DESY–19–116

On complex Gamma function integrals

S. É. Derkachov[1] **and A. N. Manashov**[2,3,1]

[1] St.Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023 St. Petersburg, Russia.

[2] Institut für Theoretische Physik, Universität Hamburg, D-22761 Hamburg, Germany

[3] Institute for Theoretical Physics, University of Regensburg, D-93040 Regensburg, Germany

Abstract. It was observed recently that relations between matrix elements of certain operators in the $\mathrm{SL}(2,\mathbb{R})$ spin chain models take the form of multidimensional integrals derived by R.A. Gustafson. The spin magnets with $\mathrm{SL}(2,\mathbb{C})$ symmetry group and $\mathrm{L}_2(\mathbb{C})$ as a local Hilbert space give rise to a new type of Γ-function integrals. In this work we present a direct calculation of two such integrals. We also analyse properties of these integrals and show that they comprise the star–triangle relations recently discussed in the literature. It is also shown that in the quasi-classical limit these integral identities are reduced to the duality relations for Dotsenko-Fateev integrals.

1. Introduction

The multidimensional integrals derived by R.A. Gustafson [1, 2] together with their q- and elliptic analogues [1–6] play an important role in different topics of physics and mathematics such as theory of multi-variable orthogonal polynomials [7], Selberg type integrals and constant term identities [8, 9], supersymmetric dualities in quantum field theory [10]. Recently, a new field — noncompact spin magnets — was added to this list [11].

Models of this type appear in gauge field theories and have been under intense investigations in the last two decades, see for a review [12, 13]. Mathematical description of such systems is well-developed and known as Quantum Inverse Scattering Method (QISM), [14–19]. Noncompact spin chains have an infinite-dimensional local Hilbert space and most conveniently can be analysed within the Separation of Variable (SoV) framework [18]. It was shown by Sklyanin that eigenfunctions of the entries of a monodromy matrix provide a suitable basis for solving the spectral problem for the spin chain Hamiltonian. Building such a basis for a generic spin chain is in itself a nontrivial problem. † Fortunately, for the noncompact magnets of interest the corresponding eigenfunctions are known explicitly [23–25].

In the $\mathrm{SL}(2,\mathbb{R})$ spin chin framework, Gustafson's integrals follow from the identities for matrix elements of the shift operator [11]. Extension of this analysis to the $\mathrm{SL}(2,\mathbb{C})$ spin magnets leads to new integral identities [26,27] which, up to the expected modifications, are the exact replicas of the $\mathrm{SL}(2,\mathbb{R})$ integrals. The analysis, however, essentially depends on the completeness of the SoV representation. Proof of completeness is a rather complicated problem, see e.g. refs. [28–30]. Completeness for the closed $\mathrm{SL}(2,\mathbb{R})$ spin chain follows from that for the Toda chain [28] while for the $\mathrm{SL}(2,\mathbb{C})$ magnets of the length $N > 2$ it is still an open question. A fruitful strategy seems to be to make use of Gustafson's integrals to prove completeness. To make it feasible one needs an independent derivation of the corresponding integrals. One of the purposes of this work is to provide such a derivation.

The paper is organized as follows: in sect. 2, after setting the notations, we prove two integral identities which are direct analogs of Gustafson integrals associated with the classical $su(N)$ and

† For a recent progress in constructing the SoV representation for compact chains see refs. [20–22]

$sp(N)$ Lie algebras [2]. We analyze analytic properties of these integrals in sect. 3 and derive two new integrals which are the SL$(2,\mathbb{C})$ versions of Gustafson integrals [1] generalizing the second Barnes Lemma. In sect. 4 it will be shown that $N = 1, 2$ integrals take, after some rewriting, the form of the star-triangle relations derived in [31–33] as special limits of the elliptic star-triangle identity [34–37]. Sect. 5 is devoted to the study of the quasi-classical limit of the Γ-integrals. We show that in this limit the integrals are equivalent to the special case of the duality relation [38] for Dotsenko-Fateev (DF) integrals [39]. In sect. 6 we present an elementary proof of the above duality relation and give some evidence in favor that similar duality relations hold for the Γ-integrals. Sect. 7 is reserved for a summary.

2. Gamma integrals

2.1. Definitions and basic properties

Let $u, \bar{u}$ be a pair of complex numbers of the form

$$u = \frac{n}{2} + \nu, \qquad \bar{u} = -\frac{n}{2} + \nu, \tag{2.1}$$

where n is an integer and ν is complex number. We will use the notations $[u] = u - \bar{u} = n$ for the discrete part and $\langle u\rangle = \nu$ for the continuous part and put $||u||^2 = -u\bar{u} = -\nu^2 + n^2/4$ so that for imaginary ν, $||u||^2 \geq 0$. The $\mathbf{\Gamma}$ function of the complex field $\mathbb{C}$ [40] is defined as

$$\mathbf{\Gamma}(u,\bar{u}) = \frac{\Gamma(u)}{\Gamma(1-\bar{u})} = \frac{\Gamma(n/2+\nu)}{\Gamma(1+n/2-\nu)} = (-1)^n \frac{\Gamma(-n/2+\nu)}{\Gamma(1-n/2-\nu)} = (-1)^{[u]}\Gamma(\bar{u}, u). \tag{2.2}$$

In what follows we will, for brevity, display only the first argument of the $\mathbf{\Gamma}$ function, i.e. $\mathbf{\Gamma}(u) \equiv \mathbf{\Gamma}(u,\bar{u})$. Hereafter the following functional relations will be useful

$$\mathbf{\Gamma}(u)\mathbf{\Gamma}(1-u) = (-1)^{[u]}, \qquad \mathbf{\Gamma}(u+1) = -u\bar{u}\,\mathbf{\Gamma}(u). \tag{2.3}$$

The $\mathbf{\Gamma}$ function appears in the generalization of Gustafson's integrals to the complex case. The corresponding integrals take the following form [26, 27]

$$I_N^{(1)} = \frac{1}{N!}\prod_{r=1}^{N}\sum_{n_r=-\infty}^{\infty}\int_{-i\infty}^{i\infty}\frac{d\nu_r}{2\pi i}\,\frac{\prod_{m=1}^{N+1}\prod_{k=1}^{N}\mathbf{\Gamma}(z_m-u_k)\mathbf{\Gamma}(u_k+w_m)}{\prod_{m<j}\mathbf{\Gamma}(u_m-u_j)\mathbf{\Gamma}(u_j-u_m)} = \frac{\prod_{k,j=1}^{N+1}\mathbf{\Gamma}(z_k+w_j)}{\mathbf{\Gamma}\left(\sum_{k=1}^{N+1}(z_k+w_k)\right)}, \tag{2.4a}$$

$$I_N^{(2)} = \frac{1}{2^N N!}\prod_{r=1}^{N}\sum_{n_r=-\infty}^{\infty}\int_{-i\infty}^{i\infty}\frac{d\nu_r}{2\pi i}\,\frac{\prod_{k=1}^{N}\prod_{m=1}^{2N+2}\mathbf{\Gamma}(z_m\pm u_k)}{\prod_{k=1}^{N}\mathbf{\Gamma}(\pm 2u_k)\prod_{k<j}\mathbf{\Gamma}(\pm u_k\pm u_j)} = \frac{\prod_{j<k}\mathbf{\Gamma}(z_j+z_k)}{\mathbf{\Gamma}(\sum_{k=1}^{2N+2}z_k)}, \tag{2.4b}$$

where we put $\mathbf{\Gamma}(a\pm b) \equiv \mathbf{\Gamma}(a+b)\mathbf{\Gamma}(a-b)$. The variables u_k, w_m, z_m have the form (2.1)

$$u_r = \frac{n_r}{2} + \nu_r, \qquad z_j = \frac{m_j}{2} + x_j, \qquad w_m = \frac{\ell_m}{2} + y_m \tag{2.5}$$

and similarly for the barred variables. However, n_r, m_i, w_m are allowed take integer or half-integer values, simultaneously. Accordingly, the sums in (2.4) go over integers or half-integers. For the first integral (2.4a) there is no difference between the integer/half-integer cases since they are related by change of variables: $[u_r], [w_m], [z_j] \mapsto [u_r]+1/2, [w_m]-1/2, [z_j]+1/2$ so that we will assume that variables $[u_k], [w_m], [z_m]$ in the integral (2.4a) are integers.

The integration contours in (2.4) separate the series of " ± " poles due to the $\mathbf{\Gamma}$ functions in the numerators. The poles are located at the points,

$$\nu_{rj}^{I,+}(p) = \frac{1}{2}|n_r - m_j| + x_j + p, \qquad \nu_{rj}^{I,-}(p) = -\frac{1}{2}|n_r + \ell_j| - y_j - p, \qquad p \geq 0, \tag{2.6}$$

where $r \in [1, N]$, $j \in [1, N+1]$ for the first integral and

$$\nu_{rj}^{II,+}(p) = \frac{1}{2}|n_r - m_j| + x_j + p, \qquad \nu_{rj}^{II,-}(p) = -\frac{1}{2}|n_r + m_j| - x_j - p, \qquad p \geq 0, \tag{2.7}$$

where $r \in [1, N]$, $j \in [1, 2N+2]$, for the second one.

