Integrable Deformations of the AdS Superstring and their Dual Gauge Theories

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We study a new class of two-dimensional field theories which are integrable deformations of the string sigma model on $\mathrm{AdS}_5 \times S^5$. We show that some of these theories are scale but not Weyl invariant. For the real deformation parameter we find the quantum spectral curve equations which encode the energy spectrum. More generally, we investigate a relation between integrable deformations of CFTs and functional equations as well as lattice integrable discretisations based on the universal R-matrix.

1 Introduction

In recent years significant progress has been made towards understanding the excitation spectrum of strings moving in five-dimensional anti-de Sitter space-time and, accordingly, the spectrum of scaling dimensions of composite operators in planar $\mathcal{N} = 4$ supersymmetric gauge theory. This progress became possible due to the fundamental insight that strings propagating in AdS space can be described by an integrable model. In certain aspects, however, the deep origin of this exact solvability has not yet been unraveled, mainly because of tremendous complexity of the corresponding model. A related question concerns robustness of integrability in the context of the gauge-string correspondence [1], as well as the relationship between integrability and the amount of global (super)symmetries preserved by the target space-time in which strings propagate. To shed further light on these important issues, one may attempt to search for new examples of integrable string backgrounds that can be solved by similar techniques. One such instance, where this program is largely promising to succeed, is to study various deformations of the string target space that preserve the integrability of the two-dimensional quantum field theory on the world sheet. Simultaneously, this should provide interesting new information about integrable string models and their dual gauge theories.

There are two known classes of integrable deformations of the $\mathrm{AdS}_5 \times S^5$ superstring. The first of these is a class of backgrounds obtained either by orbifolding $\mathrm{AdS}_5 \times S^5$ by a discrete subgroup of the corresponding isometry group or by applying a sequence of T-duality – shift – T-duality transformations (also known as γ-deformations) to this space, giving a string theory on a TsT-transformed background. Eventually all deformations of this class can be conveniently described in terms of the original string theory, where the deformations result into quasi-periodic but still integrable boundary conditions for the world-sheet fields.

The second class of deformations affects the $\mathrm{AdS}_5 \times S^5$ model on a much more fundamental level and is related to deformations of the underlying symmetry algebra. In the light-cone gauge this symmetry algebra constitutes two copies of the centrally extended Lie superalgebra $\mathfrak{psu}(2|2)$ with the same central extension for each copy. It appears that this centrally extended $\mathfrak{psu}(2|2)$, or more precisely its universal enveloping algebra, admits a natural deformation $\mathfrak{psu}_q(2|2)$ in the sense of quantum groups. This algebraic structure is the starting point for the construction of a $\mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$-invariant S-matrix, giving a quantum deformation of the $\mathrm{AdS}_5 \times S^5$ world-sheet S-matrix [2]. The deformation parameter q can be an arbitrary complex number, but in physical applications is typically taken to be either real or a root of unity.

Some time ago there was an interesting proposal on how to deform the sigma-model for strings on $\mathrm{AdS}_5 \times S^5$ with a real deformation parameter η, preserving classical integrability [3, 4]. In the following we call these models η-deformed and we will talk about η-deformations. Deformations of this type constitute a general class of deformations governed by solutions of the classical Yang–Baxter equation [5]. This class is not solely restricted to the string model in question but includes a large variety of two-dimensional integrable models based on (super)groups or their cosets.

The paper is organised as follows. In the next three sections we summarise the main results of our investigation of the η-deformed string sigma models and discuss a number of important related issues. We concentrate on three directions – the η-deformed background, the access to the spectrum of the model via the quantum spectral curve and finite-dimensional integrable systems obtained through various reductions of the sigma model. In section 5 we address a more general problem on finding perturbations of CFTs which preserve integrability and also investigate a vital question on uniqueness of solutions of functional equations that suppose to encode the spectrum of a deformed CFT. Finally, in section 6, aiming at developing direct quantisation tools for world-sheet theories, including string sigma model on $\mathrm{AdS}_5 \times S^5$ and its deformations, we consider lattice discretisations of integrable systems in the formalism of the universal R-matrix. Importantly, we outline a general program of constructing such integrable discretisations and solving the corresponding spectral problem. The results presented in sections 5 and 6 constitute a continuation of the research line of the former SFB project A8.

2 The nature of the deformed background

Recall that the Lagrangian density of the η-deformed model is given by [3]

$$\mathscr{L} = -\frac{g}{4}(1+\eta^2)\big(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}\big)\,\mathrm{str}\Big[\tilde{d}(A_\alpha)\frac{1}{1-\eta R_{\mathfrak{g}}\circ d}(A_\beta)\Big]\,.$$

We use the notations and conventions from [6], in particular g is the effective string tension. The current $A_\alpha = -\mathfrak{g}^{-1}\partial_\alpha \mathfrak{g}$, where $\mathfrak{g} \equiv \mathfrak{g}(\tau,\sigma)$ is a coset representative from $\mathrm{PSU}(2,2|4)/\mathrm{SO}(4,1)\times \mathrm{SO}(5)$. The operators d and $\tilde{d}$ acting on the currents A_α are defined as

$$d \;=\; P_1 + \frac{2}{1-\eta^2}P_2 - P_3, \qquad\qquad \tilde{d} = -P_1 + \frac{2}{1-\eta^2}P_2 + P_3\,,$$

where P_i, $i = 0,1,2,3$, are projections on the corresponding components of the $\mathbb{Z}_4$-graded decomposition of the superalgebra $\mathscr{G} = \mathfrak{psu}(2,2|4)$. The operator $R_{\mathfrak{g}}$ acts on $M \in \mathscr{G}$ as follows

$$R_{\mathfrak{g}}(M) = \mathfrak{g}^{-1}R(\mathfrak{g}M\mathfrak{g}^{-1})\mathfrak{g}\,,$$

where R is a linear operator on $\mathscr{G}$ which in this paper we define as

$$R(M)_{ij} = -i\,\tau_{ij} M_{ij}\,, \quad \tau_{ij} = \begin{cases} 1 & \text{if} \quad i<j \\ 0 & \text{if} \quad i=j \\ -1 & \text{if} \quad i>j \end{cases}\,,$$

where M is an arbitrary 8×8 matrix. This choice of R corresponds to the standard Dynkin diagram of $\mathfrak{psu}(2,2|4)$.

The η-deformed model appears to be rather involved, primarily because of fermionic degrees of freedom. The strategy is therefore to first switch off fermions and proceed by studying the corresponding bosonic action. This action can be further used to determine $2\to 2$ scattering matrix for the η-deformed model in the limit of large string tension g and to compare the corresponding result with the known q-deformed S-matrix found from quantum group symmetries, unitarity and crossing [2]. Of course, the perturbative S-matrix computed from this action will not coincide with the full world-sheet S-matrix but nevertheless will give a sufficient part of the scattering data to provide a non-trivial test for both integrability (the Yang–Baxter equation) and a comparison with the q-deformed S-matrix.

This preliminary work has been carried out in [6], where it was shown that for a particular choice of the bosonic coset element the η-deformed metric G and the B-field (NSNS background) can be written in the form

$$\begin{aligned}
\frac{1}{\tilde g} ds^2_{\mathfrak{a}} =& -\frac{dt^2\left(1+\rho^2\right)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{\left(1+\rho^2\right)\left(1-\varkappa^2\rho^2\right)} \\
&+ \frac{d\zeta^2\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\rho^2\sin^2\zeta\,, \\
\frac{1}{\tilde g} ds^2_{\mathfrak{s}} =& \frac{d\phi^2\left(1-r^2\right)}{1+\varkappa^2 r^2} + \frac{dr^2}{\left(1-r^2\right)\left(1+\varkappa^2 r^2\right)} \\
&+ \frac{d\xi^2 r^2}{1+\varkappa^2 r^4\sin^2\xi} + \frac{d\phi_1^2 r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi} + d\phi_2^2 r^2\sin^2\xi\,, \\
\frac{1}{\tilde g} B =& \varkappa\left(\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} d\psi_1\wedge d\zeta - \frac{r^4\sin 2\xi}{1+\varkappa^2 r^4\sin^2\xi} d\phi_1\wedge d\xi\right)\,,
\end{aligned}$$

where $\varkappa = \frac{2\eta}{1-\eta^2}$ and $\tilde g = g\sqrt{1+\varkappa^2}$. This metric corresponds to a product of two five-dimensional spaces with coordinates $\{t,\rho,\zeta,\psi_1,\psi_2\}$ and $\{\phi,r,\xi,\phi_1,\phi_2\}$ respectively. The range of ρ is restricted to $[0,1/\varkappa)$ to preserve the time-like nature of t, with a curvature singularity at $\rho=1/\varkappa$. At $\varkappa=0$ there is no singularity but rather the conformal boundary of anti-de Sitter space at $\rho=\infty$. The bosonic two-body S-matrix computed from this action perfectly coincides with the large tension limit of the exact S-matrix based on the $\mathfrak{psu}_q(2|2)$ symmetry [6].

