Heavy quark form factors at two loops

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We compute the two-loop QCD corrections to the heavy quark form factors in the case of the vector, axial-vector, scalar and pseudoscalar currents up to second order in the dimensional parameter $\varepsilon = (4-D)/2$. These terms are required in the renormalization of the higher-order corrections to these form factors.

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I. INTRODUCTION

Since its discovery [1,2] in 1995, the top quark has been studied extensively both in theoretical and experimental premises. These studies of the heaviest particle of the Standard Model (SM) also provide a detailed probe into some aspects of electroweak symmetry breaking (EWSB). Due to its very short lifetime, the top quark decays before hadronizing, and thus provides a window to study its production dynamics widely without accounting for the hadronization effects and, therefore, as a directly accessible quark.[1] The experiments carried out at the Tevatron and later at the LHC have already measured many observables which allow us to extract the properties of the top quark with remarkable accuracy. Compared to the Tevatron, the LHC offers an abundant rate of top quark pair and single production, hence providing a perfect ground for precision tests. Due to the combined effort from both the theoretical and experimental sides, a striking accuracy has been achieved in many observables; e.g., the uncertainties on the predictions of the inclusive production cross section of a top quark pair are now around 5% at a fixed top quark mass of $m_t = 172.5$ GeV. While these precise measurements provide a strong ground for testing the predictions within the SM, beyond the Standard Model (BSM) physics scenarios can also hide under those small uncertainties. To find a hint of BSM physics or to rule out some hypotheses, we need more precision and certainly a future linear or circular e^+e^- collider can achieve that. In order to match the experimental accuracy, precise predictions are required on the theoretical side.

In this paper, we focus on perturbative quantum chromodynamics (QCD) corrections to the form factors involving heavy quarks which are basic building blocks of various physical quantities concerning top quark pair production. The massive vector and axial vector form factors play an important role in the forward-backward asymmetry of bottom or top quark production at electron-positron colliders. Likewise the decay of a scalar or pseudoscalar particle to a pair of heavy quarks could also play a very important role in shedding light on the quantum nature of the Higgs boson. There are also static quantities like the anomalous magnetic moment, which receive contributions from such massive form factors. For these reasons, the phenomenology and the perturbative QCD corrections to these form factors have gained much attention during the last decade.

In Refs. [3,4], the first-order QCD corrections were obtained for the vector and axial-vector form factors. A massless approximation was considered to obtain the next-to-next-to-leading-order (NNLO) QCD corrections in [5] numerically, later followed by an analytic computation in [6]. Another numerical computation was performed in [7] at NNLO using a different formalism. On the other hand, the next-to-leading-order (NLO) contributions to the scalar and pseudoscalar form factors were known [8–11] for long and NNLO corrections by employing quark mass expansion to various orders, c.f. [12–19]. A series of papers followed obtaining the two-loop QCD corrections for the vector form factor [20], the axial-vector form factor [21], the anomaly contributions [22] and the scalar and pseudoscalar form factors [23]. An independent cross-check of the vector form factor has been performed in [24] with the addition of the $\mathcal{O}(\varepsilon)$ contribution, where $\varepsilon = (4-D)/2$ and D is the spacetime dimension. Recently, the calculation

[1]The reconstruction of the t-quark itself, and in particular its mass, however, requires us to study the hadronization effects.

of a subset of the three-loop master integrals [25] has made it possible to obtain the vector form factor at three loops [26] in the color-planar limit. The large β_0 limit has been considered in [27]. While the main goal is to obtain the complete three-loop corrections for the form factors, the $\mathcal{O}(\varepsilon)$ pieces at two-loop order are important ingredients. In addition, we compute the master integrals to the required order in ε with a different technique.

In the present paper, we compute the contributions to the massive form factors up to $\mathcal{O}(\varepsilon^2)$ for different currents, namely, vector, axial-vector, scalar and pseudoscalar currents, which serve as input for ongoing and future three- and four-loop calculations. We also perform the expansion of the exact results in different kinematic regions. In Sec. II, we briefly describe the theoretical formalism for all the currents and corresponding form factors, followed by their renormalization procedure and a description of the universal infrared (IR) structure. Section III contains the details about the computational technique. Here we describe especially how we have computed the master integrals. In Sec. IV, we present the results, also expanding the complete expressions in regions, which are kinematically relevant. Finally, we conclude in Sec. V. Various of the expressions are rather voluminous. A part of it is presented in the appendices and the $O(\varepsilon^2)$ terms are only given in computer readable form in a file attached to this paper.

II. THE HEAVY QUARK FORM FACTORS

We consider the decay of a virtual massive boson of momentum q into a pair of heavy quarks of mass m, momenta q_1 and q_2 and color c and d, through a vertex X_{cd}, where $X_{cd} = \Gamma^\mu_{V,cd}, \Gamma^\mu_{A,cd}, \Gamma_{S,cd}$ and $\Gamma_{P,cd}$ indicates the coupling to a vector, an axial-vector, a scalar and a pseudoscalar boson, respectively. Here $q^2 = (q_1 + q_2)^2$ is the center of mass energy squared and the dimensionless variable s is defined by

$$s = \frac{q^2}{m^2}. \tag{2.1}$$

The amplitude takes the following general form

$$\bar{u}_c(q_1) X_{cd} v_d(q_2), \tag{2.2}$$

where $\bar{u}_c(q_1)$ and $v_d(q_2)$ are the bi-spinors of the quark and the anti-quark, respectively. We denote the corresponding UV renormalized form factors by F_I, with $I = V, A, S, P$. They are expanded in the strong coupling constant $\alpha_s = g_s^2/(4\pi)$ as follows

$$F_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n F_I^{(n)}. \tag{2.3}$$

The unrenormalized form factors are denoted by $\hat{F}_I$. In the following subsections, we discuss the properties of each current, the corresponding renormalization procedure and their universal infrared structure.

A. The vector and axial-vector current

In this section, we consider the spin 1 case, i.e. the vertex Γ^μ of a Z-boson or photon coupling to a pair of heavy quarks. The general structure of Γ^μ consists of six form factors, two of which are CP odd. We consider only higher-order QCD effects and SM neutral current interactions to lowest order. Since CP invariance holds, we only take into account the four CP even form factors, $F_{V,i}(s), F_{A,i}(s) i = 1, 2$ in the following. They can be cast in the following general form

$$\Gamma^\mu_{cd} = \Gamma^\mu_{V,cd} + \Gamma^\mu_{A,cd} = -i\delta_{cd}\left[v_Q\left(\gamma^\mu F_{V,1} + \frac{i}{2m}\sigma^{\mu\nu} q_\nu F_{V,2}\right) + a_Q\left(\gamma^\mu\gamma_5 F_{A,1} + \frac{1}{2m} q^\mu \gamma_5 F_{A,2}\right)\right], \tag{2.4}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $q = q_1 + q_2$, and v_Q and a_Q are the SM vector and axial-vector coupling constants as defined by

$$v_Q = \frac{e}{\sin\theta_w \cos\theta_w}\left(\frac{T_3^Q}{2} - \sin^2\theta_w Q_Q\right), \qquad a_Q = -\frac{e}{\sin\theta_w \cos\theta_w}\frac{T_3^Q}{2}. \tag{2.5}$$

e denotes the elementary charge, θ_w the weak mixing angle, T_3^Q the third component of the weak isospin, and Q_Q the charge of the heavy quark.

To extract the form factors $F_{I,i}, I = V, A$, we multiply Γ^μ by the following projectors and perform a trace over the spinor and color indices

$$P_{V,i} = \frac{i}{v_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\left(\gamma_\mu g^1_{V,i} + \frac{1}{2m}(q_{2\mu} - q_{1\mu}) g^2_{V,i}\right)\frac{\not{q}_1 + m}{m},$$

$$P_{A,i} = \frac{i}{a_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\left(\gamma_\mu\gamma_5 g^1_{A,i} + \frac{1}{2m}(q_{1\mu} + q_{2\mu})\gamma_5 g^2_{A,i}\right)\frac{\not{q}_1 + m}{m}, \tag{2.6}$$

where,[2]

$$g^1_{V,1} = -\frac{1}{4(1-\varepsilon)}\frac{1}{(s-4)},$$
$$g^2_{V,1} = \frac{(3-2\varepsilon)}{(1-\varepsilon)}\frac{1}{(s-4)^2},$$
$$g^1_{V,2} = \frac{1}{(1-\varepsilon)}\frac{1}{s(s-4)},$$
$$g^2_{V,2} = -\frac{1}{(1-\varepsilon)}\frac{1}{(s-4)^2}\left(\frac{4}{s}+2-2\varepsilon\right),$$
$$g^1_{A,1} = -\frac{1}{4(1-\varepsilon)}\frac{1}{(s-4)},$$
$$g^2_{A,1} = -\frac{1}{(1-\varepsilon)}\frac{1}{s(s-4)},$$
$$g^1_{A,2} = \frac{1}{(1-\varepsilon)}\frac{1}{s(s-4)},$$
$$g^2_{A,2} = \frac{1}{(1-\varepsilon)}\frac{1}{s^2(s-4)}(4(3-2\varepsilon)-2s(1-\varepsilon)), \quad (2.7)$$

and N_c denotes the number of colors. Later on we will also use the Casimir operators $C_A = N_c, C_F = (N_c^2-1)/(2N_c)$, $T_F = 1/2$ for $SU(N_c)$, with $N_c = 3$ in the case of QCD.

B. The scalar and pseudoscalar current

We consider the current implied by a general neutral spin 0 particle h that couples to heavy quarks through the Yukawa interaction,

$$\mathcal{L}_{\rm int} = -\frac{m}{v}[s_Q\bar{Q}Q + ip_Q\bar{Q}\gamma_5 Q]h, \quad (2.8)$$

where $v = (\sqrt{2}G_F)^{-1/2}$ is the SM Higgs vacuum expectation value, with G_F being the Fermi constant, s_Q and p_Q are the scalar and pseudoscalar coupling, respectively, and Q and h are the heavy quark and scalar and pseudoscalar field, respectively. The vertex for $h \to \bar{Q} + Q$, $X_{cd} \equiv \Gamma_{cd}$ consists of two form factors with the following general structure,

$$\Gamma_{cd} = \Gamma_{S,cd} + \Gamma_{P,cd} = -\frac{m}{v}\delta_{cd}[s_Q F_S + ip_Q\gamma_5 F_P], \quad (2.9)$$

where F_S and F_P denote the renormalized scalar and pseudoscalar form factors, respectively. As before, the form factors can be obtained from Γ_{cd} through suitable projectors as given below and performing the trace over the spinor and color indices:

$$P_S = \frac{v}{2ms_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\left(-\frac{1}{(s-4)}\right)\frac{\not{q}_1 + m}{m},$$
$$P_P = \frac{v}{2mp_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\left(-\frac{i}{s}\gamma_5\right)\frac{\not{q}_1 + m}{m}. \quad (2.10)$$

[2]In [20], the expression for $g^2_{V,2}$ contains a typographical error.

C. Anomaly and Ward identities

Since we use dimensional regularization [28] in $D = 4-2\varepsilon$ spacetime dimensions, one important point is to define a proper description for the treatment of γ_5. In the case of the axial-vector and the pseudoscalar form factors, two types of Feynman diagrams contribute: the nonsinglet diagrams containing only open fermion lines, and the singlet diagrams where a fermion loop is attached to the axial-vector or pseudoscalar vertex. It is convenient to separate the two contributions and write

$$\Gamma^\mu_{A,cd} = \Gamma^{\mu,\text{“ns”}}_{A,cd} + \Gamma^{\mu,\text{“s”}}_{A,cd}, \qquad \Gamma_{P,cd} = \Gamma^{\text{“ns”}}_{P,cd} + \Gamma^{\text{“s”}}_{P,cd}, \quad (2.11)$$

where “ns” and “s” denote the nonsinglet and the singlet contributions, respectively.

In the nonsinglet case, we use an anticommuting γ_5 in D spacetime dimensions, with $\gamma_5^2 = 1$, as it does not lead to any spurious singularities. This approach respects chiral invariance and leaves us with the Ward identity,

$$q^\mu\Gamma^{\mu,\text{“ns”}}_{A,cd} = 2m\Gamma^{\text{“ns”}}_{P,cd}, \quad (2.12)$$

which, in terms of the form factors, takes the form

$$2F^{\text{“ns”}}_{A,1} + \frac{s}{2}F^{\text{“ns”}}_{A,2} = 2F^{\text{“ns”}}_P. \quad (2.13)$$

On the other hand, the singlet pieces for the axial-vector and the pseudoscalar vertex are related to each other through the Adler-Bell-Jackiw (ABJ) anomaly [29,30]. With this constraint, we use the following prescription as presented in [31,32], which mostly followed [28]: For a single γ_5 in a fermion loop, we use

$$\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \quad (2.14)$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor and all Lorentz indices are taken D-dimensional. Finally, the contraction of two ϵ-tensors is expressed in terms of products of D-dimensional metric tensors. This prescription of γ_5 needs a special treatment during renormalization, as discussed later.

The ABJ anomaly involves the truncated matrix element of the gluonic operator between the vacuum and a pair of heavy quark states. The gluonic operator is given by

$$G(x)\tilde{G}(x) \equiv \varepsilon_{\mu\nu\rho\sigma}G^{a,\mu\nu}(x)G^{a,\rho\sigma}(x), \quad (2.15)$$

where $G^{a,\mu\nu}$ represents the gluonic field strength tensor. Denoting its contribution by $F_{G,Q}$, we can immediately write down the anomalous Ward identity for the singlet case as follows,

$$q_\mu \Gamma^{\mu,s}_{A,cd} = 2m\Gamma^s_{P,cd} - i\left(\frac{\alpha_s}{4\pi}\right) T_F \langle G\tilde{G}\rangle_Q, \qquad (2.16)$$

which implies

$$2F^s_{A,1} + \frac{s}{2}F^s_{A,2} = 2F^s_P - i\left(\frac{\alpha_s}{4\pi}\right) T_F F_{G,Q}. \qquad (2.17)$$

D. Renormalization

The UV renormalization of the form factors has been performed in a mixed scheme. We renormalize the heavy quark mass and wave function in the on-shell (OS) renormalization scheme, while the strong coupling constant is renormalized in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme [33,34]. The corresponding renormalization constants are well known and are denoted by $Z_{m,\mathrm{OS}}$ [35–39], $Z_{2,\mathrm{OS}}$ [35–37,40] and Z_{a_s} [41–45] for the heavy quark mass, wave function and strong coupling constant, respectively. All renormalization constants follow a perturbative expansion in α_s,

$$Z_I = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n Z_I^{(n)}. \qquad (2.18)$$

For reference, we present $Z^{(n)}_{m,\mathrm{OS}}$ and $Z^{(n)}_{2,\mathrm{OS}}$ in Appendix A up to $n = 2$ and $\mathcal{O}(\varepsilon^2)$:

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F(n_l + n_h). \qquad (2.19)$$

Here n_l and n_h denote the number of light and heavy quarks, respectively. In the following, we will set $n_h = 1$.

