pseudo Goldstone modes have to obey the following spectrum:

$$m_l \sim \frac{l^2}{\mathcal{S}^2} \frac{\hbar}{r_S} \sim l^2 m_0 \,, \quad \text{with} \quad l = 1, \ldots, l_{\max} = \sqrt{\mathcal{S}} \,. \tag{83}$$

The quantum number l can be viewed as an angular momentum number, and hence each mode has an l^2 degeneracy. Thus, the total number of modes contributing to the entropy scales as $l_{\max}^2 = \mathcal{S}$. Furthermore the maximal energy gap is indeed given by (82). From these formulas we can easily read off the associated microscopic mass scale, namely the mass of the lowest pseudo Goldstone excitation expressed in Planck units:

$$m_0 \sim \left(\frac{1}{\mathcal{S}}\right)^{5/2} . \tag{84}$$

So we see that the tower of the infinite distance conjecture for the entropy in (10) and the microscopic tower of the pseudo Goldstone bosons agree for the value $\alpha = 5/2$. In the context of the swampland scenario, the emergence of the light pseudo Goldstone modes in the infinite entropy limit of non-extremal black holes means that the effective horizon field theory must contain these states. In other words, without these light states, the effective field theory on the black hole horizon would be inconsistent, i.e. it would be lying in the swampland. Notice that the argument depicted here requires only the fact that quantumly the microstates are finitely many and it costs energy to move from one microstate to another. This property does not necessarily demand the microstates degeneracy to be lifted. Thus our argument is valid also in other models, like e.g. the fuzzball proposal (see the reviews [78, 79] and references therein), although the details of the latter differ from the N-portrait model.

Let us also comment on what is happening with the pseudo Goldstone modes in the extremal limit of a charged RN black hole. Towards the extremal case with $T_H = 0$ and with $\mathcal{S}, M = |Q| \to \infty$ we expect that the Goldstone modes become massless, although the entropy of the extremal black hole is still finite. Indeed, the mass gap of the Goldstone modes is zero for extremal black holes even in the quantum regime, because their masses are proportional to the black hole temperature. As we discussed in section II, the tower of states is then given in terms of internal light Kaluza-Klein modes or light string excitations.

We underline that the non-extremal situation may be different from the extremal one. In the latter, we could identify a tower of states from a higher dimensional construction while in the former we only analysed the black hole horizon properties. The relationship between these two towers in the extremal limit is still an open question and source of investigation.

VII. CONCLUSION

We have argued that besides the physical Kaluza-Klein modes and string excitations there exists another kind of tower of modes that can be determined by the distance to the large entropy limit in the space of non-extremal black hole geometries. This tower of modes is closely related to the black hole microstates. It is given in terms of pseudo Goldstone modes, which lift the degeneracy of the black hole microstates in an N-portrait model and describe the transitions among them. These modes are part of the lower-dimensional effective field theory on the horizon of the black hole. According to the swampland distance conjecture, this tower of states is necessary to have a consistent effective theory in quantum gravity on the black hole horizon. In other words, without this tower of states, the black hole with large entropy would belong to the swampland. In fact, we believe that the entropy distance (9) and the associated tower behaviour (10) are general features of black holes in quantum gravity and their microstates. Applied to macroscopic black holes like supermassive Kerr black holes with a horizon scale that is much larger than the Planck length L_p, the associated tower of states on the black hole horizon is enormous, reflecting the fact that the entropy of macroscopic black holes is also huge. However, since these modes live on the horizon, they are hard to access by earth experiments.

For extremal, BPS black holes, we have seen that the large entropy limit at infinite distance can be related to a tower of states, which are given in terms of physical Kaluza-Klein modes of an internal Calabi-Yau manifold. This behaviour originates from the underlying attractor equations of $\mathcal{N} = 2$ supersymmetric black hole solutions. It would be therefore interesting to see if there is a possible relation between attractor equations and the geometric flow equations.

As advocated in [80] the behaviour of the "mass" of entropy carriers compared to the entropy in (82) is indeed universal and follows in any system with entropy from unitarity and also from the species bound. The invariant statement is the connection between the gap and the entropy. From the unitarity arguments, the maximum energy gap is of the order of $1/\mathcal{S}$ (for black holes in units of the horizon), but it could also be smaller. The distance in space of field theories with entropies is then given in terms of a field space metric, which is proportional to $\log\mathcal{S}$. For instance, in a Higgs field theory described in [81], the infinite entropy limit is at infinite distance in the field space. This limit is reached for a critical value of the Higgs field, where the system reaches a quantum critical point.