The next step is to clarify an important question of whether or not the η-deformed model is a type IIB string sigma model. As we will show, the answer turns out to be negative.

One way to approach this question would be to try to find an embedding of the given NSNS background into a full solution of type IIB supergravity. Given the complexity of the NSNS background, this appears however a rather difficult task. First of all the equation for the dilaton has many solutions and also many components of the RR forms seem to be switched on. Even if successful, this approach does not however guarantee that the string sigma model in the corresponding supergravity background will actually coincide with a deformed model.

Another way to proceed is to note that the Green–Schwarz (GS) action restricted to quadratic order in fermions contains all the information about the background fields. The corresponding Lagrangian has the form, see *e.g.* [7],

$$\mathscr{L}_{\Theta^2} = -\frac{g}{2}\, i\, \bar{\Theta}_I\, (\gamma^{\alpha\beta}\delta^{IJ} + \epsilon^{\alpha\beta}\sigma_3^{IJ})e_\alpha^m \Gamma_m\, D_\beta^{JK}\Theta_K,$$

where Θ_I are two Majorana–Weyl fermions of the same chirality. The operator D_α^{IJ} acting on fermions has the following expression

$$\begin{aligned} D_\alpha^{IJ} =& \delta^{IJ}\left(\partial_\alpha - \frac{1}{4}\omega_\alpha^{mn}\Gamma_{mn}\right) + \frac{1}{8}\sigma_3^{IJ} e_\alpha^m H_{mnp}\Gamma^{np} \\ & - \frac{1}{8}e^\phi \left(\epsilon^{IJ}\Gamma^p F_p^{(1)} + \frac{1}{3!}\sigma_1^{IJ}\Gamma^{pqr}F_{pqr}^{(3)} + \frac{1}{2\cdot 5!}\epsilon^{IJ}\Gamma^{pqrst}F_{pqrst}^{(5)}\right) e_\alpha^m \Gamma_m\,, \end{aligned}$$

where (e,ω,H) constitute a vielbein, the spin connection and the field strength of a B-field, while F's are RR forms and ϕ is a dilaton. Note that the dilaton and RR forms appear only through the combination $\mathcal{F}_n = e^\phi F_n$. Our approach is therefore to work out the quadratic fermionic action starting from the η-deformed action of [3] and find a field redefinition which brings this action into the GS canonical form above. This would allow us to identify the background fields and further check if they satisfy the equations of motion of type IIB supergravity and, in particular, to find a solution for the dilaton.

Performing the corresponding calculations, we arrive at the following result for non-vanishing RR forms written with *flat* indices of the tangent space [8]

$$\begin{aligned}
\mathcal{F}_1 &= -4\varkappa^2\ c_F^{-1}\ \rho^3 \sin\zeta, & \mathcal{F}_6 &= +4\varkappa^2\ c_F^{-1}\ r^3 \sin\xi, \\
\mathcal{F}_{014} &= +4\varkappa\ c_F^{-1}\ \rho^2 \sin\zeta, & \mathcal{F}_{123} &= -4\varkappa\ c_F^{-1}\ \rho, \\
\mathcal{F}_{569} &= +4\varkappa\ c_F^{-1}\ r^2 \sin\xi, & \mathcal{F}_{678} &= -4\varkappa\ c_F^{-1}\ r, \\
\mathcal{F}_{046} &= +4\varkappa^3\ c_F^{-1}\ \rho r^3 \sin\xi, & \mathcal{F}_{236} &= -4\varkappa^3\ c_F^{-1}\ \rho^2 r^3 \sin\zeta\sin\xi, \\
\mathcal{F}_{159} &= -4\varkappa^3\ c_F^{-1}\ \rho^3 r \sin\zeta, & \mathcal{F}_{178} &= -4\varkappa^3\ c_F^{-1}\ \rho^3 r^2 \sin\zeta\sin\xi, \\
\mathcal{F}_{01234} &= +4\ c_F^{-1}, & \mathcal{F}_{02346} &= -4\varkappa^4\ c_F^{-1}\rho^3 r^3 \sin\zeta\sin\xi, \\
\mathcal{F}_{01459} &= +4\varkappa^2\ c_F^{-1}\rho^2 r \sin\zeta, & \mathcal{F}_{01478} &= +4\varkappa^2\ c_F^{-1}\rho^2 r^2 \sin\zeta\sin\xi, \\
\mathcal{F}_{04569} &= +4\varkappa^2\ c_F^{-1}\rho r^2 \sin\xi, & \mathcal{F}_{04678} &= -4\varkappa^2\ c_F^{-1}\rho r.
\end{aligned}$$

Here we defined the common coefficient

$$c_F = \frac{1}{\sqrt{1+\varkappa^2}}\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2\rho^4\sin^2\zeta}\sqrt{1+\varkappa^2 r^2}\sqrt{1+\varkappa^2 r^4\sin^2\xi}.$$

For the five-form we presented here only half of all its non-vanishing components, namely those which involve the index 0. The other half is obtained from the self-duality equation for the five-form. The answer appears to be rather simple and in the limit $\varkappa \to 0$ all the components vanish except $\mathcal{F}_{01234}$ which reduces to the constant five-form flux of the $\mathrm{AdS}_5 \times S^5$ background. In the following we will use for the background found above the name "ABF background".

Inspection of the found RR couplings reveals that contrary to the natural expectations they do not obey equations of motion of type IIB supergravity. First of all for the Bianchi identities this is already obvious from the expression for the 1-form F_1. To fit the supergravity content this form must be exact $F^{(1)} = \mathrm{d}\chi$, where χ is axion. One can verify that there is no way to split off an integrating factor e^{ϕ}, such that the corresponding $F^{(1)}$ becomes exact. Concerning other equations of motion, consider, for instance, the Einstein equations which involve an unknown dilaton. One can show that to achieve vanishing of the off-diagonal components of the Einstein equations the dilaton ϕ *must* be of the form $\phi = \Phi_{\mathfrak{a}}(\rho,\zeta) + \Phi_{\mathfrak{s}}(r,\xi)$, where $\Phi_{\mathfrak{a}}$ and $\Phi_{\mathfrak{s}}$ are some functions. However, analysis of the diagonal components of the Einstein equations shows that a solution for $\Phi_{\mathfrak{a}}$ and $\Phi_{\mathfrak{s}}$ does not exist. The next surprising observation is that the RR couplings do not meet the necessary conditions of the mirror duality [9], and, as a consequence, the mirror background [10] is not reproduced in the expected limit $\eta \to 1$. Although this duality is a symmetry of the exact S-matrix, it involves rescaling of the string tension and therefore its absence in the classical Lagrangian might be explained by the order of limits problem.

While not solving the standard type IIB equations directly this ABF background still turns out to be very special: it is related by T-duality to an exact type IIB supergravity solution [11]. The latter HT background involves a non-diagonal metric $\hat{G}$, an imaginary 5-form $\hat{F}_5$ and the dilaton $\hat{\phi}$, and the T-duality applied in all 6 isometric directions acts only on the fields $\hat{G}$ and $\hat{\mathcal{F}}_5 = e^{\hat{\phi}}\hat{F}_5$ entering the corresponding GS action on a flat 2d background. The GS action for any type II solution (and thus for the HT background) should be Weyl invariant and, in particular, scale invariant. As the T-duality applied to the GS action is a simple path integral transformation, the T-duality relation between the ABF and HT backgrounds implies that the action should define a scale invariant 2d theory at least to 1-loop order.