While the renormalization of the heavy-quark wave function and the strong coupling constant can be done multiplicatively, the mass renormalization requires the explicit calculation of counterterm diagrams. Hence, the bare and renormalized vector form factors are at two loops related by

$$F_{V,i} = Z_{2,\mathrm{OS}}\hat{F}_{V,i} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(1)}_{m,\mathrm{OS}}\hat{F}^{ct,(1)}_{V,i} + \mathcal{O}(\alpha_s^3), \qquad (2.20)$$

where the unrenormalized form factors $\hat{F}_{V,i}$ are expanded in the unrenormalized strong coupling constant $\hat{\alpha}_s = \alpha_s Z_{a_s}$, and $\hat{F}^{\mathrm{ct},(1)}_{V,i}$ denotes the bare contribution from counterterm diagrams at one loop.

The nonsinglet contributions to the axial-vector form factor can be renormalized in the same way. The singlet part requires extra care due to the prescription employed for γ_5. It is infrared finite and the UV pole is renormalized by the multiplicative renormalization constant Z_J as

$$F^s_{A,i} = Z_J Z_5^{\mathrm{fin}}\hat{F}^s_{A,i}, \qquad (2.21)$$

where Z_5^{fin} is a finite renormalization constant which restores the anomalous Ward identity Eq. (2.16). We would like to remark that the Ward identities are valid for physical quantities. Therefore, it is not reasonable to study them at higher orders in ε, and neither it is to consider ε-dependent pieces of Z_5^{fin}. In the $\overline{\mathrm{MS}}$ scheme for the form factors,

$$Z_J = 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{6C_F T_F}{\varepsilon} + \mathcal{O}(\alpha_s^3) \qquad (2.22)$$

and

$$Z_5^{\mathrm{fin}} = 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 (3C_F T_F) + \mathcal{O}(\alpha_s^3). \qquad (2.23)$$

The remaining finite renormalization has to be carried out later for the corresponding observables of which the form factors form a part.

The renormalization of the quantity $F_{G,Q}$ appearing in Eq. (2.17) involves the mixing of the gluonic operator $G\tilde{G}$ with another operator, namely, $\partial_\mu\bar{\psi}\gamma^\mu\gamma_5\psi$, as discussed in [32,46,47], where ψ indicates all quark flavors including the massive one. We get,

$$F_{G,Q} = Z_{GG}\hat{F}_{G,Q} + Z_{GJ}\hat{F}_{J,Q}, \qquad (2.24)$$

where $\hat{F}_{J,Q}$ indicates the bare contribution from the second operator, while Z_{GG} and Z_{GJ} are the corresponding renormalization constants.

The renormalization of the scalar and pseudoscalar (nonsinglet) vertices also follows a similar procedure, except for the presence of the heavy quark mass in the Yukawa coupling. Thus the renormalized form factors are given by

$$F_S = Z_{m,\mathrm{OS}}Z_{2,\mathrm{OS}}\hat{F}_S + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(1)}_{m,\mathrm{OS}}\hat{F}^{ct,(1)}_S + \mathcal{O}(\alpha_s^3),$$
$$F_P^{\text{“ns”}} = Z_{m,\mathrm{OS}}Z_{2,\mathrm{OS}}\hat{F}_P^{\text{“ns”}} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(1)}_{m,\mathrm{OS}}\hat{F}_P^{ct,(1),\text{“ns”}} + \mathcal{O}(\alpha_s^3). \qquad (2.25)$$

On the other hand, the singlet piece of the pseudoscalar vertex is both IR and UV finite; hence, no additional renormalization is necessary.

E. The infrared structure

The study of the IR behavior of the form factors has attracted a lot of attention in the past few decades. A plethora of works on massless scattering amplitudes [48–52] has already provided a remarkable understanding on the universal IR pattern characterized by soft and collinear dynamics. In [53], the first step was taken to generalize this for the two-loop scattering amplitudes with massive partons. Later, in [54], following a soft-collinear effective theory (SCET) approach, the general IR structures have been presented.

The IR singularities of the massive form factors can be factorized as a multiplicative renormalization factor. Its structure is constrained by the renormalization group equation (RGE), as follows,

$$F_I = Z(\mu) F_I^{\rm fin}(\mu), \tag{2.26}$$

where $F_I^{\rm fin}$ is finite as $\varepsilon \to 0$. The RGE for Z reads

$$\frac{d}{d\ln\mu} \ln Z(\varepsilon, x, m, \mu) = -\Gamma(x, m, \mu), \tag{2.27}$$

where Γ is the corresponding anomalous dimension. Notice that Z does not carry any information regarding the vertex. Γ can be identified as the massive cusp anomalous dimension, which is by now available up to the three-loop level [55–58]. Both Z and Γ can be expanded in a perturbative series in α_s as follows,

$$Z = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n Z^{(n)}, \quad \Gamma = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \Gamma_n, \tag{2.28}$$

and we find the following solution for Eq. (2.27):

$$Z = 1 + \left(\frac{\alpha_s}{4\pi}\right)\left[\frac{\Gamma_0}{2\varepsilon}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{\varepsilon^2}\left(\frac{\Gamma_0^2}{8} - \frac{\beta_0\Gamma_0}{4}\right) + \frac{\Gamma_1}{4\varepsilon}\right] + \mathcal{O}(\alpha_s^3). \tag{2.29}$$

Equation (2.29) correctly predicts the infrared singularities for all massive form factors at the two-loop level.

III. DETAILS OF THE CALCULATION

The Feynman diagrams were generated using QGRAF [59], the output of which was then processed using Q2e/Exp [60,61] and FORM [62,63] in order to express the diagrams in terms of a linear combination of a large set of scalar integrals. These integrals were then reduced to a much smaller set of master integrals (MIs) using integration by parts identities (IBPs) [64–68] with the help of the program Crusher [69]. Since all this is common practice, we refrain from going into any detail.

After performing the reductions, all that remains to be done is to calculate the master integrals. In the following sections, we present the methods we used to achieve this.

A. The conventional differential equations method

We computed the two-loop master integrals contributing to the massive fermion form factors as Laurent expansions in the dimensional parameter ε by means of the differential equation method [70–76]. This technique has already been applied to massive form factor integrals at two and three loops in [25,77,78]. In this work, we calculate the two-loop master integrals up to a sufficiently high order in ε to obtain $\mathcal{O}(\varepsilon^2)$ accuracy in the form factors.

In this section, we briefly review the main steps of this calculation. The master integrals are classified according to their underlying topology. In particular, we distinguish the nonsinglet topologies (Fig. 1) from the singlet topology (Fig. 2) according to whether the external current does or does not couple to the external massive quark.

In the case of the nonsinglet topologies depicted in Fig. 1, it turns out that the master integrals associated to the two topologies on the right of that figure represent a subset of the ones required to calculate the topologies on the left. We, therefore, concentrate on the topologies in Figs. 1(a) and 1(b). The master integrals for both topologies can be expressed in terms of a single integral family with seven propagators given by

$$J(\nu_1,\ldots,\nu_7) = ((4\pi)^{2-\varepsilon} e^{\varepsilon\gamma_E})^2 \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}} \frac{1}{D_1^{\nu_1}\ldots D_7^{\nu_7}}, \tag{3.1}$$

where

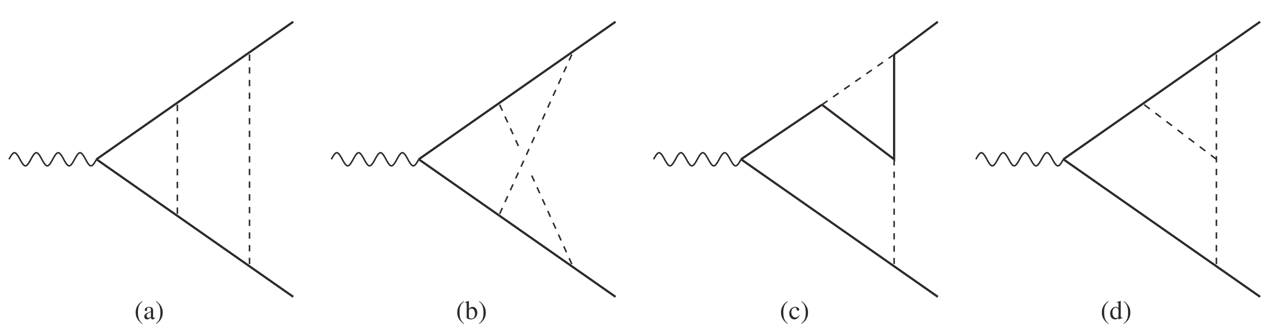

FIG. 1. The nonsinglet topologies required for the calculation of two-loop form factors. Solid lines represent massive particles in external or internal lines, while dashed lines correspond to massless propagators. The external vector current can also be replaced by an axial-vector, scalar or pseudoscalar. The master integrals associated to topologies (c) and (d) are a subset of the master integrals required for topologies (a) and (b).

TABLE I. The list of the nonsinglet master integrals identified by the indices ν_1 to ν_7. In the last column, we indicate the order k in ε to which each integral needs to be expanded in order to calculate the form factors to $\mathcal{O}(\varepsilon^2)$.

MI	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	k
J_1	1	1	0	0	0	0	0	6
J_2	0	1	0	0	1	1	0	6
J_3	1	0	0	1	0	0	1	4
J_4	1	1	1	0	0	0	0	4
J_5	1	1	1	1	0	0	0	4
J_6	1	0	0	1	0	1	0	4
J_7	1	0	0	2	0	1	0	3
J_8	1	0	0	1	0	2	0	4
J_9	0	1	1	0	1	1	0	3
J_{10}	0	1	0	1	1	1	0	4
J_{11}	1	1	0	1	0	0	0	3
J_{12}	1	0	0	1	0	1	1	3
J_{13}	1	0	0	2	0	1	1	3
J_{14}	1	0	0	1	0	2	1	3
J_{15}	1	1	0	0	1	0	1	3
J_{16}	1	1	0	0	2	0	1	3
J_{17}	1	1	0	0	1	0	2	3
J_{18}	1	1	0	1	0	0	1	3
J_{19}	1	1	0	2	0	0	1	3
J_{20}	1	1	0	1	1	1	0	2
J_{21}	1	1	0	1	1	2	0	4
J_{22}	1	1	0	1	1	1	1	3
J_{23}	1	1	0	1	1	2	1	3

$$D_1 = (l_1+q_1)^2 - m^2, \quad D_2 = (l_2+q_1)^2 - m^2,$$
$$D_3 = (l_1-q_2)^2 - m^2, \quad D_4 = (l_2-q_2)^2 - m^2,$$
$$D_5 = l_1^2, \quad D_6 = (l_1-l_2)^2, \quad D_7 = (l_1-l_2+q_2)^2 - m^2. \tag{3.2}$$

Here the q_i's with $i = 1, 2$ are the external momenta, which are taken on-shell ($q_1^2 = q_2^2 = m^2$). The MIs are, therefore, labeled by the exponents $\nu_1, \ldots, \nu_7$ of the denominators $D_1, \ldots, D_7$. The integrals corresponding to the topology in Fig. 1(a) will have $\nu_7 = 0$, while the ones corresponding to the topology in Fig. 1(b) will have $\nu_3 = 0$. There are several master integrals where ν_3 and ν_7 are both equal to zero, which are, therefore, common to both topologies.[3] The list of master integrals required to reduce these topologies is given in Table I. Notice that there are several sets of master integrals that have the same set of nonvanishing positive powers of propagators. Such a set of integrals is called a *sector*. For example, integrals J_6, J_7 and J_8 belong to the same sector, since for all of them $\nu_1, \nu_4, \nu_6 \geq 1$ and $\nu_2 = \nu_3 = \nu_5 = \nu_7 = 0$. A given sector is said to be a subsector of another sector if the propagators in the first sector are a subset of the propagators in the second.

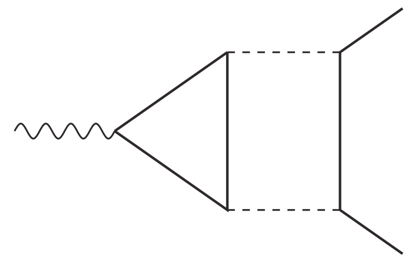

FIG. 2. The singlet topology.

In the case of the singlet topology, shown in Fig. 2, the master integrals are given by

$$K(\nu_1, \ldots, \nu_6) = ((4\pi)^{2-\varepsilon} e^{\varepsilon \gamma_E})^2 \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}} \frac{1}{D_1'^{\nu_1} \ldots D_6'^{\nu_6}}, \tag{3.3}$$

where

$$D_1' = (l_1+q_1)^2, \qquad D_2' = (l_2+q_1)^2 - m^2,$$
$$D_3' = (l_1-q_2)^2, \qquad D_4' = (l_2-q_2)^2 - m^2,$$
$$D_5' = l_1^2 - m^2, \qquad D_6' = (l_1-l_2)^2 - m^2. \tag{3.4}$$

The list of master integrals required in this case is given in Table II. From now on, instead of using the exponents ν_i to identify the master integrals, we will use the single subindex we have assigned to each integral according to the leftmost columns of Tables I and II.

All of the master integrals can be expressed in terms of harmonic polylogarithms (HPLs) [79] in the kinematic variable [80,81]:

TABLE II. The list of singlet master integrals identified by the indices ν_1 to ν_6. In the last column, we indicate the order k in ε to which each integral needs to be expanded in order to calculate the form factors to $\mathcal{O}(\varepsilon^2)$.

MI	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	k
K_1	0	1	0	0	1	0	6
K_2	0	1	0	0	1	1	4
K_3	1	1	1	0	0	0	3
K_4	1	1	1	1	0	0	3
K_5	0	1	1	0	0	1	3
K_6	0	1	1	0	0	2	3
K_7	0	1	0	1	1	0	3
K_8	1	1	1	0	1	0	3
K_9	0	1	1	0	1	1	3
K_{10}	0	1	1	0	1	2	3
K_{11}	0	1	0	1	1	1	3
K_{12}	0	1	0	1	1	2	3
K_{13}	1	1	1	1	1	0	2
K_{14}	1	1	1	1	0	1	2
K_{15}	1	1	1	1	1	1	2

[3] This is the reason we chose to group here both topologies within a single integral family, although the actual reductions performed with `Crusher` were done using two separate families.

$$x = \frac{\sqrt{q^2-4m^2}-\sqrt{q^2}}{\sqrt{q^2-4m^2}+\sqrt{q^2}} \leftrightarrow \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}. \qquad (3.5)$$

In particular, we focus on the Euclidean region, $q^2<0$, corresponding to x ranging in (0,1). A large center of mass energy $|q^2| \gg m^2$ is equivalent to the boundary $x \to 0$, while the large-mass limit $m^2 \gg |q^2|$ is mapped to the endpoint $x \to 1$.