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A. THE METRIC DISTANCE AND DIFFEOMORPHISM INVARIANCE

The metric distance (23) seems invariant under coordinate changes $x^\mu \to x^\mu(\tilde{x})$ of the manifold M, since one integrates space-time scalars with the $\sqrt{g}$ measure. On the other hand, there is a ∂_τ derivative which may contradict the previous statement if one tries to change the coordinates in a τ-dependent fashion. In this appendix we comment on those diffeomorphism-related issues.

A. An Eddington-Finkelstein puzzle

We can easily understand the need to clarify the role of diffeomorphisms by looking at (4D) space-times in Eddington-Finkelstein (EF) gauge, discussed in section V A. We computed there the distance for the metric (52), and obtained

$$\Delta_g = 4\,c\log\left(\frac{\alpha(\tau_f)}{\alpha(\tau_i)}\right)\,. \tag{85}$$

However, we could have also computed it for (51), in which case we would have obtained

$$\Delta_g = 0\,. \tag{86}$$

One thus immediately sees that the distance depends on the coordinate choice, since those two results were reached for two metrics related by the change of coordinates $r \to \alpha(\tau)\tilde{r}, v \to \alpha(\tau)\tilde{v}$.

B. Fixed charts VS evolving charts

The change between (85) and (86) can be mapped to the τ dependence of the distance Δ_g. Indeed, by changing the EF coordinates according to $(r,v)\to\alpha(\tau)(r,v)$, we performed a diffeomorphism which changes along the metric flow. The distance is not invariant under such a diffeomorphism: let us consider a change of coordinates

$$x^\mu = x^\mu(\tilde{x}^\nu,\tau) = \tilde{x}^\mu + \xi^\mu(\tilde{x}^\nu,\tau)\,, \tag{87}$$

so that the metric transforms according to

$$\tilde{g}(\tilde{x}^\mu,\tau) = \Phi^T(\tilde{x}^\mu,\tau)\,g(x^\nu(\tilde{x}^\mu,\tau),\tau)\,\Phi(\tilde{x}^\mu,\tau)\,, \tag{88}$$

with

$$\Phi^\mu{}_\nu = \frac{\partial x^\mu}{\partial\tilde{x}^\nu} = \delta^\mu{}_\nu + \frac{\partial\xi^\mu}{\partial\tilde{x}^\nu}\,. \tag{89}$$

Consequently, (23) transforms non trivially since we have

$$\begin{aligned}\partial_\tau\tilde{g} &= \Phi^T\left(\partial_\tau g + \partial_\mu g\,\partial_\tau x^\mu\right)\Phi + \partial_\tau\Phi^T g\Phi + \Phi^T g\partial_\tau\Phi\\ &= \Phi^T\left(\partial_\tau g + \mathcal{L}_{\partial_\tau\xi}g\right)\Phi\,,\end{aligned} \tag{90}$$

where $\partial_\tau\xi^\mu$ is calculated at fixed $\tilde{x}$ and $\mathcal{L}_X$ is the Lie derivative with respect to the vector X in coordinates x. We thus expect to map

$$\Delta_{\tilde{g}} = c\int_{\tau_i}^{\tau_f}\left(\frac{1}{V_M}\int d^4\tilde{x}\sqrt{\tilde{g}}\,\mathrm{tr}\left[\left(\tilde{g}^{-1}\frac{\partial\tilde{g}}{\partial\tau}\right)^2\right]\right)^{\frac{1}{2}}\,, \tag{91}$$

to

$$\Delta'_g = c\int_{\tau_i}^{\tau_f}\left(\frac{1}{V_M}\int d^4x\sqrt{g}\,\mathrm{tr}\left[\left(g^{-1}\left[\frac{\partial g}{\partial\tau} + \mathcal{L}_{\partial_\tau\xi}g\right]\right)^2\right]\right)^{\frac{1}{2}}\,, \tag{92}$$

with ξ defined in (87).

