DESY 19-122

A Simple Gradient Flow Equation for the Bounce Solution

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(Dated: July 5, 2019)

Motivated by the recent work of Chigusa, Moroi, and Shoji [1], we propose a new simple gradient flow equation to derive the bounce solution which contributes to the decay of the false vacuum. Our discussion utilizes the discussion of Coleman, Glaser, and Martin [2] and we solve a minimization problem of the kinetic energy while fixing the potential energy. The bounce solution is derived as a scale-transformed of the solution of this problem.

I. INTRODUCTION

The decay of the false vacua is one of important topic in particle physics and cosmology. The decay rate of the false vacua can be calculated from "the imaginary part" of the Euclidean path integral [3]. In the path integral formalism, we can see that the main contribution comes from the bounce solution ϕ_B which is a non-trivial solution of the equation of motion with the least action. Thus, the bounce solution has a crucial role for the decay of the false vacua. To calculate the bounce solution, we have to solve the equation of motion with the boundary condition at infinity. In general, it is not easy to calculate the bounce solution, in particular, for models with multi scalar fields.

Several algorithms to calculate the bounce action has been discussed so far, *e.g.*, gradient flow with modifications [4–6], modified actions which have the bounce solution as a local minimum [7–10], changing gradually a coefficient of the friction term (the second term of LHS of Eq. (7) [11, 12], machine learning [13] and so on. Also, public codes to calculate the bounce solution are available, such as `CosmoTransitions` [14, 15], `AnyBubble` [16], and `BubbleProfiler` [17, 18]. There are some works to discuss the bounce solution/action avoiding the direct calculation, *e.g.*, some approximations [19–21], upperbounds [22–24], lowerbounds, [24–26], and an alternative formulation [27–29].

One of the reason of technical difficulty is that the bounce solution is a saddle point of the action, *i.e.*, the bounce is not a stable solution of a simple minimization problem. Recently, Chigusa, Shoji, and Moroi [1] proposed a new method to obtain the bounce solution. They proposed a gradient flow equation whose fixed point is the bounce solution. Their flow equation has the gradient of the action and an additional term to lift up unstable direction around the bounce solution. Motivated by Ref. [1], in this paper, we propose a new simple flow equation. Coleman, Glaser, and Martin (CGM) [2] showed that the calculation of the bounce solution is equivalent to the minimization of the kinetic energy $\mathcal{T}$ while fixing the potential energy $\mathcal{V} < 0$. This minimization problem can be naturally formulated in a flow equation. In the end, the bounce solution is obtained as a scale-transformed of the solution of this problem. In Sec. II, we describe our formulation to calculate the bounce solution. In Sec. III, we discuss numerical analysis on several examples by using our flow equation, and show that our flow equation works well.

II. FORMULATION

In this paper, we focus on the Euclidean action with n scalar fields with the canonical kinetic term.

$$\mathcal{S}[\phi] = \mathcal{T}[\phi] + \mathcal{V}[\phi], \tag{1}$$

$$\mathcal{T}[\phi] = \sum_{i=1}^{n} \int d^d x \frac{1}{2} (\nabla \phi_i)^2, \tag{2}$$

$$\mathcal{V}[\phi] = \int d^d x V(\phi). \tag{3}$$

Here d is the dimension of the space, and we assume d is larger than 2. The scalar potential V satisfies $V(0) = 0$, $\partial V/\partial \phi_i = 0$, all of the eigenvalues of the Hessian of V at $\phi_i = 0$ are non-negative, and V is somewhere negative.