Let us discuss now the convergence properties of the integrals (2.4). Since the integrands are meromorphic functions and contours of integration avoid the poles it is sufficient to analyse the region of large $u_r = n_r/2 + i\nu_r$, $||u_r||^2 = \nu_r^2 + n_r^2/4 \to \infty$, only. With the help of Eq. (2.3) we simplify the denominators in the integrals (2.4) as follows

$$\prod_{1\leq i<k\leq N} \frac{1}{\mathbf{\Gamma}(u_i - u_k)\mathbf{\Gamma}(u_k - u_i)} = (-1)^{(N+1)\sum_k [u_k]} \prod_{1\leq i<k\leq N} ||u_i - u_k||^2 \tag{2.8}$$

and

$$\prod_{k=1}^{N} \frac{1}{\mathbf{\Gamma}(\pm 2u_k)} \prod_{1\leq i<m\leq N} \frac{1}{\mathbf{\Gamma}(\pm u_i \pm u_m)} = \varkappa_N 4^N \prod_{k=1}^{N} ||u_k||^2 \prod_{1\leq i<m\leq N} ||u_i - u_m||^2 ||u_i + u_m||^2 \,, \tag{2.9}$$

where $\varkappa_N = 1$ for the integer case and $\varkappa_N = (-1)^{N(N+1)/2}$ for the half-integer case. Finally, taking into account that for large u

$$\mathbf{\Gamma}(z-u)\mathbf{\Gamma}(u+w) = (-1)^{[z-u]} \frac{\mathbf{\Gamma}(u+w)}{\mathbf{\Gamma}(u-z)} = (-1)^{[z-u]} u^{z+w-1} (-\bar{u})^{\bar{z}+\bar{w}-1} \Big(1 + O\left(1/||u||\right)\Big) \tag{2.10}$$

we conclude that the integrals (2.4) converge absolutely provided

$$\mathrm{Re} \sum_{j=1}^{N+1} (x_j + y_j) < 1 \quad \text{and} \quad \mathrm{Re} \sum_{j=1}^{2N+2} x_j < 1, \tag{2.11}$$

respectively. From now on we assume that these conditions are fulfilled.

2.2. Determinant representation

In this subsection we present the integrals (2.4) as determinants of one-fold integrals. Such a representation will be quite useful in what follows. ‡ For the derivation let us denote by $\mathcal{Q}(u|z,w)$ the function

$$\mathcal{Q}(u|z,w) = \prod_{k=1}^{N+1} (-1)^{[u]} \mathbf{\Gamma}(z_k - u)\mathbf{\Gamma}(u + w_k) \tag{2.12}$$

and by $\mathcal{Q}_{ik}(z,w)$ its Mellin moments:

$$\mathcal{Q}_{ik}(z,w) = \int \mathcal{D}u \, u^{i-1} \, (-\bar{u})^{k-1} \, \mathcal{Q}(u|z,w)\,, \qquad i,k = 1,\ldots,N. \tag{2.13}$$

‡ The determinant representation for elliptic hypergeometric integrals was constructed in ref. [41].

Here we introduced a short-hand notation for the integration measure

$$\int \mathcal{D}u \equiv \sum_{n=-\infty}^{\infty} \int_{-i\infty}^{i\infty} \frac{d\nu}{2\pi i}\,. \tag{2.14}$$

Let $\boldsymbol{\mathcal{Q}}_N(z,w)$ be a $N\times N$ matrix constructed from the Mellin moments,

$$\boldsymbol{\mathcal{Q}}_N(z,w) = \begin{pmatrix} \mathcal{Q}_{11}(z,w) & \cdots & \mathcal{Q}_{1N}(z,w) \\ \vdots & \ddots & \vdots \\ \mathcal{Q}_{N1}(z,w) & \cdots & \mathcal{Q}_{NN}(z,w) \end{pmatrix}. \tag{2.15}$$

Rewriting the product on the r.h.s. of Eq. (2.8) as the product of two Vandermonde determinants

$$\prod_{1\le i<k\le N} ||u_i-u_k||^2 = \det V(u)\det V(-\bar u)\,, \qquad V_{ij}(u) = u_j^{i-1}, \qquad i,j=1,\ldots,N \tag{2.16}$$

and taking into account the symmetry of the integrand in (2.4a) with respect to the permutations $u_i \leftrightarrow u_j$ we bring the first integral into the determinant form

$$I_N^{(1)} = \left(\prod_{r=1}^{N}\int \mathcal{D}u_r\right)\det V(-\bar u)\,\prod_{k=1}^{N}\mathcal{Q}(u_k|z,w)u_k^{k-1} = \det \boldsymbol{\mathcal{Q}}_N(z,w)\,. \tag{2.17}$$

Proceeding in a similar way one gets the determinant representation for the second integral as follows

$$I_N^{(2)} = \varkappa_N \det \widetilde{\boldsymbol{\mathcal{Q}}}_N(z) = \varkappa_N \det \begin{vmatrix} \widetilde{\mathcal{Q}}_{11}(z) & \cdots & \widetilde{\mathcal{Q}}_{1N}(z) \\ \vdots & \ddots & \vdots \\ \widetilde{\mathcal{Q}}_{N1}(z) & \cdots & \widetilde{\mathcal{Q}}_{NN}(z) \end{vmatrix}, \tag{2.18}$$

where $\varkappa_N$ is a phase factor, see Eq. (2.9),

$$\widetilde{\mathcal{Q}}_{ik}(z) = 2\int \mathcal{D}u\, u^{2i-1}(-\bar u)^{2k-1}\,\widetilde{\mathcal{Q}}(u,z)\,, \qquad \text{and} \qquad \widetilde{\mathcal{Q}}(u,z) = \prod_{j=1}^{2N+2}\boldsymbol{\Gamma}(z_j\pm u)\,. \tag{2.19}$$

Let us note here that the conditions (2.11) are equivalent to the requirement of absolute convergence of the Mellin moments $\mathcal{Q}_{NN}(z,w)$ and $\widetilde{\mathcal{Q}}_{NN}(z)$, respectively.

2.3. Proof of identities (2.4)

Calculating the integral (2.4a) we will assume that the parameters z_k, w_k satisfy the conditions

$$\operatorname{Re}\sum_{k=1}^{N+1}(z_k+w_k) < 1 \qquad \text{and} \qquad \operatorname{Re}\sum_{k=1}^{N+1}(\bar z_k+\bar w_k) < 1. \tag{2.20}$$

These conditions imply the condition (2.11), but not follow from it and will be removed in the end of the calculation.

By virtue of (2.11) the integrals (2.13) are absolutely convergent for $i,k=1,\ldots N$. We evaluate the ν integral by residues closing the contour in the left half-plane. Recalling that $u=n/2+\nu$ and $\bar u=-n/2+\nu$ one gets

$$M_{ik}(n)=\int_{-i\infty}^{i\infty}\frac{d\nu}{2\pi i}u^{i-1}(-\bar u)^{k-1}\mathcal{Q}(u|z,w)=(-1)^{(N+1)n}\sum_{j=1}^{N+1}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!\bar p_j!}(-w_j-p)^{i-1}(\bar w_j+\bar p_j)^{k-1}$$
$$\times\prod_{q=1}^{N+1}\frac{\Gamma(z_q+w_j+p)}{\Gamma(1-\bar z_q-\bar w_j-\bar p_j)}\prod_{q\neq j}^{N+1}\frac{\Gamma(w_q-w_j-p)}{\Gamma(1-\bar w_q+\bar w_j+\bar p_j)},\tag{2.21}$$

where $\bar p_j=p+n+\ell_j$.

The first observation is that since the summand vanishes for $\bar p_j<0$, only the poles at $\nu^{I,-}_{rj}(p)$, Eq. (2.6), contribute to the integral and, second, that under the assumptions (2.20) the sum over p converges uniformly on n. Therefore, evaluating $\sum_n M_{ik}(n)$ we swap the summation over n and p, change the summation variable from n to $\bar p_j$ and finally suppress the index j in $\bar p_j$: $\bar p_j\to\bar p$

$$\sum_n M_{ik}(n)=\sum_{j=1}^{N+1}\sum_{p=0}^{\infty}\sum_{n=-\infty}^{\infty}(\ldots)\mapsto\sum_{j=1}^{N+1}\sum_{p=0}^{\infty}\sum_{\bar p_j=-\infty}^{\infty}(\ldots)=\sum_{j=1}^{N+1}\sum_{p=0}^{\infty}\sum_{\bar p=0}^{\infty}(\ldots).$$

The final answer can be written in the form

$$\mathcal{Q}_{ik}(z,w)=\sum_{j=1}^{N+1}(-1)^{(N+1)[w_j]}\prod_{q=1}^{N+1}\frac{\Gamma(z_q+w_j)}{\Gamma(1-\bar z_q-\bar w_j)}\prod_{\substack{q=1\\q\neq j}}^{N+1}\frac{\Gamma(w_q-w_j)}{\Gamma(1-\bar w_q+\bar w_j)}$$
$$\times\left(\sum_{p=0}^{\infty}(-w_j-p)^{i-1}\prod_{q=1}^{N+1}\frac{(z_q+w_j)_p}{(1-w_q+w_j)_p}\right)\left(\sum_{\bar p=0}^{\infty}(\bar w_j+\bar p)^{k-1}\prod_{q=1}^{N+1}\frac{(\bar z_q+\bar w_j)_{\bar p}}{(1-\bar w_q+\bar w_j)_{\bar p}}\right).\tag{2.22}$$

After substituting this expression in (2.18), it can be rewritten as follows

$$I_N^{(1)}=\frac{1}{N!}\sum_{\sigma\in S_{N+1}}(-1)^{(N+1)\sum_{s=1}^N[w_{\sigma(s)}]}\prod_{k=1}^{N}\left(\prod_{q=1}^{N+1}\frac{\Gamma(z_q+w_{\sigma(k)})}{\Gamma(1-\bar z_q-\bar w_{\sigma(k)})}\prod_{\substack{q=1\\q\neq\sigma(k)}}^{N+1}\frac{\Gamma(w_q-w_{\sigma(k)})}{\Gamma(1-\bar w_q+\bar w_{\sigma(k)})}\right)$$
$$\times\sum_{p_1,\ldots,p_N=0}^{\infty}\left(\prod_{k<m}(w_{\sigma(k)}+p_k-w_{\sigma(m)}-p_m)\right)\prod_{q=1}^{N+1}\prod_{k=1}^{N}\frac{(z_q+w_{\sigma(k)})_{p_k}}{(1-w_q+w_{\sigma(k)})_{p_k}}$$
$$\times\sum_{\bar p_1,\ldots,\bar p_N=0}^{\infty}\left(\prod_{k<m}(\bar w_{\sigma(m)}+\bar p_m-\bar w_{\sigma(k)}-\bar p_k)\right)\prod_{q=1}^{N+1}\prod_{k=1}^{N}\frac{(\bar z_q+\bar w_{\sigma(k)})_{\bar p_k}}{(1-\bar w_q+\bar w_{\sigma(k)})_{\bar p_k}}.\tag{2.23}$$