We can test this formula on the EF coordinates of section A A: there, we had

$$\xi^\mu = \left(\xi^{\tilde{v}},\xi^{\tilde{r}},\xi^{\tilde{\theta}},\xi^{\tilde{\phi}}\right) = \left((\alpha-1)\tilde{v},(\alpha-1)\tilde{r},0,0\right)\,, \tag{93}$$

hence

$$\partial_\tau\xi^\mu = \left(\partial_\tau\alpha\;\tilde{v},\partial_\tau\alpha\;\tilde{r},0,0\right) = \left(\frac{\partial_\tau\alpha}{\alpha}\;v,\frac{\partial_\tau\alpha}{\alpha}\;r,0,0\right) \tag{94}$$

and we get as expected $\Delta_{\tilde{g}} = \Delta'_g = 4c\log\left(\frac{\alpha(\tau_f)}{\alpha(\tau_i)}\right)$.

We can also check from this formula that the distance is unaffected by diffeomorphisms which are constant along the flow, since then $\partial_\tau\xi = 0$. A good example of this is the distance for the Schwarzschild black hole, which we know how to compute in Kruskal coordinates, as done in section IV B, and in dimensionless EF coordinates as discussed in section V A. The two frames being related by M-independent diffeomorphisms, the computation of the distance leads to identical results in those two frames.

C. The need for a preferred frame

The previous discussion shows that before computing any distance, one should choose the reference frame in which the Lie derivative is absent from the distance. In section IV for instance, we chose to compute the distance with respect to dimensionless coordinates. This can be understood by saying that we chose coordinates which can be transformed to compact and geodesically complete ones without ever referring to the mass scale of the problem, i.e. we used coordinates which correspond to charts which exist for any black hole/(A)dS space-times, irrespective of the mass/cosmological constant. This procedure is straightforward when only one mass scale is present.

In the case of a geometry defined by two dimensionful parameters, this procedure cannot be carried out since ratios of mass scales are dimensionless, so that there is an apparent degeneracy in choosing which combination of scales we should use when defining dimensionless coordinates. The identification of the relevant scale (assuming this exists) is also sometimes complicated by the need to use at least two charts to cover the maximally extended space-time, which is the case of the non-extremal RN black hole. Our strategy, defined in section V A, was then to try different choices for the mass parameter used to make the coordinates dimensionless, and identify a generic behaviour of the large distances in terms of large entropies.

B. THE METRIC DISTANCE WITHOUT THE VOLUME FACTOR

The distance in metric space can be sometimes encountered without the regulating volume factor of (23) (see e.g. [62, 82]),

$$\Delta''_g = c\int_{\tau_i}^{\tau_f}\left[\int_M \sqrt{g}\,\mathrm{tr}\,f^2\right]^{\frac{1}{2}}\mathrm{d}\tau\ , \tag{95}$$

where again $f = g^{-1}\dot{g}$ while the dot indicates differentiation with respect to the affine parameter τ. The corresponding geodesic equation reads

$$\dot{f} = \tfrac{1}{4}\left(\mathrm{tr}f^2 - 2f\,\mathrm{tr}f\right)\ . \tag{96}$$

The locality of this equation allows us to formally solve it. For this purpose we write (96) explicitly in components

$$\dot{f}^\mu{}_\nu = \tfrac{1}{4}\delta^\mu_\nu f^\kappa{}_\lambda f^\lambda{}_\kappa - \tfrac{1}{2}f^\mu{}_\nu f^\kappa{}_\kappa\,, \tag{97}$$

and decompose $f^\mu{}_\nu$ into its trace and and a traceless part (we now use the same notation for the scalar f as before for the matrix),

$$f^\mu{}_\nu = \tfrac{1}{d}\delta^\mu_\nu f + \bar{f}^\mu{}_\nu \qquad \text{with} \qquad \bar{f}^\mu{}_\mu = 0\,. \tag{98}$$

This allows us to rewrite (97) as

$$\begin{aligned}\dot{f} &= -\tfrac{1}{4}f^2 + \tfrac{d}{4}\bar{f}^\kappa{}_\lambda\bar{f}^\lambda{}_\kappa\,,\\ \dot{\bar{f}}^\mu{}_\nu &= -\tfrac{1}{2}\bar{f}^\mu{}_\nu f\,.\end{aligned} \tag{99}$$

