Strings and High Energy Scattering

Jochen Bartels[1], *Sven-Olaf Moch*[1], *Volker Schomerus*[2]

[1]II. Institut für Theoretische Physik, Universität Hamburg, Germany
[2]DESY, Hamburg, Germany

DOI: http://dx.doi.org/10.3204/PUBDB-2018-00782/A3

While early string theory provided an intriguing explanation for the observed Regge trajectories of meson resonances, it failed to reproduce the correct high energy behavior of scattering amplitudes. This has changed with the AdS/CFT correspondence. In fact, string theory can now produce precision results even in the multi-Regge regime. In this note we review the status of high energy scattering in planar $\mathcal{N} = 4$ super-Yang–Mills theory, at weak, strong and intermediate coupling, with a focus on its integrability and the relation to string theory.

1 Introduction

Back at the end of the 1960ies, string theory seemed to offer an intriguing explanation for the Regge trajectories of meson resonances that experimentalists had seen in pion scattering amplitudes. In fact, the observed linear trajectories of particles with mass along $M_J^2 = \alpha' J + \alpha_0, J = 0, 1, \ldots$ were interpreted as vibrational modes of a 1-dimensional object of tension $T_s = 1/\alpha'$. On the other hand it was also noticed early on that the scattering amplitudes of strings, such as the famous Veneziano amplitude, do not possess the correct high energy limit in the physical regime since they fall off exponentially with the center of mass energy and hence much faster than in nature. This fall-off behavior may be understood from the extended nature of strings which makes the interaction region in coordinate space a bit fuzzy and hence localizes the amplitudes sharply in momentum space. So, in spite of its beautiful interpretation of meson resonances, it appeared that string theory could not describe hadronic physics and the field was left to Quantum Chromo Dynamics (QCD). The latter was very efficient in particular in the high energy asymptotically free regime. On the other hand, even after decades of experience it remains difficult to use QCD for low energy physics, the regime in which string theory had seemed so promising.

Ideas in particular of ’t Hooft [1] and Polyakov [2] nurtured hopes that one may eventually be able to reconcile the two approaches to hadronic physics, i.e. combine the advantages of QCD with those of string theory. But it was only through Maldacena’s celebrated AdS/CFT correspondence [3] that a concrete route opened up which would eventually allow for precision computations of the type described below. The correspondence also showed very clearly that early attempts to model hadronic physics with strings had suffered from one problematic assumption: It had always been taken for granted that strings and particles propagate in the same space-time. For the modern string theory descriptions of 4-dimensional gauge theory to work, however, it is absolutely crucial that the strings propagate an a 5-dimensional curved geometry. In the AdS/CFT correspondence the geometry is curved by the presence of stack of D3 branes with a 4-dimensional world-volume. The curvature introduces a red-shift factor that depends on the 5^{th} coordinate, i.e. on the distance r from the stack of branes. Through this r-dependent red-shift factor, string theory can reproduce both hard high energy scattering and low energy meson resonances [4].

The gauge theory that lives on the stack of D3 branes is of course not QCD but a maximally supersymmetric cousin thereof, the $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory. According to the AdS/CFT correspondence, this theory is equivalent to type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. It was realized early on [5] that the classical world-sheet models which describe strings in this background are integrable, at least on a world-sheet of genus $g = 0$. This tied in nicely with an earlier observation by Minahan and Zarembo [6] according to which the one-loop dilation operator in a subsector of planar $\mathcal{N} = 4$ SYM theory is given by the Hamiltonian of the integrable Heisenberg spin chain. For a review of integrability in the AdS/CFT correspondence and many more references see [7]. Integrability of strings in $\mathrm{AdS}_5 \times S^5$ suggested that – within the framework of the AdS/CFT correspondence – the string theoretic description could even be used for precision computations in $\mathcal{N} = 4$ SYM theory at finite coupling.

The potential of string theoretic calculations was first seen in the context of anomalous dimensions. This is a topic with a long history in gauge theory. In particular, matrix elements of singlet local twist-two Wilson operators of spin L and their evolution equations were studied extensively. For the fields in the $\mathcal{N} = 4$ SYM multiplet, i.e., one gluon g, four Majorana fermions ψ and three complex scalars ϕ, it was realized [8], that the anomalous dimension matrix governing those evolution equations is fixed completely by the super-conformal invariance and one universal anomalous dimension $\gamma^{\mathrm{uni}}(L)$, whose argument is shifted by an integer number, depending on the field, g, ψ or ϕ. Moreover, it turns out [8, 9] that the most complicated contributions in the anomalous dimensions for the corresponding matrix elements of quark and gluon operators in QCD [10,11], that is an $\mathrm{SU}(n_c)$ gauge theory with fermions and bosons in the fundamental and adjoint representation, respectively, directly deliver the universal anomalous dimension $\gamma^{\mathrm{uni}}(L)$ in the $\mathcal{N} = 4$ SYM theory, provided the color $SU(n_c)$ invariants are adjusted accordingly. The leading coefficient of the universal anomalous dimension in the large L limit is known as the cusp anomalous dimension and it was the first interesting gauge theory quantity that has been computed with string theoretic techniques [12].

Putting all this together it seemed very natural to finally address the computation of high energy scattering amplitudes in $\mathcal{N} = 4$ SYM theory through the dual string theoretic description. Our goal was to show that string theory in an AdS geometry was not only compatible with gauge theoretic high energy scattering, as argued in [4], but that it could even produce precision results in the regime that had challenged early string theory. The aim of this review is to explain how this goal has been achieved, at least for (MHV) scattering amplitudes of up to six external gluons in planar $\mathcal{N} = 4$ SYM theory.

In order to do so, we will provide some basics on scattering amplitudes in perturbative $\mathcal{N} = 4$ SYM theory in the next section. Our discussion is centered around the well-known Bern–Dixon–Smirnov (BDS) formula and it includes a brief survey of the Beisert–Eden–Staudacher equation, as a first non-trivial example of a string theoretic precision result in gauge theory, see section 3. In the forth subsection we then zoom into the high energy regime and explain a remarkable formula from [13] that addresses high energy scattering amplitudes for $n = 6$ gluons at any loop order in the so-called leading logarithmic approximation (LLA), along with some extensions most notably from [14]. Then we turn to strong coupling. Here, our discussion begins with

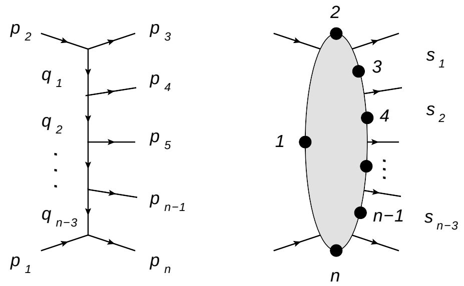

Figure 1: Kinematics of the scattering process $2 \to n-2$. On the right-hand side we show a graphical representation of the dual variables x_i. Figure taken from Ref. [19].

the geometric description of scattering amplitudes in terms of minimal area surfaces that was suggested in [15]. The latter has been reformulated in [16,17] through an auxiliary 1D integrable system which was specifically designed to solve the minimal area problem in AdS. In section 6 we will explain two beautiful insights from [18,19]. Namely, it turns out that the gauge theoretic high energy limit actually corresponds to the low energy limit of the auxiliary 1D integrable system. Moreover, the Regge cuts in gauge theory are mapped to quasi-particle excitations in 1D. When taken together, the two statements give rise to a set of Bethe ansatz equations for high energy scattering amplitudes in strongly coupled $\mathcal{N} = 4$ SYM theory [20]. One special solution of these equations determines the strong coupling limit of the six-gluon amplitude in the multi-Regge limit. In the concluding section we finally present a formula from [21] that smoothly interpolates between the weak and strong coupling results in [13] and [22], respectively.

2 The weak coupling theory

We consider the scattering of n gluons. With later kinematical limits in mind we shall think of two incoming particles whose momenta we denote by p_1, p_2 and $n-2$ outgoing particles of momentum $p_3, ..., p_n$ as shown in Fig. 1. It will be convenient to label momenta p_i by arbitrary integers i such that $p_{i+n} = p_i$. In the context of $\mathcal{N} = 4$ SYM theory it is advantageous to pass to a set of dual variables x_i such that

$$p_i = x_{i-1} - x_i. \tag{1}$$

The variables x_i inherit their periodicity $x_{i+n} = x_i$ from the periodicity of the p_i and momentum conservation. Let us also introduce the notation $x_{ij} = x_i - x_j$. The x_{ij}^2 provide a large set of Lorentz invariants $x_{ij}^2 = x_{ji}^2$. Throughout this note we use a Lorentzian metric with signature $(-, +, +, +)$. When expressed in terms of the momenta, the invariants read

$$x_{ij}^2 = (p_{i+1} + \cdots + p_j)^2 . \tag{2}$$

Lorentz symmetry along with the mass-shell conditions $p_i^2 = 0$ imply that only $3n-10$ of these variables are independent. We will not make any specific choice here. In the physical regime, all the energies p_i^0 are assumed to be positive. We will refer to the Mandelstam invariants x_{ij}^2 that are positive in the physical regime as s-like. Those that obey $x_{ij}^2 \leq 0$ in the physical regime are called t-like.