We derived a system of coupled linear differential equations for each topology by reducing the derivative with respect to x of each MI to a linear combination of the master integrals themselves, with the help of Crusher. The derivative of an integral cannot produce integrals in this linear combination with more propagators than the original one, so they are all either in the same sector as the integral to which one takes the derivative or they belong to a subsector. This means that the integrals in the subsectors need to be solved first, and the differential equations within a given sector will constitute a coupled subsystem. In the case of the singlet master integrals, there are three such coupled subsystems, namely, the subsystems formed by the set of integrals $\{K_5, K_6\}$, $\{K_9, K_{10}\}$ and $\{K_{11}, K_{12}\}$. In all other sectors, only one integral is present. The differential equations in each coupled subsystem will, therefore, need to be decoupled in order to solve them. The strategy for solving the whole system consists then in solving first the simplest sectors (with fewer propagators), and move up in the chain of subsystems, decoupling and solving each one of them until all integrals are obtained. The starting point are the integrals for which the derivative with respect to x equals zero,[4] which must be obtained not by the differential equations method but by other means.

In the case of the nonsinglet topologies, by expanding the MIs in a Laurent series around $\varepsilon = 0$, the systems greatly simplify, assuming an almost complete block-triangular form. Only one 2×2 coupled subsystem remains after the expansion in ε. We, therefore, solve the system order-by-order in ε by integrating each equation by quadratures. These steps are automated and results are efficiently simplified using a minimal set of independent HPLs by means of the Mathematica packages Sigma [82,83] and HarmonicSums [84–89]. This procedure is slightly modified for the aforementioned 2×2 coupled system, which does not assume a triangular form after expanding in ε

$$\frac{d}{dx}\begin{pmatrix} J_{22} \\ J_{23}\end{pmatrix} = \begin{bmatrix} \frac{1+x^2}{x(1-x^2)} & \frac{1-x^2}{x^2} \\ -\frac{1}{1-x^2} & \frac{4(1+x^2)}{x(1-x^2)} \end{bmatrix} \begin{pmatrix} J_{22} \\ J_{23}\end{pmatrix} + \begin{pmatrix} R_1(\varepsilon,x) \\ R_2(\varepsilon,x)\end{pmatrix}, \qquad (3.6)$$

where the inhomogeneities $R_1(\varepsilon,x)$, $R_2(\varepsilon,x)$ are determined at each order in ε by subsector MIs. The most general solution of the homogeneous system involves only logarithms and rational functions,

$$J_{22} = c_1\frac{x^2}{(1-x^2)^2} + c_2\frac{x^2\ln(x)}{(1-x^2)^2},$$
$$J_{23} = c_1\frac{x^2(1+x^2)}{(1-x^2)^4} + c_2\frac{x^3(x^2\ln(x)+\ln(x)+2)}{(1-x^2)^4}, \qquad (3.7)$$

and therefore we integrate the inhomogeneous system order-by-order in ε, using the method of variation of constants. For example, at leading order in ε, the inhomogeneous parts of the system (3.6) are

$$R_1(\varepsilon,x) = \frac{1}{\varepsilon}\left\{\frac{x^4+x^3+6x^2+x+1}{(1-x)^2(x+1)^4}\ln(x) - \frac{x^6+2x^5-25x^4-4x^3-25x^2+2x+1}{16x^2(1-x)(x+1)^3}\right\} + \mathcal{O}(\varepsilon^0), \qquad (3.8)$$

$$R_2(\varepsilon,x) = \frac{1}{\varepsilon}\left\{\frac{x\ln(x)}{(1-x)^4(x+1)^6}(2x^6+x^5-4x^4+6x^3-4x^2+x+2) - \frac{x^8+2x^7-8x^6-10x^5+22x^4-10x^3-8x^2+2x+1}{4(1-x)^3x(x+1)^5}\right\} + \mathcal{O}(\varepsilon^0). \qquad (3.9)$$

By introducing Eq. (3.7) with $c_i \to c_i(x)$ into the system (3.6), we get first-order differential equations for $c_1(x)$ and $c_2(x)$ that can be solved by quadratures using the same automated tools introduced above. The integration constants are fixed by imposing the regularity of the functions J_{22} and J_{23} in the limit of vanishing space-like momentum $q^2 \to 0$, corresponding to $x \to 1$, giving

$$J_{22} = \frac{1}{\varepsilon}\left\{-\frac{x^2}{3(1-x^2)^2}[H_0^3(x) - 3H_0(x)(2H_{0,-1}(x) - 2H_{0,1}(x) - \zeta_2) + 12H_{0,0,-1}(x) - 12H_{0,0,1}(x) + 3\zeta_3]\right\} + \mathcal{O}(\varepsilon^0),$$

$$\begin{aligned} J_{23} = \frac{1}{\varepsilon}\Bigg\{&\frac{x^3(1+x^2)}{(1-x^2)^4}\left[2H_0(x)H_{0,-1}(x) - 2H_0(x)H_{0,1}(x) - 4H_{0,0,-1}(x) + 4H_{0,0,1}(x) - \zeta_2 H_0(x) - \frac{1}{3}H_0^3(x) - \zeta_3\right] \\ &+ \frac{x^3}{(1-x^2)^3}[2H_{0,1}(x) - 2H_{0,-1}(x) + 2H_{-1}(x)H_0(x) - 2H_0(x)H_1(x) - H_0^2(x) - \zeta_2] \\ &+ \frac{x^6+2x^5-25x^4-4x^3-25x^2+2x+1}{16(1-x)^2(x+1)^4} - \frac{x^2(x^4+x^3+6x^2+x+1)}{(1-x)^3(x+1)^5}H_0(x)\Bigg\} + \mathcal{O}(\varepsilon^0), \end{aligned} \qquad (3.10)$$

[4]These are J_1, J_2 and J_3 in the case of the nonsinglet topologies, and K_1 and K_2 in the singlet case.

where the harmonic polylogarithms, $H_{a_1,a_2,\ldots,a_n}(x)$, are defined by

$$H_{a_1,a_2,\ldots,a_n}(x) = \int_0^x dy f_{a_1}(y) H_{a_2,\ldots,a_n}(y),$$
$$H_{\varnothing} = 1, \qquad a_i \in \{0,1,-1\}, \tag{3.11}$$

with

$$f_0(x) = \frac{1}{x}, \qquad f_1(x) = \frac{1}{1-x}, \qquad f_{-1}(x) = \frac{1}{1+x}, \tag{3.12}$$

and

$$H_{\underbrace{0,\ldots,0}_{\text{n times}}}(x) = \frac{1}{n!}\ln^n(x). \tag{3.13}$$

ζ_n denotes the Riemann ζ function:

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}, \qquad n \geq 2, \qquad n \in \mathbb{N}.$$

We proceed similarly at higher orders, getting expansions of the MIs J_{22} and J_{23} up to $\mathcal{O}(\varepsilon^3)$. It should be noticed that solving a system of coupled differential equations is in general far from trivial. When the solution is written in terms of multiple polylogarithms or related iterated integrals, as in the case of the two-loop massive form factor [77,78] and of the planar three-loop massive form factor [25], it is possible to properly choose the set of MIs such that the system doesn't have any coupled equation, assuming for example the *canonical* form of [74,75]. In this work, we are interested in applying the methods developed in [76,90,91], to solve the systems of coupled differential equations algorithmically, as discussed in the next sections.

In order to solve the differential equations, boundary conditions have to be determined. As it was observed in [77,78], the analytic structure of the master integrals strongly constrains the choice of the integration constants. In particular, boundary conditions of the master integrals of the nonsinglet topologies are completely determined by requiring the regularity of the functions in $x = 1$, as will be discussed in the example above. However, we cannot use the same argument for some of the master integrals of the singlet topology, which are characterized by a branch cut at $x = 1$, as occurs, for example, in the case of the integral $K_{14} = K(1,1,1,1,0,1)$ depicted in Fig. 3. We fixed the boundary conditions of such integrals by matching the general solutions of the differential equations with the asymptotic expansions of the corresponding integrals around $x \to 1$. The latter were computed by means of the large-mass expansion $m^2 \gg q^2$ [92], or with a Mellin Barnes representation, as described e.g. in [93].

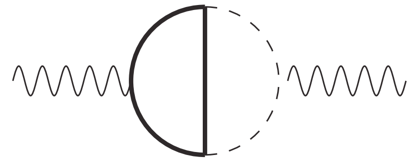

FIG. 3. The master integral K_{14}. Massive (massless) propagators are represented by solid (dashed) lines. The presence of a massless cut determines the asymptotic behaviour $K_{14} \sim \log(1-x)$.

B. Calculation of the master integrals using difference equations

In the following, we describe an alternative method for calculating the master integrals. The idea of the method is to write all integrals in terms of series expansions and then use the differential equations obeyed by the MIs to derive difference equations satisfied by the coefficients of these series. In the nonsinglet case, this can be done using the fact that the MIs are regular around $x = 1$. In terms of the variable $y = 1 - x$, we can, therefore, write

$$J_i(y) = \sum_{n=0}^{\infty} \sum_{j=-2}^{r} \varepsilon^j C_{i,j}(n) y^n, \tag{3.14}$$

where we have included the expansion in ε up to $\mathcal{O}(\varepsilon^r)$. We can introduce Eq. (3.14) in the differential equations after rewriting them in terms of the variable y, leading to a system of coupled difference equations for the different $C_{i,j}$'s.

The method of solving differential equations by introducing series expansions and then finding the solutions to the resulting recursion relations is, of course well-known. However, to the best of our knowledge, this method has not been used before to calculate master integrals in perturbative quantum field theory. We propose this method here since it can be useful also for higher-loop calculations, and we can take advantage of the powerful mathematical tools implemented in `Sigma` and `HarmonicSums` to solve the difference equations.

Whenever we are given subsystems of differential equations, it is usually more convenient to uncouple them and then insert Eq. (3.14) in the corresponding uncoupled equations, as supposed to inserting Eq. (3.14) first and then uncouple the resulting coupled difference equations. This is so because usually the latter approach will lead to difference equations of higher order, although occasionally, this might not be the case and this latter approach might turn out to be preferable.

In the singlet case, some of the master integrals will not be regular at $x = 1$, due to the presence of the logarithm $\ln(1-x) = \ln(y)$ discussed in the previous section. We, therefore, include a formal expansion around powers of this logarithm,[5]

[5]In two particular cases (integrals K_8 and K_{13}), the sum in n actually starts from $n = -1$.

$$K_i(y) = \sum_{n=0}^{\infty}\sum_{k=0}^{3}\sum_{j=-2}^{r} \varepsilon^j C_{i,j,k}(n)\ln^k(y) y^n. \tag{3.15}$$

The integrals belonging to the three multi-integral sectors in the singlet case have no logarithmic singularities at $y = 0$, and can, therefore, all be written as in Eq. (3.14). Only some of the integrals that are the sole representative of their sector in Table II need to be written according to Eq. (3.15). Since Eq. (3.14) is a particular case of Eq. (3.15),[6] we will consider for illustration purposes one of the integrals that require an expansion of the form (3.15). The integral $K_4 = K(1,1,1,1,0,0)$ is simple enough (it consists of a product of two one-loop integrals) to be calculated just through Feynman parameters, but precisely because of this simplicity, it allows us to describe the main features of the method without unnecessary complications or long formulas.

The differential equation associated to K_4 is given by

$$\frac{dK_4}{dy} + \left(\frac{2\varepsilon y}{(2-y)(1-y)} + \frac{2(1+2\varepsilon)}{y(2-y)}\right)K_4 = \frac{2-2\varepsilon}{y(2-y)}K_3. \tag{3.16}$$

As we discussed above, in order to solve the differential equations, we must first obtain all of the integrals associated to the subtopologies of the system under consideration. In the case of Eq. (3.16), this means obtaining K_3. In terms of the variable y, it is given by

$$\begin{aligned} K_3(y) = {}& -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}[-3 - \ln(1-y) + 2\ln(y)] - 7 - 3\ln(1-y) - \frac{1}{2}\ln^2(1-y) + [6 + 2\ln(1-y)]\ln(y) - 2\ln^2(y) \\ & + \varepsilon\Big[-15 + \frac{8}{3}\zeta_3 - 7\ln(1-y) - \frac{3}{2}\ln^2(1-y) - \frac{1}{6}\ln^3(1-y) + [14 + 6\ln(1-y) + \ln^2(1-y)]\ln(y) \\ & - [6 + 2\ln(1-y)]\ln^2(y) + \frac{4}{3}\ln^2(y)\Big] + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{3.17}$$

which, in expanded form, as in Eq. (3.15), can be written as

$$\begin{aligned} K_3(y) = {}& -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 + 2\ln(y) + \sum_{n=1}^{\infty}\frac{y^n}{n}\right) - 7 + 6\ln(y) - 2\ln^2(y) - \sum_{n=1}^{\infty}\frac{2}{n}y^n\ln(y) + \sum_{n=1}^{\infty}\left(\frac{1+3n}{n^2} - \frac{1}{n}S_1(n)\right)y^n \\ & + \varepsilon\Big[\frac{4}{3}\ln^3(y) - 6\ln^2(y) + 14\ln(y) - 15 + \frac{8}{3}\zeta_3 + \sum_{n=1}^{\infty}\frac{2}{n}y^n\ln^2(y) - 2\sum_{n=1}^{\infty}\left(\frac{1+3n}{n^2} - \frac{1}{n}S_1(n)\right)y^n\ln(y) \\ & + \sum_{n=1}^{\infty}\left(\frac{1+3n+7n^2}{n^3} - \frac{1+3n}{n^2}S_1(n) + \frac{1}{2n}S_1^2(n) - \frac{1}{2n}S_2(n)\right)y^n\Big] + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{3.18}$$

where

$$S_k(n) = \sum_{i=1}^{n}\frac{1}{i^k}. \tag{3.19}$$

We proceed as follows: First, we clear the denominators in Eq. (3.16), which in this case means multiplying by $(2-y)(1-y)y$. After that, we can then insert the expanded version of K_3 given in Eq. (3.18) together with the generic expanded version of K_4 according to Eq. (3.15). The resulting equation will be satisfied if the coefficients in the expansion in y, for each power in $\ln(y)$, are equal on both sides of the equation. This leads to the following system of difference equations:

$$0 = (2-n)C_{4,-2,3}(n-2) + (3n-1)C_{4,-2,3}(n-1) - 2(n+1)C_{4,-2,3}(n), \tag{3.20}$$

$$\begin{aligned} 0 = {}& (2-n)C_{4,-2,k}(n-2) + (3n-1)C_{4,-2,k}(n-1) - 2(n+1)C_{4,-2,k}(n) \\ & - (k+1)C_{4,-2,k+1}(n-2) + 3(k+1)C_{4,-2,k+1}(n-1) - 2(k+1)C_{4,-2,k+1}(n) \quad \text{for } k = 0,1,2, \end{aligned} \tag{3.21}$$

$$\begin{aligned} 0 = {}& -2C_{4,j,3}(n-2) + 4C_{4,j,3}(n-1) - 4C_{4,j,3}(n) + (2-n)C_{4,j+1,3}(n-2) \\ & + (3n-1)C_{4,j+1,3}(n-1) - 2(n+1)C_{4,j+1,3}(n) \quad \text{for } j = -2,-1,0, \end{aligned} \tag{3.22}$$