To proceed we also decompose $\bar{f}^\mu{}_\lambda\bar{f}^\lambda{}_\nu$ into its trace and traceless part,

$$\bar{f}^\mu{}_\lambda\bar{f}^\lambda{}_\nu = \tfrac{1}{d^2}\delta^\mu_\nu\bar{f}^2 + \tilde{f}^\mu{}_\nu \qquad \text{with} \qquad \tilde{f}^\mu{}_\mu = 0\,, \tag{100}$$

hence we arrive at

$$\begin{aligned}\dot{f} &= -\tfrac{1}{4}\left(f^2 - \bar{f}^2\right)\,,\\ \dot{\bar{f}} &= -\tfrac{1}{2}\bar{f}f\,.\end{aligned} \tag{101}$$

This is a system of two ordinary, scalar differential equations and has the general solution

$$f(\tau) = \frac{4(\tau-a)}{(\tau-a)^2+b^2}\,, \qquad \bar{f}(\tau) = \frac{4b}{(\tau-a)^2+b^2}\,, \tag{102}$$

where a and b are space-time dependent integration constants. We can finally use the second equation of (99) to determine also the rest of $\bar{f}^\mu{}_\nu$ which gives

$$\bar{f}^\mu{}_\nu = \frac{4}{(\tau-a)^2+b^2}b^\mu{}_\nu\,, \tag{103}$$

where we have absorbed an integration constant into the definition of $b^\mu{}_\nu$ and

$$b^\mu{}_\mu = 0 \qquad \text{and} \quad b^\mu{}_\nu b^\nu{}_\mu = \frac{b^2}{d}\,. \tag{104}$$

This shows that the most general solution of (97) is given by

$$f^\mu{}_\nu = \frac{4}{(\tau-a)^2+b^2}\left[\frac{(\tau-a)}{d}\delta^\mu_\nu + b^\mu{}_\nu\right]\,. \tag{105}$$

Notice that a, b and $b^\mu{}_\nu$ are constant with respect to τ, but can still depend non-trivially on the coordinates of the manifold M. Moreover, the combination

$$f^\mu{}_\nu f^\nu{}_\mu = \frac{16}{d\big[(\tau-a)^2+b^2\big]}\,, \tag{106}$$

which appears in the distance formula (95) does not depend on $b^\mu{}_\nu$ but only on a and b. The general solution (105) for $f^\mu{}_\nu$ can be expressed in terms of $g_{\mu\nu}$ as

$$\dot{g}_{\mu\nu} = \frac{4}{(\tau-a)^2+b^2}\left[\frac{(\tau-a)}{d}g_{\mu\nu} + g_{\mu\lambda}b^\lambda{}_\nu\right]\,. \tag{107}$$

This is a matrix valued ordinary differential equation (pointwise in M) and is formally solved by

$$g_{\mu\nu}(\tau) = \big[(\tau-a)^2+b^2\big]^{\frac{2}{d}}\,g^0_{\mu\lambda}\exp\left[\frac{4b^\lambda{}_\nu}{b}\arctan\left(\frac{\tau-a}{b}\right)\right]\,, \tag{108}$$

where the exponential is to be understood as the matrix exponential of $b^\mu{}_\nu$.

To illustrate that (95) produces different results than the distance formulae (23) with varying normalization factor V_M^{-1}, we can for example consider the special case

$$a = \text{const.} \qquad \text{and} \qquad b = b^\mu{}_\nu = 0\,. \tag{109}$$

This corresponds to the conformal rescaling

$$g_{\mu\nu}(\tau) = \alpha(\tau) g^0_{\mu\lambda}\,, \qquad \alpha(\tau) = (\tau - a)^{\frac{4}{d}}\,. \tag{110}$$

Notice, that this is the same path as (30) but parametrized differently, i.e. here (23) and (95) give rise to different proper times. Moreover, we can insert (110) into (95) and obtain

$$\Delta''_g = c \left(\int_{\mathcal{M}} \sqrt{g^0} \right)^{1/2} \frac{4}{\sqrt{d}} \left(\alpha_f^{\frac{d}{4}} - \alpha_i^{\frac{d}{4}} \right)\,. \tag{111}$$

For AdS space-times, it was argued in [35] that matching the scaling of the mass of (Kaluza-Klein) towers of states suggests a logarithmic behaviour of the distance with respect to the cosmological constant. Thus, since (23) reproduces such a logarithmic scaling whereas (95) does not, the former is preferred in this paper. On the other hand, the geodesic equation (96) is local, unlike (28). Some properties of the former can be found in [82] and references therein.