The infinite sums over $\{p\},\{\bar p\}$ can be evaluated with the help of the following Milne's $U(n)$ generalization of the Gauss summation formula [42],see also [2], Eq.(5.8),

$$\sum_{p_1,\ldots,p_N=0}^{\infty}\left(\prod_{1\le k<m\le N}(\alpha_{\sigma(k)}+p_k-\alpha_{\sigma(m)}-p_m)\right)\prod_{q=1}^{N+1}\prod_{k=1}^{N}\frac{(\beta_q+\alpha_{\sigma(k)})_{p_k}}{(1-\alpha_q+\alpha_{\sigma(k)})_{p_k}}=$$
$$=\frac{\Gamma(1-\sum_{k=1}^{N+1}(\alpha_k+\beta_k))\prod_{i=1}^{N}\Gamma(1+\alpha_{\sigma(i)}-\alpha_{\sigma(N+1)})}{\prod_{i=1}^{N+1}\Gamma(1-\beta_i-\alpha_{\sigma(N+1)})}.\tag{2.24}$$

Using (2.24) we derive the following representation for the integral $I_N^{(1)}$

$$I_N^{(1)} = \frac{1}{N!} \sum_{\sigma \in S_{N+1}} (-1)^{(N+1)\sum_{s=1}^N [w_{\sigma(s)}]} \prod_{k=1}^N \left(\prod_{q=1}^{N+1} \frac{\Gamma(z_q + w_{\sigma(k)})}{\Gamma(1 - \bar{z}_q - \bar{w}_{\sigma(k)})} \prod_{\substack{q=1 \\ q \neq \sigma(k)}}^{N+1} \frac{\Gamma(w_q - w_{\sigma(k)})}{\Gamma(1 - \bar{w}_q + \bar{w}_{\sigma(k)})} \right)$$
$$\times \prod_{1 \le m < j \le N} (w_{\sigma(m)} - w_{\sigma(j)}) \, \Gamma\left(1 - \sum_{k=1}^{N+1} (z_k + w_k)\right) \frac{\prod_{k=1}^N \Gamma(1 + w_{\sigma(k)} - w_{\sigma(N+1)})}{\prod_{k=1}^{N+1} \Gamma(1 - z_k - w_{\sigma(N+1)}))}$$
$$\times \prod_{1 \le m < j \le N} (\bar{w}_{\sigma(j)} - \bar{w}_{\sigma(m)}) \, \Gamma\left(1 - \sum_{k=1}^{N+1} (\bar{z}_k + \bar{w}_k)\right) \frac{\prod_{k=1}^N \Gamma(1 + \bar{w}_{\sigma(k)} - \bar{w}_{\sigma(N+1)})}{\prod_{k=1}^{N+1} \Gamma(1 - \bar{z}_k - \bar{w}_{\sigma(N+1)})}. \qquad (2.25)$$

After some simplifications it can be written as

$$I_N^{(1)} = \frac{\Gamma\left(1 - \sum_{k=1}^{N+1} (\bar{z}_k + \bar{w}_k)\right)}{\Gamma\left(\sum_{k=1}^{N+1} (z_k + w_k)\right)} \prod_{i,k=1}^{N+1} \frac{\Gamma(z_i + w_k)}{\Gamma(1 - \bar{z}_i - \bar{w}_k)} \frac{R_N}{N! \sin \pi \left(\sum_{k=1}^{N+1} (z_k + w_k)\right)} \qquad (2.26)$$

with

$$R_N = \sum_{\sigma \in S_{N+1}} \frac{\prod_{k=1}^{N+1} \sin \pi (z_{\sigma(k)} + w_{\sigma(N+1)})}{\prod_{k=1}^N \sin \pi (w_{\sigma(N+1)} - w_{\sigma(k)})} (-1)^{(N+1)\sum_{s=1}^N [w_{\sigma(s)}]} \prod_{1 \le k < j \le N} \frac{\sin \pi (\bar{w}_{\sigma(j)} - \bar{w}_{\sigma(k)})}{\sin \pi (w_{\sigma(j)} - w_{\sigma(k)})} . \qquad (2.27)$$

Taking into account that $w_k - \bar{w}_k$ is an integer one finds that the last product in (2.27) yields $(-1)^{(N-1)\sum_{s=1}^N [w_{\sigma(s)}]}$ that cancels the second factor in (2.27). On the last step we use the Lemma 5.10 in Ref. [2] which claims

$$\sum_{\sigma \in S_{N+1}} \frac{\prod_{k=1}^{N+1} \sin \pi (\beta_k + \alpha_{\sigma(N+1)})}{\prod_{k=1}^N \sin \pi (\alpha_{\sigma(N+1)} - \alpha_{\sigma(k)})} = N! \sin \pi \left(\sum_{k=1}^{N+1} (\alpha_k + \beta_k) \right). \qquad (2.28)$$

It results in

$$R_N = N! \sin \pi \left(\sum_{k=1}^{N+1} (z_k + w_k) \right).$$

so that we get the required result for $I_N^{(1)}$

$$I_N^{(1)} = \frac{\prod_{k,j=1}^{N+1} \boldsymbol{\Gamma}(z_k + w_j)}{\boldsymbol{\Gamma}\left(\sum_{k=1}^{N+1} (z_k + w_k)\right)} . \qquad (2.29)$$

Finally by analytic continuation in ν_k the assumptions (2.20) can be relaxed to the condition (2.11).

One sees that the crucial point in the proof of (2.4a) is the factorization of the double sum arising after taking the ν integral by residues, Eq. (2.21), into the product of two infinite sums, see Eq. (2.22). After this property is established the further analysis follows the lines of Ref. [2] with minimal modifications. We note here that the factorization of sums was noticed by Ismagilov [43,44].

The proof of the second identity (2.4b) goes along the same lines so we will not go into details and discuss only the differences. First, we assume that the parameters z_k satisfy the conditions

$$\mathrm{Re} \sum_{k=1}^{2N+2} z_k < 1 \qquad \text{and} \qquad \mathrm{Re} \sum_{k=1}^{2N+2} \bar{z}_k < 1. \qquad (2.30)$$

After taking ν-integrals by residues and some rewriting one cast $I_N^{(2)}$ into the following form

$$
\begin{aligned}
I_N^{(2)} = \frac{2^N \varkappa_N}{N!}(-1)^N \sum_{\pi} \Bigg\{ \sum_{y_1,\dots,y_N=0}^{\infty} \Bigg(\prod_{j=1}^{N} (z_{\pi(j)} + y_j) \prod_{1\le i<j\le N} \Big(z_{\pi(i)} + y_i \pm (z_{\pi(j)} + y_j) \Big) \Bigg) \\
\times \prod_{k=1}^{N} \frac{(-1)^{y_k}}{y_k!} \Gamma(2z_{\pi(k)} + y_k) \prod_{\substack{j=1 \\ j\neq\pi(k)}}^{2N+2} \Gamma(z_j \pm (z_{\pi(k)} + y_k)) \Bigg\} \times \\
\Bigg\{ \sum_{\bar y_1,\dots,\bar y_N=0}^{\infty} \Bigg(\prod_{j=1}^{N} (\bar z_{\pi(j)} + \bar y_j) \prod_{1\le i<j\le N} \Big(\bar z_{\pi(i)} + \bar y_i \pm (\bar z_{\pi(j)} + \bar y_j) \Big) \Bigg) \\
\times \prod_{k=1}^{N} \frac{1}{\bar y_k!} \frac{1}{\Gamma(1 - 2\bar z_{\pi(k)} - \bar y_k)} \prod_{\substack{j=1 \\ j\neq\pi(k)}}^{2N+2} \frac{1}{\Gamma(1 - \bar z_j \pm (\bar z_{\pi(k)} + \bar y_k))} \Bigg\}, \qquad (2.31)
\end{aligned}
$$

where π is an injective map from $(1\dots N)$ to $(1\dots 2N+2)$. The sums over $\{y\}$, $\{\bar y\}$ can be evaluated with the help of the hypergeometric series summation formula, see Refs [2, 45] which reads

$$
\begin{aligned}
\sum_{y_1,\dots,y_n=-\infty}^{\infty} \left\{ \prod_{j=1}^{n} \frac{z_j + y_j}{z_j} \prod_{1\le i<j\le n} \frac{z_i + y_i \pm (z_j + y_j)}{z_i \pm z_j} \prod_{i=1}^{2n+2} \prod_{j=1}^{n} \frac{\Gamma(w_i \pm (z_j + y_j))}{\Gamma(w_i \pm z_j)} \right\} \\
= \frac{\Gamma\left(1 - \sum_{k=1}^{2n+2} w_i\right) \prod_{i=1}^{2n+2} \prod_{j=1}^{n} \Gamma(1 - w_i \pm z_j)}{\prod_{j=1}^{N} \Gamma(1 \pm 2z_j) \prod_{1\le i<j\le n} \Gamma(1 \pm z_i \pm z_j) \prod_{1\le i<j\le 2n+2} \Gamma(1 - w_i - w_j)}. \qquad (2.32)
\end{aligned}
$$

Collecting all factors we obtain

$$
I_N^{(2)} = \frac{\prod_{1\le i<j\le 2N+2} \boldsymbol{\Gamma}(z_i + z_j)}{\boldsymbol{\Gamma}\left(1 - \sum_k z_k\right)} \times T_N\,, \qquad (2.33)
$$

where

$$
T_N = \frac{(-1)^N}{2^N N!} \frac{\prod_{1\le i<j\le 2N+2} \sin\pi(z_i + z_j)}{\sin\pi\left(\sum_k z_k\right)} \sum_{\pi} \frac{\prod_{1\le i<j\le N} \sin^2\pi(z_{\pi(i)} \pm z_{\pi(j)}) \prod_{j=1}^{N} \sin\pi(2z_{\pi(j)})}{\prod_{j=1}^{N} \prod_{\substack{i=1 \\ i\neq\pi(j)}}^{2N+2} \sin\pi(z_i \pm z_{\pi(j)})}\,. \qquad (2.34)
$$

One can show that $T_N = 1$ by taking the limit $q \to 1$ of the identities (7.11) and (7.12) in ref. [2]. Finally, the conditions (2.30) can be relaxed to (2.11) by analytic continuation in ν_k.