However, there may be a problem with Weyl invariance for the η-deformed sigma-model on a curved 2d background. The HT dilaton $\hat{\phi}$ has a term linearly depending on the isometric directions of $\hat{G}$ and $\hat{\mathcal{F}}_5$ and thus one cannot directly apply the standard T-duality transformation rules to the full HT background to get a full T-dual supergravity solution, and thus the Weyl invariance of the T-dual sigma model requires further investigation. This is of course consistent with the observation [8] that the ABF background does not satisfy the IIB supergravity equations.

In the work [12] we have found that the ABF background, while not a supergravity solution, satisfies the following two generalisations or "modifications" of the type II supergravity equations:

(i) the scale invariance conditions for the type II superstring sigma model (with equations on the R-R fields $\mathcal{F}$ being of 2nd order in derivatives);

(ii) a set of equations that are structurally similar to those of type II supergravity (with 1st-order equations for the RR fields $\mathcal{F}$) but involving, instead of derivatives of the dilaton, a certain co-vector Z_m playing now the role of the dilaton one-form and a Killing vector I^m responsible for the "modification" of the equations from their standard form.

The conditions of scale invariance for the bosonic NSNS fields have the familiar form involving the β-function for the metric and the B-field

$$\begin{aligned}\beta^G_{mn} &\equiv R_{mn} - \tfrac{1}{4} H_{mkl} H_n{}^{kl} - \mathcal{T}_{mn} = -D_m X_n - D_n X_m \ ,\\ \beta^B_{mn} &\equiv \tfrac{1}{2} D^k H_{kmn} + \mathcal{K}_{mn} = X^k H_{kmn} + \partial_m Y_n - \partial_n Y_m \ ,\end{aligned}$$

where

$$\begin{aligned}\mathcal{T}_{mn} &\equiv \tfrac{1}{2}\mathcal{F}_m\mathcal{F}_n + \tfrac{1}{4}\mathcal{F}_{mpq}\mathcal{F}_n{}^{pq} + \tfrac{1}{4\times 4!}\mathcal{F}_{mpqrs}\mathcal{F}_n{}^{pqrs} - \tfrac{1}{2}G_{mn}\big(\tfrac{1}{2}\mathcal{F}_k\mathcal{F}^k + \tfrac{1}{12}\mathcal{F}_{kpq}\mathcal{F}^{kpq}\big) \ ,\\ \mathcal{K}_{mn} &\equiv \tfrac{1}{2}\mathcal{F}^k\mathcal{F}_{kmn} + \tfrac{1}{12}\mathcal{F}_{mnklp}\mathcal{F}^{klp} \ .\end{aligned}$$

Here $\mathcal{T}_{mn}$ is the stress tensor that follows from the type IIB action upon variation over G_{mn}. For $X_m = \partial_m \phi,\ Y_m = 0$ these equations follow from the standard type IIB supergravity action.

The key observation is that indeed there exist vectors X_m and Y_m such that the equations above are satisfied for the ABF background. The vector X_m turns out to be

$$\begin{aligned} X \equiv X_m dx^m = & \, c_0 \frac{1+\rho^2}{1-\varkappa^2\rho^2} dt + c_1\, \rho^2 \sin^2\zeta\, d\psi_2 + c_2 \frac{\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} d\psi_1 \\ & + c_3 \frac{1-r^2}{1+\varkappa^2 r^2} d\varphi + c_4\, r^2 \sin^2\xi\, d\phi_2 + c_5 \frac{r^2\cos^2\xi}{1+\varkappa^2 r^4 \sin^2\xi} d\phi_1 \\ & + \frac{\varkappa^2\rho^4 \sin 2\zeta}{2(1+\varkappa^2\rho^4\sin^2\zeta)} d\zeta + \frac{1}{\rho}\big(1 - \frac{3}{1-\varkappa^2\rho^2} + \frac{2}{1+\varkappa^2\rho^4\sin^2\zeta}\big) d\rho \\ & + \frac{\varkappa^2 r^4 \sin 2\xi}{2(1+\varkappa^2 r^4\sin^2\xi)} d\xi + \frac{1}{r}\big(1 - \frac{3}{1+\varkappa^2 r^2} + \frac{2}{1+\varkappa^2 r^4\sin^2\xi}\big) dr \ , \end{aligned}$$

and it can be split in the following way

$$X_m = I_m + Z_m \ , \qquad D_m I_n + D_n I_m = 0 \ , \qquad D^m I_m = 0 \ ,$$

where $I^m = \sum_{i=1}^6 c_i (I^{(i)})^m$. The index i labels the 6 isometric directions $y^i = (t, \psi_2, \psi_1, \varphi, \phi_2, \phi_1)$ of the 10d ABF metric and c_i are arbitrary constant coefficients. The quantities $(I^{(i)})^m$ are the 6 independent commuting Killing vectors of the ABF background: the Lie derivatives of the G, B and $\mathcal{F}$-fields along I^m all vanish. If we split the coordinates as $x^m = (y^i, x^\mu)$ where $\mu = 1,2,3,4$ labels the non-isometric directions $x^\mu = (\zeta, \rho, \xi, r)$, then

$$I_m = \sum_{i=1}^6 \delta^i_m c_i G_{ii}(x^\mu) \ , \qquad I^m = \delta^i_m c_i = \text{const} \ , \qquad Z_m = \delta^\mu_m Z_\mu(x^\nu) \ .$$

For the ABF background the vector Y_m turns out to coincide with X provided we fix c_i to the following specific values

$$c_0 = c_3 = 4\varkappa \ , \qquad c_1 = c_4 = 0 \ , \qquad c_2 = -c_5 = 2\varkappa .$$

The next surprising observation is that for these specially chosen values of c_i the vector X_m satisfies also a direct generalisation of the dilaton equation

$$\beta^X \equiv R - \tfrac{1}{12} H^2_{mnk} + 4 D_k X^k - 4 X_k X^k = 0 \ .$$

The scale invariance equations for the $\mathcal{F}$-fields (to be discussed later) will not, however, have the familiar supergravity form of 1st-order equations for $\mathcal{F}$ (these should follow from the Weyl invariance conditions). Instead they will be of 2nd order, $D^2\mathcal{F} + ... = X$-dependent terms, and for $X_m = \partial_m \phi$ will be a consequence of the 1st-order supergravity equations.

Now we come to the discussion of the modified type IIB equations for the RR couplings satisfied by the ABF background [12]. Introduce $Z = Z_m dx^m$ and $I = I_m dx^m$. The equations for the one-form are

$$\begin{aligned} & D^m \mathcal{F}_m - Z^m \mathcal{F}_m - \tfrac{1}{6} H^{mnp} \mathcal{F}_{mnp} = 0 \ , \qquad I^m \mathcal{F}_m = 0 \ , \\ & (d\mathcal{F}_1 - Z \wedge \mathcal{F}_1)_{mn} - I^p \mathcal{F}_{mnp} = 0 \ . \end{aligned}$$

We have added the condition $I^m \mathcal{F}_m = 0$ as an independent equation on $\mathcal{F}_1$. Similarly, the equations satisfied for the three-form $\mathcal{F}_3$ are found to be

$$D^p \mathcal{F}_{pmn} - Z^p \mathcal{F}_{pmn} - \tfrac{1}{6} H^{pqr} \mathcal{F}_{mnpqr} - (I \wedge \mathcal{F}_1)_{mn} = 0 \ ,$$
$$(d\mathcal{F}_3 - Z \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{mnpq} - I^r \mathcal{F}_{mnpqr} = 0 \ .$$

Finally, the equations satisfied by $\mathcal{F}_5$ of the ABF background are

$$D^r \mathcal{F}_{rmnpq} - Z^r \mathcal{F}_{rmnpq} + \tfrac{1}{36} \varepsilon_{mnpqrstuvw} H^{rst} \mathcal{F}^{uvw} - (I \wedge \mathcal{F}_3)_{mnpq} = 0 \ ,$$
$$(d\mathcal{F}_5 - Z \wedge \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{mnpqrs} + \tfrac{1}{6} \varepsilon_{mnpqrstuvw} I^t \mathcal{F}^{uvw} = 0 \ .$$

These two are equivalent in view of the self-duality of $\mathcal{F}_5$.