[6] That is, the case where $C_{i,j,k}(n) = 0$ for $k > 0$.

$$
\begin{aligned}
0 =& -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2) + (3n-1)C_{4,j+1,k}(n-1) \\
&- 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2) + 3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n) \\
&\text{for } j=-2, k=1,2 \quad \text{and} \quad j=-1, k=2,
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
0 =& -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2) + (3n-1)C_{4,j+1,k}(n-1) \\
&- 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2) + 3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n) \\
&+ \frac{j-2k-7jk}{(n-1)n} \qquad \text{for } j=-2, k=0; j=-1, k=1 \quad \text{and} \quad j=0, k=2,
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
0 =& -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2) + (3n-1)C_{4,j+1,k}(n-1) \\
&- 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2) + 3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n) \\
&+ \frac{2(-1)^k(k+1)}{(n-1)n} S_1(n) - 2(-1)^k(k+1)\frac{3n^2-n-1}{(n-1)^2n^2} \qquad \text{for } j=-1, k=0 \quad \text{and} \quad j=0, k=1,
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
0 =& -2C_{4,0,0}(n-2) + 4C_{4,0,0}(n-1) - 4C_{4,0,0}(n) + (2-n)C_{4,1,0}(n-2) + (3n-1)C_{4,1,0}(n-1) - 2(n+1)C_{4,1,0}(n) \\
&- C_{4,1,1}(n-2) + 3C_{4,1,1}(n-1) - 2C_{4,1,1}(n) + \frac{2(3n^2-n-1)}{(n-1)^2n^2} S_1(n) - \frac{S_1^2(n)}{(n-1)n} + \frac{S_2(n)}{(n-1)n} \\
&- \frac{2(6n^4-6n^3+1)}{(n-1)^3n^3}.
\end{aligned} \tag{3.26}
$$

This system is triangular. So, the coefficients $C_{4,j,k}(n)$ can be obtained successively by solving Eqs. (3.20) to (3.26) one after the other, inserting the results of the $C_{4,j,k}$'s obtained at each step in subsequent equations. We start with the coefficient for which the value of j is the lowest and the value of k is the largest [in this case, $C_{4,-2,3}(n)$], and proceed to obtain the coefficients for lower values of k, keeping j fixed, until all coefficients for $k=3$ to $k=0$ are determined. After that, we increase the value of j by one and repeat the procedure until all values of j are exhausted and all coefficients are determined. All of this can be done automatically using `Sigma` and `HarmonicSums`. The results will be given in terms of harmonic sums [94],

$$
S_{b,\vec{a}}(n) = \sum_{k=1}^{n} \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \qquad S_\varnothing = 1, \qquad b, a_i \in \mathbb{Z}\backslash\{0\}, \tag{3.27}
$$

and generalized harmonic sums [88,95],

$$
S_{b,\vec{a}}(\{c,\vec{d}\},n) = \sum_{k=1}^{n} \frac{c^k}{k^b} S_{\vec{a}}(\{\vec{d}\},k), \qquad b, a_i \in \mathbb{N}\backslash\{0\}, \qquad c, d_i \in \mathbb{Z}\backslash\{0\}, \qquad S_\varnothing = 1. \tag{3.28}
$$

Since all difference equations are of second order, we need at least two initial values in order to solve them. The first few expansion coefficients of $K_4(y)$ are given by

$$
\begin{aligned}
K_4(y) =& -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-2 + y + \frac{2}{3}y^2 + \frac{y^3}{2} + 2\ln(y)\right) - 4 + 2y + \frac{5}{6}y^2 + \frac{y^3}{3} + \left(4 - 2y - \frac{4}{3}y^2 - y^3\right)\ln(y) - 2\ln^2(y) \\
&+ \varepsilon\left[-8 + \frac{8}{3}\zeta_3 + 4y + \frac{5}{3}y^2 + \frac{5}{6}y^3 + \left(-4 + 2y + \frac{4}{3}y^2 + y^3\right)\ln^2(y) + \left(8 - 4y - \frac{5}{3}y^2 - \frac{2}{3}y^3\right)\ln(y) \right. \\
&\left. + \frac{4}{3}\ln^3(y)\right] + \mathcal{O}(\varepsilon^2, y^4).
\end{aligned} \tag{3.29}
$$

These were obtained as described in the previous section. In general, the solutions to the difference equations (3.20)–(3.26) for general values of n will be valid starting from a certain value $n = n_0$, and therefore the initial values used to solve the difference equations must also be taken starting from $n \geq n_0$. There are a few cases above where we can start from $n_0 = 0$, but in most cases we must take the initial values starting from $n_0 = 1$. Later, when we formally perform the sum (3.15), the expansion terms for $n < n_0$ will have to be added separately.

We obtain the following results for the expansion coefficients,

$$C_{4,-2,k}(n) = 0 \quad \text{for } k \geq 1, n \geq 0 \quad \text{and} \quad k = 0, n \geq 1, \tag{3.30}$$

$$C_{4,-1,k}(n) = 0 \quad \text{for } k \geq 2, n \geq 0, \quad \text{and} \quad k = 1, n \geq 1, \tag{3.31}$$

$$C_{4,-1,0}(n) = \frac{2}{n+1} \quad \text{for } n \geq 1, \tag{3.32}$$

$$C_{4,0,k}(n) = 0 \quad \text{for } k = 3, n \geq 0, \quad \text{and} \quad k = 2, n \geq 1, \tag{3.33}$$

$$C_{4,0,1}(n) = -\frac{4}{n+1} \quad \text{for } n \geq 1, \tag{3.34}$$

$$C_{4,0,0}(n) = \frac{8}{n+1} + \frac{2(1-2n)}{n(n+1)} S_1(n) - \frac{2^{1-n}}{n(n+1)} S_1(\{2\}, n) \quad \text{for } n \geq 1, \tag{3.35}$$

$$C_{4,1,3}(n) = 0 \quad \text{for } n \geq 1, \tag{3.36}$$

$$C_{4,1,2}(n) = \frac{4}{n+1} \quad \text{for } n \geq 1, \tag{3.37}$$

$$C_{4,1,1}(n) = -2C_{4,0,0}(n) \quad \text{for } n \geq 1, \tag{3.38}$$

$$\begin{aligned} C_{4,1,0}(n) = {} & \frac{24}{n+1} + \frac{4n-3}{n(n+1)} S_1^2(n) - \frac{S_2(n)}{n(n+1)} - \frac{8}{n(n+1)2^n} S_1(\{2\}, n) \\ & - \frac{4S_1(n)}{n(n+1)} \left[4n - 2 + \frac{1}{2^n} S_1(\{2\}, n)\right] - \frac{6}{n(n+1)2^n} S_2(\{2\}, n) \\ & - \frac{4(n-1)}{n(n+1)} S_{1,1}\left(\left\{\frac{1}{2}, 2\right\}, n\right) + \frac{10}{n(n+1)2^n} S_{1,1}(\{2,1\}, n) \quad \text{for } n \geq 1, \end{aligned} \tag{3.39}$$

with the separate values,

$$\begin{aligned} & C_{4,-2,0}(0) = -1, \quad C_{4,-1,1}(0) = 2, \quad C_{4,-1,0}(0) = -2, \quad C_{4,0,2}(0) = -2, \quad C_{4,0,1}(0) = 4, \quad C_{4,0,0}(0) = -4, \\ & C_{4,1,3}(0) = \frac{4}{3}, \quad C_{4,1,2}(0) = -4, \quad C_{4,1,1}(0) = 8, \quad C_{4,1,0}(0) = \frac{8}{3}\zeta_3 - 8, \end{aligned} \tag{3.40}$$

We can insert the results from Eq. (3.30) to (3.40) into the expansion (3.15) and perform the sums using `Sigma`, `HarmonicSums`, `EvaluateMultiSums` and `SumProduction` [96–98]. We obtain

$$\begin{aligned} K_4(y) = {} & -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[\frac{2}{y} H_1(y) + 2\ln(y) - 4\right] - 12 - \frac{2}{y}(y-2) H_{2,1}(y) + \frac{(y-3)}{y} H_1^2(y) + \frac{8}{y} H_1(y) + \left(8 - \frac{4}{y} H_1(y)\right) \ln(y) \\ & - 2\ln^2(y) + \varepsilon\left[\frac{y-2}{y}(-8H_{2,1}(y) + 4H_{1,2,1}(y) + 6H_{2,1,1}(y) - 4H_{2,2,1}(y)) - \frac{3y-7}{3y} H_1^3(y) + \frac{4}{y}(y-3) H_1^2(y) \right. \\ & + \frac{24}{y} H_1(y) + \frac{8}{3}\zeta_3 - 32 + \left(\frac{4}{y}(y-2) H_{2,1}(y) - \frac{2}{y}(y-3) H_1^2(y) - \frac{16}{y} H_1(y) + 24\right) \ln(y) + \left(\frac{4}{y} H_1(y) - 8\right) \ln^2(y) \\ & \left. + \frac{4}{3}\ln^3(y)\right] + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.41}$$

Notice the presence of the letters

$$\frac{1}{2-y}, \qquad \frac{1}{1-y} \quad \text{and} \quad \frac{1}{y}. \tag{3.42}$$

After we go back to the original variable $x = 1 - y$, we obtain a representation in terms of the standard harmonic polylogarithms with the letters (3.12),

$$
\begin{aligned}
K_4(x) = &-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\left[\frac{2}{1-x}H_0(x)+2H_1(x)+4\right]-\frac{2(1+x)}{1-x}H_{0,-1}(x)-\frac{2+x}{1-x}H_0^2(x)\\
&+\frac{1}{1-x}(2(1+x)H_{-1}(x)-4H_1(x)-8)H_0(x)-2H_1^2(x)-8H_1(x)-12\\
&+\frac{1+x}{1-x}\zeta_2+\varepsilon\left[\left(\frac{1+x}{1-x}(2\zeta_2-4H_{0,-1}(x))-24\right)H_1(x)-\frac{4}{3}H_1^3(x)\right.\\
&-\frac{1+x}{1-x}[H_{-1}(x)(-4H_{0,-1}(x)-3H_0^2(x)-(4H_1(x)+8)H_0(x)+2\zeta_2)\\
&+2H_0(x)(H_{0,-1}(x)-\zeta_2)+8H_{0,-1}(x)+4H_{0,-1,-1}(x)+2H_{0,0,-1}(x)\\
&+2H_0(x)H_{-1}^2(x)-4\zeta_2]-\frac{4+3x}{3(1-x)}H_0^3(x)-\frac{2+x}{1-x}(2H_1(x)+4)H_0^2(x)\\
&\left.-\frac{2}{1-x}(2H_1^2(x)+8H_1(x)+12)H_0(x)-8H_1^2(x)+\frac{2(7-x)}{3(1-x)}\zeta_3-32\right]+\mathcal{O}(\varepsilon^2). \qquad (3.43)
\end{aligned}
$$

IV. RESULTS

We calculated the heavy-quark form factors F_I, $I=V, A, S, P$, up to two loops and $\mathcal{O}(\varepsilon^2)$. Due to the length of the expressions we list here only the expansion corresponding to the low-energy ($0<q^2\ll m^2$), high-energy ($|q^2|\gg m^2$) and threshold ($|q^2|\approx 4m^2$) region up to $\mathcal{O}(\varepsilon)$. The complete analytical results up to $\mathcal{O}(\varepsilon^2)$ are provided as supplemental material together with this publication [99]. We present renormalized results for all form factors but the singlet contributions to the axial-vector and pseudoscalar currents $\hat{F}_{A,i}^{\text{“2”},(s)}$ and $\hat{F}_P^{\text{“2”},(s)}$ for which we present the bare results as discussed in Sec. II D. The expansions have been obtained with the help of the Mathematica packages `Sigma` and `HarmonicSums`.

The calculations in QCD are usually performed in the $\overline{\text{MS}}$ scheme, in which besides pole subtraction universal contributions are absorbed into the singularities of the problem. The widest accepted way, and the one we also employed for our calculation, is to introduce a S_ε-factor for each loop given by

$$S_\varepsilon=\exp\left[-\varepsilon(\gamma_E-\ln(4\pi))\right] \qquad (4.1)$$

for $D=4-2\varepsilon$. At the end of the calculation one sets $S_\varepsilon=1$. In Refs. [20,21,23], another convention has been used and other terms are absorbed as well. In higher orders in ε, this leads to different expressions and also influences, beyond NLO, the evolution of the strong coupling constant α_s. As one wants to usually compare $\alpha_s(M_Z^2)$ for different observables, we were using the standard definition also used by the PDG [100]. Differences also emerge for other quantities to be renormalized. Considering just the level of NLO, there are no differences.

For convenience, we collect here the notation used in the presentation of the results. We use the dimensionless variable x, Eq. (3.5). The kinematic regions of interest correspond to $x\to 1$ ($q^2=0$), $x\to\pm 0$ ($q^2=\mp\infty$) and $x\to -1$ ($q^2=4m^2$).

Since the region $0<q^2<4m^2$ corresponds to the upper half of the unit circle in the complex plane it is convenient to define the variable ϕ by

$$x=e^{i\phi} \qquad (4.2)$$

and to expand around $\phi=0$ instead.

In the threshold region, we use the velocity of the heavy quarks as basic variable

$$\beta=\sqrt{1-\frac{4m^2}{q^2}}\leftrightarrow x=\frac{\beta-1}{\beta+1} \qquad (4.3)$$

and expand around $\beta=0$. This avoids the appearance of square roots. Furthermore, we use the abbreviations

$$
\begin{aligned}
c_1&=12\zeta_2\ln^2(2)+\ln^4(2)+24\mathrm{Li}_4\left(\frac{1}{2}\right)\\
c_2&=26\zeta_2^2\ln(2)-20\zeta_2\ln^3(2)-\ln^5(2)+120\mathrm{Li}_5\left(\frac{1}{2}\right) \qquad (4.4)
\end{aligned}
$$

and

$$
\begin{aligned}
&H_{a_1,\ldots,a_n}\equiv H_{a_1,\ldots,a_n}(x), \qquad L\equiv H_0(x)=\ln(x),\\
&\bar{H}_0(\phi)\equiv H_0(\phi)-\frac{i\pi}{2}, \qquad (4.5)
\end{aligned}
$$

with the harmonic polylogarithms $H_{a_1,\ldots,a_n}(x)$ as defined in Eq. (3.11).

To validate our results we compare them to the existing literature. Up to $\mathcal{O}(\varepsilon^0)$ we agree with all available unrenormalized results for the various form factors. Note that in Refs. [20,21,23] a different normalization for the master integrals has been used, resulting in a difference proportional to $(\Gamma(1+\varepsilon)/\exp(\gamma_E\varepsilon))^2$, where γ_E denotes the Euler-Mascheroni constant.