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[1] C. Vafa, "The String landscape and the swampland," hep-th/0509212.

[2] H. Ooguri and C. Vafa, "On the Geometry of the String Landscape and the Swampland," Nucl. Phys. B **766** (2007) 21 doi:10.1016/j.nuclphysb.2006.10.033 [hep-th/0605264].

[3] E. Palti, "The Swampland: Introduction and Review," Fortsch. Phys. **67** (2019) no.6, 1900037 doi:10.1002/prop.201900037 [arXiv:1903.06239 [hep-th]].

[4] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333 (1973). doi:10.1103/PhysRevD.7.2333 [5]

[5] S. W. Hawking, "Particle Creation by Black Holes," Commun. Math. Phys. **43**, 199 (1975) Erratum: [Commun. Math. Phys. **46**, 206 (1976)]. doi:10.1007/BF02345020, 10.1007/BF01608497

[6] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," Phys. Lett. B **379**, 99 (1996) doi:10.1016/0370-2693(96)00345-0 [hep-th/9601029].

[7] S. W. Hawking, "The Information Paradox for Black Holes," arXiv:1509.01147 [hep-th].

[8] S. W. Hawking, M. J. Perry and A. Strominger, "Soft Hair on Black Holes," Phys. Rev. Lett. **116** (2016) no.23, 231301 doi:10.1103/PhysRevLett.116.231301 [arXiv:1601.00921 [hep-th]].

[9] A. Averin, G. Dvali, C. Gomez and D. Lüst, "Gravitational Black Hole Hair from Event Horizon Supertranslations," JHEP **1606** (2016) 088 doi:10.1007/JHEP06(2016)088 [arXiv:1601.03725 [hep-th]].

[10] A. Averin, G. Dvali, C. Gomez and D. Lüst, "Goldstone origin of black hole hair from supertranslations and criticality," Mod. Phys. Lett. A **31** (2016) no.39, 1630045 doi:10.1142/S0217732316300457 [arXiv:1606.06260 [hep-th]].

[11] S. W. Hawking, M. J. Perry and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," JHEP **1705** (2017) 161 doi:10.1007/JHEP05(2017)161 [arXiv:1611.09175 [hep-th]].

[12] S. Haco, S. W. Hawking, M. J. Perry and A. Strominger, "Black Hole Entropy and Soft Hair," JHEP **1812** (2018) 098 doi:10.1007/JHEP12(2018)098 [arXiv:1810.01847 [hep-th]].

[13] A. B. Zamolodchikov, "Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory," JETP Lett. **43**, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. **43**, 565 (1986)].

[14] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP **1112**, 099 (2011) doi:10.1007/JHEP12(2011)099 [arXiv:1107.3987 [hep-th]].

[15] D. Klaewer and E. Palti, "Super-Planckian Spatial Field Variations and Quantum Gravity," JHEP **1701** (2017) 088 doi:10.1007/JHEP01(2017)088 [arXiv:1610.00010 [hep-th]].

[16] B. Heidenreich, M. Reece and T. Rudelius, "The Weak Gravity Conjecture and Emergence from an Ultraviolet Cutoff," Eur. Phys. J. C **78** (2018) no.4, 337 doi:10.1140/epjc/s10052-018-5811-3 [arXiv:1712.01868 [hep-th]].

[17] S. Andriolo, D. Junghans, T. Noumi and G. Shiu, "A Tower Weak Gravity Conjecture from Infrared Consistency," Fortsch. Phys. **66** (2018) no.5, 1800020 doi:10.1002/prop.201800020 [arXiv:1802.04287 [hep-th]].

[18] T. W. Grimm, E. Palti and I. Valenzuela, "Infinite Distances in Field Space and Massless Towers of States," JHEP **1808** (2018) 143 doi:10.1007/JHEP08(2018)143 [arXiv:1802.08264 [hep-th]].

[19] B. Heidenreich, M. Reece and T. Rudelius, "Emergence of Weak Coupling at Large Distance in Quantum Gravity," Phys. Rev. Lett. **121** (2018) no.5, 051601 doi:10.1103/PhysRevLett.121.051601 [arXiv:1802.08698 [hep-th]].