These modified equations reduce back to the standard type II equations if we drop all terms with I_m and assume that $dZ = 0$, *i.e.* if we set

$$Z_m \to \partial_m \phi \ , \qquad I_m \to 0 \ .$$

The structure of modified equations supports the interpretation of Z as a generalised "dilaton one-form", while the isometry vector I effectively drives the deformation of the standard type IIB equations.

An interesting observation is that there exist certain combinations of the modified supergravity equations that depend on Z and I only through the combination $X = Z + I$, which entered the NS-NS equations of the previous section. These are found by adding together equations of equal form degree, for example, the equation of motion for the R-R three-form and the Bianchi identity for the R-R one-form. The resulting X-dependent equations are given by

$$D^m \mathcal{F}_m - X^m \mathcal{F}_m - \tfrac{1}{6} H^{mnp} \mathcal{F}_{mnp} = 0 \ ,$$
$$D^p \mathcal{F}_{pmn} - X^p \mathcal{F}_{pmn} - \tfrac{1}{6} H^{pqr} \mathcal{F}_{mnpqr} + (d\mathcal{F}_1 - X \wedge \mathcal{F}_1)_{mn} = 0 \ ,$$
$$D^r \mathcal{F}_{rmnpq} - X^r \mathcal{F}_{rmnpq} + \tfrac{1}{36} \varepsilon_{mnpqrstuvw} H^{rst} \mathcal{F}^{uvw} + (d\mathcal{F}_3 - X \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{mnpq} = 0 \ .$$

Using the self-duality of $\mathcal{F}_5$ the last equation can be also written as

$$(d\mathcal{F}_5 - X \wedge \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{pqrlmn} - \tfrac{1}{6} \varepsilon_{pqrlmnvstu} (D^v \mathcal{F}^{stu} - X^v \mathcal{F}^{stu} - \mathcal{F}^v H^{stu}) = 0 \ .$$

As will be discussed below, these three equations are already sufficient for deriving candidates for the scale invariance equations for the $\mathcal{F}$-fields, which are 2nd order in derivatives.

Let us return to the discussion of the scale invariance conditions for the couplings of the GS sigma model and consider the equations for the RR couplings $\mathcal{F}$ that should follow from the requirement of (1-loop) UV finiteness of the 2d model. One can argue that the conditions for the G and B-field couplings should have the form

$$\beta^{\mathcal{F}}_{k_1 \ldots k_s} \equiv \frac{1}{2} D^2 \mathcal{F}_{k_1 \ldots k_s} + \ldots = X^m \partial_m \mathcal{F}_{k_1 \ldots k_s} + \sum_i \mathcal{F}_{k_1 \ldots m \ldots k_s} \partial_{k_i} X^m \ ,$$

where we have omitted possible non-linear terms such as $R\mathcal{F} + DH\mathcal{F} + \ldots$ on the l.h.s. The X-dependent Lie derivative term on the r.h.s. reflects the reparametrisation (or off-shell x^m-renormalisation) freedom.

For $X_m = \partial_m \phi$ the equation for $\beta^{\mathcal{F}}_{k_1 \ldots k_s}$ should be a consequence of stronger Weyl invariance conditions, which should be equivalent to the type II supergravity equations with $Z = X = d\phi$.

Indeed, combining ("squaring") the familiar $dF + ... = 0,\ d\star F + ... = 0$ equations leads to $d\star d\star F + \star d\star dF + ... = 0$ or $D^2F + ... = 0$, where the leading term is the Hodge–de Rham operator.

Moreover, the same equations should follow also from the modified type II equations (as, e.g., the ABF background that solves the modified equations should also be a solution of the scale invariance conditions). This should provide a non-trivial consistency check: after properly "squaring" the equations of modified supergravity, the dependence on the Z and I vectors in any candidate scale invariance equations should appear only through their sum $X = Z + I$. Note that to be a candidate for the scale invariance conditions these equations should have the following properties:

(i) vanish on the modified supergravity equations with $X = d\phi,\ Y = 0$

(ii) depend on Z and I through $X = Z + I$

(iii) depend on X through Lie derivatives.

Starting with the modified equations and properly acting with $\star d\star$ and $d\star$, we arrive at the equations, which satisfies the above properties. For $\mathcal{F}_1$ we find

$$\begin{aligned} D^2\mathcal{F}_m - R_{mn}\mathcal{F}^n + \tfrac{1}{4}(R - \tfrac{3}{4}H^2)\mathcal{F}_m \\ + \tfrac{1}{2}H^{pnk}H_{mpn}\mathcal{F}_k - \tfrac{1}{6}D_mH^{pnk}\mathcal{F}_{pnk} - \tfrac{1}{2}H^{pnk}D_p\mathcal{F}_{nkm} \\ = 2(X^pD_p\mathcal{F}_m + D_mX^p\mathcal{F}_p) + \beta^G_{mn}\mathcal{F}^n - \tfrac{1}{2}\beta^B_{nk}\mathcal{F}^{nk}{}_m\ . \end{aligned}$$

The equation for $\mathcal{F}_3$ reads as

$$\begin{aligned} & D^2\mathcal{F}_{nkm} - R_{a[n}\mathcal{F}^a{}_{km]} + R_{ab[nk}\mathcal{F}^{ab}{}_{m]} + \tfrac{1}{4}(R - \tfrac{3}{4}H^2)\mathcal{F}_{nkm} \\ & + \tfrac{1}{2}H^{abc}H_{ab[n}\mathcal{F}_{km]c} - \tfrac{1}{2}H^{abc}H_{a[nk}\mathcal{F}_{m]bc} \\ & + D^aH_{a[nk}\mathcal{F}_{m]} + H_{a[nk}D^a\mathcal{F}_{m]} - \mathcal{F}_aD^aH_{nkm} \\ & - \tfrac{1}{6}D_{[n}H^{abc}\mathcal{F}_{km]abc} - \tfrac{1}{2}H^{abc}D_a\mathcal{F}_{bcnkm} \\ & \quad = 2(X^aD_a\mathcal{F}_{nkm} + D_{[n}X^a\mathcal{F}_{km]a}) + \beta^G_{a[n}\mathcal{F}^a{}_{km]} + \beta^B_{[nk}\mathcal{F}_{m]} - \tfrac{1}{2}\beta^B_{ab}\mathcal{F}^{ab}{}_{nkm}\ , \end{aligned}$$

while the equation for $\mathcal{F}_5$ is

$$\begin{aligned} & D^2\mathcal{F}_{ijklm} - R_{a[i}\mathcal{F}^a{}_{jklm]} + R_{ab[ij}\mathcal{F}^{ab}{}_{klm]} + \tfrac{1}{4}(R - \tfrac{3}{4}H^2)\mathcal{F}_{ijklm} \\ & + \tfrac{1}{2}H^{abc}H_{ab[i}\mathcal{F}_{jklm]c} - \tfrac{1}{2}H^{abc}H_{a[ij}\mathcal{F}_{klm]bc} \\ & + D^aH_{a[ij}\mathcal{F}_{klm]} + H_{a[ij}D^a\mathcal{F}_{klm]} - \mathcal{F}_{a[ij}D^aH_{klm]} \\ & + \tfrac{1}{12}\varepsilon_{ijklmbdef}(D_aH^{abc}\mathcal{F}^{def} + H^{abc}D_a\mathcal{F}^{def} - \mathcal{F}^{abc}D_aH^{def}) = \\ & = 2(X^aD_a\mathcal{F}_{ijklm} + D_{[i}X^a\mathcal{F}_{jklm]a}) + \beta^G_{a[i}\mathcal{F}^a{}_{jklm]} + \beta^B_{[ij}\mathcal{F}_{klm]} + \tfrac{1}{12}\varepsilon_{ijklmabcde}(\beta^B)^{ab}\mathcal{F}^{cde}\ . \end{aligned}$$

This expression is consistent with the self-duality of $\mathcal{F}_5$ (in particular, the third and fourth lines are manifestly dual to each other).

These 2nd-order equations for $\mathcal{F}_1$, $\mathcal{F}_3$ and $\mathcal{F}_5$ exhibit obvious structural similarities. In particular, they contain the expected Hodge–de Rham operator terms and the vector X only enters through the reparametrisation terms.