The bounce solution which contributes to the decay of the false vacuum satisfies the equation of the motion and the boundary condition at infinity:

$$-\nabla^2 \phi_i + \frac{\partial V}{\partial \phi_i} = 0, \tag{4}$$

$$\lim_{|x| \to \infty} \phi_i(x) = 0. \tag{5}$$

Also, the bounce solution should be a non-trivial solution, *i.e.*, $\exists i, x, \ \phi_i(x) \neq 0$. Thus,

$$\mathcal{T}[\phi] > 0, \quad \mathcal{V}[\phi] < 0. \tag{6}$$

Note that $\mathcal{V}[\phi] < 0$ is required in order for the bounce solution to be an extremum under the scale transformation: $\phi_i(x) \to \phi_i(\lambda x)$. See, *e.g.*, Ref. [2]. The bounce solution has the least action among configurations which satisfy the above conditions Eqs. (4, 5, 6). It is known that the bounce solution has spherical symmetry [2, 30–32]. Therefore, Eq. (4) can be simplified as

$$-\frac{d^2\phi_i}{dr^2} - \frac{d-1}{r}\frac{d\phi_i}{dr} + \frac{\partial V}{\partial \phi_i} = 0. \tag{7}$$

In order to discuss the bounce solution, CGM [2] introduced the reduced problem, which is defined as *the problem of finding a configuration vanishing at infinity which minimizes $\mathcal{T}$ for some fixed negative $\mathcal{V}$.* The existence of the solution of this problem is ensured by CGM's theorem B in Ref. [2]. Also, CGM's theorem A ensures that the bounce solution can be obtained as a scale-transformed of a solution of the reduced problem. See the Appendix. Here we solve the CGM's reduced problem by using a gradient flow equation. We introduce functions $\varphi_i(r,\tau)$ and propose the following gradient flow equation.

$$\frac{\partial}{\partial \tau}\varphi_i(r,\tau) = \nabla^2\varphi_i - \lambda[\phi]\frac{\partial V(\varphi)}{\partial \varphi_i}, \tag{8}$$

$$\lambda[\varphi] = \frac{\displaystyle\sum_i \int_0^\infty dr r^{d-1}\frac{\partial V(\varphi)}{\partial \varphi_i}\nabla^2\varphi_i}{\displaystyle\sum_i \int_0^\infty dr r^{d-1}\left(\frac{\partial V(\varphi)}{\partial \varphi_i}\right)^2}. \tag{9}$$

Here τ is "the time" for the flow of φ and $\nabla^2\varphi_i = \partial_r^2\varphi_i + (d-1)(\partial_r\varphi)/r$. We take the initial $\varphi(r,0)$ such that

$$\mathcal{V}[\varphi]|_{\tau=0} < 0. \tag{10}$$

Note that $\lim_{r\to\infty}\varphi_i(r,\tau) = 0$ should be hold in order for $\mathcal{V}[\phi]$ to be finite. By using Eq. (8) and Eq. (9), we can show

$$\frac{d}{d\tau}\mathcal{V}[\varphi] = 0, \tag{11}$$

$$\frac{d}{d\tau}\mathcal{T}[\varphi] \le 0. \tag{12}$$

To show Eq. (12), we used the following Cauchy-Schwarz inequality:

$$\left(\sum_i \int_0^\infty dr r^{d-1}(\nabla^2\varphi_i)^2\right)\left(\sum_i \int_0^\infty dr r^{d-1}\left(\frac{\partial V(\varphi)}{\partial \varphi_i}\right)^2\right) \ \ge\ \left(\sum_i \int_0^\infty dr r^{d-1}\frac{\partial V(\varphi)}{\partial \varphi_i}\nabla^2\varphi_i\right)^2. \tag{13}$$

Also, we can see the equalities of Eq. (12) and Eq. (13) hold if and only if

$$\nabla^2\varphi_i = \lambda\frac{\partial V(\varphi)}{\partial \varphi_i}. \tag{14}$$

are satisfied. Eqs. (11, 12) tell us that $\mathcal{T}[\varphi]$ monotonously decreases while $\mathcal{V}[\varphi]$ is constant during the flow of φ. In the limit of $\tau \to \infty$, φ converges to a configuration which satisfies $\nabla^2\varphi_i - \lambda(\partial V(\varphi)/\partial\varphi_i) = 0$. Note that this fixed point cannot be the false vacuum $\varphi_i = 0$ because $\mathcal{V}[\varphi]$ in neighborhood of the false vacuum is positive and $\mathcal{V}[\varphi]$ is always negative during the flow. As long as the initial condition is not fine-tuned, φ at $\tau \to \infty$ should be stable solution under the small perturbation, *i.e.*, $\mathcal{T}[\varphi]$ should be a local minimum under the small perturbation such that $\mathcal{V}[\phi]$ is not changed. In principle, the reduced problem could have several local minima. Physically, this case happens if there exist several directions of tunneling. In this case, φ at $\tau \to \infty$ depends on the initial condition, and we can find the global minimum among those local minima. The configuration which gives the least $\mathcal{T}$ is the solution of the CGM's reduced problem.