3. Limiting cases

In this section we derive two more integrals which are replicas of the integrals, (3.2) and (5.4) in ref. [1]. We show that in the complex case these integrals are intrinsically related to the integrals (2.4). This property is not seen in the $\mathrm{SL}(2,\mathbb{R})$ setup.

We start our analysis with the integral (2.4a) and introduce the variables ζ and integer η by

$$
\zeta + \frac{\eta}{2} = \sum_{k=1}^{N+1} (z_k + w_k), \qquad \zeta - \frac{\eta}{2} = \sum_{k=1}^{N+1} (\bar z_k + \bar w_k)\,, \qquad (3.1)
$$

The r.h.s. of (2.4a) is a meromorphic function of ζ with poles located at the points $\zeta_p = |1+\eta/2|+p$, $p \in \mathbb{N}_+$. For $\eta = 0$ and $\zeta \sim 1$ the r.h.s. of (2.4a) takes the form

$$I_N^{(1)} = \frac{1}{1-\zeta} \prod_{k,j=1}^{N+1} \mathbf{\Gamma}(z_k + w_j) + \dots\,, \tag{3.2}$$

where z_k, w_k obey the constraint $\sum_{k=1}^{N+1}(z_k+w_k) = \sum_{k=1}^{N+1}(\bar{z}_k+\bar{w}_k) = 1$. At the same time, only the element $\mathcal{Q}_{NN}$ of the matrix (2.15) becomes singular at this point. The corresponding integral diverges at large ν and n as $\zeta \to 1$. Indeed, taking into account Eq. (2.10) one gets

$$\mathcal{Q}(u|z,w) = (-1)^{\sum_k [z_k]} u^{\zeta-N-1}(-\bar{u})^{\zeta-N-1}\left(1 + O\left(1/||u||\right)\right). \tag{3.3}$$

Thus for $\zeta \to 1$

$$\mathcal{Q}_{N,N}(z,w) \simeq (-1)^{\sum_k [z_k]} \frac{1}{\pi} \int_{r>\Lambda} dxdy \frac{1}{r^{4-2\zeta}} + \dots = \frac{1}{1-\zeta}(-1)^{\sum_k [z_k]} + \dots \tag{3.4}$$

and therefore

$$I_N^{(1)} \underset{\zeta\to 1}{=} \mathcal{Q}_{NN}(z,w) \times \det \widehat{\boldsymbol{\mathcal{Q}}}_{N-1}(z,w) \,+\, \text{finite terms}, \tag{3.5}$$

where $\widehat{\boldsymbol{\mathcal{Q}}}_{N-1}$ is the main $N-1$ minor of the matrix $\boldsymbol{\mathcal{Q}}_N$. Taking into account Eq. (3.4) and comparing with (3.2) one obtains the following identity

$$\det \widehat{\boldsymbol{\mathcal{Q}}}_{N-1}(z,w) = (-1)^{\sum_k [z_k]} \prod_{k,j=1}^{N+1} \mathbf{\Gamma}(z_k+w_j)\,, \tag{3.6}$$

which holds provided $\sum_{k=1}^{N+1}(z_k+w_k) = \sum_{k=1}^{N+1}(\bar{z}_k+\bar{w}_k) = 1$. The l.h.s. of (3.6) can be written in an integral form as

$$\frac{1}{(N-1)!}\left(\prod_{p=1}^{N-1}\int \mathcal{D}u_p\,(-1)^{n_p}\right) \frac{\prod_j^{N+1}\prod_{k=1}^{N-1}\mathbf{\Gamma}(z_j-u_k)\mathbf{\Gamma}(u_k+w_j)}{\prod_{k<j}\mathbf{\Gamma}(u_k-u_j)\mathbf{\Gamma}(u_j-u_k)} = (-1)^{\sum_m [z_m]} \prod_{k,j=1}^{N+1}\mathbf{\Gamma}(z_k+w_j). \tag{3.7}$$

This integral is an analog of the integral (3.2) in ref. [1]. Indeed, replacing the variable z_{N+1} by $z_{N+1} = 1 - \sum_{k=1}^{N} z_k - \sum_{m=1}^{N+1} w_m$ and using the relations (2.3) one can bring Eq. (3.7) into the form which is a replica of Eq. (3.2) in ref. [1]

$$\frac{1}{(N-1)!}\left(\prod_{p=1}^{N-1}\int \mathcal{D}u_p\right) \frac{\prod_{k=1}^{N-1}\prod_{m=1}^{N}\prod_{j=1}^{N+1}\mathbf{\Gamma}(z_m-u_k)\mathbf{\Gamma}(u_k+w_j)}{\prod_{m=1}^{N-1}\mathbf{\Gamma}(\gamma+u_m)\prod_{k<j}\mathbf{\Gamma}(u_k-u_j)\mathbf{\Gamma}(u_j-u_k)} = \frac{\prod_{k=1}^{N}\prod_{j=1}^{N+1}\mathbf{\Gamma}(z_k+w_j)}{\prod_{j=1}^{N+1}\mathbf{\Gamma}(\gamma-w_j)}, \tag{3.8}$$

where $\gamma = \sum_{k=1}^{N} z_k + \sum_{m=1}^{N+1} w_m$.

The analysis of the second integral, Eq. (2.4b), goes by exactly the same lines so we give only a brief account. Similar to (3.1) we define variables, η and ζ, by $\zeta + \eta/2 = \sum_{k=1}^{2N+2} z_k$. The l.h.s. and r.h.s. of Eq. (2.4b) have a pole at $\zeta = 1$ (for $\eta = 0$). Comparing the corresponding residues we get

$$\frac{1}{2^{N-1}(N-1)!}\prod_{k=1}^{N-1}\int_{\pm}\mathcal{D}u_k \frac{\prod_{k=1}^{N-1}\prod_{j=1}^{2N+2}\mathbf{\Gamma}(z_j\pm u_k)}{\prod_{k=1}^{N-1}\mathbf{\Gamma}(\pm 2u_k)\prod_{j<k}\mathbf{\Gamma}(\pm u_k \pm u_j)} = \pm \prod_{1\le j<k\le 2N+2}\mathbf{\Gamma}(z_j+z_k)\,, \tag{3.9}$$

where $\sum_{k=1}^{2N+2} z_k = \sum_{k=1}^{2N+2} \bar{z}_k = 1$ and the subscript $\pm$ at the integral sign indicates that the sum goes over either integer n (plus) or half-integer n (minus). Again, introducing the variable $\gamma = \sum_{k=1}^{2N+1} z_k$, one can rewrite this integral in a form identical to the integral (5.4) in ref. [1]

$$\frac{1}{2^{N-1}(N-1)!}\prod_{k=1}^{N-1}\int_{\pm}\mathcal{D}u_k \frac{\prod_{k=1}^{N-1}\prod_{j=1}^{2N+1}\boldsymbol{\Gamma}(z_j \pm u_k)}{\prod_{k=1}^{N-1}\boldsymbol{\Gamma}(\gamma \pm u_k)\boldsymbol{\Gamma}(\pm 2u_k)\prod_{j<k}\boldsymbol{\Gamma}(\pm u_k \pm u_j)} = \frac{\prod_{1\leq j<k\leq 2N+1}\boldsymbol{\Gamma}(z_j+z_k)}{\prod_{k=1}^{2N+3}\boldsymbol{\Gamma}(\gamma - z_k)}. \tag{3.10}$$

Thus one sees that in the SL$(2,\mathbb{C})$ setup the integrals (3.7), (3.8) and (3.9), (3.10) are intrinsically related to the integrals (2.4). For $N = 2$ the relation (3.10) was derived by Sarkissian and Spiridonov [46].

4. Star-triangle relation

In this section we show that the star-triangle relations with the Boltzmann weights given by a product of Γ-functions [31–33] follow in a rather straightforward way from the integrals (2.4). The star-triangle relation underlies exact solvability of various two dimensional lattice models, see refs. [47, 48] for a review. We recall here the star-triangle relation inherent in the noncompact SL$(2,\mathbb{C})$ spin chain magnets [23].

Let $s_\alpha(z) \equiv s_{\alpha,\bar{\alpha}}(z,\bar{z})$ be a function of the complex variables $z = x + iy$, $\bar{z} \equiv z^* = x - iy$,

$$\boldsymbol{s}_\alpha(z) = [z]^{-\alpha} \equiv z^{-\alpha}\bar{z}^{-\bar{\alpha}}, \qquad [\alpha] = \alpha - \bar{\alpha} \in \mathbb{Z}. \tag{4.1}$$

This is a single valued function on the complex plane and in physics literature it is usually called the propagator §. It satisfies two relations:

- the chain relation

$$\frac{1}{\pi}\int d^2z\, \boldsymbol{s}_{\alpha_1}(z_1 - z)\, \boldsymbol{s}_{\alpha_2}(z - z_2) = \frac{\boldsymbol{\Gamma}(1-\alpha_1)\boldsymbol{\Gamma}(1-\alpha_2)}{\boldsymbol{\Gamma}(2-\alpha_1-\alpha_2)}\, \boldsymbol{s}_{\alpha_1+\alpha_2-1}(z_1 - z_2) \tag{4.2}$$

- the star-triangle relation

$$\frac{1}{\pi}\int d^2z \prod_{k=1}^{3}\boldsymbol{s}_{\alpha_k}(z_k - z) = \left(\prod_{k=1}^{3}\boldsymbol{\Gamma}(1-\alpha_k)\right)\boldsymbol{s}_{1-\alpha_1}(z_2 - z_3)\boldsymbol{s}_{1-\alpha_2}(z_3 - z_1)\boldsymbol{s}_{1-\alpha_3}(z_1 - z_2), \tag{4.3}$$

  which holds provided $\alpha_1 + \alpha_2 + \alpha_3 = \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 = 2$.