In summary, we have suggested the modified supergravity equations that replace the condition of Weyl invariance and proved that they are satisfied by the background fields of the η-deformed theory. We have also derived the equations expressing the conditions of scale invariance and showed that they are satisfied by the corresponding background fields. Thus, the

η-deformed model is a new interesting example of a sigma model which is scale but not Weyl invariant.

3 Quantum spectral curve

As alluded to in the introduction, finding the excitation spectrum of the $AdS_5 \times S^5$ superstring theory – also dubbed the $AdS_5 \times S^5$ *spectral problem* – has been an important goal on the way to understanding the AdS/CFT correspondence. For the $AdS_5 \times S^5$ superstring theory string excitations can be related to scaling dimensions of local operators of planar $\mathcal{N} = 4$ SYM theory, such that finding a description of the former directly also yields a description of the latter. Apart from its consequences for the AdS/CFT correspondence, having a clear description of these sets of observables is desirable in itself: it is very rare to have so much control over the observables in an interacting quantum field theory.

Using the integrability present in both the planar gauge and string theory discussed above it is possible to give a very simple but exact description of the spectral problem. This description has gotten simpler over the years, going through various intermediate stages, and at present the simplest form known is the *quantum spectral curve* (QSC) [13]. The QSC has led to many interesting results: not only did it allow for the analysis of arbitrary states such as twist operators, it turned out to be a starting point for the study of different observables in $\mathcal{N} = 4$ SYM, such as the BFKL pomeron, the cusped Wilson line and the quark-anti-quark potential. This is remarkable, as these observables are outside of the scope of the original spectral problem. Its wide applicability suggests a deeper level to the QSC that is yet to be understood. One might also wonder whether the occurrence of such a drastic simplification to the spectral problem is unique to the $AdS_5 \times S^5$ case.

In an effort to gain more understanding of the QSC and more generally the role played by integrability in the simplification of the spectral problem a project was undertaken to construct the quantum spectral curve for the η-deformed superstring theory. More precisely, starting from the exact quantum scattering theory described by the S-matrix constructed in [2] one can follow the same path as was taken for the original $AdS_5 \times S^5$ case: the first step was already undertaken in [14] in the construction of the η-deformed Thermodynamic Bethe Ansatz equations, an infinite set of non-linear integral equations.

To understand these equations and their constructions better the Thermodynamic Bethe Ansatz (TBA) method was applied to a simpler model first: Inozemtsev's elliptic spin chain. This spin chain with elliptic long-range interactions was never analyzed in the thermal regime, despite interesting claims being made about its thermodynamic behaviour [15], namely being insensitive to the presence of supersymmetry. The TBA-equations were derived in [16], allowing for the numerical analysis necessary to confirm the insensitivity to supersymmetry. Moreover, the succesful application of this approach provides further evidence towards the integrability of the model, which has still not been established.

After these introductory remarks we come to the derivation of the η-deformed quantum spectral curve. The first step is to rewrite the TBA-equations in the form of a Y-system: a set of finite-difference equations for the unknown 2π-periodic functions $Y_{a,s}$ that can be compactly written as

$$Y^+_{a,s} Y^-_{a,s} = \frac{(1 + Y_{a-1,s})(1 + Y_{a+1,s})}{(1 + Y_{a,s-1})(1 + Y_{a,s+1})},$$

where $f^+(u) = f(u + ic)$ with c the parameter carrying the η-deformation and where the indices (a, s) take values on what is known as the Y-hook. To specify which solutions of the Y-system should be considered to describe the spectral problem one has to impose additional conditions known as discontinuity equations. These discontinuities relate the jump of the various Y functions at their infinitely many branch cuts on the complex plane. These discontinuities were derived in [17], moreover showing equivalence of the Y-system with the original TBA-equations in line with the original work in [18].

The second step consists in further simplifying the Y-system equations by the introduction of a new parametrisation known as the T-system: the Y-system equations simplify further and can now be written as

$$T^+_{a,s}T^-_{a,s} = T_{a,s+1}T_{a,s-1} + T_{a+1,s}T_{a-1,s},$$

where the (a, s) live on the T-hook. This equation is known as the Hirota equation, a ubiquitous equation in integrability. The price to pay for the further simplicity of the equations is that the additional conditions become more convoluted. The T-functions and the Hirota equations admit a huge gauge freedom that makes it hard to select a convenient gauge to work in, and moreover it seems that no single convenient gauge exists. Nevertheless, in [17] four sets of T-gauges were proposed inspired by T-system for $\mathrm{AdS}_5 \times S^5$ in [19]. Their construction is based on spectral theory for periodic functions on the complex plane, more details of which can be found in [20]. Combined with gluing conditions that relate the different gauges this gives a full description of the spectral problem. In principal this T-system can be used to analyse the spectrum of η-deformed $\mathrm{AdS}_5 \times S^5$ superstring theory, but like in the $\mathrm{AdS}_5 \times S^5$ case a further simplification exists.

Using the solution theory of the Hirota equation [21] one can reparametrise one of the T-gauges into so-called $\mathbf{P}$ functions, which can be regarded as the first step in the construction of the quantum spectral curve. Working out all the constraints ultimately yields five independent functions $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}^3, \mathbf{P}^4, \mu_{12}$ which describe all the T-gauges. All these functions are 2π (anti)-periodic (at least just above the real axis) and have branch cuts: the $\mathbf{P}$ have only one on the real axis whereas μ_{12} has an infinite ladder that goes all the way to imaginary infinity. This analytic structure is illustrated in Fig. 1. All of these branch cuts are of square-root type though, allowing for a relatively large amount of control. Introducing auxiliary functions $\mathbf{P}_a, \mathbf{P}^a$ and μ_{ab} the equations that these objects need to satisfy – known as the $\mathbf{P}\mu$ system – become particularly pleasing. Written for $a, b = 1, \ldots, 4$ it reads

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a, \quad \tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b, \quad \mathbf{P}_a\mathbf{P}^a = 0, \quad \mathrm{Pf}(\mu) = 1,$$

where the tilde indicates the second sheet evaluation of the function involved, the summation convention is followed and $\mathrm{Pf}(\mu)$ is the Pfaffian of the antisymmetric matrix μ. The form of these equations exactly coincide with the $\mathbf{P}\mu$ system derived for the undeformed $\mathrm{AdS}_5 \times S^5$ superstring, consistent with the similarities between the representation theory of the $\mathrm{AdS}_5 \times S^5$ superstring and its η-deformed counterpart. These equations form one of the many equivalent ways to write the QSC-equations. Another important set of equations one can derive is the dual $\mathbf{Q}\omega$ system, which also has the same form as in the undeformed case.

As before, these equations do not give a full description of the spectral problem, which need to be supplemented by boundary conditions that encode which solution of the $\mathbf{P}\mu$ system corresponds to which state in the η-deformed string theory. Clearly, this is also where the difference between the undeformed and deformed becomes most pronounced. In the undeformed case, the extra boundary conditions come in the form of asymptotics, that is prescribed limiting behaviour for all the functions in the $\mathbf{P}\mu$ system as one sends $u \to \infty$. Clearly, such a condition

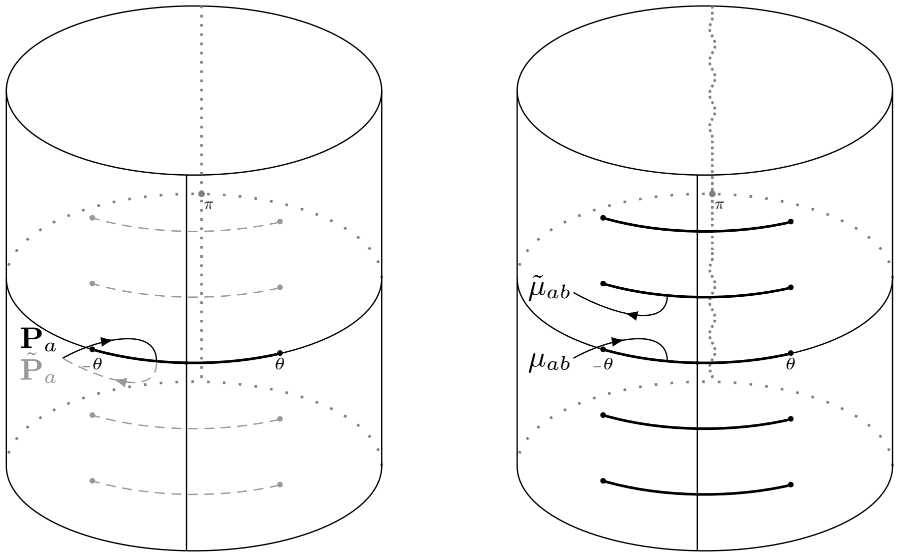


Figure 1: The analytic structure of the $\mathbf{P}_a$ (a) and the μ_{ab} (b) on the cylinder. The thick lines indicate branch cuts between $\pm\theta$ on the first sheet. The continuation of short-cutted μ_{ab} can be expressed on the first sheet using its $2ic$ periodicity. The squiggly line in (b) indicates that for generic θ outside the physical strip the μ_{ab} cannot be put on (a finite cover of) the cylinder. Figure taken from Ref. [17].