Let $\phi_i(r)(\equiv \lim_{\tau\to\infty}\varphi_i(r,\tau))$ be the solution of the reduced problem, and derive the bounce solution. The bounce solution $\phi_B(r)$ can be obtained by a scale transformation of ϕ as

$$\phi_B(r) = \phi(\lambda^{1/2}r). \tag{15}$$

The above λ is calculated as $\lim_{\tau\to\infty}\lambda[\varphi]$. Although the CGM's theorem A ensures that this ϕ_B is the bounce solution, let us see this more explicitly. We can immediately see that i) ϕ_B satisfies the EOM (Eq. (4)) and ii) $\lim_{r\to\infty}\phi_B(r) = 0$ because $\mathcal{V}[\phi_B]$ is finite. Also, we can see that iii) $\mathcal{S}$ has only one unstable direction around ϕ_B. Since ϕ_B is a scale-transformed of ϕ, ϕ_B is the global minimum of the action $\mathcal{S}$ if the potential energy $\mathcal{V}$ is fixed. The direction in which $\mathcal{S}$ decreases is the direction which changes $\mathcal{V}[\phi]$, *i.e.*, the scale transformation. Therefore, ϕ_B which is defined in Eq. (15) is the bounce solution.

An essential point of our method is that the negative eigenmode around the bounce solution can be related to the scale transformation. By fixing the potential energy $\mathcal{V}$, we freeze fluctuation in this direction. Note that a method which is proposed in Ref. [4] also utilizes this property.

III. EXAMPLE

In the previous section, we have seen the CGM's reduced problem can be solved by the flow equation Eq. (8) and Eq. (9), and the bounce solution can be obtained from Eq. (15). In this section, we discuss numerical results for several example, show that our method works well.

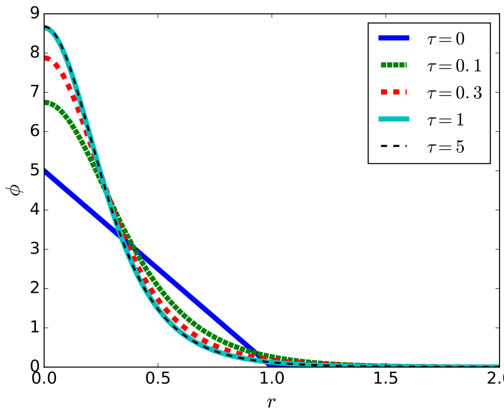

FIG. 1. A flow of the field configuration with the potential Eq. (16) with $d = 4$ and the initial condition Eq. (17).

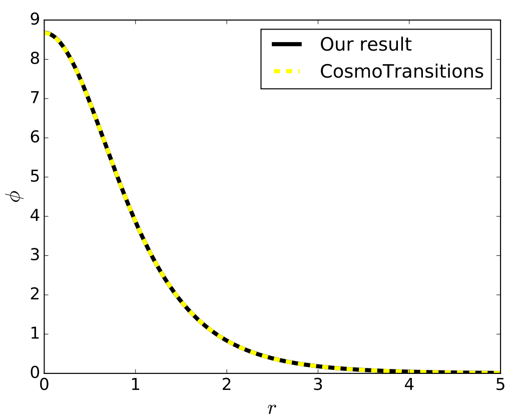

FIG. 2. The black line is obtained from Eq. (15) in the limit of large τ. The yellow dotted line is calculated by `CosmoTransitions`.

First, let us take the following single scalar potential in $d = 4$ Euclidean space.