Let us recall that in planar $\mathcal{N}=4$ super-Yang–Mills theory the full color ordered maximally helicity violating amplitude takes the following form

$$A_n \sim A_n^{(0)} e^{F_n^{\rm BDS}(s,t,\epsilon;a)+R_n(u;a)} \,. \tag{3}$$

Each of the terms on the right hand side has a rich story to tell. Here, we only mention that the tree level factor $A_n^{(0)}$ is given by a surprisingly simple formula that was found by Parke and Taylor [23]. The function $F_n^{\rm BDS}$ was introduced by Bern, Dixon and Smirnov in [24]. It depends on $3n-10$ Mandelstam variables which we denote collectively as s,t, according to our split into s- and t-like invariants. In addition, F_n are functions of the cut-off ϵ and of the 't Hooft coupling a. BDS designed the F_n to incorporate all terms that are singular in the cut-off ϵ along with a relatively simple finite term. The latter was included to reproduce the correct one-loop amplitude. Explicit formulas for F_n can be found in [24] and we will also discuss some more features below. By definition, the so-called finite remainder functions R_n contain the part of the amplitude that is not captured by the tree level factor $A_n^{(0)}$ and the BDS formula. The remainder functions are not known in general. So, most of our discussion below will concern the functions R_n.

Before we get there, we want to go into a bit more detail about the BDS amplitudes F_n. Bern, Dixon and Smirnov had actually suggested that its dependence on the Mandelstam variables s,t was determined entirely by one-loop computations. The dependence on the coupling a, on the other hand, enters their expression for F_n only through a few numbers, most prominently the so-called cusp anomalous dimension $\gamma_c(a)$, i.e. somewhat symbolically, their proposal for F_n reads

$$F_n^{\rm BDS}(s,t,\epsilon;a) \sim \gamma_c(a) M_n^{(1)}(s,t,\epsilon) + \ldots \ , \tag{4}$$

see [24] for the explicit formula. The factorization of the dependence on the kinematical variables and the coupling constant is the central feature of the BDS formula. Obviously, the functions M_n were easy to work out since they just contain one-loop contributions. We will not give explicit expressions here. What is much less obvious is that also the dependence of $\gamma_c(a)$ on the coupling a has been determined. In fact it is known to any loop order as a solution to the remarkable Beisert–Eden–Staudacher (BES) non-linear integral equation [12]. We will discuss this is the next section. Here, we only state the first few orders of its weak coupling expansion.

$$\gamma_c(a) = 4a - 4\zeta_2 a^2 + \frac{44}{5}\zeta_2^2 a^3 - \left(\frac{876}{35}\zeta_2^2 + 4\zeta_3^2\right) a^4 + \mathcal{O}(a^5)\,. \tag{5}$$

Let us now turn to the main actors of our review, the finite remainder functions R_n. By construction, the BDS ansatz is one-loop exact and hence R_n can only start from two loops. Moreover, since F_n contains all singular terms, R_n is finite at all loops. It was argued in [25] to be invariant under dual conformal transformations, i.e. conformal transformations in momentum space. Hence, R_n depends on the Mandelstam invariants only through conformal invariant cross ratios. Since the four-dimensional conformal group has 15 generators, there are $3n-15$ such cross ratios which we denote as u. For the discussion of the multi-Regge limit we adopt the following choice [18, 19]

$$u_{1\sigma} = \frac{x^2_{\sigma+1,\sigma+5}x^2_{\sigma+2,\sigma+4}}{x^2_{\sigma+2,\sigma+5}x^2_{\sigma+1,\sigma+4}} \quad , \quad u_{2\sigma} = \frac{x^2_{\sigma+3,n}x^2_{1,\sigma+2}}{x^2_{\sigma+2,n}x^2_{1,\sigma+3}} \quad , \quad u_{3\sigma} = \frac{x^2_{2,\sigma+3}x^2_{1,\sigma+4}}{x^2_{2,\sigma+4}x^2_{1,\sigma+3}}\,, \tag{6}$$

where $\sigma = 1, ..., n-5$. Note that for $n < 6$ one cannot form any cross ratios and hence the remainder functions R_n must be trivial for $n = 4, 5$. In the case of $n = 6$ external gluons, however, there exits 3 independent cross ratios which we shall simply denote by u_1, u_2, u_3. And indeed it has been argued in [26] that R_6 must be a non-vanishing function of the cross ratios u_a in order to correct for the unphysical analytical structure of the BDS ansatz.

The remainder functions R_n are not known in general, but quite a few results have been obtained within the last decade. One line of calculations aims at exact multi-loop results at fixed order perturbation theory. Another approach computes, in the Regge limit, amplitudes at all loop orders in perturbation theory, restricted to the leading logarithmic approximation, making use of analytic properties, unitarity and conventional perturbation theory. The first approach started with a computation of R_6 at two loops in [27] which was turned into a very compact formula for R_6 in [28]. We refrain from stating an explicit expression here, but stress that this R_6 possesses non-trivial branch cuts. We will discuss these results in more detail in section 4. Computations of $2 \to 4$ scattering amplitudes in the multi-Regge limit started in [13, 26], see also section 4. In the meantime, the determination of remainder functions for arbitrary kinematics has been pushed to higher orders and a larger number of external gluons, see e.g. [29, 30] and [31, 32].

3 Interlude: Anomalous dimensions

In our discussion of the BDS Ansatz for planar amplitudes above the entire dependence on the coupling constant a entered through some functions that were independent of the kinematic data. The most important of these functions is the cusp anomalous dimensions γ_c. We want to pause our main story for a moment and discuss in a bit more detail what is actually known about the cusp anomalous dimension and how these results were obtained. While a large part of this section is devoted to the weak coupling expansion we will also review the BES equation along with some strong coupling results, thereby providing a first view on the remarkable interplay between gauge and string theory we promoted in the introduction.

The earliest results on the cusp anomalous dimension were obtained from the study of anomalous dimensions of twist-two spin-L Wilson operators for large spins, i.e. in the limit $L \to \infty$. To begin with, we will describe the setup in QCD before going back to $\mathcal{N} = 4$ SYM theory. In QCD the relevant set of (flavor-singlet) operators for quark and gluon fields, ψ and g, is given by

$$O^{\psi}_{\{\mu_1,\ldots,\mu_L\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \ldots D_{\mu_L\}}\,\psi\,, \tag{7}$$

$$O^{g}_{\{\mu_1,\ldots,\mu_L\}} = F_{\nu\{\mu_1} D_{\mu_2} \cdots D_{\mu_{L-1}} F_{\mu_L\}}{}^{\nu}\,, \tag{8}$$

where $F_{\mu\nu}$ denotes the field strength, D_μ the covariant derivative and $\{\ldots\}$ symmetrization of the indices μ_i. The local operators are subject to renormalization (in a minimal subtraction scheme) as $[O^i] = Z^{ij}\,O^j$ where the square brackets $[\ldots]$ denote renormalized operators and the corresponding anomalous dimensions $\gamma_{ij}(L)$ which are obtained from $\gamma_{ij} = \mu d/(d\mu) \ln Z^{ij}$ can be expressed in terms of harmonic sums. These are subject to the recursive definition

$$S_{m_1,m_2,\ldots,m_n}(L) = \sum_{k=1}^{L} \mathrm{sgn}(m_1)^k\, k^{-m_1}\, S_{m_2,\ldots,m_n}(k)\,, \tag{9}$$

and their weight w is given by $w = \sum_{i=1}^{n} \mathrm{abs}(m_i)$. In the weak coupling expansion of $\gamma_{ij}(L)$ at l-loop order harmonic sums up to weight $w \le 2l-1$ contribute and the terms with maximal weight, $w = 2l-1$, are referred to as terms of leading transcendentality. Upon adjusting the color SU(n_c) invariants of the QCD results and, at the same time, keeping only terms of leading transcendentality, one extracts the universal anomalous dimension $\gamma^{\mathrm{uni}}(L)$ of $\mathcal{N} = 4$ SYM theory, as has been verified up to three loops [8–11]. Interestingly, $\gamma^{\mathrm{uni}}(L)$ has an additional property, see, e.g., [33], that is it consists of certain combinations of harmonic sums which are reciprocity-respecting, i.e., invariant under the replacement $L \to 1-L$. Explicit expressions for $\gamma^{\mathrm{uni}}(L)$ in the $\mathcal{N} = 4$ SYM theory have now been constructed up to seven loops [34–37], while the complete computation of the four-loop anomalous dimensions $\gamma_{ij}(L)$ in QCD is still a formidable task, see [38] for first results on the flavor non-singlet part in the planar limit.