In fact these two relations are equivalent: Eq. (4.3) is reduced to Eq. (4.2) in the limit $z_3 \to \infty$ and,vice versa, Eq. (4.3) can be derived from Eq. (4.2) using SL$(2,\mathbb{C})$ transformation. The relation (4.3) underlies integrability of the noncompact SL$(2,\mathbb{C})$ spin chain magnets.

In refs. [31–33] new solutions of the star-triangle relation have been derived. They arise as certain limits of the elliptic star-triangle relation [34–37]. Below we show that these star-triangle relations can be derived from the integrals (2.4).

As in the previous sections we first consider the relation associated with the integral (2.4a). To this end we define the propagator

$$\boldsymbol{S}_\alpha(u) = (-1)^{[\alpha/2+u]}\boldsymbol{\Gamma}\left(\frac{1-\alpha}{2}+u\right)\boldsymbol{\Gamma}\left(\frac{1-\alpha}{2}-u\right) = (-1)^{[\beta/2+u]}\frac{\Gamma\left(\frac{1-\alpha}{2}+u\right)}{\Gamma\left(\frac{1+\bar{\alpha}}{2}-\bar{u}\right)}\frac{\Gamma\left(\frac{1-\alpha}{2}-u\right)}{\Gamma\left(\frac{1+\bar{\alpha}}{2}+\bar{u}\right)}. \tag{4.4}$$

§ Let us stress here that $[z]^\alpha$ denotes the power function while $[\alpha]$ without any superscript stands for the "integer" part of α, $[\alpha] = \alpha - \bar{\alpha}$. We hope that this somewhat unfortunate notation will not lead to confusion.

The variables u and α have the form

$$u = n/2 + i\nu \qquad \bar{u} = -n/2 + i\nu, \qquad \alpha = m + \sigma \qquad \bar{\alpha} = -m + \sigma\,, \tag{4.5}$$

where $[u] = n$ and $\frac{1}{2}[\alpha] = m$ are either both integer or half-integer numbers, $\langle u\rangle = \nu$ is real and $\langle\alpha\rangle = \sigma$ is a complex number. Under these conditions the arguments of the $\boldsymbol{\Gamma}$-functions in (4.4) has the form (2.1). Slightly abusing the terminology we will call u (α) integer or half-integer depending on the character of n (m) and refer to this property as parity. Also, in order to avoid possible misunderstanding due to our agreement to indicate only "holomorphic" arguments of functions, $f(\alpha) \equiv f(\alpha, \bar{\alpha})$, we accept that, whenever $\bar{x}$ is not defined from a context, $x + \alpha \equiv (x + \alpha, x + \bar{\alpha})$, e.g. $\boldsymbol{\Gamma}(1/2 + z) \equiv \boldsymbol{\Gamma}(1/2 + z, 1/2 + \bar{z})$.

The propagator $\boldsymbol{S}_\alpha$ inherits many properties of $\boldsymbol{s}_\alpha$:

- for integer(half-integer) α the propagator is an even(odd) function of u

$$\boldsymbol{S}_\alpha(-u) = (-1)^{[\alpha]}\boldsymbol{S}_\alpha(u). \tag{4.6}$$

- for imaginary $\langle\alpha\rangle = \sigma$, $\boldsymbol{S}_\alpha(u)(\boldsymbol{S}_\alpha(u))^\dagger = 1$, i.e. the propagator reduces to a phase factor, while for $[\alpha] = 0$ and $\langle\alpha\rangle$ real, $\boldsymbol{S}_\alpha(u)$ is real and positive.

The chain relation for the propagator $\boldsymbol{S}$ follows from the first integral (2.4a) for $N = 1$. Namely, substituting

$$z_1 = (1-\alpha_1)/2 + z \quad w_1 = (1-\alpha_1)/2 - z, \quad z_2 = (1-\alpha_2)/2 + w, \quad w_2 = (1-\alpha_2)/2 - w \tag{4.7}$$

one easily derives

$$\sum_{n=-\infty}^{\infty}\int_{-i\infty}^{i\infty}\frac{d\nu}{2\pi i}\boldsymbol{S}_{\alpha_1}(z-u)\,\boldsymbol{S}_{\alpha_2}(u-w) = \frac{\boldsymbol{\Gamma}(1-\alpha_1)\boldsymbol{\Gamma}(1-\alpha_2)}{\boldsymbol{\Gamma}(2-\alpha_1-\alpha_2)}\,\boldsymbol{S}_{\alpha_1+\alpha_2-1}(z-w), \tag{4.8}$$

where $u = n/2 + \nu$ and sum goes over integers. The parity of the α_1 and z (α_2 and w) are always the same. The integral is well defined provided $\mathrm{Re}\langle\alpha_k\rangle < 1\,, k = 1, 2$ and $\mathrm{Re}\langle\alpha_1 + \alpha_2\rangle > 1$: the poles of the $\boldsymbol{\Gamma}$ functions in the integral (4.8) are separated by the integration contour if $\mathrm{Re}\langle\alpha_k\rangle < 1$ and the integral converges at large u if $\mathrm{Re}\langle\alpha_1 + \alpha_2\rangle > 1$.

The star-triangle relation for $\boldsymbol{S}_\alpha$ can be obtained from the integral identity (3.7) for $N = 2$. Let us make the following substitution

$$z_i \mapsto \frac{1-\alpha_i}{2} + z_i, \qquad \bar{z}_i \mapsto \frac{1-\bar{\alpha}_i}{2} + \bar{z}_i, \qquad w_i \mapsto \frac{1-\alpha_i}{2} - z_i, \qquad \bar{w}_i \mapsto \frac{1-\bar{\alpha}_i}{2} - \bar{z}_i \tag{4.9}$$

in this equation. The condition $\sum_k(z_k + w_k) = 1$ gives rise to the following restriction on the indices: $\sum_k \alpha_k = \sum_k \bar{\alpha}_k = 2$ and after some algebra one derives

$$\sum_{n=-\infty}^{\infty}\int_{-i\infty}^{i\infty}\frac{d\nu}{2\pi i}\prod_{k=1}^{3}\boldsymbol{S}_{\alpha_k}(z_k-u) = \left(\prod_{k=1}^{3}\boldsymbol{\Gamma}(1-\alpha_k)\right)\boldsymbol{S}_{1-\alpha_1}(z_2-z_3)\boldsymbol{S}_{1-\alpha_2}(z_3-z_1)\boldsymbol{S}_{1-\alpha_3}(z_1-z_2), \tag{4.10}$$

where, again, α_k and z_k are integer or half-integer simultaneously. For the special choice of the parameters, $\alpha_k = \bar{\alpha}_k$, this relation coincides with the star-triangle relation derived in [31].

Proceeding with the second integral (2.4b) we define the propagator as the product of four $\boldsymbol{\Gamma}$ functions

$$\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) = \boldsymbol{\Gamma}\left(\frac{1-\alpha}{2} \pm z_1 \pm z_2\right) = \frac{\Gamma\left(\frac{1-\alpha}{2} \pm z_1 \pm z_2\right)}{\Gamma\left(\frac{1+\bar{\alpha}}{2} \pm \bar{z}_1 \pm \bar{z}_2\right)}\,. \tag{4.11}$$

Here, again the requirement for the arguments of $\mathbf{\Gamma}$ functions, $\frac{1-\alpha}{2} \pm z_1 \pm z_2$, to be integers imposes obvious restrictions on the relative parity of all variables. The propagator $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2)$ is an even function of z_1, z_2 and invariant under $z_1 \leftrightarrow z_2$ permutation. Therefore, for each variable $z_k = n_k/2+\nu_k$ one can restrict n_k to positive (negative) values. Note that unlike the previous case the propagator $\boldsymbol{\mathcal{D}}_\alpha$ is not shift invariant. Also, up to a phase factor depending on the parity of arguments, $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) \sim \boldsymbol{S}_\alpha(z_1 - z_2)\boldsymbol{S}_\alpha(z_1 + z_2)$.

The chain relation for the propagator $\boldsymbol{\mathcal{D}}_\alpha$ follows from the identity (2.4b). Indeed, after the substitution $z_{1(2)} = (1-\alpha_1)/2 \pm z$, $z_{3(4)} = (1-\alpha_2)/2 \pm w$ the integral (2.4b) takes the form

$$2\int_\pm \mathcal{D}u\, ||u||^2\, \boldsymbol{\mathcal{D}}_{\alpha_1}(z,u)\boldsymbol{\mathcal{D}}_{\alpha_2}(u,w) = \pm\frac{\mathbf{\Gamma}(1-\alpha_1)\mathbf{\Gamma}(1-\alpha_2)}{\mathbf{\Gamma}(2-\alpha_1-\alpha_2)}\boldsymbol{\mathcal{D}}_{\alpha_1+\alpha_2-1}(z,w)\,. \tag{4.12}$$

Here the subscripts $\pm$ indicate that the sum goes over all integers ("$+$") or half-integers ("$-$") and we also recall that for $u = n/2 + i\nu$, $||u||^2 = \nu^2 + n^2/4$.

