Schrödinger's equation for Conformal Symmetry

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Based on work with M. Isachenkov, E. Sobko, P. Liendo, Y. Linke;

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HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

0.1 CFT and Conformal Symmetry

CFT is everywhere: 2nd order phase transitions, IR dynamics of many interesting QFTs, AdS/CFT correspondence ....

Understanding of perturbative & non-perturbative dynamics is based on the study of both local and non-local observables

Primary fields $\Phi_{\Delta,\ell}(x)$

weights of SO(1,1) & SO(d)

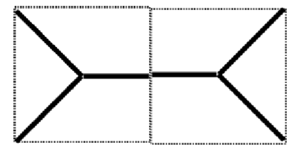
'‘t Hooft, Wilson line, surface, defect, interface operators …

Analyis and construction of their correlators relies on mathematics of conformal symmetry G = SO(1,d+1) … yet …

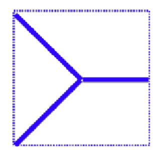
our present knowledge of conformal symmetry is incomplete

0.2 Conformal Partial Waves

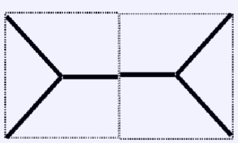
… are the CFT-analogues of plane waves in Fourier theory

e.g. 4-pt fcts $\langle \prod_{i=1}^{4} \Phi_i(x_i) \rangle = \sum_{\Phi} \gamma_{12\Phi}\gamma_{34\Phi}$  **(u$_1$,u$_2$)**

CPW

$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) \rangle = \gamma_{123}$ 

3J symbol

 = $G_{\Delta,l}(z,\bar{z}) \sim$

$$\mu = \frac{x_{10}}{x_{10}^2} - \frac{x_{20}}{x_{20}^2} \qquad \tilde{\mu} = \frac{x_{40}}{x_{40}^2} - \frac{x_{30}}{x_{30}^2}$$

$$\sim \int d^d x_0 \, x_{10}^{l+a-\Delta} x_{20}^{l-a-\Delta} x_{30}^{l-b+\Delta-d} x_{40}^{l+b+\Delta-d} \, (|\mu||\tilde{\mu}|)^l \, Y_l^d \left(\frac{\mu \cdot \tilde{\mu}}{|\mu||\tilde{\mu}|} \right)$$

Zonal spherical functions

0.2 Conformal Partial Waves

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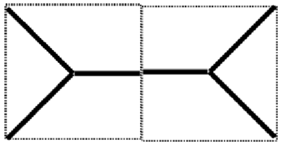
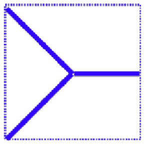
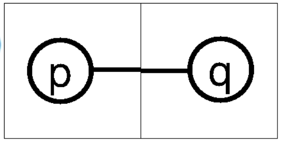
e.g. 4-pt fcts $$\langle \prod_{i=1}^{4} \Phi_i(x_i) \rangle = \sum_{\Phi} \gamma_{12\Phi}\gamma_{34\Phi}$$  **(u_1, u_2)**

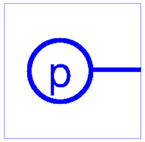
CPW

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) \rangle = \gamma_{123}$$ 

3J symbol

Defect 2-pt fcts $$\langle D_p(\mathcal{X}_p)\, D_q(\mathcal{X}_q) \rangle = \sum_{\Phi} \alpha_{\Phi}^{(p)} \alpha_{\Phi}^{(q)}$$ 

CPW

$$\langle D_p(\mathcal{X})\Phi(x) \rangle = \alpha_{\Phi}^{(p)}$$ 

What kind of functions are the CPWs ?

How do they depend on Conformal cross ratios u and parameters of field Φ

0.3 Main Results and Plan

CPWs are wave functions of integrable N-particle Schrödinger problem in coordinate space and in weight/momentum space.

