Topological Field Theories from and for 4d SUSY Gauge Theories

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AGT-correspondences give profound relations between certain families of $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions and conformal field theories in two dimensions. Subsequent investigations of the AGT-correspondences revealed a new type of topological field theory associated to these theories that captures completely the dependence of important physical quantities on the gauge coupling constants, including perturbative and non-perturbative corrections.

The topological field theories and chiral data associated to $\mathcal{N} = 2$ supersymmetric field theories are the central objects of study in this project. We present first steps towards a precise mathematical framework for them in which loop and surface operators play a particularly important role.

1 Introduction

Progress made in the recent years has seen several instances where topological quantum field theories (TQFT) served as powerful tools for the study of the non-perturbative dynamics of usual quantum field theories (QFT). At the same time it has developed into an active area of mathematical research, characterised by a profound interplay between various structures of algebraic nature, category theory, and topology.

A striking example for the relevance of TQFT for the study of non-perturbative phenomena in QFT is the work of Kapustin and Witten [1] relating the S-duality conjecture in $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory to the geometric Langlands program. Mathematical work on the geometric Langlands program thereby provides checks of the S-duality conjecture, and helps to elaborate its quantum field theoretical consequences. Ideas and methods of TQFT play an important role in the approach of Kapustin and Witten.

Even more recent work by Nekrasov and Witten [2] has proposed a reinterpretation of the famous AGT-correspondence (after Alday, Gaiotto and Tachikawa) in terms of TQFT (or some generalisation thereof). The AGT-correspondence relates partition functions of certain four-dimensional $\mathcal{N} = 2$ supersymmetric quantum field theories to correlation functions in two-dimensional conformal field theories. This gives exact results for partition functions and expectation values of certain observables encoding detailed information on the non-perturbative dynamics of these four-dimensional quantum field theories. The work of Nekrasov and Witten reinterprets the partition functions appearing in the AGT-correspondence in terms of objects in a modified version of the four-dimensional $\mathcal{N} = 2$-supersymmetric quantum field theories which are expected to possess topological invariance while capturing crucial information on the original QFT.

It should be noted that the term TQFT has been used above in a rather loose sense. Important mathematical work has led to precise definitions of certain classes of TQFT, see [3] for a review. The modified versions of QFT mentioned above, often called topologically twisted QFT, are in many cases expected to represent generalisations of TQFT in a sense which remains to be clarified.

The progress achieved within our project has on the one hand shed some light on the relations between the approach to the geometric Langlands correspondence of Beilinson and Drinfeld, Kapustin and Witten, and the AGT-correspondence [4]. On the other hand, within a PhD project, we have developed algebraic tools towards an explicit description of the chiral data associated to $\mathcal{N} = 2$ gauge theories.

2 TQFT in two and four dimensions

In this section we will briefly describe the relations between four- and two-dimensional TQFT which play a basic role in both [1] and [2].

2.1 Two-dimensional topological sigma models

Nonlinear two-dimensional sigma models with $\mathcal{N} = (2, 2)$ supersymmetry can be modified in basically two natural ways to get two-dimensional field theories which are topologically invariant in the sense that they depend only on the topology of the two-dimensional surfaces on which these theories are defined. The modification is called a topological twist. It modifies the Lagrangian of the theory in such a way that the twisted theory coincides with an important subsector of the untwisted theory on surfaces with trivial canonical bundle. From a given two-dimensional sigma model with $\mathcal{N} = (2, 2)$ supersymmetry one can obtain two TQFT in this way, called A- and B-model respectively.

Considering two-dimensional surfaces with boundaries one gets the so-called open topological sigma models. In many cases these sigma models are expected to be examples of two-dimensional TQFT as axiomatised in the mathematical literature, or generalisations thereof. Two-dimensional TQFT are characterised by the collection of boundary conditions $\mathsf{A}, \mathsf{B}, \dots$ which can be associated to the boundary components of two-dimensional surfaces, and by the vector spaces V_{AB} associated to intervals $I \simeq [0, \pi]$ decorated with particular boundary conditions A and B at the two ends 0 and π, respectively. A TQFT furthermore associates to two-dimensional surfaces having boundaries with fixed choices of boundary conditions a number called the partition function.

It turns out to be useful to regard the collection of defining data as a category having the boundary conditions as its objects, and the spaces V_{AB} as the spaces of morphisms $\mathrm{Hom}(\mathsf{A}, \mathsf{B})$. The strip $I \times \mathbb{R}$ can be mapped to the punctured half-plane, defining a variant of the state-operator correspondence relating elements of V_{AB} to boundary-changing operators $\mathcal{O}_{\mathsf{AB}}$, as depicted in Fig. 1 below. The composition of morphism thereby gets related to the product of boundary-changing operators, see Fig. 2.

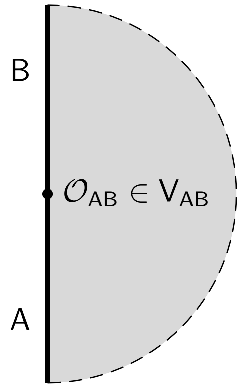

Figure 1: Morphisms in the category of boundary conditions correspond to boundary-changing operators in two-dimensional TQFT. Figure taken from Ref. [5].

2.2 Two-dimensional reductions of four-dimensional TQFT

TQFT in higher dimensions have a much richer structure. Precise mathematical definitions have been given only fairly recently, based on the mathematical framework of higher categories, see [3] for a review. A very readably discussion of the physical motivation for considering higher categories can be found in [5].

There exist higher-dimensional analogs of the topological twisting procedure expected to turn supersymmetric quantum field theories into TQFT, or generalisations thereof. Topological twists of the four-dimensional, $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM) have been investigated in [1]. There is a one-parameter family of such theories labelled by a parameter t.