Of particular interest for further studies of the relations between the $\mathcal{N} = 4$ SYM theory and QCD are the so-called wrapping corrections, which complement the so-called asymptotic Bethe ansatz and control the high energy behavior for $L \to 0$. These occur for the first time at four loops [34] as $\gamma^{\mathrm{uni}}(L)|_{\mathrm{wrap}} \simeq a^4\, S_1(L)^2\, f^{\mathrm{wrap}}(L)$ with a function

$$f^{\mathrm{wrap}}(L) \;=\; 5\zeta_5 - 2S_{-5}(L) + 4S_{-2}(L)\zeta_3 - 4S_{-2,-3}(L) + 8S_{-2,-2,1}(L) + 4S_{3,-2}(L) - 4S_{4,1}(L) + 2S_5(L)\ . \tag{10}$$

cf. [39], where $f^{\mathrm{wrap}}(L)$ falls off as $1/L^2$ for $L \to \infty$, so that $\gamma^{\mathrm{uni}}(L)|_{\mathrm{wrap}}$ is compatible with Eq. (5) for the cusp anomalous dimension. Likewise, in QCD quartic Casimir invariants occur at four loops for the first time. They are proportional to $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ for representation labels x, y with generators T_r^a

$$d_r^{abcd} \;=\; \frac{1}{6}\,\mathrm{Tr}\,(\,T_r^a T_r^b T_r^c T_r^d \;+\; \text{five } bcd \text{ permutations}\,)\,, \tag{11}$$

so that one has, e.g., $d_{AA}^{(4)}/n_A = n_c^2(n_c^2+36)/24$ with normalization $n_A = (n_c^2-1)$ for the adjoint representation 'A' in an $SU(n_c)$ gauge theory. Such terms are effectively 'leading-order' and therefore scheme-independent and subject to particular relations, such as those of an $\mathcal{N} = 1$ SYM theory upon properly adjusting all $SU(n_c)$ color factors. Moreover, they consist entirely of reciprocity-respecting combinations of harmonic sums and rational polynomials in L. Using these insights and a recent computation [40] of fixed moments up to $L \le 16$ of those quartic color factors of the QCD anomalous dimensions it has been possible to reconstruct analytic expressions in L from the solution of Diophantine equations. For example, the four-loop contribution to the gluon-gluon anomalous dimension proportional to $d_{AA}^{(4)}/n_A$ and ζ_5 is given by

$$\gamma_{gg}^{(3)}(L)\Big|_{a^4\,\zeta_5\,d_{AA}^{(4)}/n_A} \;=\; 640\,\Big(12\,\eta^2 - 4\,\nu^2 - \,S_1(L)(4\,S_1(L) + 8\,\eta - 8\,\nu - 11) - 7\nu + \frac{12032}{3}\,\eta - \frac{48064}{9} - \frac{32}{9}\,L\,(L+1)\Big) \tag{12}$$

with abbreviations for the reciprocity respecting combinations $\eta = 1/L - 1/(L+1)$ and $\nu = 1/(L-1) - 1/(L+2)$.

Interestingly, the leading terms in the limit $L \to \infty$ of Eq. (12) behave as $L\,(L+1)\zeta_5$ and $\ln(L)^2\zeta_5$. This contradicts the single logarithmic rise of the anomalous dimensions as $\ln(L)$ for large L, which is a universal property of massless gauge theories. E.g., the cusp anomalous dimension in Eq. (5) for a $\mathcal{N}=4$ SYM theory derives from the factorization $\gamma^{\text{uni}}(L) \simeq \ln(L)\gamma_c(a)$. Therefore, it has been conjectured [40] that the terms proportional to $L\,(L+1)\zeta_5$ and $\ln(L)^2\zeta_5$ are, in fact, the first glimpse of the wrapping corrections at four loops in Eq. (10). While the particular combination $a^4\,S_1(L)^2\,f^{\text{wrap}}(L)$ is known from $\gamma^{\text{uni}}(L)$ [34], the other instance is also known from the computation of the three-loop QCD Wilson coefficients in deep-inelastic scattering [39], which contain the term $C_{\text{ns}}(L) \simeq a^3(n_c^2-1)/n_c^3\,\{L(L+1)\,f^{\text{wrap}}(L)\}$. Thus, completion of the ζ_5 in Eq. (12) as indicated would allow to recover the expected behavior for large spins, $L \to \infty$.

Similar relations between quantities in QCD and $\mathcal{N}=4$ SYM theory are also expected for other cases. One such case are the energy-energy correlations (EEC), proposed back in 1978 in [41], which measure the correlations of the energies E_a and E_b of partons a and b as a function of an angular variable ξ in electron-positron collisions

$$\Sigma(\xi) = \sum_{a,b} \int \mathrm{dPS}\, \frac{E_a E_b}{Q^2}\, \sigma\big(e^+ + e^- \to a + b + X\big)\, \delta\left(\xi - \cos\theta_{ab}\right) . \tag{13}$$

Due to the involved phase space integration of the cross section σ with the measure dPS in Eq. (13), the analytical expressions for the NLO QCD corrections have been unavailable for long in contrast to $\mathcal{N}=4$ SYM theory, where the corresponding results have been derived [42]. Quite recently significant progress towards the EEC at NLO in QCD has been reported [43] and full analytical results have been published [44], which in particular display again the correspondence between QCD and $\mathcal{N}=4$ SYM theory regarding contributions of leading transcendentality, i.e. the polylogarithms of highest weight in the angular variable ξ.

In the case of planar $N=4$ SYM theory all-loop results on anomalous dimensions can be obtained systematically with the help of methods from integrable systems. The first example of such results concerned the dependence of $\gamma_c(a)$ on the coupling a, which is known to any loop order as a solution to the BES non-linear integral equation [12]. The latter is easiest to appreciate from the string theory perspective. String theory in an AdS_5 is classically integrable, i.e. the classical equations of motion possess an infinite number of conservation laws. Many solutions have been constructed explicitly, the most famous of which is known as the Gubser–Klebanov–Polyakov (GKP) string [45]. It describes a folded string that rotates in a three-dimensional subspace of AdS_5. The conservation laws can be thought of as moments of some charge density function $Q_a(\tau)$ of the GKP string. From the solutions one can certainly compute all these charge densities at infinite coupling $a=\infty$. But it is possible to do a lot better: One can actually compute the charge density function $Q_a(\tau)$ for any value of the coupling a by solving the following infinite set of coupled integral equations [12],

$$\int_0^\infty \frac{d\tau}{\tau}\, J_\nu(\tau)\, Q_a((-1)^{\nu+1}\tau) = \frac{1}{2}\delta_{\nu,1} . \tag{14}$$

Here $\nu = 1, 2, \ldots$ runs through all positive integers, and $J_\nu(\tau)$ are Bessel functions of first kind. In order to reconstruct the infinite set of charge densities q_a^ν from the charge density function $Q_a(\tau)$ one expands the latter as

$$Q_a(\tau) = \sum_{\nu=1}^{\infty} q_a^\nu B_\nu(a,\tau)\,, \qquad \text{where} \qquad B_\nu(a,\tau) = (-1)^\nu \frac{2\nu J_\nu(2g\tau)}{1-e^{(-1)^\nu\tau}} . \tag{15}$$

The cusp anomalous dimension $\gamma_c(a)$ is claimed to coincide with the first coefficient in this expansion, i.e. $\gamma_c(a) = q_a^1$. It is actually surprisingly simple to construct the weak and strong coupling expansions of the cusp anomalous dimension from this description, at least when compared to perturbative gauge theory computations at weak coupling, see [12, 46] and [47]. The first few orders at weak coupling were displayed in Eq. (5) already. At strong coupling the leading terms reads

$$\gamma_c(a) = 2\sqrt{a} - \frac{3\ln 2}{2\pi} + \mathcal{O}(1/a)\,, \tag{16}$$

The determination of the cusp anomalous dimension was the first instance where the string theoretic description of gauge theories paid off. In fact, when the BES proposal appeared, $\gamma_c(a)$ was only known to three loops from [9, 10].

4 High energy scattering at weak coupling

In gauge theory, scattering amplitudes in the high energy regime are of particular relevance. They describe the behavior of typical collider kinematics with two highly energetic incoming particles and a final state in which two highly energetic outgoing particles are accompanied by a certain number of lower energy particles. Remarkably, this regime is not only most relevant but also computationally more accessible than generic kinematics. From S-matrix theory, general Regge theory [48] and from a vast number of studies of the high energy behavior in QED and in nonabelian gauge theories it is known that, in the multi-Regge limit, signatured scattering amplitudes show remarkable structural simplicity. For example, for the $2 \to n-2$ process with Regge pole exchanges the amplitudes factorize into impact factors and production vertices. This factorization is expected to hold also beyond the leading order approximations. The Regge limit, therefore, provides possibilities of testing higher loop calculations in perturbation theory. Furthermore, it has been known for a long time that integrable Heisenberg spin chains enter the expressions for scattering amplitudes in the so-called multi-Regge (high energy) limit [49, 50]. To leading logarithmic order, this is even true for usual QCD.