[20] R. Blumenhagen, D. Kläwer, L. Schlechter and F. Wolf, "The Refined Swampland Distance Conjecture in Calabi-Yau Moduli Spaces," JHEP **1806** (2018) 052 doi:10.1007/JHEP06(2018)052 [arXiv:1803.04989 [hep-th]].

[21] R. Blumenhagen, "Large Field Inflation/Quintessence and the Refined Swampland Distance Conjecture," PoS CORFU **2017** (2018) 175 doi:10.22323/1.318.0175 [arXiv:1804.10504 [hep-th]].

[22] S. J. Lee, W. Lerche and T. Weigand, "Tensionless Strings and the Weak Gravity Conjecture,"

JHEP **1810** (2018) 164 doi:10.1007/JHEP10(2018)164 [arXiv:1808.05958 [hep-th]].
[23] S. J. Lee, W. Lerche and T. Weigand, "A Stringy Test of the Scalar Weak Gravity Conjecture," Nucl. Phys. B **938** (2019) 321 doi:10.1016/j.nuclphysb.2018.11.001 [arXiv:1810.05169 [hep-th]].
[24] T. W. Grimm, C. Li and E. Palti, "Infinite Distance Networks in Field Space and Charge Orbits," JHEP **1903** (2019) 016 doi:10.1007/JHEP03(2019)016 [arXiv:1811.02571 [hep-th]].
[25] D. Klaewer, D. Lüst and E. Palti, "A Spin-2 Conjecture on the Swampland," Fortsch. Phys. **67** (2019) no.1-2, 1800102 doi:10.1002/prop.201800102 [arXiv:1811.07908 [hep-th]].
[26] Q. Bonnefoy, E. Dudas and S. Lüst, "On the weak gravity conjecture in string theory with broken supersymmetry," Nucl. Phys. B **947** (2019) 114738 [arXiv:1811.11199 [hep-th]].
[27] P. Corvilain, T. W. Grimm and I. Valenzuela, "The Swampland Distance Conjecture for Kähler moduli," JHEP **1908** (2019) 075 doi:10.1007/JHEP08(2019)075 [arXiv:1812.07548 [hep-th]].
[28] S. J. Lee, W. Lerche and T. Weigand, "Modular Fluxes, Elliptic Genera, and Weak Gravity Conjectures in Four Dimensions," JHEP **1908** (2019) 104 doi:10.1007/JHEP08(2019)104 [arXiv:1901.08065 [hep-th]].
[29] A. Joshi and A. Klemm, "Swampland Distance Conjecture for One-Parameter Calabi-Yau Threefolds," JHEP **1908** (2019) 086 doi:10.1007/JHEP08(2019)086 [arXiv:1903.00596 [hep-th]].
[30] F. Marchesano and M. Wiesner, "Instantons and infinite distances," JHEP **1908** (2019) 088 doi:10.1007/JHEP08(2019)088 [arXiv:1904.04848 [hep-th]].
[31] A. Font, A. Herraez and L. E. Ibanez, "The Swampland Distance Conjecture and Towers of Tensionless Branes," JHEP **1908** (2019) 044 doi:10.1007/JHEP08(2019)044 [arXiv:1904.05379 [hep-th]].
[32] S. J. Lee, W. Lerche and T. Weigand, "Emergent Strings, Duality and Weak Coupling Limits for Two-Form Fields," arXiv:1904.06344 [hep-th].
[33] C. Bachas, "Massive AdS Supergravitons and Holography," JHEP **1906** (2019) 073 doi:10.1007/JHEP06(2019)073 [arXiv:1905.05039 [hep-th]].
[34] D. Erkinger and J. Knapp, "Refined swampland distance conjecture and exotic hybrid Calabi-Yaus," JHEP **1907** (2019) 029 doi:10.1007/JHEP07(2019)029 [arXiv:1905.05225 [hep-th]].
[35] D. Lüst, E. Palti and C. Vafa, "AdS and the Swampland," arXiv:1906.05225 [hep-th].
[36] P. Agrawal, G. Obied and C. Vafa, "H_0 Tension, Swampland Conjectures and the Epoch of Fading Dark Matter," arXiv:1906.08261 [astro-ph.CO].
[37] C. Gomez, "Gravity as universal UV completion: Towards a unified view of Swampland conjectures," arXiv:1907.13386 [hep-th].
[38] F. Marchesano and J. Quirant, "A Landscape of AdS Flux Vacua," arXiv:1908.11386 [hep-th].