A key idea in the work of Kapustin and Witten is to use effective representations of the topologically twisted four-dimensional SYM on space-times of the form $\Sigma \times C$, with C being a Riemann surface, provided by two-dimensional topologically twisted sigma models on $\Sigma = \mathbb{R} \times I$. These sigma models are related to the original four-dimensional theory by a variant of the Kaluza–Klein reduction described in [1], leading to sigma models which have the Hitchin moduli spaces $\mathcal{M}_{\rm H}(C, G)$ as target spaces. The space $\mathcal{M}_{\rm H}(C, G)$ can be described as the moduli space of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E} = (E, \bar{\partial}_{\mathcal{E}})$ is a holomorphic structure on a smooth G-bundle E on C, and $\varphi \in H^0(C, \mathrm{End}(\mathcal{E}) \otimes K)$ can be locally represented as a matrix-valued one-form. In the following we will freely use several standard definitions and results concerning Hitchin's moduli spaces. A very brief summary is collected in Appendix A in the form of a glossary. If a glossary entry exists for a term, its first occurrence will appear with a superscript as in term$^{g)}$.

The two-dimensional description is not expected to capture all of the structures of the four-dimensional theory, but it is believed to represent correctly an important part of its structure.

Similar ideas are used in the work of Nekrasov and Witten, where the starting points are theories from a class of four-dimensional $\mathcal{N} = 2$-supersymmetric theories often referred to as class $\mathcal{S}$ in the literature [6, 7]. The members of this class are labelled by the pair of data $(C, \mathfrak{g})$, where C is a Riemann surface and $\mathfrak{g}$ is a semi-simple Lie algebra of ADE-type. On four-dimensional spacetimes which can locally be described in the form $\mathbb{R} \times I \times S^1 \times S^1$ one preserves enough of the supersymmetries of the theory to define topologically twisted versions of the class $\mathcal{S}$ theories depending on two parameters ϵ_1 and ϵ_2. One may furthermore argue

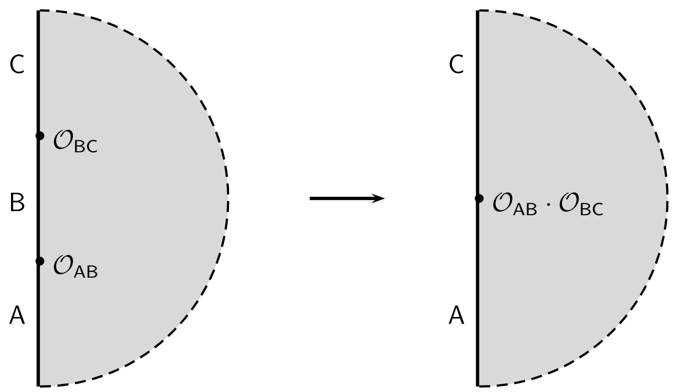

Figure 2: The composition of morphisms corresponds to the product of boundary-changing operators. Figure taken from Ref. [5].

that the resulting four-dimensional theories with topological invariance can be described by an effective two-dimensional topological sigma model on $\mathbb{R} \times I$.

This means that in the both cases one ends up with sigma models having the same target spaces $\mathcal{M}_{\mathrm{H}}(C, G)$. It should be noted, however, that such sigma models have $\mathcal{N} = (4, 4)$ supersymmetry related to the fact that the spaces $\mathcal{M}_{\mathrm{H}}(C, G)$ admit hyperkähler structures. Instead of getting just one A-model and one B-model one may now define a one-parameter family of A- and B-models depending on which part of the supersymmetries are preserved by the topological twist. The relevant parameter can be identified with the hyperkähler parameter determining a particular complex stucture on $\mathcal{M}_{\mathrm{H}}(C, G)$.

It therefore depends sensitively on the precise reduction procedure which particular topological sigma model is ultimately found as an effective description of the original four-dimensional QFT. And indeed, it turns out that the sigma models appearing in the works [1] and [2] are not identical, despite the fact that they have the same target space. Yet, there are reasons to expect that these sigma models should be more closely related than it may appear.

3 What is the geometric Langlands correspondence?

The geometric Langlands correspondence can be schematically formulated as a correspondence between two types of geometric objects naturally assigned to a Riemann surface C and a simple complex Lie group G,

$$\boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G} \quad \leftrightarrow \quad \boxed{{}^{L}G\text{-local systems}} \tag{3.1}$$

The following objects appear in this correspondence:

Bun_G is the moduli space of holomorphic G-bundles on C. The $\mathcal{D}$-modules on Bun_G appearing in this context can be described more concretely as systems of partial differential equations taking the form of eigenvalue equations $D_i f = E_i f$ for a family of commuting global differential operators D_i on Bun_G. The differential operators D_i can be regarded as quantum counterparts of the Hamiltonians of Hitchin's integrable system. The representation of the $\mathcal{D}$-modules as eigenvalue equations $D_i f = E_i f$ is not canonical, it depends on additional choices. Part of the content of the geometric Langlands correspondence describes the eigenvalues E_i geometrically as coordinates on a suitable parameter space, parameterising the objects on the right side of (3.1), which will be explained next.

The group ${}^L G$ is the Langlands dual of G which has as Dynkin diagram the transpose of the Dynkin diagram of G. The ${}^L G$ *local systems*$^{g)}$ appearing in this correspondence can be represented by pairs $(\mathcal{E}, \nabla')$ composed of a holomorphic ${}^L G$-bundle $\mathcal{E}$ with a holomorphic connection ∇', or equivalently by the representations ρ of the fundamental group $\pi_1(C)$ defined from the holonomies of $(\mathcal{E}, \nabla')$.

We will mostly be interested in the case of *irreducible* ${}^L G$ local systems. A more ambitious version of the geometric Langlands correspondence has been formulated in [8] extending it to certain classes of reducible local systems.

Some of the original approaches to the geometric Langlands correspondence start from the cases where the ${}^L G$-local systems are *opers*$^{g)}$, pairs $(\mathcal{E}, \nabla')$ in which ∇' is gauge-equivalent to a certain standard form. The space of opers forms a Lagrangian subspace in the moduli space of all local systems.

The CFT-based approach of Beilinson and Drinfeld constructs for each oper an object in the category of $\mathcal{D}$-modules on Bun_G as conformal blocks of the affine Lie algebra $\hat{\mathfrak{g}}_k$ at the critical level $k = -h^\vee$. The Ward-identities characterising the conformal blocks equip the sheaves of conformal blocks with a $\mathcal{D}$-module structure. The universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}}_k)$ has a large center at $k = -h^\vee$, isomorphic to the space of ${}^L\mathfrak{g}$-opers on the formal disc [9]. This can be used to show that the $\mathcal{D}$-module structure coming from the Ward identities can be described by the system of eigenvalue equations $D_i f = E_i f$ for the quantised Hitchin Hamiltonians, with eigenvalues E_i parameterising the choices of opers [10].