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{1}{3}\phi^3. \tag{16}$$

We take the initial configuration at $\tau = 0$ as

$$\varphi(r, 0) = \begin{cases} 5(1-r) & (0 \leq r \leq 1) \\ 0 & (r > 1) \end{cases}. \tag{17}$$

The flow of this field configuration is shown in Fig. 1. We can see the convergence of the configuration. By using this result, we can obtain the bounce solution from Eq. (15). We compare our bounce solution with the result by `CosmoTransitions` [15] in Fig. 2. We can see that two results agree well and our method works.

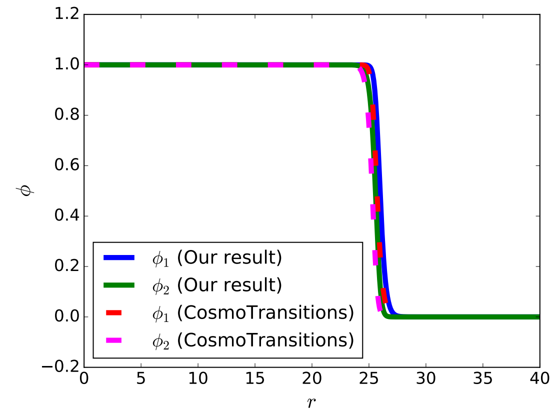

FIG. 3. The bounce solution in r-ϕ plane is shown by solid lines. The dashed lines are results of `CosmoTransitions`. We take the potential Eq. (18) with $c = 2$ in $d = 4$ space.

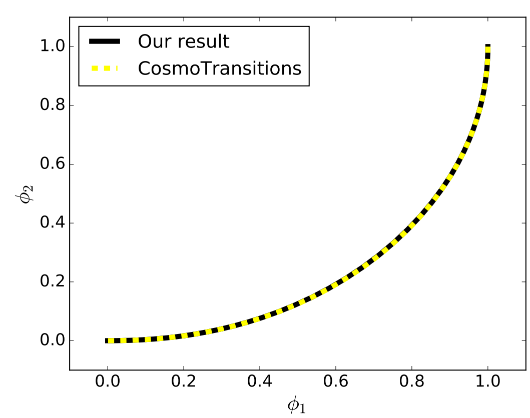

FIG. 4. The same bounce solution as Fig. 3 in ϕ_1-ϕ_2 plane.

Next, let us discuss a case with two scalar fields. We take the following potential.

$$\begin{aligned} V =& (\phi_1^2 + 5\phi_2^2)(5(\phi_1 - 1)^2 + (\phi_2 - 1)^2) \\ &+ c\left(\frac{1}{4}\phi_2^4 - \frac{1}{3}\phi_2^3\right). \end{aligned} \tag{18}$$

Again, we compare our bounce solutions with the results by `CosmoTransitions`. The case with $c = 2$ is shown in Figs. 3 and 4, and $c = 80$ in Figs. 5 and 6. We can see that our result agrees with that of `CosmoTransitions`.

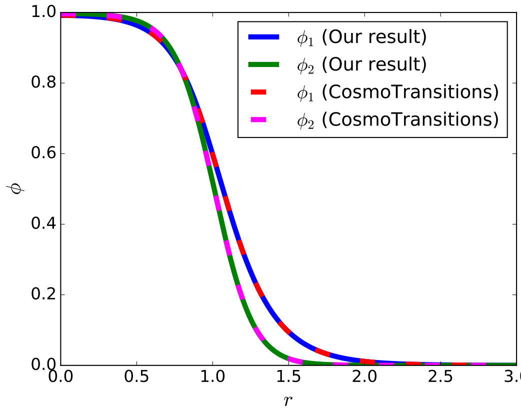

FIG. 5. The bounce solution in r-ϕ plane is shown by solid lines. The dashed lines are results of `CosmoTransitions`. We take the potential Eq. (18) with $c = 80$ in $d = 4$ space.

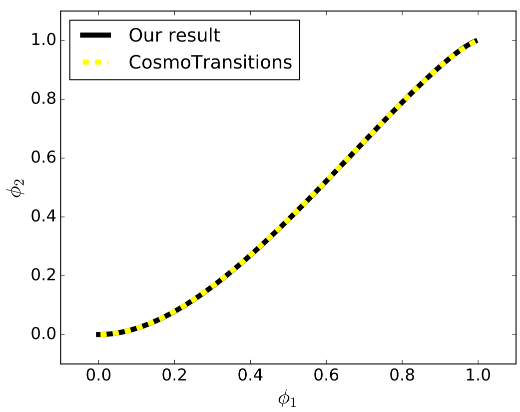

FIG. 6. The same bounce solution as Fig. 5 in ϕ_1-ϕ_2 plane.