At $\mathcal{O}(\varepsilon)$ we can compare our results for the vector form factors with the results given in Ref. [24] and find a difference

$$-C_F C_A\left\{\varepsilon\left[\frac{1037x^3}{(1+x)^6}\right]\right\}, \tag{4.6}$$

which has been reported in [26] already. In addition, we compared our analytic results as well as the corresponding expansions with the results for the color-planar limit given in [26] and found agreement.

Comparing the renormalized results, we noticed that the renormalization in Refs. [20,21,23] has been performed using a different scheme as described above, resulting in a difference proportional to ζ_2 at $\mathcal{O}(\varepsilon^0)$. For the vector form factor we agree with the renormalized results in the color-planar limit given in Ref. [26] and up to the term mentioned in Eq. (4.6) above with the results given in Ref. [24].

A. Low-energy region $0 < q^2 \ll m^2$

The low-energy limit of the space-like form factors is given by $x \to 1$. To facilitate the expansion of the HPLs in the region, we use the variable x as defined in Eq. (4.2) and expand around $\phi = 0$. In the following, we present the series expansion of the one and two-loop form factors, denoted by $\bar{F}$, for all the currents up to fourth order in ϕ.

1. *Vector form factor*

For the vector form factors we find

$$\begin{aligned}\bar{F}_{V,1}^{(1)} = C_F\Bigg\{&\frac{1}{\varepsilon}\left[-\frac{2}{3}\phi^2-\frac{2}{45}\phi^4\right]+\left[-\frac{1}{2}\phi^2-\frac{17}{120}\phi^4\right]+\varepsilon\left[\phi^2\left(-2-\frac{\zeta_2}{3}\right)+\phi^4\left(-\frac{1}{4}-\frac{\zeta_2}{45}\right)\right]\\&+\varepsilon^2\left[\phi^2\left(-4-\frac{\zeta_2}{4}+\frac{2\zeta_3}{9}\right)+\phi^4\left(-\frac{2}{3}-\frac{17\zeta_2}{240}+\frac{2\zeta_3}{135}\right)\right]\Bigg\}.\end{aligned} \tag{4.7}$$

$$\begin{aligned}\bar{F}_{V,1}^{(2)} = C_F^2\Bigg\{&\frac{2}{9\varepsilon^2}\phi^4+\frac{1}{3\varepsilon}\phi^4+\left[\phi^2\left(-\frac{47}{36}-\frac{175}{9}\zeta_2+48\ln(2)\zeta_2-12\zeta_3\right)+\phi^4\left(\frac{14473}{6480}-\frac{34243\zeta_2}{5400}+\frac{68}{5}\ln(2)\zeta_2-\frac{17}{5}\zeta_3\right)\right]\\&+\varepsilon\left[\phi^2\left(\frac{11713}{216}-8c_1-\frac{6763}{54}\zeta_2+\frac{578}{3}\ln(2)\zeta_2+\frac{504}{5}\zeta_2^2-\frac{1567}{18}\zeta_3\right)\right.\\&\left.+\phi^4\left(\frac{3508637}{194400}-\frac{34c_1}{15}-\frac{758317\zeta_2}{20250}+\frac{10139}{180}\ln(2)\zeta_2+\frac{714}{25}\zeta_2^2-\frac{295457\zeta_3}{10800}\right)\right]\Bigg\}\\&+C_FC_A\Bigg\{\frac{1}{\varepsilon^2}\left[\frac{11}{9}\phi^2+\frac{11}{135}\phi^4\right]+\frac{1}{\varepsilon}\left[\phi^2\left(-\frac{94}{27}+\frac{4\zeta_2}{3}\right)+\phi^4\left(-\frac{91}{405}+\frac{4\zeta_2}{45}\right)\right]\\&+\left[\phi^2\left(-\frac{2579}{324}+\frac{155}{18}\zeta_2-24\ln(2)\zeta_2+\frac{26}{3}\zeta_3\right)+\phi^4\left(-\frac{36239}{19440}+\frac{7447\zeta_2}{2160}-\frac{34}{5}\ln(2)\zeta_2+\frac{169}{90}\zeta_3\right)\right]\\&+\varepsilon\left[\phi^2\left(-\frac{134327}{1944}+4c_1+\frac{1297}{27}\zeta_2-\frac{289}{3}\ln(2)\zeta_2-\frac{608}{15}\zeta_2^2+\frac{1487}{36}\zeta_3\right)\right.\\&\left.+\phi^4\left(-\frac{278341}{23328}+\frac{17c_1}{15}+\frac{110029\zeta_2}{6480}-\frac{10139}{360}\ln(2)\zeta_2-\frac{613}{45}\zeta_2^2+\frac{12041}{864}\zeta_3\right)\right]\Bigg\}\\&+C_Fn_lT_F\Bigg\{\frac{1}{\varepsilon^2}\left[-\frac{4}{9}\phi^2-\frac{4}{135}\phi^4\right]+\frac{1}{\varepsilon}\left[\frac{20}{27}\phi^2+\frac{4}{81}\phi^4\right]+\left[\phi^2\left(\frac{283}{81}+\frac{16\zeta_2}{9}\right)+\phi^4\left(\frac{3139}{4860}+\frac{16\zeta_2}{135}\right)\right]\\&+\varepsilon\left[\phi^2\left(\frac{8827}{486}+\frac{181\zeta_2}{27}+\frac{32\zeta_3}{9}\right)+\phi^4\left(\frac{95527}{29160}+\frac{1777\zeta_2}{1620}+\frac{32\zeta_3}{135}\right)\right]\Bigg\}\\&+C_FT_F\Bigg\{\left[\phi^2\left(-\frac{1099}{81}+9\zeta_2\right)+\phi^4\left(-\frac{21019}{4860}+\frac{53\zeta_2}{20}\right)\right]\\&+\varepsilon\left[\phi^2\left(-\frac{635}{18}+\frac{937}{27}\zeta_2-\frac{154}{3}\ln(2)\zeta_2+\frac{1601}{54}\zeta_3\right)+\phi^4\left(-\frac{8293}{648}+\frac{9821}{810}\zeta_2-\frac{283}{18}\ln(2)\zeta_2+\frac{29651\zeta_3}{3240}\right)\right]\end{aligned} \tag{4.8}$$

$$\bar{F}_{V,2}^{(1)} = C_F\left\{\left[2+\frac{1}{3}\phi^2+\frac{7}{180}\phi^4\right]+\varepsilon\left[8+\frac{5}{3}\phi^2+\frac{41}{180}\phi^4\right]+\varepsilon^2\left[16+\zeta_2+\phi^2\left(4+\frac{\zeta_2}{6}\right)+\phi^4\left(\frac{19}{30}+\frac{7\zeta_2}{360}\right)\right]\right\}. \quad (4.9)$$

$$\begin{aligned}\bar{F}_{V,2}^{(2)} = {}& C_F^2\left\{\frac{1}{\varepsilon}\left[-\frac{4}{3}\phi^2-\frac{14}{45}\phi^4\right]+\left[-31+40\zeta_2-48\ln(2)\zeta_2+12\zeta_3+\phi^2\left(-\frac{77}{5}+\frac{122}{5}\zeta_2-\frac{184}{5}\ln(2)\zeta_2+\frac{46}{5}\zeta_3\right)\right.\right.\\
&+\phi^4\left(-\frac{4931}{1260}+\frac{2963}{350}\zeta_2-\frac{1478}{105}\ln(2)\zeta_2+\frac{739}{210}\zeta_3\right)\Bigg]+\varepsilon\left[-\frac{1243}{6}+8c_1+\frac{944}{3}\zeta_2-384\ln(2)\zeta_2\right.\\
&-\frac{504}{5}\zeta_2^2+176\zeta_3+\phi^2\left(-\frac{19666}{225}+\frac{92c_1}{15}+\frac{3704}{25}\zeta_2-\frac{14164}{75}\ln(2)\zeta_2-\frac{1932}{25}\zeta_2^2+\frac{7201}{75}\zeta_3\right)\\
&\left.\left.+\phi^4\left(-\frac{9903863}{396900}+\frac{739c_1}{315}+\frac{10057561\zeta_2}{220500}-\frac{125887\ln(2)\zeta_2}{2205}-\frac{739}{25}\zeta_2^2+\frac{1376111\zeta_3}{44100}\right)\right]\right\}\\
&+C_FC_A\left\{\left[\frac{317}{9}-12\zeta_2+24\ln(2)\zeta_2-6\zeta_3+\phi^2\left(\frac{1699}{270}-\frac{137}{15}\zeta_2+\frac{92}{5}\ln(2)\zeta_2-\frac{23}{5}\zeta_3\right)\right.\right.\\
&+\phi^4\left(\frac{11927}{22680}-\frac{21269\zeta_2}{6300}+\frac{739}{105}\ln(2)\zeta_2-\frac{739}{420}\zeta_3\right)\Bigg]+\varepsilon\left[\frac{12881}{54}-4c_1-\frac{313}{3}\zeta_2+192\ln(2)\zeta_2+\frac{252}{5}\zeta_2^2-72\zeta_3\right.\\
&+\phi^2\left(\frac{485453}{8100}-\frac{46c_1}{15}-\frac{8983}{150}\zeta_2+\frac{7082}{75}\ln(2)\zeta_2+\frac{966}{25}\zeta_2^2-\frac{6281}{150}\zeta_3\right)\\
&\left.\left.+\phi^4\left(\frac{50620531}{4762800}-\frac{739c_1}{630}-\frac{4335431\zeta_2}{220500}+\frac{125887\ln(2)\zeta_2}{4410}+\frac{739}{50}\zeta_2^2-\frac{1224967\zeta_3}{88200}\right)\right]\right\}\\
&+C_Fn_lT_F\left\{\left[-\frac{100}{9}-\frac{62}{27}\phi^2-\frac{253}{810}\phi^4\right]+\varepsilon\left[-\frac{1922}{27}-12\zeta_2+\phi^2\left(-\frac{1405}{81}-2\zeta_2\right)+\phi^4\left(-\frac{13147}{4860}-\frac{7\zeta_2}{30}\right)\right]\right\}\\
&+C_FT_F\left\{\left[\frac{476}{9}-32\zeta_2+\phi^2\left(\frac{622}{27}-14\zeta_2\right)+\phi^4\left(\frac{4841}{810}-\frac{109\zeta_2}{30}\right)\right]\right.\\
&+\varepsilon\left[\frac{2254}{27}-\frac{308}{3}\zeta_2+192\ln(2)\zeta_2-112\zeta_3+\phi^2\left(\frac{4247}{81}-\frac{490}{9}\zeta_2+84\ln(2)\zeta_2-49\zeta_3\right)\right.\\
&\left.\left.+\phi^4\left(\frac{16753}{972}-\frac{2203}{135}\zeta_2+\frac{109}{5}\ln(2)\zeta_2-\frac{763}{60}\zeta_3\right)\right]\right\}. \qquad (4.10)\end{aligned}$$

Note, that $\bar{F}_{V,2}$ is UV and IR finite in this limit and the leading term agrees with the computation of the anomalous magnetic moment in [101].

2. Axial-vector form factor

For the axial-vector form factor we present the renormalized results for the nonsinglet contributions and the unrenormalized one for the singlet parts.

$$\begin{aligned}\bar{F}_{A,1}^{(1),\text{“ns”}} = {}& C_F\left\{\frac{1}{\varepsilon}\left[-\frac{2}{3}\phi^2-\frac{2}{45}\phi^4\right]+\left[-2-\frac{5}{6}\phi^2-\frac{13}{72}\phi^4\right]+\varepsilon\left[\phi^2\left(-\frac{7}{3}-\frac{\zeta_2}{3}\right)+\phi^4\left(-\frac{29}{90}-\frac{\zeta_2}{45}\right)\right]\right.\\
&\left.+\varepsilon^2\left[-\zeta_2+\phi^2\left(-4-\frac{5\zeta_2}{12}+\frac{2\zeta_3}{9}\right)+\phi^4\left(-\frac{7}{10}-\frac{13\zeta_2}{144}+\frac{2\zeta_3}{135}\right)\right]\right\}. \qquad (4.11)\end{aligned}$$