There exists an extension of the Beilinson–Drinfeld construction of the geometric Langlands correspondence described in [10, Section 9.6] from the case of opers to general irreducible local systems. It is based on the fact that such local systems are always gauge-equivalent to opers with certain extra singularities [11]. The construction of Beilinson and Drinfeld associates to such opers conformal blocks of $\hat{\mathfrak{g}}_{-h^\vee}$ with certain degenerate representations induced from the finite-dimensional representations of $\mathfrak{g}$ inserted at the extra singularities.

We may in this sense regard the geometric Langlands correspondence for general irreducible local systems as an extension of the correspondence that exists for ordinary, non-singular opers. Let us remark that the construction of Beilinson–Drinfeld plays an important role in the outline given in [12] for a proof of the strengthened geometric Langlands conjecture formulated in [8].

4 TQFT interpretation of the Langlands correspondence

The variant of the geometric Langlands correspondence proposed by Kapustin and Witten [1] is based on the consideration of $\mathcal{N} = 4$ SYM theory with gauge group G_c, a compact real form of G, on four-manifolds of the form $\Sigma \times C$, where C is a Riemann surface. Compactification on C allows one to represent the topologically twisted version of $\mathcal{N} = 4$ SYM effectively by a topologically twisted $2d$ sigma-model with target being the Hitchin moduli space $M_H(G)$ on Σ. The complete integrability$^{g)}$ of the Hitchin moduli space, as is manifest in the description of $M_H(G)$ as a torus fibration, allows one to describe the consequences of the S-duality of $\mathcal{N} = 4$ SYM theory as the SYZ mirror symmetry relating the $2d$ sigma-models with target $M_H(G)$ and $M_H({}^L G)$, respectively.

In order to relate this to the geometric Langlands correspondence, Kapustin and Witten consider the cases when $\Sigma = R \times I$, $I = [0,\pi]$. Supersymmetric boundary condition of $\mathcal{N} = 4$ SYM theory will upon compactification on C define boundary conditions of the N=(4,4) sigma model with target $\mathcal{M}_H(G)$ on Σ. Let $\mathfrak{B}$ be the category having as objects boundary conditions B called branes preserving the maximal number of supersymmetries, with morphisms being the spaces $\mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_1, \mathsf{B}_2)$ of the sigma model on the strip $\mathbb{R} \times I$, having associated boundary conditions B_1 and B_2 to the boundaries $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\pi\}$, respectively.

A distinguished role is played by the so-called canonical coisotropic brane B_{cc} [1,2]. The vector space $\mathcal{A}_{\mathrm{cc}} = \mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\mathrm{cc}}, \mathsf{B}_{\mathrm{cc}})$ has a natural algebra structure with product corresponding to the composition of morphisms depicted in Fig. 2. The spaces $\mathcal{H}(\mathsf{B}) = \mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\mathrm{cc}}, \mathsf{B})$ are left modules over the algebra $\mathcal{A}_{\mathrm{cc}}$ with action corresponding to the composition of morphisms $\mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\mathrm{cc}}, \mathsf{B}_{\mathrm{cc}}) \times \mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\mathrm{cc}}, \mathsf{B}) \to \mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\mathrm{cc}}, \mathsf{B})$. Kapustin and Witten argue that the algebra $\mathcal{A}_{\mathrm{cc}}$ contains the algebra of global differential operators on Bun_G. It follows that the spaces $\mathcal{H}(\mathsf{B})$ represent $\mathcal{D}$-modules on Bun_G.

In order to describe the dimensional reduction of topologically twisted $\mathcal{N} = 4$ SYM on $\mathbb{R} \times I \times C$ one may find it natural to consider boundary conditions that are purely topological, not depending on the complex structure on C. This point of view motivated Ben-Zvi and Nadler [13] to propose the Betti geometric Langlands conjecture as a purely topological variant of the geometric Langlands correspondence formulated in [8] that captures some aspects of the approach of Kapustin and Witten while having good chances to be realised within a mathematically precise framework for TQFT.

5 TQFT interpretation of the AGT-correspondence

Alday, Gaiotto and Tachikawa discovered a relation between the instanton partition functions of certain $\mathcal{N} = 2$ supersymmetric gauge theories and conformal blocks of the Virasoro algebra [14]. This discovery has stimulated a lot of work leading in particular to various generalisations of such relations. We will now briefly outline the role of topological sigma models in the approach of Nekrasov and Witten to the AGT-correspondence.

5.1 The approach of Nekrasov and Witten

In an attempt to explain the relations discovered in [14] using TQFT-methods Nekrasov and Witten [2] considered four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories of class $\mathcal{S}$ obtained from the maximally supersymmetric six-dimensional QFT on manifolds of the form $\mathcal{M}^4 \times C$ by compactification on the Riemann surface C. For the case associated to the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ one has weakly coupled Lagrangian descriptions of the resulting theory associated to the choice of a pants decomposition σ of C [6]. For four-manifolds $\mathcal{M}^4$ which can be represented as a fibered product locally of the form $\mathbb{R} \times I \times S^1 \times S^1$ it is argued in [2] that (i) an Ω-deformation with parameters ϵ_1, ϵ_2 can be defined, and (ii) an effective representation is obtained by compactification on $S^1 \times S^1$ in terms of a sigma-model with target $\mathcal{M}_H(G)$ on $R \times I$. The coupling parameter of this sigma model is ϵ_1/ϵ_2.

The end points of the interval I in the representation $\mathcal{M}^4 \simeq \mathbb{R} \times I \times S^1 \times S^1$ correspond to points where $\mathcal{M}^4$ is perfectly regular. One must therefore have distinguished boundary condition in the sigma-model with target $\mathcal{M}_H(G)$ on $R \times I$ describing the compactification of a class $\mathcal{S}$ theory on $\mathcal{M}^4$. When the compactification yields a sigma model with target $\mathcal{M}_H(G)$, it is argued in [2] that the corresponding boundary conditions are described by a variant $\mathsf{B}_{\rm cc}$ of the canonical coisotropic brane at $\mathbb{R}\times\{0\}$, and a new type of brane called the "brane of opers", here denoted by $\mathsf{B}_{\rm op}$, respectively.[1] The brane $\mathsf{B}_{\rm op}$ is the mirror dual of $\mathsf{B}_{\rm cc}$, and it is proposed in [2] that the brane $\mathsf{B}_{\rm op}$ is a Lagrangian brane supported on the variety of opers within $M_H(G)$.