is impossible for 2π-periodic functions and one should consider other options. Using the TBA-equations it proved possible to prove that at least some of the functions carry information about the quantum labels of an excited state in their asymptotics as one considers the limit $u \to i\infty$, i.e. moves infinitely far up the cylinder. Postulating that this limit should yield the charges also for the other functions in the QSC one can deduce a fairly simple set of asymptotics [17]: for $z = e^{-iu/2}$ one finds that

$$\mathbf{P}_a \simeq A_a z^{-\tilde{M}_a}, \quad \mathbf{Q}_i \simeq B_i z^{\hat{M}_i} \quad \mathbf{P}^a \simeq A^a z^{\tilde{M}_a}, \quad \mathbf{Q}^i \simeq B^i z^{-\hat{M}_i},$$

where

$$\tilde{M} = \frac{1}{2}\left\{J_1 + J_2 - J_3 + 2, J_1 - J_2 + J_3, -J_1 + J_2 + J_3, -J_1 - J_2 - J_3 - 2\right\},$$
$$\hat{M} = \frac{1}{2}\left\{\Delta - S_1 - S_2 + 2, \Delta + S_1 + S_2, -\Delta - S_1 + S_2, -\Delta + S_1 - S_2 - 2\right\}.$$

The six labels $(\Delta, J_1, J_2, J_3, S_1, S_2)$ are the quantum labels for states in the η-deformed string theory. The prefactors A_a, A^a, B_i and B^i are explicitly known trigonometric functions of the quantum labels.

This defines the η-deformed QSC, which can be used for further analysis of the η-deformed string theory. Particularly interesting questions are (1) what is the field of numbers involved in the actual computation of string energies, (2) can the deformed BFKL eigenvalue be computed and possibly shed light on the thermal BFKL theory through the mirror duality of the η-deformed string theory and (3) which operator is related to the Hagedorn temperature computation through the same mirror duality.

4 New and old integrable models

The string sigma model on $\mathrm{AdS}_5 \times S^5$ and its deformations provide a source for a number of known as well as new finite-dimensional integrable models that can be obtained by applying various reduction schemes. At first glance this concerns a particle limit, *i.e.* the limit of vanishing string length. Studying this limit, as well as other finite-dimensional reductions, is principally important since in spite of the progress relying on the surmised quantum integrability, quantization of the $\mathrm{AdS}_5 \times S^5$ superstring from first principles is still an open problem. Especially for light string states, for which the $\mathfrak{psu}(2,2|4)$ charges take finite values, it has been a renowned challenge to determine the spectrum beyond the leading order [22], $E \sim \lambda^{1/4}$, where λ is the 't Hooft coupling. A way to proceed was offered in [23], where, arguing about supersymmetric effects heuristically, an investigation of the isometry group orbits of a pulsating string allowed to reproduce the first quantum corrections of order $\lambda^{-1/4}$. For this the crucial step has been to reformulate the system as a massive $\mathrm{AdS}_5 \times S^5$ particle with the mass term determined by the stringy non-zero-modes. But since quantization of massive as well as massless AdS superparticles is not yet understood, to attack this question we utilised the gauge invariant Kirillov–Kostant–Souriau method of coadjoint orbits .

In particular, in [24] we applied the orbit method quantization to the AdS_2 superparticle on $\mathrm{OSP}(1|2)/\mathrm{SO}(1,1)$, yielding a Holstein–Primakoff-like realization of the superisometries $\mathfrak{osp}(1|2)$. However, for the massless case the κ-symmetry transformation leaves only one physical real fermion, rendering the model quantum inconsistent. This problem was overcome in [25] by studying the AdS_3 superparticle on $\mathrm{OSP}(1|2) \times \mathrm{OSP}(1|2)/\mathrm{SL}(2,\mathbb{R})$. Here, calculation of the symplectic form as well as of the Noether charges naturally split up into left and right chiral sectors, yielding a quantum realization of $\mathfrak{osp}_l(1|2) \oplus \mathfrak{osp}_r(1|2)$. For the massless particle it was then found that the superisometry algebra extends to the corresponding superconformal algebra $\mathfrak{osp}(2|4)$, with its 19 charges realized by all possible real quadratic combinations of the phase space variables.

With the goal to prepare the formalism for general semi-symmetric spaces, in [26] we devised orbit method quantization of the massive bosonic AdS_N particle in a scheme manifestly exposing the coset nature of AdS_N. Apart from reproducing previous results, this led to a new quantization scheme in terms of dual oscillator variables. Furthermore, we also generalized the results of [24] and [25] and proposed an ansatz for the AdS_4 superparticle.

In [27] and [28] we continued probing the integrability of sigma models on η-deformed backgrounds. Building on previous work for the η-deformed Neumann model [29], in [27] we studied generalized bosonic string solutions on $\mathrm{AdS}_5 \times S^5$, yielding a novel η-deformed Neumann–Rosochatius model. By constructing a 4×4 Lax representation we obtained a set of abelian integrals of motion, ensuring Liouville integrability of the system. We furthermore established how these generalize the integrals of motion of the known limiting cases, *i.e.*, of the η-deformed Neumann and geodesic systems as well as of the undeformed Neumann–Rosochatius model.

As sigma models on η-deformed backgrounds enjoy a quantum deformed symmetry algebra, $\mathrm{U}_q(\mathfrak{psu}(2,2|4))$ in the case of $(\mathrm{AdS}_5 \times S^5)_\eta$, it is interesting to track how the non-abelian symmetry charges behave under the η-deformation. In [28] this question was posed in the simplest possible setup, the geodesic motion on the squashed sphere $(\mathrm{S}^2)_\eta$, the manifold of the Fateev sausage model [30]. New local integrals of motion were found, which by construction form an $\mathfrak{sl}(2)$ Poisson algebra and therefore amount to maximal superintegrability of the system. With this observation we devised a canonical map to an auxiliary sphere, by which we completely solve the geodesics problem.

5 Integrable deformations of CFTs and functional equations

While the discussion in sections 1–4 focusses on the analysis of concrete world-sheet theories directly related to the AdS/CFT correspondence, here we take a step back and look at some general questions that arise in the above procedure:

1. Give a quantum world-sheet theory, how can one describe its integrable structure? And how can one detect deformations which preserve all or part of the integrability?

2. How much ambiguity is there in solutions to functional equations, such as T- and Y-systems? What additional conditions make their solution unique?

We will not answer any of these questions for the models discussed in sections 1–4, but we will outline a framework which is convenient to discuss the questions in point 1, and we will answer question 2 at least for a much simpler type of Y-system than those arising in AdS/CFT.

The fundamental example of our approach is the integrable structure of the free boson, perturbed and unperturbed, captured in terms of non-local conserved charges, as developed by Bazhanov, Lukyanov and Zamolodchikov [31, 32]. There, the authors construct mutually commuting families of conserved charges by path-ordered integrals of free boson vertex operators which depend on a spectral parameter λ. They argue that the large-λ expansion encodes the values of all local conserved charges, i.e. conserved charges obtained by integrating conserved currents of the model.

The setting in which we discuss question 1 is that of two-dimensional conformal quantum field theory in euclidean signature. We allow the world sheet to be decorated with one-dimensional line defects, across which the bulk fields of the theory may have discontinuities. The line defects themselves are topological in the sense that they can be deformed without affecting the value of a correlator, as long as they are not taken across field insertions. The most basic example here is the critical Ising model, where across the line defect the sign of the spin-spin coupling is inverted (this is called a disorder line). When taken across such a disorder line, the Ising spin field changes sign, while the energy field (the field dual to the temperature perturbation) is continuous, see e.g. [33].