$$
\begin{aligned}
\bar{F}_{A,1}^{(2),\text{“ns”}} = C_F^2\Bigg\{&\frac{1}{\varepsilon^2}\left[\frac{2}{9}\phi^4\right]+\frac{1}{\varepsilon}\left[\frac{4}{3}\phi^2+\frac{29}{45}\phi^4\right]+\Bigg[-\frac{29}{3}+32\zeta_2-32\ln(2)\zeta_2+8\zeta_3\\
&+\phi^2\left(-\frac{1121}{180}-\frac{217}{45}\zeta_2+\frac{128}{5}\ln(2)\zeta_2-\frac{32}{5}\zeta_3\right)+\phi^4\left(\frac{95341}{45360}-\frac{5423\zeta_2}{5400}+\frac{376}{105}\ln(2)\zeta_2-\frac{94}{105}\zeta_3\right)\Bigg]\\
&+\varepsilon\Bigg[-\frac{1889}{18}+\frac{16c_1}{3}+56\zeta_2-\frac{160}{3}\ln(2)\zeta_2-\frac{336}{5}\zeta_2^2+\frac{232}{3}\zeta_3\\
&+\phi^2\left(-\frac{18997}{5400}-\frac{64c_1}{15}-\frac{49259\zeta_2}{1350}+\frac{6958}{75}\ln(2)\zeta_2+\frac{1344}{25}\zeta_2^2-\frac{14777}{450}\zeta_3\right)\\
&+\phi^4\left(-\frac{5022433}{9525600}-\frac{188c_1}{315}-\frac{44327\zeta_2}{20250}+\frac{721523\ln(2)\zeta_2}{44100}+\frac{188}{25}\zeta_2^2-\frac{131151\zeta_3}{19600}\right)\Bigg]\Bigg\}\\
&+C_FC_A\Bigg\{\frac{1}{\varepsilon^2}\left[\frac{11}{9}\phi^2+\frac{11}{135}\phi^4\right]+\frac{1}{\varepsilon}\left[\phi^2\left(-\frac{94}{27}+\frac{4\zeta_2}{3}\right)+\phi^4\left(-\frac{91}{405}+\frac{4\zeta_2}{45}\right)\right]\\
&+\Bigg[-\frac{143}{9}-8\zeta_2+16\ln(2)\zeta_2-4\zeta_3+\phi^2\left(-\frac{19813}{1620}+\frac{317}{90}\zeta_2-\frac{64}{5}\ln(2)\zeta_2+\frac{88}{15}\zeta_3\right)\\
&+\phi^4\left(-\frac{413831}{136080}+\frac{14479\zeta_2}{10800}-\frac{188}{105}\ln(2)\zeta_2+\frac{197}{315}\zeta_3\right)\Bigg]+\varepsilon\Bigg[-\frac{887}{54}-\frac{8c_1}{3}-\frac{59}{3}\zeta_2+\frac{80}{3}\ln(2)\zeta_2+\frac{168}{5}\zeta_2^2-\frac{68}{3}\zeta_3\\
&+\phi^2\left(-\frac{3362201}{48600}+\frac{32c_1}{15}+\frac{1753}{150}\zeta_2-\frac{3479}{75}\ln(2)\zeta_2-\frac{1276}{75}\zeta_2^2+\frac{16777}{900}\zeta_3\right)\\
&+\phi^4\left(-\frac{281936227}{28576800}+\frac{94c_1}{315}+\frac{583651\zeta_2}{378000}-\frac{721523\ln(2)\zeta_2}{88200}-\frac{698}{225}\zeta_2^2+\frac{5002453\zeta_3}{1058400}\right)\Bigg]\Bigg\}\\
&+C_Fn_lT_F\Bigg\{\frac{1}{\varepsilon^2}\left[-\frac{4}{9}\phi^2-\frac{4}{135}\phi^4\right]+\frac{1}{\varepsilon}\left[\frac{20}{27}\phi^2+\frac{4}{81}\phi^4\right]+\left[\frac{28}{9}+\phi^2\left(\frac{361}{81}+\frac{16\zeta_2}{9}\right)+\phi^4\left(\frac{3901}{4860}+\frac{16\zeta_2}{135}\right)\right]\\
&+\varepsilon\left[-\frac{34}{27}+12\zeta_2+\phi^2\left(\frac{9229}{486}+\frac{235\zeta_2}{27}+\frac{32\zeta_3}{9}\right)+\phi^4\left(\frac{105253}{29160}+\frac{431\zeta_2}{324}+\frac{32\zeta_3}{135}\right)\right]\Bigg\}\\
&+C_FT_F\Bigg\{\left[\frac{460}{9}-32\zeta_2+\phi^2\left(\frac{491}{81}-3\zeta_2\right)+\phi^4\left(\frac{1343}{4860}-\frac{3\zeta_2}{20}\right)\right]+\varepsilon\Bigg[\frac{3998}{27}-\frac{412}{3}\zeta_2+192\ln(2)\zeta_2-112\zeta_3\\
&+\phi^2\left(\frac{4931}{162}-\frac{653}{27}\zeta_2+\frac{62}{3}\ln(2)\zeta_2-\frac{667}{54}\zeta_3\right)+\phi^4\left(\frac{10963}{1944}-\frac{1436}{405}\zeta_2+\frac{97}{90}\ln(2)\zeta_2-\frac{2101\zeta_3}{3240}\right)\Bigg]\Bigg\}. \qquad (4.12)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\bar{F}}_{A,1}^{(2),\text{“s”}} = C_FT_F\Bigg\{&-\frac{6}{\varepsilon}+\left[-\frac{37}{3}+\frac{64}{3}\zeta_2+\phi^2\left(-\frac{85}{18}+\frac{136\zeta_2}{45}\right)+\phi^4\left(-\frac{341}{540}+\frac{2554\zeta_2}{4725}\right)\right]\\
&+\varepsilon\Bigg[-\frac{817}{18}+\frac{266}{9}\zeta_2-128\ln(2)\zeta_2+\frac{224}{3}\zeta_3+\phi^2\left(-\frac{3659}{540}+\frac{4876}{675}\zeta_2-\frac{272}{15}\ln(2)\zeta_2+\frac{476}{45}\zeta_3\right)\\
&-\frac{1}{2}i\phi^3\zeta_2+\phi^4\left(-\frac{14281}{16200}+\frac{985837\zeta_2}{1984500}-\frac{5108\ln(2)\zeta_2}{1575}+\frac{1277}{675}\zeta_3\right)\Bigg]\Bigg\}. \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
\bar{F}_{A,2}^{(1),\text{“ns”}} = C_F\Bigg\{&\left[\frac{14}{3}+\frac{11}{15}\phi^2+\frac{103}{1260}\phi^4\right]+\varepsilon\left[8+\frac{31}{15}\phi^2+\frac{389}{1260}\phi^4\right]\\
&+\varepsilon^2\left[16+\frac{7}{3}\zeta_2+\phi^2\left(4+\frac{11\zeta_2}{30}\right)+\phi^4\left(\frac{47}{70}+\frac{103\zeta_2}{2520}\right)\right]\Bigg\}. \qquad (4.14)
\end{aligned}
$$

$$\bar{F}_{A,2}^{(2),\text{``ns''}} = C_F^2\left\{\frac{1}{\varepsilon}\left[-\frac{28}{9}\phi^2-\frac{94}{135}\phi^4\right]+\left[-\frac{23}{5}+\frac{176}{5}\zeta_2-\frac{176}{5}\ln(2)\zeta_2+\frac{44}{5}\zeta_3\right.\right.$$
$$+\phi^2\left(-\frac{11111}{945}+\frac{592}{45}\zeta_2-\frac{88}{7}\ln(2)\zeta_2+\frac{22}{7}\zeta_3\right)+\phi^4\left.\left(-\frac{251113}{56700}+\frac{208091\zeta_2}{66150}-\frac{298}{105}\ln(2)\zeta_2+\frac{149}{210}\zeta_3\right)\right]$$
$$+\varepsilon\left[\frac{15527}{450}+\frac{88c_1}{15}+\frac{28688}{225}\zeta_2-\frac{6512}{25}\ln(2)\zeta_2-\frac{1848}{25}\zeta_2^2+\frac{3388}{25}\zeta_3\right.$$
$$+\phi^2\left(\frac{376}{11025}+\frac{44c_1}{21}+\frac{3776}{135}\zeta_2-\frac{235856\ln(2)\zeta_2}{3675}-\frac{132}{5}\zeta_2^2+\frac{466972\zeta_3}{11025}\right)$$
$$\left.\left.+\phi^4\left(-\frac{47414267}{17860500}+\frac{149c_1}{315}+\frac{34009847\zeta_2}{6945750}-\frac{25891\ln(2)\zeta_2}{2205}-\frac{149}{25}\zeta_2^2+\frac{1220729\zeta_3}{132300}\right)\right]\right\}$$
$$+C_FC_A\left\{\left[\frac{7663}{135}-\frac{752}{45}\zeta_2+\frac{88}{5}\ln(2)\zeta_2-\frac{22}{5}\zeta_3+\phi^2\left(\frac{5039}{378}-\frac{422}{75}\zeta_2+\frac{44}{7}\ln(2)\zeta_2-\frac{11}{7}\zeta_3\right)\right.\right.$$
$$\left.+\phi^4\left(\frac{27793}{12600}-\frac{56827\zeta_2}{44100}+\frac{149}{105}\ln(2)\zeta_2-\frac{149}{420}\zeta_3\right)\right]+\varepsilon\left[\frac{871991}{4050}-\frac{44c_1}{15}-\frac{19517}{675}\zeta_2+\frac{3256}{25}\ln(2)\zeta_2+\frac{924}{25}\zeta_2^2\right.$$
$$-\frac{14846}{225}\zeta_3+\phi^2\left(\frac{20526143}{396900}-\frac{22c_1}{21}-\frac{142673\zeta_2}{15750}+\frac{117928\ln(2)\zeta_2}{3675}+\frac{66}{5}\zeta_2^2-\frac{70838\zeta_3}{3675}\right)$$
$$\left.\left.+\phi^4\left(\frac{548179231}{71442000}-\frac{149c_1}{630}-\frac{5017571\zeta_2}{3087000}+\frac{25891\ln(2)\zeta_2}{4410}+\frac{149}{50}\zeta_2^2-\frac{118921\zeta_3}{29400}\right)\right]\right\}$$
$$+C_Fn_lT_F\left\{\left[-\frac{412}{27}-\frac{466}{135}\phi^2-\frac{2761}{5670}\phi^4\right]+\varepsilon\left[-\frac{5630}{81}-28\zeta_2+\phi^2\left(-\frac{7427}{405}-\frac{22\zeta_2}{5}\right)\right.\right.$$
$$\left.\left.+\phi^4\left(-\frac{104863}{34020}-\frac{103\zeta_2}{210}\right)\right]\right\}+C_FT_F\left\{\left[-\frac{412}{27}+\frac{32}{3}\zeta_2+\phi^2\left(-\frac{14}{27}+\frac{2\zeta_2}{5}\right)+\phi^4\left(\frac{4601}{5670}-\frac{103\zeta_2}{210}\right)\right]\right.$$
$$+\varepsilon\left[-\frac{9230}{81}+\frac{724}{9}\zeta_2-64\ln(2)\zeta_2+\frac{112}{3}\zeta_3+\phi^2\left(-\frac{10703}{405}+\frac{146}{9}\zeta_2-\frac{12}{5}\ln(2)\zeta_2+\frac{7}{5}\zeta_3\right)\right.$$
$$\left.\left.+\phi^4\left(-\frac{119839}{34020}+\frac{1238}{945}\zeta_2+\frac{103}{35}\ln(2)\zeta_2-\frac{103}{60}\zeta_3\right)\right]\right\}. \quad (4.15)$$

$$\bar{\bar{F}}_{A,2}^{(2),\text{``s''}} = C_FT_F\left\{\left[\frac{24}{\phi^2}+\frac{20}{3}+\frac{16}{15}\zeta_2+4i\phi\zeta_2+\phi^2\left(-\frac{8}{15}-\frac{184}{315}\zeta_2+\frac{4\bar{H}_0(\phi)}{3}\right)+\frac{13}{15}i\phi^3\zeta_2\right.\right.$$
$$\left.+\phi^4\left(-\frac{461}{6300}-\frac{1174\zeta_2}{4725}+\frac{13\bar{H}_0(\phi)}{45}\right)\right]+\varepsilon\left[\frac{1}{\phi^2}(124-128\zeta_2)+\frac{3394}{45}-\frac{11264}{225}\zeta_2\right.$$
$$-\frac{32}{5}\ln(2)\zeta_2+\frac{56}{15}\zeta_3+i\phi(-10\zeta_2+8\ln(2)\zeta_2-8\zeta_2\bar{H}_0(\phi))+\phi^2\left(\frac{3173}{135}-\frac{221449\zeta_2}{33075}\right.$$
$$\left.+\frac{368}{105}\ln(2)\zeta_2-\frac{92}{45}\zeta_3-4\bar{H}_0(\phi)-\frac{4}{3}\bar{H}_0^2(\phi)\right)+i\phi^3\left(-\frac{9}{5}\zeta_2+\frac{26}{15}\ln(2)\zeta_2-\frac{26}{15}\zeta_2\bar{H}_0(\phi)\right)$$
$$\left.\left.+\phi^4\left(\frac{5566123}{1701000}-\frac{1063927\zeta_2}{1984500}+\frac{2348\ln(2)\zeta_2}{1575}-\frac{587}{675}\zeta_3-\frac{31\bar{H}_0(\phi)}{45}-\frac{13}{45}\bar{H}_0^2(\phi)\right)\right]\right\}. \quad (4.16)$$

3. Scalar form factor

For the scalar form factor in the low-energy limit we obtain

$$\bar{F}_S^{(1)} = C_F\left\{\frac{1}{\varepsilon}\left[-\frac{2}{3}\phi^2-\frac{2}{45}\phi^4\right]+\left[-6-\frac{1}{3}\phi^2-\frac{31}{180}\phi^4\right]+\varepsilon\left[-8+\phi^2\left(-\frac{7}{3}-\frac{\zeta_2}{3}\right)+\phi^4\left(-\frac{49}{180}-\frac{\zeta_2}{45}\right)\right]\right.$$
$$\left.+\varepsilon^2\left[-16-3\zeta_2+\phi^2\left(-4-\frac{\zeta_2}{6}+\frac{2\zeta_3}{9}\right)+\phi^4\left(-\frac{7}{10}-\frac{31\zeta_2}{360}+\frac{2\zeta_3}{135}\right)\right]\right\}. \quad (4.17)$$

$$
\begin{aligned}
\bar{F}_S^{(2)} &= C_F^2\bigg\{\frac{2}{9\varepsilon^2}\phi^4+\frac{1}{\varepsilon}\bigg[4\phi^2+\frac{22}{45}\phi^4\bigg]+\bigg[33+\phi^2\bigg(\frac{62}{9}-\frac{71}{9}\zeta_2+28\ln(2)\zeta_2-7\zeta_3\bigg)\\
&+\phi^4\bigg(\frac{5743}{1620}-\frac{7001\zeta_2}{5400}+\frac{11}{3}\ln(2)\zeta_2-\frac{11}{12}\zeta_3\bigg)\bigg]+\varepsilon\bigg[\frac{135}{2}-120\zeta_2+192\ln(2)\zeta_2-48\zeta_3\\
&+\phi^2\bigg(\frac{1297}{27}-\frac{14c_1}{3}-\frac{2723}{54}\zeta_2+98\ln(2)\zeta_2+\frac{294}{5}\zeta_2^2-\frac{725}{18}\zeta_3\bigg)\\
&+\phi^4\bigg(\frac{193531}{24300}-\frac{11c_1}{18}-\frac{141179\zeta_2}{20250}+\frac{5971}{300}\ln(2)\zeta_2+\frac{77}{10}\zeta_2^2-\frac{29381\zeta_3}{3600}\bigg)\bigg]\bigg\}\\
&+C_FC_A\bigg\{\frac{1}{\varepsilon^2}\bigg[\frac{11}{9}\phi^2+\frac{11}{135}\phi^4\bigg]+\frac{1}{\varepsilon}\bigg[\phi^2\bigg(-\frac{94}{27}+\frac{4\zeta_2}{3}\bigg)+\phi^4\bigg(-\frac{91}{405}+\frac{4\zeta_2}{45}\bigg)\bigg]\\
&+\bigg[-\frac{185}{3}+\phi^2\bigg(-\frac{650}{81}+\frac{47}{18}\zeta_2-14\ln(2)\zeta_2+\frac{37}{6}\zeta_3\bigg)+\phi^4\bigg(-\frac{2389}{972}+\frac{11897\zeta_2}{10800}-\frac{11}{6}\ln(2)\zeta_2+\frac{229}{360}\zeta_3\bigg)\bigg]\\
&+\varepsilon\bigg[-\frac{1463}{6}+21\zeta_2-96\ln(2)\zeta_2+24\zeta_3+\phi^2\bigg(-\frac{32507}{486}+\frac{7c_1}{3}+\frac{1873}{108}\zeta_2-49\ln(2)\zeta_2-\frac{293}{15}\zeta_2^2+\frac{629}{36}\zeta_3\bigg)\\
&+\phi^4\bigg(-\frac{2663879}{291600}+\frac{11c_1}{36}+\frac{212591\zeta_2}{81000}-\frac{5971}{600}\ln(2)\zeta_2-\frac{2873}{900}\zeta_2^2+\frac{101327\zeta_3}{21600}\bigg)\bigg]\bigg\}\\
&+C_Fn_lT_F\bigg\{\frac{1}{\varepsilon^2}\bigg[-\frac{4}{9}\phi^2-\frac{4}{135}\phi^4\bigg]+\frac{1}{\varepsilon}\bigg[\frac{20}{27}\phi^2+\frac{4}{81}\phi^4\bigg]+\bigg[\frac{52}{3}+\phi^2\bigg(\frac{316}{81}+\frac{16\zeta_2}{9}\bigg)+\phi^4\bigg(\frac{883}{1215}+\frac{16\zeta_2}{135}\bigg)\bigg]\\
&+\varepsilon\bigg[\frac{206}{3}+36\zeta_2+\phi^2\bigg(\frac{4808}{243}+\frac{154\zeta_2}{27}+\frac{32\zeta_3}{9}\bigg)+\phi^4\bigg(\frac{2560}{729}+\frac{1037\zeta_2}{810}+\frac{32\zeta_3}{135}\bigg)\bigg]\bigg\}\\
&+C_FT_F\bigg\{\bigg[\frac{52}{3}+8i\phi\zeta_2+\phi^2\bigg(-\frac{1417}{81}+\frac{91}{9}\zeta_2+\frac{8}{3}\bar{H}_0(\phi)\bigg)+\frac{17}{15}i\phi^3\zeta_2\\
&+\phi^4\bigg(-\frac{25076}{6075}+\frac{6071\zeta_2}{2700}+\frac{17}{45}\bar{H}_0(\phi)\bigg)\bigg]+\varepsilon\bigg[\frac{350}{3}-60\zeta_2+i\phi(16\ln(2)\zeta_2-16\bar{H}_0(\phi)\zeta_2)\\
&+\phi^2\bigg(-\frac{787}{162}+23\zeta_2-58\ln(2)\zeta_2+\frac{1811}{54}\zeta_3+\frac{8}{3}\bar{H}_0(\phi)-\frac{8}{3}\bar{H}_0(\phi)^2\bigg)\\
&+i\phi^3\bigg(\frac{17}{15}\zeta_2+\frac{34}{15}\ln(2)\zeta_2-\frac{34}{15}\zeta_2\bar{H}_0(\phi)\bigg)+\phi^4\bigg(-\frac{241487}{121500}+\frac{35681\zeta_2}{6750}-\frac{1997}{150}\ln(2)\zeta_2+\frac{125491\zeta_3}{16200}\\
&+\frac{6}{5}\bar{H}_0(\phi)-\frac{17}{45}\bar{H}_0(\phi)^2\bigg)\bigg]\bigg\}. \qquad (4.18)
\end{aligned}
$$