In the multi-Regge limit, the s-like variables are much larger than the t-like ones which are kept finite. The precise characterization of the limit in terms of Mandelstam invariants can be found in [19]. Here we shall mostly focus on the multi-Regge limit of the remainder functions R_n which depends on the Mandelstam invariants only through the cross ratios u, see Eq. (6) for a complete set of such cross ratios. In the multi-Regge limit, the cross ratios $u_{1\sigma}$ tend to $u_{1\sigma} \sim 1$ while the remaining ones tend to zero, i.e. $u_{2\sigma}, u_{3\sigma} \sim 0$. Cross ratios with the same index σ approach their limit values such that the following ratios remain finite

$$\left[\frac{u_{2\sigma}}{1-u_{1\sigma}}\right]^{\mathrm{MRL}} =: \frac{1}{|1+w_\sigma|^2} \quad , \quad \left[\frac{u_{3\sigma}}{1-u_{1\sigma}}\right]^{\mathrm{MRL}} =: \frac{|w_\sigma|^2}{|1+w_\sigma|^2}\,. \tag{17}$$

Through these equations we have introduced the $n-5$ complex parameters w_σ. Here and in the following the superscript MRL instructs us to evaluate the expression in square brackets in multi-Regge kinematics.

We are going to evaluate the multi-Regge limit for functions which possess branch cuts and so in order to make it well-defined, we need to specify the sheet on which the limit is actually performed. There exist 2^{n-4} different sheets or regions, depending on the sign of the energies p_i^0 for $i = 4, \ldots, n-1$. Different regions can be reached from the one in which all p_i^0 are positive by analytic continuations. We will put the sign of these p_i^0 into an array $\varrho = (\mathrm{sgn}(p_i^0))$ with $n-4$

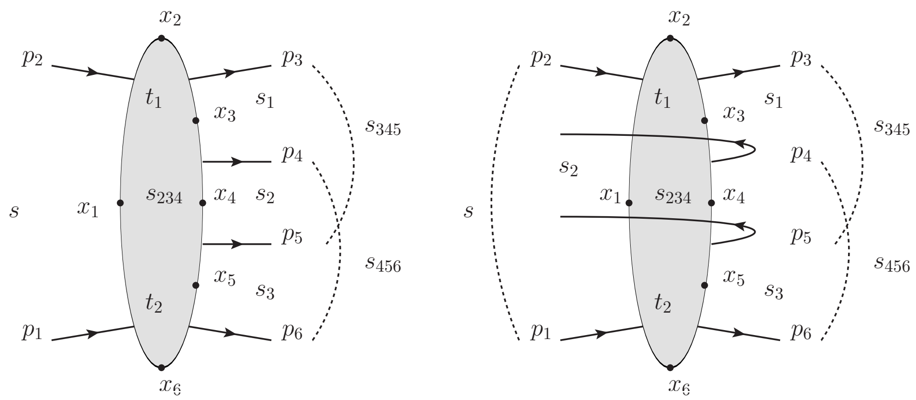

Figure 2: Kinematic configuration for the multi-Regge limit before (on the left) and after (on the right) the analytical continuation with the kinematic invariants. Momenta are denoted by p_i while dual coordinates x_i label cusps of a polygon. Figures taken from Ref. [18].

entries. The region with $\varrho_0 = (+, +, \ldots, +)$ is the one in which all outgoing particles possess non-negative energy $p_i^0 \geq 0$. If we perform the multi-Regge limit of the remainder functions R_n in this region, the result turns out to vanish

$$[R_n(u, a)]^{\mathrm{MRL}}_{++\cdots+} = 0\,. \tag{18}$$

In other words, on the sheet ϱ_0 the BDS formula is actually multi-Regge exact. So if it was only for this region, the multi-Regge limit would not be able to see the difference between a vanishing and non-vanishing remainder function.

As we have anticipated in the introduction, however, there exists other regions in which the Regge limit of the remainder functions does not vanish. Of course, the non-vanishing terms must be associated with the cut contributions that are picked up when we analytically continue from the region ϱ_0 into a new region ϱ. Hence, the multi-Regge limit is able to detect that the remainder functions are non-zero, in spite of Eq. (18). Let us discuss this in a bit more detail at the example of the 2-loop 6-gluon remainder function $R_6^{(2)}$. In this case it turns out that only one of the 2^2 regions gives a non-trivial result, namely the one with $\varrho = (--)$. The continuation into this region is depicted in Fig. 2. As one can read off from the figure, upon continuation four of the s-like invariants become negative, namely

$$x_{24}^2\,,\; x_{46}^2\,,\; x_{25}^2 \;\; \text{and} \;\; x_{36}^2 \tag{19}$$

while all other invariants x_{ij}^2 have the same sign as in the physical regime. Each of the three cross ratios $u_a = u_{a1}$ that we introduced in Eq. (6) contains two of the sign changing invariants from the list (19). Hence the cross ratios possess the same sign after continuation. But while u_2 and u_3 contain a ratio of the cross ratios from Eq. (19), u_1 involves a product. Hence, upon continuation into the $\varrho = (--)$ region, we can keep the cross ratios u_2 and u_3 fixed while u_1 must perform a full rotation around $u_1 = 0$. We shall do this by continuing the variable $u = u_1$ along a full circle

$$u(\varphi) \;=\; e^{-2i\varphi} u \tag{20}$$

where $\varphi \in [0, \pi]$. Because of the branch cuts, the behavior of the remainder functions R_n can depend very drastically on the sheet on which it is considered. Before we state the result for the R_6 at two loops, let us briefly look at the functions $Li_2(1-1/u)$ as an illustrative example. Upon analytic continuation of u along the circle (20), this function behaves as

$$Li_2\left(1 - \frac{1}{e^{-2\pi i}u}\right) = Li_2\left(1 - \frac{1}{u}\right) + 2\pi i \log\left(1 - \frac{1}{u}\right) \ . \tag{21}$$

The second term on the right hand side is the cut contribution that arises when we pass through the branch cut of the di-logarithm Li_2. When we send u to $u = 1$, the first term actually vanishes, in complete analogy to the behavior (18) of the remainder functions on the physical sheet. The second term on the right hand side of equation (21), however, is non-zero in the limit. In fact, it is actually singular.

A similar computation can be performed for the two loop remainder function $R_6^{(2)}$. As we stated in section 2, there exits a nice and relatively simple analytical formula for this function due to Goncharov et al. [28]. Lipatov and Prygarin continued this expression along the path (20) into the only non-trivial multi-Regge region for $n = 6$ [51]. It turns out that the cut contributions that are picked up during the analytic continuation do not vanish in the Regge limit. After taking the Regge limit, their result takes the following form

$$\frac{1}{2\pi i}\left[R_6^{(2)}\right]_{--}^{\mathrm{MRL}} \sim a^2 \ln(1-u) \ln|1+w|^2 \ln\left|1+\frac{1}{w}\right|^2 + g_0^{(2)}(w) \ . \tag{22}$$

Here $u = u_1$ is the cross ratio that goes to $u \sim 1$ in the Regge limit and $w = w_1$. The function $g_0^{(2)}(w)$ is also known explicitly. Since we continued a two-loop result, all terms come with a factor a^2. Let us note that the first term actually diverges as we send $u \to 1$ while the remaining terms are finite. One refers to the first term as the leading logarithmic (LL) contribution. The other term is next to leading (NLL). More generally one may show that the Regge limit of the l-loop remainder function $R_6^{(l)}$ in the region $\varrho = (--)$ contains terms which diverge as $\ln^k(1-u)$ with $k = 0, \ldots, l-1$. These are referred to as $N^{l-1-k}LL$ contributions.

As long as the remainder function is only known to a few loop orders, one cannot repeat the computation that lead to Eq. (22) for higher orders. But there exists a remarkable formula due to [13] that encodes at least the leading logarithmic terms to all loop orders. The result of Bartels et al. takes the following form

$$\left[e^{R_6+i\delta_6}\right]_{--}^{\mathrm{MRL}} \sim \sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{w}{w^*}\right)^{\frac{k}{2}} \int \frac{d\nu}{\nu^2+\frac{k^2}{4}} |w|^{2i\nu} \Phi_4 \cdot \left((u-1)\frac{|w|}{|1+w|^2}\right)^{-\omega_a(\nu,k)} \cdot \Phi_5 \tag{23}$$

where δ_6 contains the known cut contributions of he BDS Ansatz and we omitted the so-called Regge pole contributions, see [13] for the full result. The right hand side involves two functions of the coupling, the impact factors $\Phi_4 = \Phi_4(\nu, k)$ and $\Phi_5 = \Phi_5(\nu, k)$ related to the production of particles 4 and 5, resp., and the so-called BFKL eigenvalue $\omega = \omega(\nu, k)$. They all possess a power series expansion in a. In order to construct the LL contributions of the remainder function at any loop order it is sufficient to know Φ and ω in leading order. Bartels et al. showed that the impact factors Φ are trivial at leading order in a while the BFKL eigenvalue is given by the expression

$$\omega_a(\nu, k) = 2a\psi(1) - a\psi\left(1 + i\nu + \frac{|k|}{2}\right) - a\psi\left(1 - i\nu + \frac{|k|}{2}\right) + \frac{a}{2}\frac{|k|}{\nu^2+\frac{k^2}{4}} + \mathcal{O}(a^2) \, . \tag{24}$$