In [2] it is furthermore proposed that the space $\mathcal{H}=\mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\rm cc},\mathsf{B}_{\rm op})$ can be identified with the space of Virasoro conformal blocks. In order to motivate this identification, Nekrasov and Witten note that the algebra $\mathcal{A}_{\rm cc}^{\hbar}(G)=\mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\rm cc},\mathsf{B}_{\rm cc})$ with $\hbar=\epsilon_1/\epsilon_2$ is isomorphic to the algebra of Verlinde line operators acting on the space of Virasoro conformal blocks. Mirror symmetry produces a dual description of $\mathcal{H}(G)\simeq\mathcal{H}({}^{\rm L}G)$ as the space $\mathrm{Hom}_{\mathcal{M}_H({}^LG)}(\mathsf{B}'_{\rm op},\mathsf{B}'_{\rm cc})$, with $\mathsf{B}'_{\rm op}$ and $\mathsf{B}'_{\rm cc}$ being close relatives of $\mathsf{B}_{\rm op}$ and $\mathsf{B}_{\rm cc}$, respectively, with modified SUSY invariance properties. In the dual representation one has an obvious right action of the algebra $\check{\mathcal{A}}_{\rm cc}^{1/\hbar}({}^LG)=\mathrm{Hom}_{\mathcal{M}_H({}^LG)}(\mathsf{B}'_{\rm cc},\mathsf{B}'_{\rm cc})$ with action related to the composition of morphisms

$$\mathrm{Hom}_{\mathcal{M}_H({}^LG)}(\mathsf{B}'_{\rm op},\mathsf{B}'_{\rm cc})\times\mathrm{Hom}_{\mathcal{M}_H({}^LG)}(\mathsf{B}'_{\rm cc},\mathsf{B}'_{\rm cc})\to\mathrm{Hom}_{\mathcal{M}_H({}^LG)}(\mathsf{B}'_{\rm op},\mathsf{B}'_{\rm cc}).$$

The existence of (almost) commuting actions of $\mathcal{A}_{\rm cc}^{\hbar}(G)$ and $\check{\mathcal{A}}_{\rm cc}^{1/\hbar}({}^LG)$ is a characteristic feature of the space of Virasoro conformal blocks.

5.2 The other way around

It is no accident that the work of Nekrasov and Witten [2] has many elements in common with the approach Kapustin and Witten [1]. A common root can be found in the fact that both the class $\mathcal{S}$-theories and $\mathcal{N}=4$ SYM [15] can be obtained as compactifications of the six-dimensional $(2,0)$-theory on six-manifolds $\mathcal{M}^6=\mathcal{M}^4\times C$, where C is a Riemann surface, and $\mathcal{M}^4$ is a four-manifold locally represented as a circle fibration locally of the form $\mathbb{R}\times I\times S^1\times S^1$. Compactification on C yields class $\mathcal{S}$-theories [7], while compactification on $S^1\times S^1$ yields $\mathcal{N}=4$ SYM on $\mathbb{R}\times I\times C$, the set-up considered in [1] as was further discussed in [15].

One should note, however, that different topological twists are used in the two compactifications, making the comparison of the results somewhat subtle. This fact can nevertheless be used to relate supersymmetric boundary conditions in the 2d sigma model with target $\mathcal{M}_H$ arising from compactification of class $\mathcal{S}$-theories to boundary conditions in $\mathcal{N}=4$ SYM on C. These boundary conditions have been classified in the work of Gaiotto and Witten [16]. The canonical coisotropic brane is related to the pure Neumann boundary conditions in $\mathcal{N}=4$ SYM by the compactification described above [1,17]. Exchanging the two circles in $S^1\times S^1$ gets related to the S-duality of $\mathcal{N}=4$ SYM which implies relations between its boundary conditions studied in [18]. This led [17] to relate the brane $\mathsf{B}_{\rm op}$, the mirror dual of the canonical coisotropic brane in [2], to the boundary condition descending from the so-called Nahm pole boundary conditions in $\mathcal{N}=4$ SYM.

6 Towards a unified picture

One of the goals in our project has been to clarify the relations between the gauge-theoretic approach to the geometric Langlands correspondence, the CFT-based approach of Beilinson

[1] The branes denoted $\mathsf{B}_{\rm cc}$ in this context are similar but not identical with the brane considered in [1]. The paper [2] used the notation $B_{\mathcal{N}'}$ for the brane denoted $\mathsf{B}_{\rm op}$ here.

and Drinfeld, and the AGT-correspondence. The results have been announced in [4], with more detailed descriptions being in preparation. In a parallel development [19, 20], similar ingredients have been used to outline a large web of relations between $\mathcal{N}=4$ SYM, vertex algebras, and braided tensor categories.

6.1 Dirichlet boundary condition and affine Lie algebra symmetry

As mentioned above, a key ingredient in the approach of Beilinson and Drinfeld to the geometric Langlands correspondence is the current algebra of WZW conformal field theory at the critical level. The approach of Kapustin and Witten does not by itself reveal the origin of this crucial aspect of the geometric Langlands correspondence. The work of Nekrasov and Witten, on the other hand, relates conformal blocks of the Virasoro algebra to states of an open topological sigma model. A key ingredient has to be added also in this approach to understand the affine Lie algebra symmetry in this context.