IV. CONCLUSION

In this paper, motivated by a recent work of Chigusa, Shoji, and Moroi [1], we proposed a new simple gradient flow equation which is defined in Eq. (8) and Eq. (9). Our flow equation solves the CGM's reduced problem [2], *i.e.*, the minimization problem of kinetic energy $\mathcal{T}$ while fixing potential energy $\mathcal{V}$. This minimization problem can be naturally formulated in a flow equation, and the bounce solution can be obtained as a scale-transformed of this solution as Eq. (15).

ACKNOWLEDGEMENTS

The author thanks Takeo Moroi for useful discussions.

Appendix A: The CGM's theorem A

In this Appendix, we briefly summarize the theorem A in Ref. [2]. We denote the solution of the reduced problem for given $\mathcal{V}$ as $\phi_{(\mathcal{V})}$. This theorem ensures that the bounce solution is given by a scale transformation of $\phi_{(\mathcal{V})}$.

$\phi_{(\mathcal{V}_0)}$ is a stationary point of $\mathcal{T}[\phi]+\lambda(\mathcal{V}[\phi]-\mathcal{V}_0)$, where λ is the Lagrange multiplier. Thus, $\phi_{(\mathcal{V}_0)}$ satisfies

$$-\nabla^2\phi_{(\mathcal{V}_0)i} + \lambda\frac{\partial V}{\partial\phi_i} = 0. \tag{A1}$$

Here λ should be appropriately chosen for the value of $\mathcal{V}_0$. We define the following configuration ϕ_B:

$$\phi_B(x) = \phi_{(\mathcal{V}_0)}(\lambda^{1/2}x). \tag{A2}$$

We can see this is the bounce solution. First, by using Eqs. (A1, A2), we can check that ϕ_B satisfies the EOM Eq. (4). Next, let us show the action of any non-trivial solution of Eq. (4) is equal to or larger than $\mathcal{S}[\phi_B]$. Let $\tilde{\phi}$ be a non-trivial solution of Eq. (4). The action of $\tilde{\phi}$ is extremized under the scale transformation of $\tilde{\phi}$. Therefore,

$$(d-2)\mathcal{T}[\tilde{\phi}] + d\mathcal{V}[\tilde{\phi}] = 0. \tag{A3}$$

There exists a solution of the reduced problem for $\mathcal{V} = \mathcal{V}[\tilde{\phi}]$, and the kinetic energy is not larger than $\mathcal{T}[\tilde{\phi}]$:

$$\mathcal{T}[\phi_{(\mathcal{V}[\tilde{\phi}])}] \leq \mathcal{T}[\tilde{\phi}]. \tag{A4}$$

$\mathcal{T}[\phi_B]$ and $\mathcal{V}[\phi_B]$ are given as

$$\mathcal{T}[\phi_B] = \lambda^{1-d/2}\mathcal{T}[\phi_{(\mathcal{V}[\phi])}], \tag{A5}$$

$$\mathcal{V}[\phi_B] = \lambda^{-d/2}\mathcal{V}[\phi]. \tag{A6}$$

Here $\lambda \geq 1$ because of $(d-2)\mathcal{T}[\phi_B] + d\mathcal{V}[\phi_B] = 0$ and Eqs. (A3, A4). Thus, by using Eqs. (A4, A5), we can show

$$\mathcal{T}[\phi_B] \leq \mathcal{T}[\tilde{\phi}]. \tag{A7}$$

$\mathcal{S} = (2/d)\mathcal{T}$ is satisfied for solutions of Eq. (4). Then,

$$\mathcal{S}[\phi_B] \leq \mathcal{S}[\tilde{\phi}]. \tag{A8}$$

Thus, ϕ_B has the least action among the non-trivial solutions of the EOM.

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