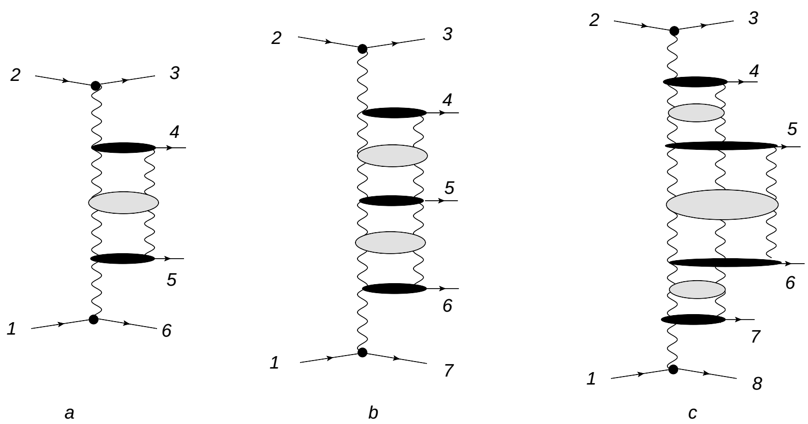

Figure 3: Illustration of the remainder functions R_6 (a), and pieces of R_7 (b) of R_8 (c): the wavy lines denote reggeized gluons. R_6 and R_7 consist of the bound state of two gluons (BFKL), R_8 contains, as a new piece, a three-gluon state. The grey circles denote Green's function; their bound state poles are given by the eigenvalues of the spin chain Hamiltonian (see text).

It is not too difficult to reconstruct the LL term in Eq. (22) from formula (23) and in fact to carry these computations to higher loop orders and even beyond the leading logarithmic order, see [14] for an extensive discussion.

The so-called BFKL eigenvalues $\omega(\nu, k)$ are the lowest eigenvalues of a non-compact SL(2,$\mathbb{C}$) Heisenberg Hamiltonian on an spin chain of length two, see [49,50,52]. The parameters ν, k label irreducible representations of SL(2,$\mathbb{C}$). Explicit expression for $\omega(\nu, k)$ in NLL were first given in [53]. Later, these were extended to NNLL using input from the amplitude bootstrap [54] and finally to all loops in [21], see below.

In the meantime, the LL calculations have been extended to $2 \to 5$ [55, 56] and even $2 \to 8$ [57] scattering processes, see Fig.3. Results for the $2 \to 5$ scattering amplitude include (a) a list of kinematic regions where the multi-Regge limit of the remainder function R_7 is nonzero and (b) an all order expression for the relevant pieces of R_7, analogous to the Eq. (23) we displayed for R_6. When compared to R_6, the multi-Regge limit of the remainder function R_7 contains a new production vertex for the centrally produced gluon, the central emission vertex. The other building blocks, i.e. the BFKL eigenvalues and the impact factors Φ are the same as for R_6. All these elements possess an expansion in the coupling a, and they are universal, i.e. they also appear in processes with higher numbers of legs, see also [58] and [59].

The remainder function R_8 for $n = 8$ external gluons contains, for the first time, a new eigenvalue of the Heisenberg Hamiltonian which extends the spin chain to length three. While in leading order this new Hamiltonian is just the sum of two length two Hamiltonians, the NLL approximation gives rise to a term which represents a new three body interaction between reggeized gluons [60]. Again, the detailed composition of R_8 depends upon the kinematic region. The next extensions of the spin chain is expected to be seen in the $n = 10$ point scattering process, $2 \to 8$, the $n = 12$ point process, $2 \to 10$ etc. One of the challenges will be to find, beyond the LL approximation, the eigenvalues of this spin chain: The three body interaction found in [60] raises some doubts whether they are simply obtained from the sum of two body interactions. This issue certainly deserves further investigation.

5 The strong coupling theory

At strong coupling, scattering amplitudes in the planar limit of $\mathcal{N} = 4$ SYM possess a geometric interpretation, namely as the area of a minimal two-dimensional surface that approaches the boundary of AdS_5 along a light-like polygon [15]. The latter is made up from the light-like four-momenta of the external gluons. In other words, the corners the polygon are given by the variables x_i we introduced in Eq. (1). Hence, the polygon encodes all the kinematic data of the process. The two-dimensional surface is drawn into AdS_5 by the gravitational field that is created by the D3-branes. As it stands, this beautiful geometric description of scattering processes does not seem to be very helpful in computing high energy limits of the kind we considered in the weakly coupled theory.

However, the minimal area problem was shown to possess an intriguing reformulation in which the area is reproduced by the free energy of a 1-dimensional integrable quantum system [16, 17]. The particle content and interactions of the latter are designed so as to solve the original geometric minimal area problem. The 1-dimensional quantum system contains a number of mass parameters and chemical potentials which match precisely the number of kinematic invariants in the scattering process, i.e. there are $3n - 15$ such parameters. The basic excitations turn out to interact through integrable $2 \mapsto 2$ scattering phases $S_{AB}(\theta)$ which depend on the rapidity θ of the 1-dimensional scattering process. As in any other quantum field theory, the vacuum of this 1-dimensional interacting quantum system is a complicated state that is determined by the quantum dynamics of the fundamental excitations, the external parameters and the interaction. More precisely, the rapidity densities of the various particles must be determined self-consistently as a function of the masses and chemical potentials. This is done by solving a system of coupled non-linear integral equations which involve both the external parameters and the scattering phases. Roughly, such equations take the form

$$\log Y_A(\theta) = -m_A \cosh\theta + \mu_A + \sum_B \int_{\infty}^{\infty} d\theta'\, K_{AB}(\theta - \theta' + i\phi_{AB}) \log\left(1 + Y_B(\theta')\right) \ . \tag{25}$$

Here the indices A, B run over the various particles and $\phi_{AB} = \phi_A - \phi_B$. The parameters $\mathfrak{m}_A = m_A \exp(i\phi_A)$ and μ_A play the role of the (complex) mass parameters and chemical potentials. The integration kernels

$$K_{AB}(\theta) = \partial_\theta \log S_{AB}(\theta) \tag{26}$$

are directly related to the scattering phases $S_{AB}(\theta)$. Once the density of excitations has been found, it can be used to determine the total energy of the system as

$$\mathcal{E}(\mathfrak{m}, \mu) \sim \sum_A \int d\theta\, m_A \cosh\theta \log(1 + Y_A(\theta)) \ . \tag{27}$$

The right hand side depends on m and μ through the explicit factor in the integrand as well as through the dependence of the rapidity densities Y_A on the external parameters. As we stressed before, the number of mass parameters and chemical potentials in the auxiliary 1-dimensional system was designed to match the number of cross-ratios u that describe the gauge theory scattering process. It is actually possible to provide a precise relation between the two sets of parameters,

$$u_{\alpha\sigma} = \frac{Y_{A(\alpha,\sigma)}}{1 + Y_{A(\alpha,\sigma)}} \left(\theta = i\frac{\pi}{4} r(\alpha,\sigma) - i\phi_{A(\alpha,\sigma)}\right) \ . \tag{28}$$

The dependence of the particle label $A = A(\alpha,\sigma)$ and the integer $r = r(\alpha,\sigma)$ on the indices $\alpha\sigma$ of the cross ratio can be found in Eq. (3.16) of [19]. The expressions on the right hand side of Eqs. (28) are functions of the masses $\mathfrak{m}_A$ and chemical potentials μ_A. These may be inverted at least numerically to determine the external parameters $\mathfrak{m}_A$ and μ_A of the 1-dimensional quantum system in terms of the kinematic data of the gauge theoretic scattering process.

According to [16, 17], the free energy we have just described provides the most non-trivial contribution to the remainder functions R_n at strong coupling. Explicit formulas for all the other terms can be found in the original literature. As we have stressed in the second section, the remainder function has a rather intricate analytic structure as a function of the kinematical variables u, at least at weak coupling. It turns our that the same is true at strong coupling [18], i.e. that the energy possesses interesting branch cuts as a function or the mass parameters and chemical potentials. From the point of view of the auxiliary 1-dimensional quantum system the basic mechanism goes back to an observation of Dorey and Tateo [61, 62]. As we move $\mathfrak{m}_A$ and μ_A through the complex plane to some new values m'_A and μ'_A the solutions $Y_B(\theta)$ are going to change. In particular, the solutions of the equation $Y_B(\theta_*) = -1$ will move through the space of complex rapidities θ. By inserting the relation (26) into Eq. (25) we can see that solutions of $Y_B(\theta_*) = -1$ are associated with poles in the integrand of the nonlinear integral equation. When these poles cross the integration contour, the equation picks up some residue contribution and hence assumes the new form

$$\begin{aligned}\log Y_A(\theta) \quad = \quad & -m'_A \cosh\theta + \mu'_A + \sum_{B,i} \sigma_i \log S_{AB}(\theta - \theta_{B,i} + i\phi_{AB}) \\ & + \sum_B \int_\infty^\infty d\theta' K_{AB}(\theta - \theta' + i\phi_{AB}) \log(1 + Y_B)\end{aligned} \tag{29}$$

with sign factors σ_i depending on whether the solution $\theta_{B,i}$ of $Y_B(\theta_i) = -1$ crosses from the lower half of the complex plane into the upper or vice versa. Here, the index $i = 1, \ldots, q_B$ was introduced in order to enumerate the crossing solutions and we denote by q_B the total number of them. Whenever such crossing happens, there is appears a new contribution to the total energy of the system,

$$\mathcal{E}(\mathfrak{m}', \mu') \sim \sum_{B,i} m'_B \sinh\theta_{B,i} + \sum_A \int d\theta\, m'_A \cosh\theta \log(1 + Y_A(\theta)) \ . \tag{30}$$

One may interpret these changes to the system as excitations that have been produced while we continued the system parameter $\mathfrak{m}_A, \mu_A$. Given the relation (28) between the mass parameters and chemical potentials of the 1-dimensional quantum system and the kinematical variables of the gauge theory, it is tempting to conjecture that the cut contributions of gauge theory amplitudes are related to the energy of excitations above the ground state in the 1-dimensional quantum system [18]. We will provide very strong evidence in the next section.