Next, substituting $z_{2i-1} = (1-\alpha_i)/2 + z_i$, $z_{2i} = (1-\alpha_i)/2 - z_i$, for $i = 1, 2, 3$ in Eq. (3.10) for $N = 2$ one gets

$$2\int_\pm \mathcal{D}u\, ||u||^2 \prod_{k=1}^{3} \boldsymbol{\mathcal{D}}_{\alpha_k}(z_k, u) = \left(\prod_{k=1}^{3}\mathbf{\Gamma}(1-\alpha_k)\right)\boldsymbol{\mathcal{D}}_{1-\alpha_1}(z_2,z_3)\boldsymbol{\mathcal{D}}_{1-\alpha_2}(z_1,z_3)\boldsymbol{\mathcal{D}}_{1-\alpha_3}(z_1,z_2), \tag{4.13}$$

where $\sum_k \alpha_k = \sum_k \bar{\alpha}_k = 2$. As was mentioned earlier the parity of all variables have to be coordinated so that (4.13) encompasses four different identities

- I All α_k are integer / have positive parity
  - (a) z_k, u – integer / positive parity
  - (b) z_k, u – half-integer / negative parity
- II α_1 and α_2, α_3 have positive and negative parity, respectively
  - (a) u and z_1 have positive and z_2, z_3 negative parity
  - (b) u and z_1 have negative and z_2, z_3 positive parity

The variant I a corresponds to the star-triangle relation obtained by A.P. Kels in [32,33]. An extension of the star-triangle relation of [32] following from relation (3.10) with N=2 was also considered by G.A.Sarkissian and V.P.Spiridonov [46].

In the first two cases if $0 < \alpha = \bar{\alpha} < 1$ the functions $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2)$ are real: $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) > 0$ for integer z_1, z_2 (the case I a) and $-\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) > 0$ for half-integer z_1, z_2 (the case I b). In both cases these functions can be interpreted as the Boltzmann weights of lattice integrable models, for more details see ref. [32, 33].

5. Quasi-classical limit

Let us consider the identities (4.8), (4.10) when the external variables become large. We replace the variables $z_k(w_k)$ by $Lz_k = L(x_k + iy_k)$ in (4.8), (4.10) and take the limit $L \to \infty$. The variables x_k, y_k can be considered as continuous in this limit, so that $z_k \in \mathbb{C}$, $\bar{z}_k = z_k^*$. It is not hard to check that the leading contribution to the integrals (4.8), (4.10) comes from the region where $u_k \sim L$, so that we replace $u_k \mapsto Lu_k$ as well. Moreover in this limit $\boldsymbol{S}_\alpha$ turns into $\boldsymbol{s}_\alpha$:

$$\boldsymbol{S}_\alpha(Lz) \underset{L\to\infty}{\mapsto} L^{-2\langle\alpha\rangle}\, z^{-\alpha}\bar{z}^{-\bar{\alpha}} = L^{-2\langle\alpha\rangle}\, \boldsymbol{s}_\alpha(z) \tag{5.1}$$

and $\int \mathcal{D}u$ can be replaced by the integral over complex plain as follows

$$\sum_{n=-\infty}^{\infty}\int_{-i\infty}^{\infty}\frac{d\nu}{2\pi i}\to L^2\,\frac{1}{\pi}\int d^2u,, \tag{5.2}$$

where $u=u_x+iu_y$ and $d^2z=du_xdu_y$. Taking into account (5.1) and (5.2) it is easy to check that Eqs. (4.8), (4.10) turn in this limit into the chain-relation (4.2) and the star-triangle relation (4.3).

In what follows we study the quasi-classical limits the integrals (2.4a) and (3.7) for general N. First of all we rewrite these identities in term of the propagator $\boldsymbol{S}_\alpha$. To this end we make the substitution

$$z_k\mapsto\frac{1-\alpha_k}{2}+z_k,\qquad w_k\mapsto\frac{1-\alpha_k}{2}-z_k,\qquad k=1,\dots,N+1. \tag{5.3}$$

in Eq. (2.4a) and rewrite it in the following form

$$\frac{1}{N!}\left(\prod_{k=1}^{N}\int\mathcal{D}u_k\right)\prod_{1\le i<j\le N}||u_i-u_j||^2\prod_{k=1}^{N}\prod_{m=1}^{N+1}\boldsymbol{S}_{\alpha_m}(z_m-u_k)=$$
$$=(-1)^{\sum_{m=1}^{N}m[\alpha_{m+1}]}\frac{\prod_{k=1}^{N+1}\boldsymbol{\Gamma}(1-\alpha_k)}{\boldsymbol{\Gamma}\left(N+1-\sum_{k=1}^{N+1}\alpha_k\right)}\prod_{1\le i<k\le N+1}\boldsymbol{S}_{\alpha_i+\alpha_k-1}(z_i-z_k)\,. \tag{5.4}$$

Similarly, the identity (3.7) can be represented as follows

$$\frac{1}{(N-1)!}\left(\prod_{k=1}^{N-1}\int\mathcal{D}u_k\right)\prod_{1\le i<j\le N-1}||u_i-u_j||^2\prod_{k=1}^{N}\prod_{m=1}^{N+1}\boldsymbol{S}_{\alpha_m}(z_m-u_k)=$$
$$=(-1)^{\sum_{m=1}^{N}m[\alpha_{m+1}]}\left(\prod_{k=1}^{N+1}\boldsymbol{\Gamma}(1-\alpha_k)\right)\prod_{1\le i<k\le N+1}\boldsymbol{S}_{\alpha_i+\alpha_k-1}(z_i-z_k) \tag{5.5}$$

and where $\sum_k\alpha_k=\sum_k\bar\alpha_k=N$.

In the quasi-classical limit these identities are reduced to the following two-dimensional integrals with power functions

$$\frac{1}{N!}\left(\prod_{k=1}^{N}\int d^2u_k\right)\prod_{1\le i<j\le N}|u_i-u_j|^2\prod_{k=1}^{N}\prod_{m=1}^{N+1}[z_m-u_k]^{-\alpha_m}=$$
$$=\pi^N(-1)^{\sum_{m=1}^{N}m[\alpha_{m+1}]}\frac{\prod_{k=1}^{N+1}\boldsymbol{\Gamma}(1-\alpha_k)}{\boldsymbol{\Gamma}\left(N+1-\sum_{k=1}^{N+1}\alpha_k\right)}\prod_{1\le i<k\le N+1}[z_i-z_k]^{1-\alpha_i-\alpha_k} \tag{5.6}$$

and

$$\frac{1}{(N-1)!}\left(\prod_{k=1}^{N-1}\int d^2u_k\right)\prod_{1\le i<j\le N-1}|u_i-u_j|^2\prod_{k=1}^{N}\prod_{m=1}^{N+1}[z_m-u_k]^{-\alpha_m}=$$
$$=\pi^{N-1}(-1)^{\sum_{m=1}^{N}m[\alpha_{m+1}]}\left(\prod_{k=1}^{N+1}\boldsymbol{\Gamma}(1-\alpha_k)\right)\prod_{1\le i<k\le N+1}[z_i-z_k]^{1-\alpha_1-\alpha_k}. \tag{5.7}$$

It is easy to check that in the quasi-classical limit the propagator $\boldsymbol{\mathcal{D}}_\alpha(z,w)$ turns into $[z^2-w^2]^{-\alpha}$. Therefore Eqs. (2.4b) and Eq. (3.9) do not produce new identities in this limit but reduce to the integrals (5.6) and (5.7) after the appropriate change of variables.

One can take a different point of view on the integrals (2.4a) and (2.4b) and consider them as the "quantized" version of the integrals (5.6). For $\alpha_k=\bar\alpha_k$ the integral (5.6) is a special case of a duality relation [38] for the Dotsenko–Fateev (DF) integrals [39], see next section, and one can hope that there exists a "quantized" version of the duality relation in a general situation.

6. Dotsenko-Fateev integrals

In this section we give an elementary proof of the following integral relation

$$\frac{1}{\pi^n n!}\int\prod_{k=1}^n d^2y_k\prod_{i<k}^n[y_i-y_k]\prod_{i=1}^n\prod_{j=1}^{n+m+1}[y_i-z_j]^{-\alpha_j}=(-1)^{\sum_{k=1}^{n+m}k[\alpha_{k+1}]}\frac{\prod_{j=1}^{n+m+1}\boldsymbol{\Gamma}(1-\alpha_j)}{\boldsymbol{\Gamma}(1+n-\sum_{j=1}^{n+m+1}\alpha_j)}$$
$$\times\prod_{i<j}^{n+m+1}[z_j-z_i]^{1-\alpha_i-\alpha_j}\,\frac{1}{\pi^m m!}\int\prod_{k=1}^m d^2u_k\prod_{i<k}^m[u_i-u_k]\prod_{i=1}^m\prod_{j=1}^{n+m+1}[u_i-z_j]^{-1+\alpha_j},\quad(6.1)$$

which is essentially the duality relation [38, 49] for the DF integrals [39] provided the parameters satisfy the constraint $\alpha_k=\bar\alpha_k$, see also [50]. For $m=0$ this identity coincides with (5.6).

To evaluate the l.h.s. of Eq. (6.1) we go over to the symmetric variables $x_k=x_k(y_1,\ldots,y_n)$ defined as follows

$$\prod_{i=1}^n(y_i+t)=x_1+x_2t+x_3t^2+\cdots+x_nt^{n-1}+t^n.\qquad(6.2)$$

It is evident that every point in x-space has $n!$ preimages in y-space and it is not hard to calculate the corresponding Jacobian $\left|\frac{\partial x_k}{\partial y_i}\right|=\prod_{i<j}^n[y_i-y_k]$. Introducing the notation $t_j=-z_j$ one gets for the l.h.s. of Eq. (6.1)

$$\text{LHS}=\pi^{-n}\int\prod_{k=1}^n d^2x_k\prod_{j=1}^{n+m+1}[x_1+x_2t_j+x_3t_j^2+\cdots+x_nt_j^{n-1}+t_j^n]^{-\alpha_j}.\qquad(6.3)$$