One now observes that in addition to bulk fields, there are new fields in a CFT with line defects, namely fields which are localised on the line defect itself. These fields behave differently from bulk fields in that they are not local (they have monodromies) when moved around other field insertions. Geometrically this is very intuitive, as in moving a defect field around another field tangles up the defect lines and there is no reason for the resulting value of the correlator to be equal to the initial one.

Consider now a world sheet which is a cylinder, with a topological defect line with defect condition X wrapped around it. This defines an operator D_X on the state space $\mathcal{H}$ of the CFT. Since X is a topological defect condition, this operator commutes with the Hamiltonian of the CFT:

$$[D_X, H_{\text{CFT}}] = 0 \quad , \quad H_{\text{CFT}} = \tfrac{1}{2\pi}(L_0 + \bar{L}_0 - \tfrac{c}{12}) \ .$$

The operator D_X is thus a *conserved charge* for the conformal field theory, albeit one which typically does not arise by integrating a conserved current. We can now ask the following natural question:

> Is it possible to deform the CFT and at the same time deform some of the conserved charges D_X such that they remain conserved for the perturbed theory?

Let us start with the perturbation of the CFT. Fix a bulk field Φ, which we assume to spinless, that is, of chiral/anti-chiral conformal weight (h, h) (so its total scaling dimension is $2h$). We assume Φ to be relevant or marginal ($h \leq 1$), so that the perturbation does not affect the UV fixed point, which is our starting CFT. The perturbed Hamiltonian is

$$H_{\text{pert}}(\mu) = H_{\text{CFT}} + \mu \int_0^{2\pi} \Phi(\theta) d\theta \ ,$$

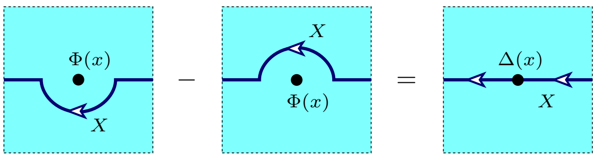
where the integral is around the cylinder, and where $\mu \in \mathbb{C}$ is the strength of the perturbation. On the defect X we consider a perturbation by a chiral defect field ψ_X and an anti-chiral defect field $\bar{\psi}_X$. We demand that their conformal weights are $(h, 0)$ and $(0, h)$, respectively, with h the same value as for the bulk perturbation. We perturb the defect X by the defect field

$$\lambda \psi_X(\theta) + \tilde{\lambda} \bar{\psi}_X(\theta) \ ,$$

where $\lambda, \tilde{\lambda} \in \mathbb{C}$ give the strengths of the perturbations. Write $D_X(\lambda, \tilde{\lambda})$ for the perturbed defect operator (defined by expanding an exponentiated integral of the above defect field, see [34,35]). We would like to understand when

$$[D_X(\lambda, \tilde{\lambda}), H_{\text{pert}}(\mu)] = 0 \ ,$$

that is, when $D_X(\lambda, \tilde{\lambda})$ is a conserved charge for the perturbed Hamiltonian. In this setup, there is a surprisingly simple sufficient condition, the *commutation condition* which has to hold locally in correlators of the unperturbed CFT, and which guarantees the vanishing of the above commutator to all orders in the perturbing parameters [34,35]:



The left hand side is the difference between passing the topological defect line X above and below the perturbing field Φ. The defect field $\Delta(x)$ on the right hand side is the difference of placing the two defect fields ψ_X and $\bar{\psi}_X$ in either order on the defect line,

$$\Delta(x) = \lim_{\epsilon \to 0} \left(\psi_X(x+\epsilon)\bar{\psi}_X(x) - \bar{\psi}_X(x+\epsilon)\psi_X(x) \right) \ .$$

Finally, the perturbing parameters have to satisfy the relation (the constant depends on normalisation choices and is conventional, see [35] for details):

$$\mu = -2i\, \lambda \tilde{\lambda} \ .$$

This simple observation is the starting point of a host of interesting structural insights:

- For a fixed value of μ, so for a fixed perturbation of the CFT, a solution to the commutation condition gives rise to a *one-parameter-family* of conserved charges, parametrised by $\zeta \in \mathbb{C} \setminus \{0\}$ via $\lambda = \zeta$, $\tilde{\lambda} = \mu/\zeta$.
- One important class of solutions to the commutation condition is the case $\mu = 0$, where the CFT is not perturbed at all. Then the condition is $\Delta = 0$, which in turn can be guaranteed by simply choosing $\bar{\psi}_X = 0$ (or $\psi_X = 0$). In this way, one can investigate the integrable structure of a CFT.
- The commutation condition can be expressed in terms of representation theoretic data obtained from the conformal field theory [35], and in examples can be related to representations of certain quantum groups. This recovers results of [31, 32].
- For example by using the relation to quantum groups, or by direct computation, one can establish that for certain choices of perturbing defect fields and defect labels X, the various conserved charges $D_X(\lambda, \tilde{\lambda})$ commute with each other, for different values of X and λ, $\tilde{\lambda}$, provided one keeps fixed the value of μ. Furthermore, one finds in examples that the $D_X(\lambda, \tilde{\lambda})$ satisfy functional relations of T-system type.

Since the $D_X(\lambda, \tilde{\lambda})$ mutually commute, they can be simultaneously diagonalised[1] on the state space $\mathcal{H}$, the same T-system functional relations are also satisfied by the eigenvalues. One arrives at a question in complex analysis: given a system of functional relations between functions which are analytic in a certain domain, what can we say about the solutions?

To address this question, it has turned out to be useful to rewrite T-system functional equations as Y-system functional equations. In a simple variant, these look as follows:

$$Y_n(x + is)Y_n(x - is) = \prod_{m=1}^{N} (1 + Y_m(x))^{G_{nm}} \quad . \tag{Y}$$

Here, $s > 0$ is a parameter, G is the adjacency matrix of a Dynkin diagram with N nodes, and the equation has to hold for $n = 1, \dots, N$ and all $x \in \mathbb{R}$. The functions Y_n are required to be analytic in the strip $\mathbb{R} \times (-is, is)$ and to have a continuous extension to the closure $\mathbb{R} \times [-is, is]$. Note that this is the smallest connected domain on which one can make sense of the above functional relation.

We have the following remarkable uniqueness and existence result, proven in [36], where also numerous references to the physical literature can be found on which the method used in the proof is based:

Theorem: Let $a_1, \dots, a_n : \mathbb{R} \times [-is, is] \to \mathbb{C}$ be valid asymptotics (see below). Then there exists a unique solution $Y_1, \dots, Y_N$ to (Y) which satisfies the above analyticity conditions, as well as, for $n = 1, \dots, N$,

1. $Y_n(\mathbb{R}) \subseteq \mathbb{R}_{>0}$, *(real & positive)*
2. $Y_n(z) \neq 0$ for all $z \in \mathbb{R} \times [-is, is]$. *(no roots)*
3. $\log Y_n(z) - a_n(z)$ is bounded on $\mathbb{R} \times [-is, is]$. *(asymptotics)*

[1] In this exposition we ignore all question of convergence and existence of integrals in perturbative expansions, as well as analytic questions such as discreetness of spectra – these points need a careful treatment in each example.

The notion of “valid asymptotics” is somewhat technical (see [36] for details), but the most important examples are, for $n = 1, \ldots, N$,

$$a)\ a_n(z) = 0 \quad , \quad b)\ a_n(z) = e^{\gamma z/s} w_n \quad , \quad c)\ a_n(z) = \cosh(\gamma z/s) w_n \ ,$$

where w is the Perron–Frobenius eigenvector of G and $\gamma > 0$ is related to the corresponding eigenvalue by $Gw = 2\cos(\gamma)w$.

The physical interpretation of this mathematical result is that the Y_n describe the ground state eigenvalues of the corresponding conserved charges in (a) the unperturbed case (one proves that the $Y_n(x)$ are independent of x in this case); (b) the massless case $\mu = 0$; (c) for a massive perturbation. Moreover, it is shown in [36] that the unique solution can be obtain by solving a non-linear integral equation of TBA-type.