4. Pseudoscalar form factor

The nonsinglet part of the pseudoscalar form factor can be obtained by using Eq. (2.13)

$$
\begin{aligned}
\bar{F}_P^{(1),\text{“ns”}} &= \bar{F}_{A,1}^{(1)}+\bigg(\frac{\phi^2}{4}-\frac{\phi^4}{48}\bigg)\bar{F}_{A,2}^{(1)}+\mathcal{O}(\phi^6),\\
\bar{F}_P^{(2),\text{“ns”}} &= \bar{F}_{A,1}^{(2)}+\bigg(\frac{\phi^2}{4}-\frac{\phi^4}{48}\bigg)\bar{F}_{A,2}^{(2)}+\mathcal{O}(\phi^6). \qquad (4.19)
\end{aligned}
$$

The unrernormalized singlet contribution to the pseudoscalar form factor reads

$$\bar{\tilde{F}}_P^{(2),s} = C_F T_F\Bigg\{\left[\frac{8}{3}+\frac{64}{3}\zeta_2+12i\phi\zeta_2+\phi^2\left(-\frac{68}{9}+4\bar{H}_0(\phi)+\frac{148\zeta_2}{45}\right)+2i\phi^3\zeta_2+\phi^4\left(-\frac{689}{540}+\frac{1759\zeta_2}{4725}+\frac{2}{3}\bar{H}_0(\phi)\right)\right]$$
$$+\varepsilon\Bigg[\frac{88}{9}-\frac{352}{9}\zeta_2-128\ln(2)\zeta_2+\frac{224}{3}\zeta_3+i\phi(-36\zeta_2+24\ln(2)\zeta_2-24\zeta_2\bar{H}_0(\phi))$$
$$+\phi^2\left(\frac{5227}{135}-\frac{3182}{675}\zeta_2-\frac{296}{15}\ln(2)\zeta_2+\frac{518}{45}\zeta_3-12\bar{H}_0(\phi)-4\bar{H}_0(\phi)^2\right)+i\phi^3(-7\zeta_2+4\ln(2)\zeta_2-4\zeta_2\bar{H}_0(\phi))$$
$$+\phi^4\left(\frac{101609}{16200}-\frac{255167\zeta_2}{496125}-\frac{3518\ln(2)\zeta_2}{1575}+\frac{1759\zeta_3}{1350}-2\bar{H}_0(\phi)-\frac{2}{3}\bar{H}_0(\phi)^2\right)\Bigg]\Bigg\}. \tag{4.20}$$

B. High energy region $|q^2| \gg m^2$

We now present the expansion of all the form factors in the asymptotic limit i.e. for $x \to 0^+$ up to $\mathcal{O}(x^2)$. The expanded form factors are denoted by $\mathcal{F}_I$. We use the abbreviation $L = \ln(x)$ in the following. The correct analytic continuation to negative values of x is given by $L \to L + i\pi$.

1. Vector form factor

$$\mathcal{F}_{V,1}^{(1)} = C_F\Bigg\{\frac{1}{\varepsilon}[-2-2L-4Lx^2]+[-4+2\zeta_2-3L-L^2+2(-2+L)x+x^2(1-8L-2L^2+4\zeta_2)]$$
$$+\varepsilon\Bigg[-8+2\zeta_2+4\zeta_3+L(-8+\zeta_2)-\frac{3L^2}{2}-\frac{L^3}{3}+x(-10-2\zeta_2+6L+L^2)$$
$$+x^2\left(4+8\zeta_2+8\zeta_3+L(-19+2\zeta_2)-4L^2-\frac{2L^3}{3}\right)\Bigg]$$
$$+\varepsilon^2\Bigg[-16+6\zeta_2+\frac{14}{5}\zeta_2^2+\frac{20}{3}\zeta_3+L\left(-16+\frac{3\zeta_2}{2}+\frac{14\zeta_3}{3}\right)+L^2\left(-4+\frac{\zeta_2}{2}\right)-\frac{L^3}{2}-\frac{L^4}{12}$$
$$+x\left(-26-8\zeta_2-4\zeta_3+L(16-\zeta_2)+3L^2+\frac{L^3}{3}\right)+x^2\Bigg(17+\frac{39}{2}\zeta_2+\frac{28}{5}\zeta_2^2+16\zeta_3+L\left(-46+4\zeta_2+\frac{28\zeta_3}{3}\right)$$
$$+L^2\left(-\frac{19}{2}+\zeta_2\right)-\frac{4L^3}{3}-\frac{L^4}{6}\Bigg)\Bigg]\Bigg\}. \tag{4.21}$$

$$\mathcal{F}_{V,1}^{(2)} = C_F^2\Bigg\{\frac{1}{\varepsilon^2}[2+4L+2L^2+8L(1+L)x^2]+\frac{1}{\varepsilon}[8+14L+8L^2+2L^3-4\zeta_2-4L\zeta_2-4(-2+L)(1+L)x$$
$$+x^2(-2+30L+32L^2+8L^3-8\zeta_2-16L\zeta_2)]$$
$$+\Bigg[46+\frac{85L}{2}+\frac{55L^2}{2}+\frac{20L^3}{3}+\frac{7L^4}{6}+39\zeta_2-4L^2\zeta_2-48\ln(2)\zeta_2-\frac{118}{5}\zeta_2^2-44\zeta_3-32L\zeta_3$$
$$+x\left(-22+13L-37L^2-\frac{28L^3}{3}-\frac{L^4}{3}-30\zeta_2+36L\zeta_2+8L^2\zeta_2+288\ln(2)\zeta_2+\frac{128}{5}\zeta_2^2-88\zeta_3-48L\zeta_3\right)$$
$$+x^2\Bigg(\frac{1307}{2}-365L+214L^2+32L^3+\frac{26L^4}{3}-980\zeta_2-376L\zeta_2-84L^2\zeta_2-576\ln(2)\zeta_2$$
$$-\frac{1756}{5}\zeta_2^2+808\zeta_3+496L\zeta_3\Bigg)\Bigg]$$

$$+\varepsilon\left[4+8c_1+\frac{479L}{4}+\frac{153L^2}{2}+\frac{137L^3}{6}+\frac{11L^4}{3}+\frac{L^5}{2}+163\zeta_2+17L\zeta_2-\frac{8}{3}L^3\zeta_2-24\ln(2)\zeta_2-160\zeta_2^2\right.$$
$$-\frac{106}{5}L\zeta_2^2-\frac{346}{3}\zeta_3-\frac{284}{3}L\zeta_3-\frac{112}{3}L^2\zeta_3-12\zeta_2\zeta_3-18\zeta_5$$
$$+x\left(210-48c_1-\frac{335L}{2}-96L^2-\frac{131L^3}{3}-6L^4-\frac{L^5}{5}-580\zeta_2+282L\zeta_2+34L^2\zeta_2+\frac{40}{3}L^3\zeta_2+1728\ln(2)\zeta_2\right.$$
$$\left.+44\zeta_2^2-\frac{668}{5}L\zeta_2^2-156\zeta_3+300L\zeta_3+4L^2\zeta_3-208\zeta_2\zeta_3+480\zeta_5\right)$$
$$+x^2\left(-\frac{1951}{4}+96c_1+464L+185L^2+178L^3+\frac{74L^4}{3}+\frac{58L^5}{15}-1164\zeta_2-2206L\zeta_2-72L^2\zeta_2\right.$$
$$\left.\left.\left.-\frac{476}{3}L^3\zeta_2-4224\ln(2)\zeta_2+3212\zeta_2^2+\frac{5992}{5}L\zeta_2^2-2000\zeta_3-\frac{7792}{3}L\zeta_3-\frac{268}{3}L^2\zeta_3+1656\zeta_2\zeta_3-6252\zeta_5\right)\right]\right\}$$
$$+C_FC_A\left\{\frac{1}{\varepsilon^2}\left[\frac{11}{3}+\frac{11L}{3}+\frac{22Lx^2}{3}\right]+\frac{1}{\varepsilon}\left[-\frac{49}{9}-\frac{67L}{9}+2\zeta_2+2L\zeta_2-2\zeta_3+x^2\left(4-\frac{188L}{9}-4L^2-\frac{4L^3}{3}-4\zeta_2-8\zeta_3\right)\right]\right.$$
$$+\left[-\frac{1595}{27}-\frac{2545L}{54}-\frac{233L^2}{18}-\frac{11L^3}{9}-\frac{7}{9}\zeta_2-\frac{22}{3}L\zeta_2+2L^2\zeta_2+24\ln(2)\zeta_2-\frac{3}{5}\zeta_2^2+\frac{134}{3}\zeta_3+26L\zeta_3\right.$$
$$+x\left(-\frac{904}{9}+\frac{341L}{9}-\frac{25L^2}{3}+\frac{8L^3}{3}+\frac{L^4}{6}+\frac{494}{3}\zeta_2+44L\zeta_2+8L^2\zeta_2-144\ln(2)\zeta_2+28\zeta_2^2-200\zeta_3-72L\zeta_3\right)$$
$$+x^2\left(\frac{8723}{18}-\frac{8968L}{27}+\frac{931L^2}{9}-\frac{94L^3}{9}-\frac{7L^4}{3}-\frac{6848}{9}\zeta_2-\frac{1076}{3}L\zeta_2-72L^2\zeta_2\right.$$
$$\left.\left.+288\ln(2)\zeta_2-\frac{1408}{5}\zeta_2^2+\frac{5188}{3}\zeta_3+808L\zeta_3\right)\right]$$
$$+\varepsilon\left[-\frac{28745}{162}-4c_1-\frac{70165L}{324}-\frac{3337L^2}{54}-\frac{565L^3}{54}-\frac{11L^4}{12}-\frac{71}{27}\zeta_2-\frac{575}{18}L\zeta_2-\frac{11}{2}L^2\zeta_2+\frac{4}{3}L^3\zeta_2\right.$$
$$+12\ln(2)\zeta_2+\frac{637}{5}\zeta_2^2+\frac{88}{5}L\zeta_2^2+\frac{1577}{9}\zeta_3+\frac{260}{3}L\zeta_3+26L^2\zeta_3-2\zeta_2\zeta_3-157\zeta_5$$
$$+x\left(-\frac{12683}{27}+24c_1+\frac{5639L}{54}+\frac{134L^2}{9}-\frac{17L^3}{3}+\frac{11L^4}{6}+\frac{L^5}{10}+\frac{4448}{9}\zeta_2+115L\zeta_2+26L^2\zeta_2\right.$$
$$\left.+\frac{16}{3}L^3\zeta_2-864\ln(2)\zeta_2-\frac{2944}{5}\zeta_2^2-\frac{482}{5}L\zeta_2^2+272\zeta_3+44L\zeta_3-38L^2\zeta_3+8\zeta_2\zeta_3+768\zeta_5\right)$$
$$+x^2\left(-\frac{25015}{27}-48c_1+\frac{155567L}{162}-\frac{18413L^2}{54}+\frac{3260L^3}{27}-\frac{53L^4}{6}-\frac{19L^5}{15}-\frac{41933}{54}\zeta_2-\frac{6002}{9}L\zeta_2-125L^2\zeta_2-52L^3\zeta_2\right.$$
$$\left.\left.\left.+2112\ln(2)\zeta_2+\frac{19996}{5}\zeta_2^2+1052L\zeta_2^2+\frac{154}{9}\zeta_3-\frac{5168}{3}L\zeta_3+476L^2\zeta_3+148\zeta_2\zeta_3-8178\zeta_5\right)\right]\right\}$$
$$+C_Fn_lT_F\left\{\frac{1}{\varepsilon^2}\left[-\frac{4}{3}-\frac{4L}{3}-\frac{8Lx^2}{3}\right]+\frac{1}{\varepsilon}\left[\frac{20}{9}+\frac{20L}{9}+\frac{40Lx^2}{9}\right]+\left[\frac{424}{27}+\frac{418L}{27}+\frac{38L^2}{9}+\frac{4L^3}{9}-\frac{28}{9}\zeta_2+\frac{8}{3}L\zeta_2-\frac{16}{3}\zeta_3\right.\right.$$
$$\left.+x\left(\frac{200}{9}-\frac{148L}{9}-\frac{4L^2}{3}+\frac{8}{3}\zeta_2\right)+x^2\left(-\frac{68}{9}+\frac{1064L}{27}+\frac{88L^2}{9}+\frac{8L^3}{9}-\frac{176}{9}\zeta_2+\frac{16}{3}L\zeta_2-\frac{32}{3}\zeta_3\right)\right]$$
$$+\varepsilon\left[\frac{5204}{81}+\frac{5813L}{81}+\frac{562L^2}{27}+\frac{94L^3}{27}+\frac{L^4}{3}-\frac{176}{27}\zeta_2+\frac{74}{9}L\zeta_2+2L^2\zeta_2-\frac{96}{5}\zeta_2^2-\frac{280}{9}\zeta_3-\frac{16}{3}L\zeta_3\right.$$
$$+x\left(\frac{3808}{27}-\frac{2654L}{27}-\frac{184L^2}{9}-\frac{4L^3}{3}+\frac{584}{9}\zeta_2-4L\zeta_2+16\zeta_3\right)$$