[39] A. Bedroya and C. Vafa, "Trans-Planckian Censorship and the Swampland," arXiv:1909.11063 [hep-th].
[40] A. Kehagias, D. Lüst and S. Lüst, "Swampland, Gradient Flow and Infinite Distance," arXiv:1910.00453 [hep-th].
[41] S. J. Lee, W. Lerche and T. Weigand, "Emergent Strings from Infinite Distance Limits," arXiv:1910.01135 [hep-th].
[42] T. W. Grimm, F. Ruehle and D. van de Heisteeg, "Classifying Calabi-Yau threefolds using infinite distance limits," arXiv:1910.02963 [hep-th].
[43] H. Geng, "Distance Conjecture and De-Sitter Quantum Gravity," arXiv:1910.03594 [hep-th].
[44] T. W. Grimm, C. Li and I. Valenzuela, "Asymptotic Flux Compactifications and the Swampland," arXiv:1910.09549 [hep-th].
[45] R. Blumenhagen, M. Brinkmann and A. Makridou, "Quantum Log-Corrections to Swampland Conjectures," arXiv:1910.10185 [hep-th].
[46] A. Kehagias and A. Riotto, "A Note on the Swampland Distance Conjecture," arXiv:1911.09050 [hep-th].
[47] L. A. Anchordoqui, I. Antoniadis, D. Lüst, J. F. Soriano and T. R. Taylor, "H_0 tension and the String Swampland," arXiv:1912.00242 [hep-th].
[48] A. Font, A. Herraez and L. E. Ibanez, "On scale separation in type II AdS flux vacua," arXiv:1912.03317 [hep-th].
[49] M. Scalisi, "Inflation, Higher Spins and the Swampland," arXiv:1912.04283 [hep-th].
[50] R.S.Hamilton, "Three manifolds with positive Ricci curvature," Jour. Diff. Geom. 17 (1982), 255-306.
[51] G. Perelman, "The Entropy formula for the Ricci flow and its geometric applications," math/0211159 [math-dg].
[52] G. Dvali and C. Gomez, "Black Hole's Quantum N-Portrait," Fortsch. Phys. **61** (2013) 742 [arXiv:1112.3359 [hep-th]];
[53] "Black Hole's 1/N Hair," Phys. Lett. B **719** (2013) 419 [arXiv:1203.6575 [hep-th]].
[54] G. Dvali and C. Gomez, "Quantum Compositeness of Gravity: Black Holes, AdS and Inflation," JCAP **1401** (2014) 023 doi:10.1088/1475-7516/2014/01/023 [arXiv:1312.4795 [hep-th]].
[55] G. Dvali and C. Gomez, "Black Hole Macro-Quantumness," arXiv:1212.0765 [hep-th]
[56] G. Dvali, C. Gomez and D. Lüst, "Black Hole Quantum Mechanics in the Presence of Species," Fortsch. Phys. **61** (2013) 768 [arXiv:1206.2365 [hep-th]].
[57] G. Dvali and C. Gomez, "Black Holes as Critical Point of Quantum Phase Transition," Eur. Phys. J. C **74**, 2752 (2014) [arXiv:1207.4059 [hep-th]];
[58] G. Dvali, C. Gomez and D. Lüst, "Classical Limit of Black Hole Quantum N-Portrait and BMS Symmetry," Phys. Lett. B **753** (2016) 173 doi:10.1016/j.physletb.2015.11.073 [arXiv:1509.02114 [hep-th]].
[59] S. Ferrara, R. Kallosh and A. Strominger, "N=2 extremal black holes," Phys. Rev. D **52** (1995) R5412 doi:10.1103/PhysRevD.52.R5412 [hep-th/9508072].
[60] S. Ferrara and R. Kallosh, "Supersymmetry and attractors," Phys. Rev. D **54** (1996) 1514 doi:10.1103/PhysRevD.54.1514 [hep-th/9602136].
[61] K. Behrndt, G. Lopes Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, "Classical and quantum N=2 supersymmetric black holes," Nucl. Phys. B **488** (1997) 236 doi:10.1016/S0550-3213(97)00028-X [hep-th/9610105].
[62] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[63] G. W. Gibbons and S. W. Hawking, "Cosmological

Event Horizons, Thermodynamics, and Particle Creation," Phys. Rev. D **15** (1977) 2738.