Gaiotto and Witten have classified 1/2 BPS boundary conditions of $\mathcal{N}=4$ SYM in [16] using the data (ρ, H, T), where $\rho : \mathfrak{sl}_2 \to \mathfrak{g}$ is an embedding of $\mathfrak{sl}_2$ into the Lie algebra $\mathfrak{g}$ of the gauge group G_c, H is a subgroup of the commutant in G_c of the image of ρ, and T is a three-dimensional SCFT with $\mathcal{N}=4$ supersymmetry and at least H global symmetry. We will only need two of the simplest of these boundary conditions. In the following we will first briefly review the so-called Nahm pole boundary condition studied in [17] which is associated to a triple $(\rho, \mathrm{Id}, T_\emptyset)$, where ρ is a *principal* $\mathfrak{sl}_2$-embedding, and $T_\emptyset$ stands for the trivial three-dimensional SCFT. We will then discuss the even simpler case where ρ is replaced by the trivial embedding mapping $\mathfrak{sl}_2$ to $0 \in \mathfrak{g}$, which will be of particular interest for us.

It is for our purposes sufficient to describe the Nahm pole boundary conditions for the solutions of the BPS-equations [1] characterising field configuration in $\mathcal{N}=4$ SYM preserving certain supersymmetries. Restricting attention to solutions to the BPS-equations on $\mathbb{R}\times\mathbb{R}_+\times C$ which are invariant under translations along $\mathbb{R}$, one gets a system of differential equations of the form

$$[\,\mathcal{D}_z\,,\,\mathcal{D}_{\bar{z}}\,]=0\,,\quad [\,\mathcal{D}_y\,,\,\mathcal{D}_z\,]=0\,,\quad [\,\mathcal{D}_y\,,\,\mathcal{D}_{\bar{z}}\,]=0\,, \tag{6.2a}$$

$$\sum_{i=1}^{3}[\,\mathcal{D}_i\,,\,\mathcal{D}_i^\dagger\,]=0\,.\,, \tag{6.2b}$$

where the notations $z=x_2+ix_3$ and $y=x_1$ have been used, and the differential operators $\mathcal{D}_i$ are of the form[2]

$$\begin{aligned}\mathcal{D}_z&=\zeta\partial_z+\mathcal{A}_z\,, &\quad \mathcal{A}_z&=\zeta A_z+\phi_z\,,\\ \mathcal{D}_{\bar{z}}&=\partial_{\bar{z}}+\mathcal{A}_{\bar{z}}\,, &\quad \mathcal{A}_{\bar{z}}&=A_{\bar{z}}+\zeta\phi_{\bar{z}}\,,\end{aligned}\qquad \mathcal{D}_y=\partial_y+A_y-i\phi_y\,. \tag{6.3}$$

The parameter ζ determines the supersymmetries that are preserved. It is proposed in [17] that the space of solutions to (6.2) modulo compact gauge transformations is isomorphic to the moduli space of the solutions to the “F-term” equations (6.2a) modulo complex gauge transformations. Equations $[\mathcal{D}_z, \mathcal{D}_{\bar{z}}]=0$ determine a flat complex connection on C at each fixed y. The remaining equations in (6.2a) imply that the y-dependence of this flat connection is represented by complex gauge transformations.

[2] Our conventions differ slightly from [17].

Boundary conditions of Nahm pole type are defined in [17] by demanding that the solutions to (6.2) have a singular behaviour of the form

$$\mathcal{A}_z \underset{y\to 0}{\sim} \mathfrak{t}_-\, y^{-1} + \mathcal{O}(y^0)\,, \qquad \mathcal{A}_{\bar{z}} \underset{y\to 0}{\sim} \mathcal{O}(y^0)\,, \qquad \mathcal{A}_1 \underset{y\to 0}{\sim} \mathfrak{t}_3\, y^{-1} + \mathcal{O}(y^0)\,, \tag{6.4}$$

with $\mathfrak{t}_+ = \mathfrak{t}_1 + i\mathfrak{t}_2$, and $\mathfrak{t}_i$, $i = 1, 2, 3$, being the generators of a principal $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}$. By a gauge transformation we may always set $\mathcal{A}_{\bar{z}}$ to zero, allowing us to represent the flat connection on C we get at each y as a local system $(\mathcal{E}_y, \nabla'_y)$ consisting of a holomorphic bundle and a holomorphic connection $\nabla'_y = dz(\partial_z + \mathcal{A}_z(z; y))$. In the case $\mathfrak{g} = \mathfrak{sl}_2$, we may reformulate the first condition in (6.4) as the condition that there exists a basis of sections $s = \{s_1, s_2\}$ with respect to which $\mathcal{A}$ has the form $\mathcal{A} = g\tilde{\mathcal{A}}g^{-1} + gdg^{-1}$, with

$$\tilde{\mathcal{A}}_z \underset{y\to 0}{\sim} \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} + \mathcal{O}(y^1)\,, \qquad g \underset{y\to 0}{\sim} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} + \mathcal{O}(y^0). \tag{6.5}$$

Horizontal sections $(d + \mathcal{A})s = 0$ will then have a first component s_1 vanishing as $y^{1/2}$. As explained in [17], this implies that the local system $\lim_{y\to 0}(\tilde{\mathcal{E}}_y, \tilde{\nabla}'_y)$ on C is an oper.

The Nahm pole boundary condition has the feature that it breaks G_c maximally since the commutant of the principal $\mathfrak{sl}_2$-embedding is trivial. At the opposite extreme, associated to the trivial $\mathfrak{sl}_2$-embedding, one gets a similar boundary condition associated to a triple $(0, \mathrm{Id}, T_\emptyset)$ by fixing the boundary value of the gauge field $\mathcal{A}_{\bar{z}}$.

In the reduction to two dimensions having fixed $\mathcal{A}_{\bar{z}}$ at the boundary of I implies having fixed a holomorphic bundle on C, leaving the $(1, 0)$ part of the complex gauge field unconstrained. The moduli space $\mathcal{M}_{dR}(G)$ of pairs $(\mathcal{E}, \nabla'_\zeta)$ is isomorphic to the Hitchin moduli space $\mathcal{M}_H(G)$ via the non-abelian Hodge (NAH) correspondence[g)]. Fixing $\mathcal{E}$ therefore defines a submanifold in $\mathcal{M}_H(G)$ which is Lagrangian with respect to the holomorphic symplectic form Ω_ζ, and holomorphic w.r.t. to the complex structure I_ζ. For $\zeta = i$ one has $\Omega_\zeta = \Omega_J$, $I_\zeta = J$, leading to the identification of the brane coming from the reduction of the zero Nahm pole boundary condition as an (A,B,A)-brane in the A-model with the symplectic structure ω_I used in [2].