6 High energy scattering at strong coupling

Given the special features of the multi-Regge limit in gauge theory one may wonder about the nature of the corresponding limit for string theory on AdS_5. An important hint actually comes from the fact (18) that the remainder functions vanishes when the multi-Regge limit is performed on the main sheet. Recall that at strong coupling the remainder functions is computed from the free energy $\mathcal{E}$ of the auxiliary 1-dimensional quantum system. As we argued before, due to quantum fluctuations the latter is a highly non-trivial function of the system parameters. But there is a way to turn off fluctuations and force the free energy to zero, namely by making the fundamental excitations of the 1-dimensional system infinitely massive so that it costs too much energy to produce them. This intuition is indeed correct as was shown in [18] for $n = 6$ gluons and then generalized to any number of external gluons in [19]. In other words, the high energy limit of gauge theory is directly related to the low energy limit of the auxiliary 1-dimensional quantum system.

As we have seen in section 4, on the gauge theory side the multi-Regge limit can become non-trivial if it is performed after continuation to a different multi-Regge region. According to our discussion in the previous section, such an analytic continuation can produce quasi-particle excitations of the ground state in the 1-dimensional auxiliary quantum system. The energy of these excitations contributes to the remainder function, see Eq. (30). In general the equations (29) for the rapidities are difficult to solve since the new quasi-particle excitations are dressed by clouds of quantum fluctuations. But as we take the multi-Regge limit, i.e. send all the mass parameters m_A to infinity, we freeze quantum fluctuations and the energy of the system is simply a sum of the bare quasi-particle energies. This physical picture suggests some drastic simplifications in the multi-Regge limit, even at strong coupling.

Our very qualitative discussion in the previous two paragraphs can actually be turned into an exact mathematical statement by analyzing the relation (28) between the cross ratios and the mass parameters. Indeed, it is possible to prove that the cross ratios possess the correct limiting behavior, i.e.

$$(u_{1\sigma}, u_{2\sigma}, u_{3\sigma}) \to (1, 0, 0) \quad \text{if} \quad \mathfrak{m}_A e^{i\frac{\pi}{4}s_A} \to \infty \,. \tag{31}$$

This result of [19] instructs us to send the masses to infinity along certain directions in the space of complex mass parameters. The value of the integer s_A can be found in the original paper. To be more precise let us stress that the parameters m_A are actually dimensionless and should rather be thought of as products $m_A = M_A L$ of a physical mass M_A and the system size L. Sending m_A to infinity is then achieved be making the physical mass M_A large or by going to the limit of large system size. In such a limit, the first term on the right hand side (25) goes to minus infinity and hence the function $Y(\theta)$ that appears in the logarithm on the left hand side must approach zero. This in turn implies that $\log(1 + Y_B(\theta')) \sim 0$ so that we can neglect the integral on the right hand side of the non-linear integral equations. This leaves us with the first two terms on the right hand side hand side of Eq. (29). The fact that we can drop the integral term from the Eqs. (25) and (29) in the low energy limit is the mathematical realization of what we referred to as freezing of fluctuations above. It clearly turns the complicated non-linear integral equations into a much simpler system.

With a little bit of additional massaging we can actually bring the resulting system into a more standard form. To this end we evaluate the first line of Eq. (29) at the points $\theta = \theta_{B,j}$, use that $Y(\theta_{B,j}) = -1$ and exponentiate both sides. As a result we obtain a set of $Q = \sum_A q_A$ algebraic Bethe ansatz equations,

$$e^{m'_A \cosh\theta_{A,j} - \mu'_A} = \prod_{(B,i)} S^{\sigma_i}_{AB}(\theta_{A,j} - \theta_{B,i} + i\phi'_{AB})\,, \tag{32}$$

for Q unknown rapidities $\theta_{A,j}$. We called these equations the *Regge Bethe ansatz* in [20]. Let us stress once again that these equations are fully explicit once we insert the known expressions for the phase shifts S_{AB} from [16]. Hence the system (32) can be solved for the rapidities $\theta_{A,j}$ of the excitations. Once the rapidities have been determined one can compute the energy as a sum of Q quasi-particle energies, one summand for each solution of $Y(\theta_{B,i}) = -1$ that has crossed the contour. From there it is an easy step to obtain the multi-Regge limit of the remainder function $[R_n^{(\infty)}]^{\mathrm{MRL}}$ at infinite coupling. The only tricky issue that remains is to know which solution of the Regge Bethe ansatz equations actually corresponds to a given multi-Regge region ϱ. At the moment there exists no general prescription that would relate discrete set of regions to the discrete set of solutions to Eqs. (32). But a few examples have been worked out.

This includes the case of $n = 6$ external gluons, see [18, 22]. In this case the index A runs through $A = 1, 2, 3$ and the corresponding mass parameters $\mathfrak{m}_A$ and chemical potentials μ_A are determined by a single real mass parameter m, an angle ϕ and a chemical potential μ, see the original literature for concrete formulas. In order to understand which solution of the Regge Bethe ansatz equation is relevant for the $\varrho = (--)$ region the system parameters m, ϕ and μ were continued along curves that kept u_2 and u_3 fixed while moving u_1 around the origin of the complex u_1 plane as prescribed in Eq. (20). Along the entire curve one can solve the non-linear integral equations and follow the solutions of $Y_A(\theta_*) = -1$. It turns out that only a single pair of such solutions for the function $Y_A = Y_3$ actually crosses the real line, see Fig. 4. This implies that the continuation produces $\mathcal{Q} = 2$ excitations or, more precisely, that $q_3 = 2$ while $q_1 = 0 = q_2$. The rapidities $\theta_{A,1} = \theta_{3,1}$ and $\theta_{A,2} = \theta_{3,2}$ of these excitations at the end of the path may be read off from Fig. 4, but they can also be found analytically. In fact, in the limit $m_3 = m \to \infty$, the two roots $\theta_{3,1}$ and $\theta_{3,2}$ must take the values [22]

$$\theta_{3,1} = i\frac{\pi}{4} + i\phi' , \qquad \theta_{3,2} = -i\frac{\pi}{4} + i\phi' \quad , \tag{33}$$

which is consistent with the position of endpoints in Fig. 4 since the plot was produced for $\phi = \phi' = 0$. Once the roots are known their values can be inserted to compute the remainder function in multi-Regge kinematics,

$$\left[e^{R_6^{(\infty)}+i\delta_6}\right]^{\mathrm{MRL}}_{(--)} \sim \left((u-1)\frac{|w|}{|1+w|^2}\right)^{-\omega_\infty} , \tag{34}$$

where

$$\omega_\infty = \sqrt{\frac{a}{2}}\left(\sqrt{2} - \log(1+\sqrt{2})\right) . \tag{35}$$

Let us point out that expression in brackets is the same function of the kinematic variables that appears in the weak coupling result (23). As we will discuss shortly, the exponent ω_∞ can be considered as the universal leading term in the strong coupling expansion of the all order (N^∞LL) BFKL eigenvalues $\omega_a(\nu, k)$ which turn out not to depend on the quantum numbers ν and k.