[64] L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon," Phys. Rev. Lett. **116** (2016) no.9, 091101 doi:10.1103/PhysRevLett.116.091101 [arXiv:1511.08687 [hep-th]].

[65] M. Blau and M. O'Loughlin, "Horizon Shells and BMS-like Soldering Transformations," JHEP **1603** (2016) 029 doi:10.1007/JHEP03(2016)029 [arXiv:1512.02858 [hep-th]].

[66] L. Donnay, G. Giribet, H. A. Gonzalez and M. Pino, "Extended Symmetries at the Black Hole Horizon," JHEP **1609** (2016) 100 doi:10.1007/JHEP09(2016)100 [arXiv:1607.05703 [hep-th]].

[67] D. Lüst, "Supertranslations and Holography near the Horizon of Schwarzschild Black Holes," Fortsch. Phys. **66** (2018) no.2, 1800001 doi:10.1002/prop.201800001 [arXiv:1711.04582 [hep-th]].

[68] A. Averin, "Schwarzschild/CFT from soft black hole hair?," JHEP **1901** (2019) 092 doi:10.1007/JHEP01(2019)092 [arXiv:1808.09923 [hep-th]].

[69] L. Donnay and G. Giribet, "Cosmological horizons, Noether charges and entropy," Class. Quant. Grav. **36** (2019) no.16, 165005 doi:10.1088/1361-6382/ab2e42 [arXiv:1903.09271 [hep-th]].

[70] L. Donnay and C. Marteau, "Carrollian Physics at the Black Hole Horizon," Class. Quant. Grav. **36** (2019) no.16, 165002 doi:10.1088/1361-6382/ab2fd5 [arXiv:1903.09654 [hep-th]].

[71] A. Averin, "Entropy Counting from Schwarzschild/CFT and Soft Hair," arXiv:1910.08061 [hep-th].

[72] H. Ooguri, E. Palti, G. Shiu and C. Vafa, "Distance and de Sitter Conjectures on the Swampland," Phys. Lett. B **788** (2019) 180 doi:10.1016/j.physletb.2018.11.018 [arXiv:1810.05506 [hep-th]].

[73] W. Y. Wen, "Hawking Radiation As Stimulated Emission," arXiv:1910.06055 [hep-th].

[74] G. Dvali and C. Gomez, "Quantum Exclusion of Positive Cosmological Constant?," Annalen Phys. **528** (2016) 68 doi:10.1002/andp.201500216 [arXiv:1412.8077 [hep-th]].

[75] G. Dvali, C. Gomez and S. Zell, "Quantum Break-Time of de Sitter," JCAP **1706** (2017) 028 doi:10.1088/1475-7516/2017/06/028 [arXiv:1701.08776 [hep-th]].

[76] G. Dvali and C. Gomez, "On Exclusion of Positive Cosmological Constant," Fortsch. Phys. **67** (2019) no.1-2, 1800092 doi:10.1002/prop.201800092 [arXiv:1806.10877 [hep-th]].

[77] G. Dvali, C. Gomez and S. Zell, "Quantum Breaking Bound on de Sitter and Swampland," Fortsch. Phys. **67** (2019) no.1-2, 1800094 doi:10.1002/prop.201800094 [arXiv:1810.11002 [hep-th]].

[78] S. D. Mathur, "The Fuzzball proposal for black holes: An Elementary review," Fortsch. Phys. **53** (2005) 793 doi:10.1002/prop.200410203 [hep-th/0502050].

[79] I. Bena and N. P. Warner, "Black holes, black rings and their microstates," Lect. Notes Phys. **755** (2008) 1 doi:10.1007/978-3-540-79523-01 [hep-th/0701216].

[80] G. Dvali, "Area Law Saturation of Entropy Bound from Perturbative Unitarity in Renormalizable Theories," arXiv:1906.03530 [hep-th].

[81] G. Dvali, "Cosmological Relaxation of Higgs Mass Before and After LHC and Naturalness," arXiv:1908.05984 [hep-ph].

[82] Olga Gil-Medrano and Peter W. Michor, "The Riemannian manifold of all Riemannian metrics," arXiv:math/9201259.