One may then argue that the $\mathcal{H}^{(2)}_x = \mathrm{Hom}_{\mathcal{M}_H(G)}(\mathsf{B}_{\mathrm{cc}}, \mathsf{L}^{(2)}_x)$ can be identified with the space of conformal blocks of the affine Lie algebra $\hat{\mathfrak{g}}_k$ at level $k = -h^\vee - \frac{\epsilon_2}{\epsilon_1}$ on C. Different arguments leading to this identification have been presented in [4] and in [20], respectively.

From the point of view of class $\mathcal{S}$-theories it has been pointed out in [4] that the zero Nahm pole boundary condition corresponds to the presence of a surface operator of co-dimension two. Indeed, as was argued in [21], the presence of a co-dimension two surface operator naturally introduces additional background data which can be geometrically represented as the choice of a holomorphic bundle on C.

6.2 Conformal blocks as triangle partition functions

We now describe yet another key extension of the TQFT formalism proposed in [4]. Using the reduction of class $\mathcal{S}$-theories to the two-dimensions we will in the following motivate a description of the four-dimensional partition functions of class $\mathcal{S}$-theories within the two-dimensional sigma model with target $\mathcal{M}_H(G)$. This description will use yet another type of boundary condition denoted $\mathsf{L}^{(1)}_a$.

Following [2] we will consider topologically twisted class $\mathcal{S}$-theories on hemispheres $B^4_{\epsilon_1\epsilon_2}$ with Omega-deformation. The topologically twisted class $\mathcal{S}$-theory associates a vector space

$\mathcal{H}_{\rm top} = Z(M^3_{\epsilon_1\epsilon_2})$ to $M^3_{\epsilon_1\epsilon_2} = \partial B^4_{\epsilon_1\epsilon_2}$, here identified with the cohomology of Q, the supercharge that can be preserved on $B^4_{\epsilon_1\epsilon_2}$. One may use the path integral over the 4d hemisphere $B^4_{\epsilon_1\epsilon_2}$ to define a vector $\Psi \in \mathcal{H}_{\rm top}$. Wave-functions $\Psi(a)$ of the vector Ψ may be identified with the partition functions $Z(B^4_{\epsilon_1,\epsilon_2}; \mathsf{B}_a)$ defined by imposing suitable Q-invariant boundary conditions B_a labelled by parameters a at $M^3_{\epsilon_1\epsilon_2}$. Such boundary conditions are determined by the boundary conditions at the infinity of $\mathbb{R}^4_{\epsilon_1\epsilon_2}$ used to define the Nekrasov partition functions, fixing in particular the zero modes of the scalars in the vector multiplets to have values collected in the vector $a = (a_1, \dots, a_{3g-3+n})$. The boundary conditions B_a define a family of boundary states β_a, allowing us to represent $Z(B^4_{\epsilon_1,\epsilon_2}; \mathsf{B}_a)$ as an overlap $\langle \beta_a, \Psi \rangle$.

In the reduction of the class $\mathcal{S}$-theory to a $2d$ topological sigma model one should get the following representation of the 4d TQFT data introduced above:

- The vector space $\mathcal{H}_{\rm top} \simeq Z(S^3_{\epsilon_1,\epsilon_2}) \to Z(I)$.
- The vector $\Psi = Z(B^4_{\epsilon_1,\epsilon_2}) \to Z(T_{\epsilon_1,\epsilon_2}) \in \mathcal{H}_{\rm top}$, where $T_{\epsilon_1,\epsilon_2}$ is the open triangle with "upper" side removed, topologically equivalent to $\mathbb{R}_- \times I$, partially compactified by adding a point at the infinity of $\mathbb{R}_-$. The boundary of $T_{\epsilon_1,\epsilon_2}$ is $\{0\} \times I$.
- The partition function $Z(B^4_{\epsilon_1,\epsilon_2}; \mathsf{B}_a) \to Z(T_{\epsilon_1,\epsilon_2}; \mathsf{B}_a)$ gets associated to a triangle $T_{\epsilon_1,\epsilon_2}$ with a boundary condition $\mathsf{L}^{(1)}_a$ assigned to the upper side $\{0\} \times I$. $\mathsf{L}^{(1)}_a$ is defined from the boundary condition B_a assigned to $M^3_{\epsilon_1\epsilon_2}$ by the reduction to one dimension.

This means that the instanton partition functions $\mathcal{Z}(a; x; \tau; \epsilon_1, \epsilon_2)$ get represented by partition functions of the sigma model on a triangle which has sides coloured by $(\mathsf{B}_{\rm cc}, \mathsf{L}^{(1)}_a, \mathsf{L}^{(2)}_x)$.

The two-dimensional description of the boundary conditions $\mathsf{L}^{(1)}_a$ has been indentified in [4] as the family of Lagrangian submanifolds defined by identifying the parameters a with complex Fenchel–Nielsen coordinates for Hitchin's moduli spaces[g)].

6.3 Geometric Langlands: CFT versus gauge theory

Having established the interpretation of conformal blocks in terms of TQFT prepares the groundwork for understanding the relations between the Beilinson–Drinfeld approach to the geometric Langlands correspondence and the work of Kapustin and Witten [4]. To understand the full picture it is important, however, to note that the geometric Langlands correspondence can be regarded as a limiting case of a one-parameter family of relations between $\mathcal{D}$-modules on Bun_G and $\mathcal{D}$-modules on $\mathrm{Bun}_{^LG}$ related to a generalisation of WZW conformal field theory having affine Lie algebra symmetry $\hat{\mathfrak{g}}_k$ with generic level k [22, 23]. This one-parameter generalisation of the geometric Langlands correspondence has a counterpart on the gauge theory side discussed in [1, 24].

Part of our work in [4] is a careful discussion of how this continuation with respect to the level can be understood from the point of view of the topological sigma model. The discussion is based on the hyperkähler structure of the Hitchin moduli space. We refer to [4] for further details.