Hyperbolic Calogero-Sutherland model for BC_N root system in u_i

Rational Ruiijsenaars - Schneider model in Δ, ℓ

[Isachenkov, VS] [Isachenkov,Liendo,Linke,VS] …

→ ``Euclidean'' Heckman-Opdam hypergeometric functions and degenerations of virtual Koornwinder polynomials

1. **Review.** ***CPWs and the Calogero-Sutherland potential***
2. **Extension.** ***Defects blocks and the N-particle CS model***
3. **Integrability.** ***Bi-spectral duality: weights ↔ coordinates***

Review

Talks at IGST 2016 [VS], IGST 2017 [Sobko]

1.1 Conformal Partial Waves

… are G invariants in TP of 4 principal series representations

= sections of a vector bundle on 2-sided coset space K\G/K

with fiber V $^{SO(d-2)}$ **Space of tensor structures** **2 – dimensional [cross ratios]**

$$\left(\bigotimes \Gamma^{\Delta_i,\ell_i}_{G/H}\right)^G = \{\, f : G \longrightarrow V \mid f(k_l g k_r) = \pi(k_l \otimes k_r^{-1}) f(g) \,\}$$

$$V = \bigotimes_{i=1}^{4} V_{\Delta_i,\ell_i}$$

Principal series reps induced from fd irrep of K = SO(1,1) x SO(d) on $V_{\Delta,\ell}$

$$\Gamma^{\Delta,\ell}_{G/H} = \{\, f : G \longrightarrow V_{\Delta,\ell} \mid f(gnk) = \pi(k^{-1}) f(g) \,\}$$

$V_{\Delta,\ell}$ - valued functions on the coset space $G/H \cong \mathbb{R}^d$ $\quad H = K \ltimes N$

1.2 The Casimir Equation

Eigenvalue equation for the quadratic Casimir element C_2 of the conformal group G on space Γ of conformal partial waves

$$m^{1/2}(u)\mathcal{D}_2 m^{-1/2}(u) = -\frac{1}{2}\frac{d^2}{du_1^2} - \frac{1}{2}\frac{d^2}{du_2^2} + V(u_1, u_2)$$

m is volume of K x K orbit through u

[M. Isachenkov, VS, E. Sobko]

Scalar CPWs: $\Delta_2 - \Delta_1 = 2a \qquad \Delta_3 - \Delta_4 = 2b$

$$V(u_1, u_2) = \sum_{i=1}^{2}\left(\frac{(a+b)^2 - 1/4}{2\sinh^2 u_i} - \frac{ab}{2\sinh^2 \frac{u_i}{2}}\right)$$

$$+\frac{(d-2)(d-4)}{16\sinh^2 \frac{u_1-u_2}{2}} + \frac{(d-2)(d-4)}{16\sinh^2 \frac{u_1+u_2}{2}} + \frac{d^2-2d+2}{8}$$

Calogero-Sutherland model

= 2 Poeschl-Teller particles

with interaction

1.2 The Casimir Equation (contd)

$G_{\Delta,l}$ in Dolan-Osbon conventions $\leftrightarrow$ CS eigenfunctions $\psi_{\lambda_1,\lambda_2}$

$$\psi_{\lambda_1,\lambda_2}(u_1,u_2) := \prod_i \frac{(z_i-1)^{\frac{a+b}{2}+\frac{1}{4}}}{z_i^{\frac{1}{2}+\frac{\epsilon}{2}}} |z_1-z_2|^{\frac{\epsilon}{2}} G_{\Delta,l}(z_1,z_2)$$

$$\lambda_1 = \frac{1}{2}+\frac{\epsilon}{2}-\frac{1}{2}(\Delta-l)$$

$$\lambda_2 = \frac{1}{2}-\frac{1}{2}(\Delta+l)$$

$$z_i = -\sinh^{-2}\frac{u_i}{2}$$

$z_1 = z \qquad z_2 = z$

[Isachenkov,VS]

u_i radial coordinates [Hogervorst,Rychkov]

$$\mathrm{Cas}_d^2 G(z,\bar z) = \frac{1}{2} C_{\Delta,l} G(z,\bar z)$$

$$C_{\Delta,l} = \Delta(\Delta-d)+l(l+d-2)$$

where

$$\mathrm{Cas}_d^2 := D^2 + \overline{D}^2 + \epsilon\left[\frac{z\bar z}{\bar z - z}\left(\bar\partial - \partial\right) + \left(z^2\partial - \bar z^2\bar\partial\right)\right]$$