Then using the momentum space representation for the propagators

$$[z]^{-\alpha}=\frac{1}{\pi}i^{\alpha-\bar\alpha}\boldsymbol{\Gamma}(1-\alpha)\int d^2k\,e^{i(kz+\bar z\bar k)}\,[k]^{\alpha-1}\qquad(6.4)$$

in Eq. (6.3) and carrying out x_k integrals one gets

$$\text{LHS}=\frac{i^{\sum_j(\alpha_j-\bar\alpha_j)}}{\pi^{m+1}}\prod_{k=1}^{n+m+1}\boldsymbol{\Gamma}(1-\alpha_k)\prod_{i=1}^{n+m+1}\int d^2k_i\,[k_i]^{\alpha_k-1}\prod_{k=1}^n\delta^{(2)}\left(\sum_j k_jt_j^{k-1}\right)\,e^{i\sum_j(k_jt_j^n+\bar k_j\bar t_j^n)}$$
$$=\frac{\prod_{k=1}^{n+m+1}\boldsymbol{\Gamma}(1-\alpha_k)}{\pi^m\boldsymbol{\Gamma}\left(n+1-\sum_j\alpha_j\right)}\prod_{i=1}^{n+m+1}\int\frac{d^2k_i}{[k_i]^{1-\alpha_k}}\prod_{k=1}^n\delta^{(2)}\left(\sum_j k_jt_j^{k-1}\right)\,\delta^{(2)}\left(\sum_j k_jt_j^n-1\right).\quad(6.5)$$

In order to get the last δ-function one represents $e^{i\sum_j(k_jt_j^n+\bar k_j\bar t_j^n)}$ as $\int d\lambda e^{i(\lambda+\bar\lambda)}\delta^{(2)}(\lambda-\sum_j k_jt_j^n)$ and rescales $k_j\to\lambda k_j$. The delta functions cut out a m dimensional surface in $n+m+1$ dimensional space which is defined by the linear equations

$$\sum_{j=1}^{n+m+1} k_j t_j^{k-1} = 0, \qquad k=1,\ldots,n \qquad \text{and} \qquad \sum_{j=1}^{n+m+1} k_j t_j^{n} = 1\,. \tag{6.6}$$

The solutions can be written in the form

$$k_j(x_1,\ldots,x_m) = \lambda_j(x_1 + x_2 t_j + \ldots x_m t_j^{m-1} + t_j^m)\,, \tag{6.7}$$

where the vector λ_j is the solution of the linear equations,

$$\sum_{j=1}^{n+m+1} \lambda_j t_j^{k-1} = 0, \qquad k=1,\ldots,m+n \qquad \text{and} \qquad \sum_{j=1}^{n+m+1} \lambda_j t_j^{n+m} = 1\,, \tag{6.8}$$

or $\Delta_{ij}\lambda_j = \delta_{i,n+m+1}$, where $\Delta_{ij} = t_j^{i-1}$, $i,j=1,\ldots,n+m+1$ is the Vandermonde matrix

$$\Delta = \begin{pmatrix} 1 & \ldots & 1 \\ t_1 & \ldots & t_{n+m+1} \\ \vdots & \ddots & \vdots \\ t_1^{n+m} & \ldots & t_{n+m+1}^{n+m} \end{pmatrix}. \tag{6.9}$$

The solution has the form

$$\lambda_j = (\Delta^{-1})_{j,n+m+1} = \Delta_j / \det\Delta = \prod_{k\neq j} \frac{1}{t_k - t_j}, \tag{6.10}$$

where Δ_k are the cofactors of the last row of Δ, $\det\Delta = \sum_{k=1}^{n+m+1} t_k^{n+m}\Delta_k$. Making change of variables

$$\begin{aligned} k_j &= \lambda_j(x_1 + x_2 t_j + \ldots x_m t_j^{m-1} + t_j^m) + s_j & j&=1,\ldots,n+1 \\ k_j &= \lambda_j(x_1 + x_2 t_j + \ldots x_m t_j^{m-1} + t_j^m), & j&=n+2,\ldots,n+m+1 \end{aligned} \tag{6.11}$$

and taking into account that

$$\prod_{k=1}^{n} \delta^2\left(\sum_{j=1}^{n+m+1} k_j t_j^{k-1}\right) \delta^2\left(\sum_{j=1}^{n+m+1} k_j t_j^{n} - 1\right) = \prod_{k=1}^{n+1} \delta^2\left(\sum_{j=1}^{n+1} s_j t_j^{k-1}\right) = \frac{\prod_{j=1}^{n+1}\delta^2(s_j)}{\prod_{1\le i<j\le n+1}[t_i - t_j]}$$

and

$$\prod_{j=1}^{n+m+1} d^2 k_j = \left(\prod_{n+2\le i<j\le n+m+1} [t_i - t_j] \prod_{k=n+2}^{n+m+1}[\lambda_k]\right) \prod_{j=1}^{m} d^2 x_j \prod_{j=1}^{n+1} d^2 s_j \tag{6.12}$$

one gets after some algebra

$$\begin{aligned} \text{LHS} = \frac{\prod_{k=1}^{n+m+1}\boldsymbol{\Gamma}(1-\alpha_k)}{\boldsymbol{\Gamma}\left(n+1-\sum_j \alpha_j\right)} &\left(\prod_{j<k}(-1)^{\alpha_k - \bar\alpha_k}[t_j - t_k]^{1-\alpha_k-\alpha_j}\right) \\ &\times \frac{1}{\pi^m} \prod_{j=1}^{n+m+1}\prod_{k=1}^{m} \int d^2x_k [x_1 + x_2 t_j + \ldots x_m t_j^{m-1} + t_j^m]^{\alpha_j - 1}. \end{aligned} \tag{6.13}$$

Finally going over to the variables u_k, $\prod_{i=1}^m (u_i+t) = x_1+x_2t+x_3t^2+\cdots+x_nt^{m-1}+t^m$ and changing $t_j \to -z_j$ one gets the r.h.s. of Eq. (6.1). We have learned from the discussions with A. Litvinov that similar proof of the duality relation (6.1) is presented in his lectures on conformal field theory [51].

Finally, we make a conjecture that the duality relation (6.1) admits a generalization to the "quantized" cases. Namely, these relations take the form

$$\frac{1}{n!}\prod_{k=1}^{n}\int \mathcal{D}u_k \prod_{i<j}||u_i-u_j||^2 \prod_{i=1}^{n}\prod_{j=1}^{n+m+1}(-1)^{[u_i]}\mathbf{\Gamma}(z_j-u_i)\mathbf{\Gamma}(u_i+w_j) = \frac{\prod_{i,j=1}^{n+m+1}\mathbf{\Gamma}(z_i+w_j)}{\mathbf{\Gamma}(\sum_j(z_j+w_j)-m)}$$
$$\times(-1)^{m\sum_j[z_j-w_j]}\frac{1}{m!}\prod_{k=1}^{m}\int \mathcal{D}u_k \prod_{i<j}||u_i-u_j||^2 \prod_{i=1}^{m}\prod_{j=1}^{n+m+1}(-1)^{[u_i]}\mathbf{\Gamma}(z'_j-u_i)\mathbf{\Gamma}(w'_j+u_i), \quad (6.14)$$

where $z'_j = 1/2 - w_j$, $w'_j = 1/2 - z_j$ and

$$\frac{2^n}{n!}\prod_{k=1}^{n}\int_{\pm} \mathcal{D}u_k||u_k||^2 \prod_{1\le i<j\le n}||u_i\pm u_j||^2 \prod_{i=1}^{n}\prod_{j=1}^{2(n+m+1)}\mathbf{\Gamma}(z_j\pm u_i) = \varkappa_{n+m}\frac{\prod_{i<j}^{2(n+m+1)}\mathbf{\Gamma}(z_i+z_j)}{\mathbf{\Gamma}(\sum_j z_j - m)}$$
$$\times\frac{2^m}{m!}\prod_{k=1}^{m}\int_{\pm} \mathcal{D}u_k||u_k||^2 \prod_{1\le i<j\le m}||u_i\pm u_j||^2 \prod_{i=1}^{m}\prod_{j=1}^{2(n+m+1)}\mathbf{\Gamma}(1/2-z_j\pm u_i), \quad (6.15)$$

where $\varkappa_k = 1, (-1)^{k(k+1)/2}$ for the integer and half-integer cases,respectively. For $m = 0$ these integrals are equivalent to the integrals (2.4) and in the quasi-classical limit both of them reproduce the duality relation (6.1). For $n = m = 1$ the relations (6.14) and (6.15) follow from the star-triangle relations (4.10) and (4.13). For few first n and m the integrals (6.14), (6.15) go through numerical tests. Closing this section we note that quite similar duality relations were observed recently in the so-called conformal Fishnet model [52].

7. Summary

In refs. [26, 27] a generalization of Gustafson integrals to the complex case have been obtained. The derivation of these integrals rely on the completeness of the SoV representation for the $SL(2,\mathbb{C})$ magnets, that is not yet proven. In this work we presented a direct calculation of two $\mathbf{\Gamma}$ function integrals. We expect that these integral identities will be helpful in proving the completeness of the SoV representation for the $SL(2,\mathbb{C})$ spin chains.

The complex integrals are, up to appropriate modification of the Γ function and integration measure, exact copies of the integrals obtained by R.A. Gustafson [2]. However, the analytic properties of these integrals are different. We have shown that several, apparently distinct in the $SL(2,\mathbb{R})$ context integrals are intrinsically related to each other in the $SL(2,\mathbb{C})$ formulation.

We have also shown that the complex $\mathbf{\Gamma}$ integrals for the lowest N underlie the star-triangle identities derived in refs. [31–33] and in the quasi-classical limit are reduced to the special ($m = 0$) case of the duality relation for the DF integrals [38, 39, 49]. We also conjecture that these duality relations, after appropriate modifications, hold for the integrals with complex $\mathbf{\Gamma}$ functions.

Acknowledgments

We are grateful to A. V. Litvinov and V. P. Spiridonov for fruitful discussions. This study was supported by the Russian Science Foundation project No 19-11-00131 and Deutsche Forschungsgemeinschaft (A. M.), grant MO 1801/1-3.

References

[1] R. A. Gustafson, "Some q-beta and Mellin-Barnes integrals with many parameters associated to the classical group," *SIAM. J. Math. Anal.*, vol. 23, no. 3, p. 525, 1992.

[2] R. A. Gustafson, "Some q-beta and Mellin-Barnes integrals on compact Lie groups and Lie algebras," *Trans. Amer. Math. Soc.*, vol. 341, no. 1, pp. 69–119, 1994.