In summary, the above constructions and results indicate that a promising approach to obtain non-perturbative information about a perturbed CFT is to first try to establish functional relations satisfied by families of mutually commuting perturbed defect operators, and to then try to prove existence and uniqueness statements for the functional equations satisfied by their eigenvalues, similar to the theorem above.

6 Integrable lattice discretisation from the universal R-matrix

Establishing that the integrability of the world-sheet sigma model for strings on AdS persists at the quantum level is a hard probem that has remained elusive up to now. Most importantly, one has to make sure that renormalisation of ultraviolet divergencies does not spoil integrability. A promising strategy to reach this goal is based on the use of lattice regularisations for integrable two-dimensional quantum field theories. There are some well-known examples including the Sine–Gordon model where lattice regularisations have been constructed which manifestly preserve integrability. However, up to now there does not exist a sufficiently general framework to construct integrable lattice regularisations for all integrable models of our interest, and in particular for the sigma models relevant for string theory on $\mathrm{AdS}_5 \times S^5$.

In a part of our project, carried out jointly with the postdoc Carlo Meneghelli, we have described a systematic approach for the construction of large families of integrable lattice regularisations [37]. This approach has been fully realised in the examples of affine Toda field theories, prototypical examples of integrable quantum field theories sharing some qualitative features with the sigma models relevant for the study of string theory on AdS spaces. The results of the recent paper [38] offer very encouraging hints that the generalisation of the approach developed in [37] to integrable sigma models is getting within our reach.

The approach taken in [37] is based on two main ingredients.

(A) In all known examples one can view integrability as the consequence of powerful algebraic structures organising the algebras of observables of the field theories in question. The relevant algebraic structures are often referred to as quantum groups. It is in many cases possible to identify the quantum group relevant for integrability of a given quantum field theory from its Lagrangian description, or alternatively from its description as perturbed conformal field theory [35, 39].

This step is performed in [39] by considering the light-cone representation of the dynamics. The interaction terms generate a non-commutative algebra which can often be identified with a subalgebra of the relevant quantum group from which the full quantum group can be reconstructed by a standard construction (quantum double).

(B) The main proposal made in [37] is that the corresponding integrable lattice regularisation can then be constructed by following a systematic procedure reducing the main steps to problems in quantum group representation theory. The main ingredients are the so-called Lax-Matrix, R-matrices and the Baxter Q-operators. The proposal of [37] offers a recipe for the construction of these key ingredients by breaking it up into two steps: First finding representations of the relevant quantum group organising the algebra of observables on the lattice, and then evaluating the known universal R-matrix on these representations. The power of this approach has been illustrated in [37] by working it out in full detail in the example of the affine Toda field theories.

In the following we will describe this approach in a bit more detail. The algebraic structures called quantum groups are characterised to a large extend by an algebra structure (non-commutative product operation) and a co-product, essentially a rule for how to act with the algebra on tensor products of its representations. The co-product will generically not be symmetric with respect to exchange of the tensor factors in a tensor product $R_1 \otimes R_2$. A useful description of this asymmetry is provided by the R-matrices, operators $\mathsf{R}_{R_1R_2} : R_1 \otimes R_2 \to R_1 \otimes R_2$ relating the quantum group action on $R_1 \otimes R_2$ to the one defined from the action on $R_2 \otimes R_1$ by subsequent permutation of tensor factors. Basic results in quantum group theory assert the existence of a universal object of the form $R = \sum_{\imath\in\mathcal{I}} x_\imath \otimes y_\imath$, with $\{x_\imath; \imath \in \mathcal{I}\}$ and $\{y_\imath; \imath \in \mathcal{I}\}$ being suitable sets of generators for the quantum group, such that

$$\mathsf{R}_{R_1R_2} = (\pi_{R_1} \otimes \pi_{R_2})(R) = \sum_{\imath\in\mathcal{I}} \pi_{R_1}(x_\imath) \otimes \pi_{R_2}(y_\imath),$$

with $\pi_R(x)$ being the operator representing the quantum group element x within the representation R.

Two types of quantum representations are relevant in the context of integrable lattice models. Most basic is a representation π_q of the quantum group on the physical Hilbert space of the lattice model, often referred to as quantum space. It then turns out to be useful to consider one-parameter families of auxiliary representations $\pi_{a,\lambda}$ allowing us to define useful generating functions as

$$M(\lambda) = (\pi_{a,\lambda} \otimes \pi_q)(R). \tag{1}$$

If, for example, the auxiliary representations $\pi_{a,\lambda}$ are finite-dimensional one may view $M(\lambda)$ as a matrix having matrix elements which are operators acting on quantum space. The matrix $M(\lambda)$ turns out to be related to the monodromy matrix of the Lax connection in the corresponding classically integrable model.

It is known that infinite-dimensional representations $\pi_{a,\lambda}$ can also be of interest in this context. This requires in particular that it is possible to define a partial trace over the space $\mathcal{H}_a$ on which the representation $\pi_{a,\lambda}$ is realised

$$Q(\lambda) = \mathrm{Tr}_{H_a}(M(\lambda)). \tag{2}$$

Some choices for $\pi_{a,\lambda}$ will produce particularly useful families of operators $Q(\lambda)$, distinguished by two main properties:

- By specialising the parameter λ one can obtain from $Q(\lambda)$ evolution operators generating a lattice version of the physical time-evolution.
- The operators $Q(\lambda)$ and $Q(\mu)$ associated to any two values of the parameter always commute with each other, $[Q(\lambda), Q(\mu)] = 0$.

This implies that $Q(\lambda)$ represents a generating function for the conserved quantities of the integrable lattice model constructed in this framework.

Having identified the relevant quantum group in step (A) of this program, it remains to

1) find suitable representations π_q and $\pi_{a,\lambda}$, and
2) calculate $Q(\lambda)$ from (1) and (2).

It was shown in [37] in the example of the affine Toda theories that taking the first step 1) is often very simple. It turns out that the relevant representations can be found among the simplest possible representations the relevant quantum groups have. Given that the operators $\pi_q(x)$ represent physical observables, one gets important constraints on the representation π_q from the requirement that the behavior of $\pi_q(x)$ under hermitian conjugation should reflect the reality properties of the corresponding physical observable. It was found in [37] that such requirements single out a unique choice for the representation π_q to be used for the models of interest.

In order to complete this program it remains to perform step 2) above, the calculation of $Q(\lambda)$. A possible starting point is provided by the known explicit formulae for the universal R-matrices R, taking the form of infinite products. These formulae are very complicated. Somewhat unexpectedly, it has turned out that the representations the $\pi_{a,\lambda}$ and π_q we found to be relevant in this context have very useful special features simplifying the evaluation of $M(\lambda)$ via (1) enormously. As a result we have obtained fairly simple formulae representing the operators $Q(\lambda)$ as integral operators with explicitly known kernels.

In this way one not only obtains all the key ingredients for the construction of integrable lattice regularisation. The algebraic structures of the quantum group imply that $Q(\lambda)$ satisfies a system of functional equations. The known representation of $Q(\lambda)$ as an integral operator enables us to determine the analytic properties of the eigenvalues of $Q(\lambda)$. Taken together, functional equations and analytic properties lead to a complete mathematical characterisation of the set of functions $q(\lambda)$ representing the possible eigenvalues of $Q(\lambda)$. This constitutes the necessary groundwork for the solution of the spectral problem in these integrable quantum field theories.

The models studied in [37] are not yet the models of our ultimate interest from the point of view of applications to AdS/CFT. It was for a long time believed that the step to be taken to treat integrable sigma models in a similar way is big, requiring to overcome the problem of non-ultralocality of the Poisson brackets for the Lax matrices describing the integrable structures of nonlinear sigma models on the classical level. More recently at least two possible ways out have become visible. For some integrable nonlinear sigma models a modified zero curvature representation of the classical equations of motion has been found leading to fully ultralocal Poisson brackets [40]. It may be hoped that this approach can be generalised considerably. There furthermore exist proposals for dual descriptions of various nonlinear sigma models (see [38,41] for recent progress containing further references) which should be accessible with only a modest generalisation of the approach in [37,39]. These observations give us hope that a full derivation of the integrability of string theory on $\mathrm{AdS}_5 \times S^5$ and its deformations is getting within reach.

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