For the case of $n = 7$ external gluons, a similar analysis has been carried out for three of the four multi-Regge regions in which one expects a non-trivial result, namely for the regions $\varrho = (--+), (+--)$ and $(---)$. The results for the first two regions can essentially be copied from the study of $n = 6$ gluons that we sketched in the previous paragraph. For the region $\varrho = (---)$ things are a little more interesting. When $n = 7$ there are six functions Y_A which are labeled by Y_{as} with $a = 1, 2, 3$ and $s = 1, 2$. When we continue the system parameters to get to the region $\varrho = (---)$ four solutions of $Y_A(\theta_*) = -1$ cross the real line, namely two

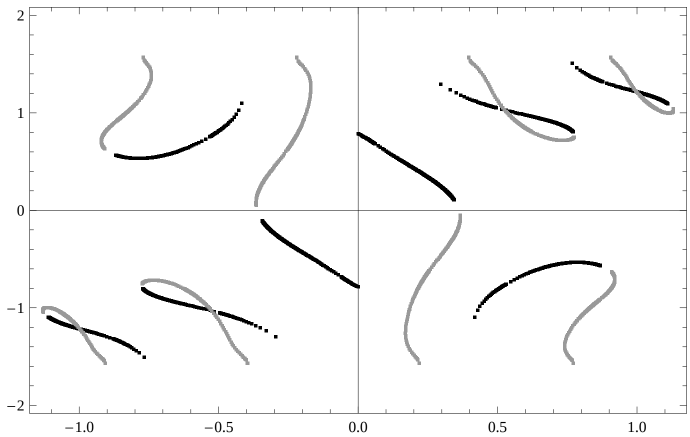

Figure 4: Solutions of the equation $Y_3(\theta_*) = -1$ in the complex θ plane for a family of parameters that implements the path (20) with $u_2 = u_3$ kept constant. The condition $u_2 = u_3$ corresponds to setting $\phi = 0$. The curves start in the light grey portions and proceed to the dark grey part. The change of color signals the point at which the pair of solutions crosses the real line. Figure taken from Ref. [18].

solutions of $Y_{31}(\theta_*) = -1$ and two solutions of $Y_{12}(\theta_*) = -1$. Hence there are four Bethe ansatz equations for four Bethe roots. These can be solved to obtain the following result for the strong coupling limit in the $\varrho = (- - -)$ region

$$\left[e^{R_7^{(\infty)}+i\delta_7}\right]^{\mathrm{MRL}}_{(---)} \sim \left((u_{11}-1)\frac{|w_1|}{|1+w_1|^2}(u_{12}-1)\frac{|w_2|}{|1+w_2|^2}\right)^{-\omega^\infty} , \tag{36}$$

where ω^∞ is once again given by Eq. (35). This is in fact nicely consistent with the weak coupling analysis. There is one more region $\varrho = (- + -)$ for which the remainder function is expected to possess a non-trivial multi-Regge limit. But in this case no solution of the Bethe ansatz has been identified that could give a reliable strong coupling prediction. We will comment a bit more on this issue below.

7 Interpolation and Outlook

In this work we have reviewed results on the multi-Regge limit of the remainder function in planar $\mathcal{N} = 4$ SYM theory. In particular we discussed this limit both at weak and strong coupling. The techniques and resulting formulas were completely different. As an example we provided expressions for the BFKL eigenvalues $\omega_a(\nu, k)$ in LLA, see Eq. (24), and for the quantity ω_∞ at strong coupling. Given the completely different calculational schemes for the two quantities it is stunning fact that Basso, Caron-Huot and Sever were able to smoothly interpolate between both expressions [21]. Based on the understanding of the flux tube in $\mathcal{N} = 4$ SYM theory that was achieved in [63–69] Basso and al. proposed the following formula for the BFKL eigenvalue $\omega_a(\nu,k)$ at any value of the coupling a,

$$\omega_a(u,k) = -\int_0^\infty \frac{d\tau}{\tau}\left(\frac{1}{2}(Q_a(-\tau)+Q_a(\tau))\cos(ut)e^{-|k|\tau/2} - Q_a(\tau)\right), \qquad (37)$$

$$\nu_a(u,k) = u + \int_0^\infty \frac{d\tau}{4\tau}(Q_a(-\tau)-Q_a(\tau))\sin(u\tau)e^{-|k|\tau/2}. \qquad (38)$$

Here, $Q_a(\tau)$ is the generating function for the infinite set of charge densities of the GKP string that is determined by the BES equation (14), including its entire dependence on the coupling a. E.g. we displayed an explicit result for the first charge q_a^1 up to forth order in Eq. (5) and a some terms of the strong coupling expansion in Eq. (16). With such expressions one can then go ahead and calculate the integrals $\omega(u,k)$ and $\nu_a(u,k)$, invert the second expression to compute $u = u(\nu,k)$ and eliminate u from $\omega_a(u,k)$ to obtain $\omega_a(\nu,k)$. It is quite straightforward to re-derive expressions such as Eq. (24) in leading order as well as higher order corrections at weak coupling. At strong coupling, it turns out that the solution becomes independent of ν and k and it indeed coincides with our formula (35).

Similar interpolation equations for those BFKL eigenvalues that correspond to Heisenberg spin chains of more than two sites at weak coupling have not been published. The first region in which these are expected to occur is $\varrho = (-++-)$ for $n = 8$. Regions in which $-$ signs are separated by $+$ signs are generally less studied than those in which all $-$ signs appear on consecutive legs. The first example of such a region appears for $n = 7$, namely the region $\varrho = (-+-)$. As we have commented above, there exists no satisfactory strong coupling prediction for this region. It would certainly be interesting to determine the relevant solution of our Regge Bethe ansatz. Let us point out that Basso et al. also determined the impact factor Φ to all loops [21]. On the other hand, the amplitudes for higher number of external gluons contain additional building blocks, such as the central emission vertex we discussed briefly in section 4. It would certainly be interesting to construct all these elements for arbitrary coupling. While the relevant equations have not appeared in the literature yet, there is a concrete path towards a complete understanding of scattering amplitudes for planar $\mathcal{N} = 4$ SYM theory, at least in the multi-Regge limit. And even the complete all loop results for $n = 6$ demonstrate that finally string theory can cope with high energy scattering, not just qualitatively but even in precision calculations and it does so with astonishing efficiency.

Acknowledgements

It is a great pleasure to thank all our collaborators for their contributions to the joint research we reviewed here, especially Jan Kotanski and Martin Sprenger. We are particularly grateful to Lev Lipatov for sharing his deep insights over so many years. Finally, we acknowledge the support from the SFB 676 on "Particles, Strings and the Early Universe".

References

[1] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, Nucl. Phys. **B72** (1974) 461. [,337(1973)].

[2] A. M. Polyakov, *Thermal Properties of Gauge Fields and Quark Liberation*, Phys. Lett. **72B** (1978) 477–480.

[3] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. **38** (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[4] J. Polchinski and M. J. Strassler, *Hard scattering and gauge / string duality*, *Phys. Rev. Lett.* **88** (2002) 031601, [hep-th/0109174].

[5] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the AdS(5) $\times$ S^5 superstring*, *Phys. Rev.* **D69** (2004) 046002, [hep-th/0305116].

[6] J. A. Minahan and K. Zarembo, *The Bethe ansatz for N=4 superYang-Mills*, *JHEP* **03** (2003) 013, [hep-th/0212208].

[7] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3–32, [1012.3982].

[8] A. V. Kotikov and L. N. Lipatov, *DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory*, *Nucl. Phys.* **B661** (2003) 19–61, [hep-ph/0208220]. [Erratum: Nucl. Phys.B685,405(2004)].

[9] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, *Three loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model*, *Phys. Lett.* **B595** (2004) 521–529, [hep-th/0404092]. [Erratum: Phys. Lett.B632,754(2006)].

[10] S. Moch, J. A. M. Vermaseren and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case*, *Nucl. Phys.* **B688** (2004) 101–134, [hep-ph/0403192].

[11] A. Vogt, S. Moch and J. A. M. Vermaseren, *The Three-loop splitting functions in QCD: The Singlet case*, *Nucl. Phys.* **B691** (2004) 129–181, [hep-ph/0404111].

[12] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, *J. Stat. Mech.* **0701** (2007) P01021, [hep-th/0610251].

[13] J. Bartels, L. N. Lipatov and A. Sabio Vera, *N=4 supersymmetric Yang Mills scattering amplitudes at high energies: The Regge cut contribution*, *Eur. Phys. J.* **C65** (2010) 587–605, [0807.0894].

[14] L. J. Dixon, C. Duhr and J. Pennington, *Single-valued harmonic polylogarithms and the multi-Regge limit*, *JHEP* **10** (2012) 074, [1207.0186].

[15] L. F. Alday and J. M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064, [0705.0303].

[16] L. F. Alday, D. Gaiotto and J. Maldacena, *Thermodynamic Bubble Ansatz*, *JHEP* **09** (2011) 032, [0911.4708].

[17] L. F. Alday, J. Maldacena, A. Sever and P. Vieira, *Y-system for Scattering Amplitudes*, *J. Phys.* **A43** (2010) 485401, [1002.2459].

[18] J. Bartels, J. Kotanski and V. Schomerus, *Excited Hexagon Wilson Loops for Strongly Coupled N=4 SYM*, *JHEP* **01** (2011) 096, [1009.3938].

[19] J. Bartels, V. Schomerus and M. Sprenger, *Multi-Regge Limit of the n-Gluon Bubble Ansatz*, *JHEP* **11** (2012) 145, [1207.4204].

[20] J. Bartels, V. Schomerus and M. Sprenger, *The Bethe roots of Regge cuts in strongly coupled $\mathcal{N} = 4$ SYM theory*, *JHEP* **07** (2015) 098, [1411.2594].

[21] B. Basso, S. Caron-Huot and A. Sever, *Adjoint BFKL at finite coupling: a short-cut from the collinear limit*, *JHEP* **01** (2015) 027, [1407.3766].