[3] R. A. Gustafson, "Some q-beta integrals on SU(n) and Sp(n) that generalize the Askey-Wilson and Nasrallach-Rahman integrals," *SIAM J. Math Anal.*, vol. 25, no. 2, p. 441, 1994.

[4] V. P. Spiridonov, "Theta hypergeometric integrals," *Algebra i Analiz*, vol. 15, no. 6, pp. 161–215, 2003.

[5] V. P. Spiridonov and S. O. Warnaar, "Inversions of integral operators and elliptic beta integrals on root systems," *Adv. Math.*, vol. 207, p. 91, 2006.

[6] V. P. Spiridonov, "Short proofs of the elliptic beta integrals," *Ramanujan J.*, vol. 13, no. 1-3, pp. 265–283, 2007.

[7] J. V. Stokman, "On bc type basic hypergeometric orthogonal polynomials," *Trans. Amer. Math. Soc.*, vol. 352, pp. 1527–1579, 2000.

[8] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, vol. 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[9] P. J. Forrester and S. O. Warnaar, "The importance of the selberg integral," *Bull. Amer.Math. Soc. (N.S.)*, vol. 45, pp. 489–534, 2008.

[10] V. P. Spiridonov and G. S. Vartanov, "Elliptic Hypergeometry of Supersymmetric Dualities," *Commun. Math. Phys.*, vol. 304, pp. 797–874, 2011, 0910.5944.

[11] S. E. Derkachov and A. N. Manashov, "Spin chains and Gustafsons integrals," *J. Phys.*, vol. A50, no. 29, p. 294006, 2017, 1611.09593.

[12] A. V. Belitsky, V. M. Braun, A. S. Gorsky, and G. P. Korchemsky, "Integrability in QCD and beyond," *Int. J. Mod. Phys.*, vol. A19, pp. 4715–4788, 2004.

[13] N. Beisert *et al.*, "Review of AdS/CFT Integrability: An Overview," *Lett. Math. Phys.*, vol. 99, pp. 3–32, 2012, 1012.3982.

[14] L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, "The Quantum Inverse Problem Method. 1," *Theor. Math. Phys.*, vol. 40, pp. 688–706, 1980. [Teor. Mat. Fiz.40,194(1979)].

[15] P. P. Kulish and E. K. Sklyanin, "Quantum spectral transform method. Recent developments," *Lect. Notes Phys.*, vol. 151, pp. 61–119, 1982.

[16] P. P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin, "Yang-Baxter Equation and Representation Theory. 1.," *Lett. Math. Phys.*, vol. 5, pp. 393–403, 1981.

[17] E. K. Sklyanin, "Boundary Conditions for Integrable Quantum Systems," *J. Phys.*, vol. A21, pp. 2375–289, 1988.

[18] E. K. Sklyanin, "Quantum inverse scattering method. Selected topics," 1991, hep-th/9211111.

[19] L. D. Faddeev, "How the algebraic Bethe ansatz works for integrable models," in *Symétries quantiques (Les Houches, 1995)*, pp. 149–219, North-Holland, Amsterdam, 1998.

[20] J. M. Maillet and G. Niccoli, "On quantum separation of variables," *J. Math. Phys.*, vol. 59, no. 9, p. 091417, 2018, 1807.11572.

[21] N. Kitanine, J. M. Maillet, G. Niccoli, and V. Terras, "The open XXX spin chain in the SoV framework: scalar product of separate states," *J. Phys.*, vol. A50, no. 22, p. 224001, 2017, 1606.06917.

[22] N. Kitanine, J. M. Maillet, G. Niccoli, and V. Terras, "The open XXZ spin chain in the SoV framework: scalar product of separate states," *J. Phys.*, vol. A51, no. 48, p. 485201, 2018, 1807.05197.

[23] S. E. Derkachov, G. P. Korchemsky, and A. N. Manashov, "Noncompact Heisenberg spin magnets from high-energy QCD: 1. Baxter Q operator and separation of variables," *Nucl. Phys.*, vol. B617, pp. 375–440, 2001, hep-th/0107193.

[24] A. V. Belitsky, S. E. Derkachov, and A. N. Manashov, "Quantum mechanics of null polygonal Wilson loops," *Nucl. Phys.*, vol. B882, pp. 303–351, 2014, 1401.7307.

[25] S. E. Derkachov and A. N. Manashov, "Iterative construction of eigenfunctions of the monodromy matrix for $SL(2,\mathbb{C})$ magnet," *J. Phys.*, vol. A47, no. 30, p. 305204, 2014, 1401.7477.

[26] S. E. Derkachov, A. N. Manashov, and P. A. Valinevich, "Gustafson integrals for $\mathrm{SL}(2,\mathbb{C})$ spin magnet," *J. Phys.*, vol. A50, no. 29, p. 294007, 2017, 1612.00727.

[27] S. E. Derkachov, A. N. Manashov, and P. A. Valinevich, "$\mathrm{SL}(2,\mathbb{C})$ Gustafson Integrals," *SIGMA*, vol. 14, p. 030, 2018, 1711.07822.

[28] K. K. Kozlowski, "Unitarity of the SoV Transform for the Toda Chain," *Commun. Math. Phys.*, vol. 334, no. 1, pp. 223–273, 2015, 1306.4967.

[29] S. E. Derkachov, K. K. Kozlowski, and A. N. Manashov, "On the separation of variables for the modular XXZ magnet and the lattice Sinh-Gordon models," 2018, 1806.04487.

[30] G. Schrader and A. Shapiro, "On b-Whittaker functions," 2018, 1806.00747.

[31] V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, "Exact solution of the Faddeev-Volkov model," *Phys. Lett.*, vol. A372, pp. 1547–1550, 2008, 0706.3077.

[32] A. P. Kels, "A new solution of the star-triangle relation," *J. Phys.*, vol. A47, p. 055203, 2014, 1302.3025.

[33] A. P. Kels, "New solutions of the startriangle relation with discrete and continuous spin variables," *J. Phys.*, vol. A48, no. 43, p. 435201, 2015, 1504.07074.

[34] V. P. Spiridonov, "On the elliptic beta function," *Uspekhi Mat. Nauk*, vol. 56, no. 1(337), pp. 181–182, 2001.

[35] V. V. Bazhanov and S. M. Sergeev, "A Master solution of the quantum Yang-Baxter equation and classical discrete integrable equations," *Adv. Theor. Math. Phys.*, vol. 16, no. 1, pp. 65–95, 2012, 1006.0651.

[36] V. P. Spiridonov, "Elliptic beta integrals and solvable models of statistical mechanics," *Contemp. Math.*, vol. 563, pp. 181–211, 2012, 1011.3798.

[37] S. E. Derkachov and V. P. Spiridonov, "Yang-Baxter equation, parameter permutations, and the elliptic beta integral," *Russ. Math. Surveys*, vol. 68, pp. 1027–1072, 2013, 1205.3520.

[38] P. Baseilhac and V. A. Fateev, "Expectation values of local fields for a two-parameter family of integrable models and related perturbed conformal field theories," *Nucl. Phys.*, vol. B532, pp. 567–587, 1998, hep-th/9906010.

[39] V. S. Dotsenko and V. A. Fateev, "Four Point Correlation Functions and the Operator Algebra in the Two-Dimensional Conformal Invariant Theories with the Central Charge c ¡ 1," *Nucl. Phys.*, vol. B251, pp. 691–734, 1985.

[40] I. M. Gel'fand, M. I. Graev, and V. S. Retakh, "Hypergeometric functions over an arbitrary field," *Russian Math. Surveys*, vol. 59, no. 5, p. 831, 2004.

[41] E. M. Rains and V. P. Spiridonov, "Determinants of elliptic hypergeometric integrals," *Funct. Analysis and its Appl*, vol. 43, no. 4, pp. 297–311, 2009, 0712.4253.

[42] S. Milne, "A q-analog of the gauss summation theorem for hypergeometric series in u(n)," *Advances in Mathematics*, vol. 72, no. 1, pp. 59 – 131, 1988.

[43] R. S. Ismagilov, "On Racah operators," *Funktsional. Anal. i Prilozhen.*, vol. 40, no. 3, pp. 69–72, 2006.

[44] R. S. Ismagilov, "On Racah operators for representations of the principal series of the group $SL(2,\mathbb{C})$," *Mat. Sb.*, vol. 198, no. 3, pp. 77–90, 2007.

[45] R. A. Gustafson, "The Macdonald identities for affine root systems of classical type and hypergeometric series very-well-poised on semisimple Lie algebras," in *Ramanujan International Symposium on Analysis (Pune, 1987)*, pp. 185–224, Macmillan of India, New Delhi, 1989.

[46] G.A. Sarkissian and V.P. Spiridonov, in preparation; V. P. Spiridonov, "The endless beta integrals," *Elliptic integrable systems, special functions and quantum field theory 16-20 June 2019.* https://agenda.albanova.se/conferenceTimeTable.py?confId=6142.

[47] R. J. Baxter, *Exactly solved models in statistical mechanics.* 1982.

[48] V. V. Bazhanov, A. P. Kels, and S. M. Sergeev, "Quasi-classical expansion of the star-triangle relation and integrable systems on quad-graphs," *J. Phys.*, vol. A49, no. 46, p. 464001, 2016, 1602.07076.

[49] V. A. Fateev and A. V. Litvinov, "Multipoint correlation functions in Liouville field theory and minimal Liouville gravity," *Theor. Math. Phys.*, vol. 154, pp. 454–472, 2008, 0707.1664.

[50] S. Pasquetti and M. Sacchi, "3d dualities from 2d free field correlators: recombination and rank stabilization," 2019, 1905.05807.

[51] A. V. Litvinov, "Lectures on conformal field theory, lecture 7," http://www.mathnet.ru/rus/person27650.

[52] S. Derkachov, V. Kazakov, and E. Olivucci, "Basso-Dixon Correlators in Two-Dimensional Fishnet CFT," *JHEP*, vol. 04, p. 032, 2019, 1811.10623.