$$\epsilon = d-2 \qquad 2a = \Delta_2-\Delta_1 \qquad 2b = \Delta_3-\Delta_4$$

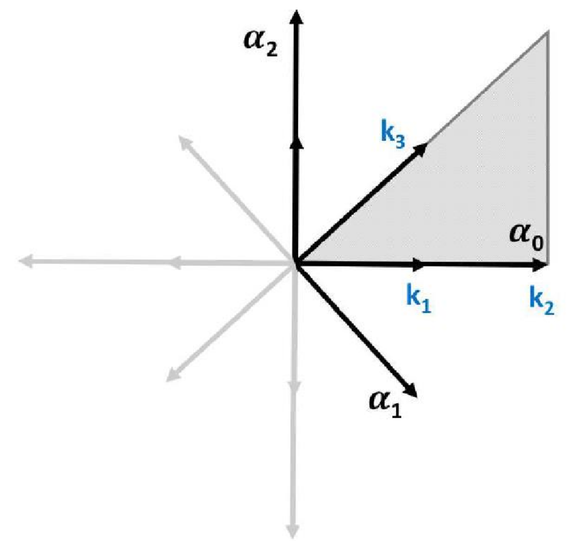
$$D^2 = z^2(1-z)\partial^2 - (a+b+1)z^2\partial \; - abz$$

[Dolan, Osborn]

1.3 Calogero-Sutherland Models

[Calogero 71] [Sutherland 72]

Integrable multi-particle generalization of Poeschl-Teller model



Associated with non-reduced root system – here with BC_N

Eigenvalue problem ↔ hypergeometrics

The scattering problem for particles in a Weyl chamber is solved

$$\Psi_\lambda(u) \sim e^{\lambda\, u} \qquad \text{for} \qquad u_i \to \infty$$

[Heckman,Opdam]

Harish-Chandra functions: single plane plane waves asymptotics

Much is known: Poles in space of momenta λ, series representations ….

Extensions

- Spinning blocks
- Defect blocks ←
- Superconformal blocks
- Thermal blocks
- Multi-point blocks

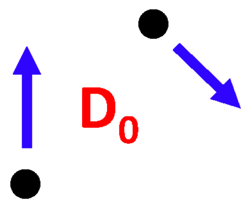
2.1 Conformal Defect Operators

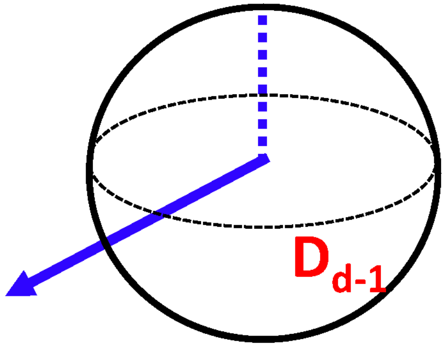
Isometries of a p-dimensional conformal defect form subgroup

$$G_p = SO(1, p+1) \times SO(d-p) \subset G$$

p = 0: isometries of pair of points (dilations, rotations); $G_0 = K$

Conformal defect possesses *dim* G/G_p = (p+2)(d-p) parameters





e.g. D_0: 2d parameters

D_{d-1}: d+1 parameters

2.2 Conformal Partial Waves

Space of CPWs for two scalar defects D_p and D_q can be realized as

as functions on the 2-sided coset space $G_p \backslash G / G_q$

$$\Gamma^{p,q;d} = \{\, f : G \to \mathbb{C} \,|\, f(h_L g h_R) = f(g); h_L \in G_p \,,\, h_R \in G_q\}$$

number N of ``cross ratios''

$$dim\, G_p \backslash G / G_q = dim\, G - dim\, G_p - dim\, G_q + dim\, B =$$

$$N = min(d-p, q+2) \qquad \leftrightarrow \text{[Gadde]}$$

$$B = SO(p-q) \times SO(|d-p-q-2|) \subset G_{p,q}$$

2.3 The Casimir Equation

Eigenvalue equation for the quadratic Casimir element C_2 of the conformal group G on space Γ of conformal partial waves

$$m^{1/2}(u)\mathcal{D}_2 m^{-1/2}(u) = -\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{du_i^2} + V(u)$$

m is volume of $G_p \times G_q$ orbit through $u = (u_1, \dots, u_N)$

Scalar CPWs:

[Isachenkov,Liendo,Linke,VS]

$$V(u) = \sum_{i=1}^{N}\left(\frac{(p-q)(p-q-2)}{8\sinh^2 u_i} + \frac{(N-d/2)(N-d/2+p-q-1)}{8\sinh^2 \frac{u_i}{2}}\right)$$

$$-\sum_{i<j}\left(\frac{1}{16\sinh^2 \frac{u_i-u_j}{2}} + \frac{1}{16\sinh^2 \frac{u_i+u_j}{2}}\right) + C$$

$$N = min(d-p, q+2)$$

2.4 Some Applications

All defect blocks for any value of N were constructed in terms of multi-variate hypergeometrics. [Liendo,Linke,Isachenkov,VS]