7 Boundary line operators

Part of the rich structure of an four-dimensional TQFT is the category of boundary line operators, one-dimensional extended objects supported on three-dimensional boundary components of the four-dimensional space-time. The category $\mathcal{L}_\hbar(\mathsf{B})$ of boundary line operators depends on the choice of a boundary condition B and the parameter $\hbar = \epsilon_1/\epsilon_2$. The structure of a four-dimensional TQFT includes the structure of a braided tensor category on $\mathcal{L}_\hbar(\mathsf{B})$. For TQFT like topologically twisted $\mathcal{N} = 4$ SYM one may expect to find a rich family of categories $\mathcal{L}_\hbar(\mathsf{B})$ when the boundary conditions are varied. The resulting picture remains largely unexplored. Very subtle are in particular the cases where $\hbar = \epsilon_1/\epsilon_2$ is a rational number. In this case one can see indications both from topologically twisted $\mathcal{N} = 4$ SYM [20] and from the representation theory of vertex algebras that the precise relation between the category $\mathcal{L}_\hbar(\mathsf{B})$ and representation categories of suitable vertex algebras must be very interesting.

In the following we will briefly describe results obtained within the project A10 that can be regarded as first steps in this direction. These will be followed by a short description of a result from [4] giving a dual interpretation of the boundary line operators representing the Hecke functors in the geometric Langlands correspondence according to [1] in the context of the AGT-correspondence.

7.1 Non semi-simple braided tensor categories

For special non-generic values of the quotient ϵ_1/ϵ_2, the analytic continuation of the usual 3-point-functions becomes singular, as the representation theory becomes non-semisimple. It is expected that the situation is related to certain logarithmic vertex algebra models, starting with the triplet algebra $\mathcal{W}_{p,p'}$ for $\mathfrak{g} = \mathfrak{sl}_2$.

These vertex algebras are recently an intense subject of study, and their nonsemisimple representation category is conjectured to be equivalent to the representation category of a small quantum group [25–27], i.e. a finite-dimensional quasi-triangular Hopf algebra.

However, this equivalence cannot be an equivalence of *monoidal* categories, as the respective quantum groups may not even admit a braiding, if the deformation parameter is an even root of unity. It was pointed out [28] for the example $\mathfrak{g} = \mathfrak{sl}_2, q = i$ that one should consider instead a quasi-Hopf algebra related to $u_q(\mathfrak{sl}_2)$ to get an equivalence of monoidal categories. The 3-cocycle involved in this quasi-Hopf algebra also appears in the corresponding conformal field theory.

Within this project, a PhD student has analyzed systematically [29] the existence and nondegeneracy of braidings for quantum groups $u_q(\mathfrak{g})$ at even order root of unity, which is the case relevant for conformal field theory. This produces many braided tensor categories, including examples that are definitely new. However, only few of them are modular tensor categories, i.e. they obey a non-degeneracy condition on the braiding, and are thus candidates for the chiral data of a conformal field theory. However, given the very explicit form of the results in [29], it was possible [30] to construct explicitly a large family of quasi-Hopf algebra relatives of $u_q(\mathfrak{g})$, which have representation categories that are indeed modular categories. (In fact, these categories appear as a non-semisimple variant of a modularization of the former categories.)

At present, it is still out of reach to prove for a general reductive Lie algebra $\mathfrak{g}$ that the representation categories of these quasi-Hopf algebras are braided equivalent to representation categories of vertex algebras and are thus realized in conformal field theory. Still, it has been shown that the previously mentioned quasi-Hopf algebra is reproduced for $\mathfrak{g} = \mathfrak{sl}_2$. Moreover, for all $\mathfrak{g}$ the 3-cocycle precisely coincides with the one on the CFT side. Put differently, both the Hopf algebra and the vertex algebra admit a functor to the same quadratic space defined by the root lattice of $\mathfrak{g}$. It is an even more challenging question at the time of writing to what extent these algebraic data appear in descriptions of four-dimensional supersymmetric field theories.

7.2 The Hecke eigenvalue property

A beautiful feature of the approach of Kapustin and Witten is an alternative derivation of the so-called Hecke-eigenvalue property of the $\mathcal{D}$-modules appearing the geometric Langlands correspondence. Part of the work [4] was yet another interpretation of the Hecke-eigenvalue property in relation to the AGT-correspondence, as we will now briefly review.

The reduction of Wilson- and 't Hooft line operators in $\mathcal{N} = 4$ SYM with support on $\mathbb{R} \times \{x\} \times P$, $x \in I$, $P \in C$, to the two dimensional TQFT defines natural functors on the category of branes, inducing modifications of the spaces $\mathcal{H}(\mathsf{B})$. The functors defined in this way are identified in [1] with the Hecke functors in the geometric Langlands correspondence. For some branes B one may represent for each fixed $P \in C$ the resulting modification as the tensor product of $\mathcal{H}(\mathsf{B})$ with a finite-dimensional representation V of ${}^LG$. One says that the brane B satisfies the Hecke eigenvalue property if the family of modifications obtained by varying the point $P \in C$ glues into a local system.

A family of branes F_μ is identified in [1] having this property. The branes F_μ are supported on fibers of Hitchin's torus fibration[g)]. S-duality of $\mathcal{N} = 4$ SYM gets represented within the sigma model with target $\mathcal{M}_H(G)$ as a variant of SYZ mirror symmetry, relating the branes F_μ to branes in the dual sigma model with target $\mathcal{M}_H({}^LG)$ represented by skyscraper sheaves $\check{\mathsf{F}}_\mu$ having pointlike support at $\mu \in \mathcal{M}_H({}^LG)$.

Part of the results presented in [4] is a dual interpretation of the Hecke eigenvalue property in the context of class $\mathcal{S}$ theories. The Wilson- and 't Hooft line operators have a dual representation in this context as surface operators of a specific type defined by coupling certain two-dimensional quantum field theories on a two-dimensional subspace to the four-dimensional class $\mathcal{S}$-theories. The relevant two-dimensional quantum field theories have $(2, 2)$ supersymmetry and can be described as gauged linear sigma models (GLSM). It was observed in [4] that the tt^* connection [31] of the GLSM turns in the limit $\epsilon_2 \to 0$ into the oper connection appearing on one side of the geometric Langlands correspondence. The limit $\epsilon_2 \to 0$ furthermore implies a factorisation of the partition functions into a four-dimensional part and a two-dimensional part. This factorisation directly expresses the Hecke eigenvalue property of the geometric Langlands correspondence in the dual picture in terms of class $\mathcal{S}$ theories [4].