[22] J. Bartels, J. Kotanski, V. Schomerus and M. Sprenger, *The Excited Hexagon Reloaded*, 1311.1512.

[23] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.

[24] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D72** (2005) 085001, [hep-th/0505205].

[25] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *Dual superconformal symmetry of scattering amplitudes in N=4 super-Yang-Mills theory*, *Nucl. Phys.* **B828** (2010) 317–374, [0807.1095].

[26] J. Bartels, L. N. Lipatov and A. Sabio Vera, *BFKL Pomeron, Reggeized gluons and Bern-Dixon-Smirnov amplitudes*, *Phys. Rev.* **D80** (2009) 045002, [0802.2065].

[27] V. Del Duca, C. Duhr and V. A. Smirnov, *The Two-Loop Hexagon Wilson Loop in N = 4 SYM*, *JHEP* **05** (2010) 084, [1003.1702].

[28] A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, *Phys. Rev. Lett.* **105** (2010) 151605, [1006.5703].

[29] L. J. Dixon, J. M. Drummond, C. Duhr, M. von Hippel and J. Pennington, *Bootstrapping six-gluon scattering in planar N=4 super-Yang-Mills theory*, *PoS* **LL2014** (2014) 077, [1407.4724].

[30] B. Basso, A. Sever and P. Vieira, *Hexagonal Wilson loops in planar $\mathcal{N} = 4$ SYM theory at finite coupling*, *J. Phys.* **A49** (2016) 41LT01, [1508.03045].

[31] J. Golden and M. Spradlin, *An analytic result for the two-loop seven-point MHV amplitude in $\mathcal{N} = 4$ SYM*, *JHEP* **08** (2014) 154, [1406.2055].

[32] L. J. Dixon, J. Drummond, T. Harrington, A. J. McLeod, G. Papathanasiou and M. Spradlin, *Heptagons from the Steinmann Cluster Bootstrap*, *JHEP* **02** (2017) 137, [1612.08976].

[33] B. Basso and G. P. Korchemsky, *Anomalous dimensions of high-spin operators beyond the leading order*, *Nucl. Phys.* **B775** (2007) 1–30, [hep-th/0612247].

[34] Z. Bajnok, R. A. Janik and T. Lukowski, *Four loop twist two, BFKL, wrapping and strings*, *Nucl. Phys.* **B816** (2009) 376–398, [0811.4448].

[35] T. Lukowski, A. Rej and V. N. Velizhanin, *Five-Loop Anomalous Dimension of Twist-Two Operators*, *Nucl. Phys.* **B831** (2010) 105–132, [0912.1624].

[36] C. Marboe, V. Velizhanin and D. Volin, *Six-loop anomalous dimension of twist-two operators in planar $\mathcal{N} = 4$ SYM theory*, *JHEP* **07** (2015) 084, [1412.4762].

[37] C. Marboe and V. Velizhanin, *Twist-2 at seven loops in planar $\mathcal{N} = 4$ SYM theory: full result and analytic properties*, *JHEP* **11** (2016) 013, [1607.06047].

[38] S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*, *JHEP* **10** (2017) 041, [1707.08315].

[39] J. A. M. Vermaseren, A. Vogt and S. Moch, *The Third-order QCD corrections to deep-inelastic scattering by photon exchange*, *Nucl. Phys.* **B724** (2005) 3–182, [hep-ph/0504242].

[40] S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*, *Phys. Lett.* **B782** (2018) 627–632, [1805.09638].

[41] C. L. Basham, L. S. Brown, S. D. Ellis and S. T. Love, *Energy Correlations in electron-Positron Annihilation in Quantum Chromodynamics: Asymptotically Free Perturbation Theory*, *Phys. Rev.* **D19** (1979) 2018.

[42] A. V. Belitsky, S. Hohenegger, G. P. Korchemsky, E. Sokatchev and A. Zhiboedov, *Energy-Energy Correlations in N=4 Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **112** (2014) 071601, [1311.6800].

[43] O. Gituliar and S. Moch, *Fuchsia and Master Integrals for Energy-Energy Correlations at NLO in QCD*, *Acta Phys. Polon.* **B48** (2017) 2355, [1711.05549].

[44] L. J. Dixon, M.-X. Luo, V. Shtabovenko, T.-Z. Yang and H. X. Zhu, *Analytical Computation of Energy-Energy Correlation at Next-to-Leading Order in QCD*, *Phys. Rev. Lett.* **120** (2018) 102001, [1801.03219].

[45] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *A Semiclassical limit of the gauge / string correspondence*, *Nucl. Phys.* **B636** (2002) 99–114, [hep-th/0204051].

[46] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, *The Four-Loop Planar Amplitude and Cusp Anomalous Dimension in Maximally Supersymmetric Yang-Mills Theory*, *Phys. Rev.* **D75** (2007) 085010, [hep-th/0610248].

[47] B. Basso, G. P. Korchemsky and J. Kotanski, *Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling*, *Phys. Rev. Lett.* **100** (2008) 091601, [0708.3933].

[48] R. C. Brower, C. E. DeTar and J. H. Weis, *Regge Theory for Multiparticle Amplitudes*, *Phys. Rept.* **14** (1974) 257.

[49] L. N. Lipatov, *Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models*, *JETP Lett.* **59** (1994) 596–599, [hep-th/9311037]. [Pisma Zh. Eksp. Teor. Fiz.59,571(1994)].

[50] L. D. Faddeev and G. P. Korchemsky, *High-energy QCD as a completely integrable model*, *Phys. Lett.* **B342** (1995) 311–322, [hep-th/9404173].

[51] L. N. Lipatov and A. Prygarin, *Mandelstam cuts and light-like Wilson loops in N=4 SUSY*, *Phys. Rev.* **D83** (2011) 045020, [1008.1016].

[52] L. N. Lipatov, *Integrability of scattering amplitudes in N=4 SUSY*, *J. Phys.* **A42** (2009) 304020, [0902.1444].

[53] V. S. Fadin and L. N. Lipatov, *BFKL equation for the adjoint representation of the gauge group in the next-to-leading approximation at N=4 SUSY*, *Phys. Lett.* **B706** (2012) 470–476, [1111.0782].

[54] L. J. Dixon, J. M. Drummond, C. Duhr and J. Pennington, *The four-loop remainder function and multi-Regge behavior at NNLLA in planar $N = 4$ super-Yang-Mills theory*, *JHEP* **06** (2014) 116, [1402.3300].

[55] J. Bartels, A. Kormilitzin and L. Lipatov, *Analytic structure of the $n = 7$ scattering amplitude in $\mathcal{N} = 4$ SYM theory in the multi-Regge kinematics: Conformal Regge pole contribution*, *Phys. Rev.* **D89** (2014) 065002, [1311.2061].

[56] J. Bartels, A. Kormilitzin and L. N. Lipatov, *Analytic structure of the $n = 7$ scattering amplitude in $\mathcal{N} = 4$ theory in multi-Regge kinematics: Conformal Regge cut contribution*, *Phys. Rev.* **D91** (2015) 045005, [1411.2294].

[57] J. Bartels, *in preparation*, .

[58] T. Bargheer, G. Papathanasiou and V. Schomerus, *The Two-Loop Symbol of all Multi-Regge Regions*, *JHEP* **05** (2016) 012, [1512.07620].

[59] V. Del Duca, S. Druc, J. Drummond, C. Duhr, F. Dulat, R. Marzucca et al., *The seven-gluon amplitude in multi-Regge kinematics beyond leading logarithmic accuracy*, *JHEP* **06** (2018) 116, [1801.10605].

[60] J. Bartels, V. S. Fadin, L. N. Lipatov and G. P. Vacca, *NLO Corrections to the kernel of the BKP-equations*, *Nucl. Phys.* **B867** (2013) 827–854, [1210.0797].

[61] P. Dorey and R. Tateo, *Excited states by analytic continuation of TBA equations*, *Nucl. Phys.* **B482** (1996) 639–659, [hep-th/9607167].

[62] P. Dorey and R. Tateo, *Excited states in some simple perturbed conformal field theories*, *Nucl. Phys.* **B515** (1998) 575–623, [hep-th/9706140].

[63] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *An Operator Product Expansion for Polygonal null Wilson Loops*, *JHEP* **04** (2011) 088, [1006.2788].

[64] B. Basso, *Exciting the GKP string at any coupling*, *Nucl. Phys.* **B857** (2012) 254–334, [1010.5237].

[65] B. Basso and A. Rej, *Bethe ansätze for GKP strings*, *Nucl. Phys.* **B879** (2014) 162–215, [1306.1741].

[66] B. Basso, A. Sever and P. Vieira, *Spacetime and Flux Tube S-Matrices at Finite Coupling for N=4 Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **111** (2013) 091602, [1303.1396].

[67] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and Flux tube S-matrix II. Extracting and Matching Data*, *JHEP* **01** (2014) 008, [1306.2058].

[68] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and Flux-tube S-matrix III. The two-particle contributions*, *JHEP* **08** (2014) 085, [1402.3307].

[69] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and Flux-tube S-matrix IV. Gluons and Fusion*, *JHEP* **09** (2014) 149, [1407.1736].