For N = 2 we found complete set of relations with 4-point blocks extending results by [Billo,Goncalves,Lauria,Meineri] [Liendo,Meneghelli]

We found a Lorentzian inversion formula extending [Caron-Huot]

→ Computation of bulk-defect OPE coefficients for large spins.

work in progress

related with [Alday et al.] [Caron-Huot][Lemos,Liendo,Meineri,Sarkar]

Integrability

3.1 Dependence on weights/momenta

Dolan & Osborn noticed that scalar blocks obey shift equations

$$\begin{aligned}
&\Big(\frac{x+z}{xz}-1\Big)F_{\lambda_1\lambda_2}(x,z) = F_{\lambda_1\lambda_2-1}(x,z) + \frac{\lambda_-(\lambda_- -1+\varepsilon)}{(\lambda_- +\frac{1}{2}\varepsilon)(\lambda_- -1+\frac{1}{2}\varepsilon)}F_{\lambda_1-1\lambda_2}(x,z)\\
&-\tfrac{1}{4}\big((2\lambda_1+c)(2\lambda_1+c-2)+(2\lambda_2+c-\varepsilon)(2\lambda_2+c-2-\varepsilon)\\
&\qquad\qquad -\varepsilon(\varepsilon-2)\big)\hat{A}_{\lambda_1\lambda_2-\frac{1}{2}\varepsilon}F_{\lambda_1\lambda_2}(x,z)\\
&+\frac{(\lambda_1+\lambda_2+c-1)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-1-\frac{1}{2}\varepsilon)}B_{\lambda_1}F_{\lambda_1+1\lambda_2}(x,z)\\
&+\frac{\lambda_-(\lambda_- -1+\varepsilon)}{(\lambda_- +\frac{1}{2}\varepsilon)(\lambda_- -1+\frac{1}{2}\varepsilon)}\,\frac{(\lambda_1+\lambda_2+c-1)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-1-\frac{1}{2}\varepsilon)}B_{\lambda_2-\frac{1}{2}\varepsilon}F_{\lambda_1\lambda_2+1}(x,z)\,,
\end{aligned}$$

Eq. (5.1) from hep-th/0309180 [Dolan,Osborn]

3.2 Ruijsenaars-Schneider model

After multiplication with some factor c = c(λ_1, λ_2) one obtains

$$\sum_{j=1,2} \left[v_j(\lambda)(\psi_{\lambda+e_j} - \psi_\lambda) + v_j(-\lambda)(\psi_{\lambda-e_j} - \psi_\lambda) \right] = 4 \sum_{j=1,2} \sinh^2 u_i/2 \; \psi_\lambda$$

$$v_j(\lambda) = \frac{(\lambda_j + a + 1/2)(\lambda_j - b + 1/2)}{\lambda_j(\lambda_j + 1/2)} \frac{(\lambda_1 + \lambda_2 + \epsilon/2)(\lambda_1 - \lambda_2 + \epsilon/2)}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)}$$

2nd order difference eq: rational Ruijsenaars-Schneider model

Comments: This generalizes to wave functions $\psi_\lambda(u)$ of the BC_N Calogero-Sutherland model and hence to defect blocks.

note $\psi_{\lambda+e_i} = e^{\partial_i} \psi_\lambda$

Rational RS contains exponential of $\partial_i = \frac{\partial}{\partial \lambda_i}$ & is rational in λ_i

Hyperbolic CS is polynomial in $\partial_i = \frac{\partial}{\partial u_i}$ & exponential of u_i

3.3 Hyperbolic RS model

Rational Ruijsenaars-Schneider model possesses integrable hyperbolic/trigonometric deformation deformation parameter q

Dependence of its wave functions $\psi_\lambda^q(u)$ on coordinates u determined by dual hyperbolic Ruijsenaars-Schneider model

Casimir differential equation obtained by degeneration q → 1

Wave functions of this bi-spectrally self-dual RS model are (virtual) Koornwinder polynomials (functions)

Conformal partial waves obtained by degeneration

4 Outlook and Conclusions

Integrable quantum mechanics provides a new approach to CPWs

that is powerful **by embedding into modern theory of multivariate hypergeometric functions** ↔ **SUSY gauge theory**

Series expansions, recurrence relations, integral formulas

...

is flexible **Applies to conformal defects, spinning correlators superconformal symmetry** [VS,Sobko][Buric,VS,Sobko]

↔ [Cornagliotto,Lemos,VS] ...

Many aspects need to be further developed