A Hitchin's moduli spaces

We assume that $G = SL(2)$, and that C is a Riemann surface with genus g and n punctures.

Hitchin moduli space $\mathcal{M}_H(G)$ [32]. Moduli space of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E} = (E, \bar{\partial}_{\mathcal{E}})$ is a holomorphic structure on a smooth vector bundle E, and $\varphi \in H^0(C, \mathrm{End}(\mathcal{E}) \otimes K)$. The moduli space of such pairs modulo natural gauge transformations is denoted by $\mathcal{M}_H(G)$.

Hitchin's integrable system [32]. Given $(\mathcal{E}, \varphi)$ one constructs the spectral curve $\Sigma = \{(u, v); v^2 = \frac{1}{2}\mathrm{tr}(\varphi^2)\} \subset T^*C$, and the line bundle $\mathcal{L}$ representing the cokernel of $\varphi - v$. One may reconstruct $(\mathcal{E}, \varphi)$ from $(\Sigma, \mathcal{L})$ as $\mathcal{E} = \pi_*(\mathcal{L})$ and $\varphi = \pi_*(v)$. This describes $\mathcal{M}_H(G, C)$ as a torus fibration over the base $\mathcal{B} \simeq H^0(C, K^2)$, with fibres representing the choices of $\mathcal{L}$ identified with the Jacobian of Σ if $G = GL(2)$, and with the Prym variety if $G = SL(2)$. Natural coordinates for the base $\mathcal{B}$ are provided by Hitchin's Hamiltonians, defined by expanding $\frac{1}{2}\mathrm{tr}(\varphi^2) = \sum_{r=1}^{3g-3+n} \vartheta_r H_r$, with $\{\vartheta_r, r = 1, \dots, 3g - 3 + n\}$ being a basis for $H^0(C, K^2)$.

Local systems. Pairs $(\mathcal{E}, \nabla'_\epsilon)$, where $\mathcal{E}$ is a holomorphic vector bundle as above, and ∇'_ϵ is a holomorphic ϵ-connection, satisfying $\nabla'_\epsilon(fs) = \epsilon(\partial f)s + f\nabla'_\epsilon s$ for functions f and smooth sections s of E. The moduli space of such pairs is denoted $\mathcal{M}_{dR}(G)$. Local systems are here often identified with the corresponding *flat bundles*, systems of local trivialisations with constant transitions functions, or the representations of the fundamental group (modulo conjugation) obtained as holonomy of $(\mathcal{F}, \nabla'_\epsilon)$, leading to the isomorphism between $\mathcal{M}_{dR}(G)$ and the

Character variety $\mathcal{M}_B(G)$: The space of representations of $\pi_1(C)$ into G, modulo overall conjugation, as algebraic variety described as a GIT quotient $\mathbb{C}[\mathrm{Hom}(\pi_1(C), G]^G$.

Opers. Special local systems, where $\mathcal{E} = \mathcal{E}_{\mathrm{op}}$, the unique extension $0 \to K^{1/2} \to \mathcal{E}_{\mathrm{op}} \to K^{-1/2} \to 0$ allowing a holomorphic connection ∇'_ϵ of the form $\nabla'_\epsilon = dz\big(\epsilon\partial_z + \big(\begin{smallmatrix} 0 & u \\ 1 & 0 \end{smallmatrix}\big)\big)$.

Non-Abelian Hodge (NAH) correspondence [32,33]. Given a Higgs pair $(\mathcal{E}, \varphi)$, there exists a unique harmonic metric h on E satisfying $F_{\mathcal{E},h} + R^2[\varphi, \varphi^{\dagger_h}] = 0$ where $F_{\mathcal{E},h}$ is the curvature of the unique h-unitary connection $D_{\mathcal{E},h}$ having $(0,1)$-part $\bar{\partial}_{\mathcal{E}}$. One may then form the corresponding two-parameter family of flat connections $\nabla_{\zeta,R} = \zeta^{-1}R\varphi + D_{\mathcal{E},h} + R\zeta\varphi^{\dagger_h}$. Decomposing $\nabla_{\zeta,R}$ into the $(1,0)$ and $(0,1)$-parts defines a pair $(\mathcal{F}, \nabla'_\epsilon)$ consisting of $\mathcal{F} = (E, \bar{\partial}_{\mathcal{F}})$ and the ϵ-connection $\nabla'_\epsilon = \epsilon\nabla' = \epsilon\partial_{\mathcal{E},h} + \varphi$, with $\epsilon = \zeta/R$, holomorphic in the complex structure defined by $\bar{\partial}_{\mathcal{F}}$.

Hyperkähler structure [32]. There exists a $\mathbb{P}^1$ worth of complex structures I_ζ and holomorphic symplectic structures Ω_ζ. The latter are defined as $\Omega_\zeta = \frac{1}{2}\int_C \mathrm{tr}(\delta\mathcal{A}_\zeta \wedge \delta\mathcal{A}_\zeta)$. A triplet of symplectic forms $(\omega_I, \omega_J, \omega_K)$ can be defined by expanding Ω_ζ as $\Omega_\zeta = \frac{1}{2\zeta}(\omega_J + i\omega_K) + i\omega_I + \frac{1}{2}\zeta(\omega_J - i\omega_K)$. The corresponding complex structures are $I_\zeta = \frac{1}{1+|\zeta|^2}((1-|\zeta|^2)I - i(\zeta - \bar{\zeta})J - (\zeta + \bar{\zeta})K)$.

Complex Fenchel–Nielsen coordinates [34]. Darboux coordinates for $\mathcal{M}_B(G)$ associated to pants decompositions σ of C obtained by cutting along closed curves γ_i, $i = 1, \ldots, 3g-3+n$. The complex length coordinates parameterise the trace functions $L_i = \mathrm{tr}(\rho(\gamma_i))$ as $L_r = 2\cosh(a_r/2)$. One may define canonically conjugate coordinates κ_r such that the natural Poisson structure gets represented as $\{a_r, \kappa_s\} = \delta_{r,s}$, $\{a_r, a_s\} = 0 = \{\kappa_r, \kappa_s\}$.

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