$$+x^2\left(-\frac{2698}{27}+\frac{18238L}{81}+\frac{1406L^2}{27}+\frac{224L^3}{27}+\frac{2L^4}{3}-\frac{2974}{27}\zeta_2+\frac{184}{9}L\zeta_2+4L^2\zeta_2-\frac{192}{5}\zeta_2^2-\frac{896}{9}\zeta_3-\frac{32}{3}L\zeta_3\right)\Bigg]\Bigg\}$$
$$+C_FT_F\Bigg\{\Bigg[\frac{1532}{27}+\frac{530L}{27}+\frac{38L^2}{9}+\frac{4L^3}{9}-\frac{8}{3}\zeta_2+4L\zeta_2+x\left(-\frac{784}{9}-\frac{436L}{9}-\frac{52L^2}{3}-88\zeta_2\right)$$
$$+x^2\left(\frac{1568}{9}+\frac{9604L}{27}+\frac{808L^2}{9}-\frac{16L^3}{9}+\frac{2096}{3}\zeta_2-8L\zeta_2\right)\Bigg]$$
$$+\varepsilon\Bigg[\frac{4138}{27}+\frac{191L}{3}+\frac{562L^2}{27}+\frac{94L^3}{27}+\frac{L^4}{3}-\frac{1616}{27}\zeta_2+6L\zeta_2+2L^2\zeta_2+\frac{224}{3}\ln(2)\zeta_2-\frac{8}{5}\zeta_2^2-\frac{184}{3}\zeta_3-\frac{56}{9}L\zeta_3$$
$$+x\left(-\frac{1148}{27}-\frac{5246L}{27}-\frac{472L^2}{9}-12L^3-\frac{2192}{9}\zeta_2-4L\zeta_2+320\ln(2)\zeta_2-\frac{352}{3}\zeta_3\right)$$
$$+x^2\left(\frac{215}{27}+\frac{32696L}{27}+\frac{9452L^2}{27}+\frac{1772L^3}{27}-\frac{4L^4}{3}+\frac{47870}{27}\zeta_2+40L\zeta_2-4L^2\zeta_2\right.$$
$$\left.-\frac{9344}{3}\ln(2)\zeta_2-\frac{8}{5}\zeta_2^2+\frac{13120}{9}\zeta_3+\frac{176}{9}L\zeta_3\right)\Bigg]\Bigg\}. \tag{4.22}$$

$$\mathcal{F}_{V,2}^{(1)}=C_F\Bigg\{-4Lx+\varepsilon[x(-16L-2L^2+4\zeta_2)+8(-1+L)x^2]$$
$$+\varepsilon^2\left[x\left(-8L^2-\frac{2L^3}{3}+\frac{1}{3}L(-96+6\zeta_2)+16\zeta_2+8\zeta_3\right)+x^2(-40+32L+4L^2-8\zeta_2)\right]\Bigg\}. \tag{4.23}$$

$$\mathcal{F}_{V,2}^{(2)}=C_F^2\Bigg\{\frac{1}{\varepsilon}[8L(1+L)x]+\Bigg[x(62L+34L^2+8L^3+60\zeta_2-48L\zeta_2-192\ln(2)\zeta_2+16\zeta_3)$$
$$+x^2\left(-232+232L-200L^2-\frac{64L^3}{3}-\frac{4L^4}{3}+752\zeta_2+208L\zeta_2+64L^2\zeta_2+384\ln(2)\zeta_2+\frac{1056}{5}\zeta_2^2-864\zeta_3-448L\zeta_3\right)\Bigg]$$
$$+\varepsilon\Bigg[x\left(-20+32c_1+249L+74L^2+\frac{94L^3}{3}+\frac{14L^4}{3}+492\zeta_2-116L\zeta_2-16L^2\zeta_2-1152\ln(2)\zeta_2\right.$$
$$\left.-\frac{1192}{5}\zeta_2^2+128\zeta_3-192L\zeta_3\right)$$
$$+x^2\left(-640-64c_1+392L-308L^2-\frac{728L^3}{3}-\frac{40L^4}{3}-\frac{4L^5}{5}+2200\zeta_2+1296L\zeta_2+264L^2\zeta_2\right.$$
$$\left.+\frac{256}{3}L^3\zeta_2+3072\ln(2)\zeta_2-\frac{14176}{5}\zeta_2^2-\frac{4848}{5}L\zeta_2^2+2672\zeta_3+1488L\zeta_3-80L^2\zeta_3-1088\zeta_2\zeta_3+4608\zeta_5\right)\Bigg]\Bigg\}$$
$$+C_FC_A\Bigg\{\Bigg[x\left(12-\frac{346L}{9}+\frac{2L^2}{3}-\frac{244}{3}\zeta_2+96\ln(2)\zeta_2+80\zeta_3\right)$$
$$+x^2\left(-\frac{616}{3}+\frac{232L}{3}-72L^2+\frac{32L^3}{3}+\frac{2L^4}{3}+656\zeta_2+368L\zeta_2+64L^2\zeta_2-192\ln(2)\zeta_2+\frac{1104}{5}\zeta_2^2-1456\zeta_3-544L\zeta_3\right)\Bigg]$$
$$+\varepsilon\Bigg[x\left(78-16c_1-\frac{8057L}{27}-\frac{250L^2}{9}+\frac{2L^3}{3}-\frac{1768}{9}\zeta_2-38L\zeta_2-8L^2\zeta_2+576\ln(2)\zeta_2+\frac{1504}{5}\zeta_2^2-264\zeta_3+48L\zeta_3\right)$$
$$+x^2\left(\frac{2672}{9}+32c_1-\frac{5912L}{9}-4L^2-\frac{152L^3}{3}+\frac{20L^4}{3}+\frac{2L^5}{5}+664\zeta_2+656L\zeta_2+160L^2\zeta_2+\frac{160}{3}L^3\zeta_2\right.$$
$$\left.-1536\ln(2)\zeta_2-\frac{16032}{5}\zeta_2^2-\frac{4104}{5}L\zeta_2^2+608\zeta_3+1168L\zeta_3-248L^2\zeta_3-224\zeta_2\zeta_3+5760\zeta_5\right)\Bigg]\Bigg\}$$

$$+ C_F n_l T_F \left\{ \left[-\frac{32}{3}(-1+L)x^2 + x\left(\frac{200L}{9} + \frac{8L^2}{3} - \frac{16}{3}\zeta_2\right) \right] + \varepsilon\left[x\left(\frac{3844L}{27} + \frac{296L^2}{9} + \frac{8L^3}{3} - \frac{592}{9}\zeta_2 + 8L\zeta_2 - 32\zeta_3\right) + x^2\left(\frac{1472}{9} - \frac{1184L}{9} - 16L^2 + 32\zeta_2\right) \right] \right\}$$
$$+ C_F T_F \left\{ \left[x\left(\frac{272}{3} + \frac{200L}{9} + \frac{8L^2}{3} - 16\zeta_2\right) + x^2\left(-\frac{544}{3} - \frac{992L}{3} - 64L^2 - 512\zeta_2\right) \right] + \varepsilon\left[x\left(\frac{1528}{9} + \frac{3844L}{27} + \frac{296L^2}{9} + \frac{8L^3}{3} + \frac{256}{9}\zeta_2 + 8L\zeta_2 + 128\ln(2)\zeta_2 - \frac{256}{3}\zeta_3\right) + x^2\left(-\frac{2368}{9} - \frac{8672L}{9} - 336L^2 - \frac{128L^3}{3} - \frac{4000}{3}\zeta_2 + 2304\ln(2)\zeta_2 - 1088\zeta_3\right) \right] \right\}. \quad (4.24)$$

We note that in the asymptotic limit, the magnetic part of the vector form factors vanish.

2. Axial-vector form factor

$$\mathcal{F}_{A,1}^{(1),\text{“ns”}} = C_F \left\{ \frac{1}{\varepsilon}[-2 - 2L - 4Lx^2] + [-4 - 3L - L^2 + 2\zeta_2 + 2(-2+3L)x + x^2(1 - 8L - 2L^2 + 4\zeta_2)] + \varepsilon\left[-8 - 8L - \frac{3L^2}{2} - \frac{L^3}{3} + 2\zeta_2 + L\zeta_2 + 4\zeta_3 + x(-10 + 6L + 3L^2 - 6\zeta_2) + x^2\left(12 - 27L - 4L^2 - \frac{2L^3}{3} + 8\zeta_2 + 2L\zeta_2 + 8\zeta_3\right) \right] + \varepsilon^2\left[-16 - 16L - 4L^2 - \frac{L^3}{2} - \frac{L^4}{12} + 6\zeta_2 + \frac{3}{2}L\zeta_2 + \frac{1}{2}L^2\zeta_2 + \frac{14}{5}\zeta_2^2 + \frac{20}{3}\zeta_3 + \frac{14}{3}L\zeta_3 + x(-26 + 16L + 3L^2 + L^3 - 8\zeta_2 - 3L\zeta_2 - 12\zeta_3) + x^2\left(25 - 46L - \frac{27L^2}{2} - \frac{4L^3}{3} - \frac{L^4}{6} + \frac{55}{2}\zeta_2 + 4L\zeta_2 + L^2\zeta_2 + \frac{28}{5}\zeta_2^2 + 16\zeta_3 + \frac{28}{3}L\zeta_3\right) \right] \right\} \quad (4.25)$$

$$\mathcal{F}_{A,1}^{(2),\text{“ns”}} = C_F^2 \left\{ \frac{1}{\varepsilon^2}[2 + 4L + 2L^2 + 8L(1+L)x^2] + \frac{1}{\varepsilon}[8 + 14L + 8L^2 + 2L^3 - 4\zeta_2 - 4L\zeta_2 - 4(1+L)(-2+3L)x + x^2(-2 + 30L + 32L^2 + 8L^3 - 8\zeta_2 - 16L\zeta_2)] + \left[46 + \frac{85L}{2} + \frac{55L^2}{2} + \frac{20L^3}{3} + \frac{7L^4}{6} + 39\zeta_2 - 4L^2\zeta_2 - 48\ln(2)\zeta_2 - \frac{118}{5}\zeta_2^2 - 44\zeta_3 - 32L\zeta_3 + x\left(-22 + 19L - 19L^2 - 12L^3 + \frac{L^4}{3} - 66\zeta_2 + 44L\zeta_2 + 8L^2\zeta_2 + 288\ln(2)\zeta_2 + \frac{144}{5}\zeta_2^2 - 104\zeta_3 - 80L\zeta_3\right) + x^2\left(\frac{1195}{2} - 357L + 182L^2 + \frac{64L^3}{3} + \frac{14L^4}{3} - 228\zeta_2 - 216L\zeta_2 - 52L^2\zeta_2 - 576\ln(2)\zeta_2 - \frac{1308}{5}\zeta_2^2 + 488\zeta_3 + 432L\zeta_3\right) \right] + \varepsilon\left[4 + 8c_1 + \frac{479L}{4} + \frac{153L^2}{2} + \frac{137L^3}{6} + \frac{11L^4}{3} + \frac{L^5}{2} + 163\zeta_2 + 17L\zeta_2 - \frac{8}{3}L^3\zeta_2 - 24\ln(2)\zeta_2 - 160\zeta_2^2 - \frac{106}{5}L\zeta_2^2 - \frac{346}{3}\zeta_3 - \frac{284}{3}L\zeta_3 - \frac{112}{3}L^2\zeta_3 - 12\zeta_2\zeta_3 - 18\zeta_5 + x\left(190 - 48c_1 - \frac{173L}{2} - 26L^2 - \frac{53L^3}{3} - \frac{22L^4}{3} + \frac{L^5}{5} - 856\zeta_2 + 166L\zeta_2 - 2L^2\zeta_2 + \frac{8}{3}L^3\zeta_2 + 1728\ln(2)\zeta_2 + \frac{1124}{5}\zeta_2^2 - 84L\zeta_2^2 - 628\zeta_3 + 228L\zeta_3 - 52L^2\zeta_3 + 80\zeta_2\zeta_3 + 864\zeta_5\right) + x^2\left(-\frac{543}{4} + 96c_1 + 792L - 163L^2 + \frac{518L^3}{3} + \frac{58L^4}{3} + \frac{22L^5}{15} + 1708\zeta_2 - 510L\zeta_2 + 136L^2\zeta_2 - \frac{188}{3}L^3\zeta_2 - 2688\ln(2)\zeta_2 + \frac{3596}{5}\zeta_2^2 + \frac{2328}{5}L\zeta_2^2 + 832\zeta_3 - \frac{5008}{3}L\zeta_3 + \frac{452}{3}L^2\zeta_3 - 328\zeta_2\zeta_3 - 5868\zeta_5\right) \right] \right\}$$

$$
\begin{aligned}
&+C_FC_A\Bigg\{\frac{1}{\varepsilon^2}\left[\frac{11}{3}+\frac{11L}{3}+\frac{22Lx^2}{3}\right]+\frac{1}{\varepsilon}\left[-\frac{49}{9}-\frac{67L}{9}+2\zeta_2+2L\zeta_2-2\zeta_3+x^2\left(4-\frac{188L}{9}-4L^2-\frac{4L^3}{3}-4\zeta_2-8\zeta_3\right)\right]\\
&+\Bigg[-\frac{1595}{27}-\frac{2545L}{54}-\frac{233L^2}{18}-\frac{11L^3}{9}-\frac{7}{9}\zeta_2-\frac{22}{3}L\zeta_2+2L^2\zeta_2+24\ln(2)\zeta_2-\frac{3}{5}\zeta_2^2+\frac{134}{3}\zeta_3+26L\zeta_3\\
&+x\left(-\frac{796}{9}+\frac{241L}{3}-9L^2-\frac{L^4}{6}+158\zeta_2+36L\zeta_2+8L^2\zeta_2-144\ln(2)\zeta_2+\frac{132}{5}\zeta_2^2-168\zeta_3-56L\zeta_3\right)\\
&+x^2\left(\frac{7523}{18}-\frac{9760L}{27}+\frac{283L^2}{9}-\frac{46L^3}{9}-\frac{L^4}{3}-\frac{2528}{9}\zeta_2-\frac{164}{3}L\zeta_2-24L^2\zeta_2+288\ln(2)\zeta_2-\frac{544}{5}\zeta_2^2+\frac{1060}{3}\zeta_3+328L\zeta_3\right)\Bigg]\\
&+\varepsilon\Bigg[-\frac{28745}{162}-4c_1-\frac{70165L}{324}-\frac{3337L^2}{54}-\frac{565L^3}{54}-\frac{11L^4}{12}-\frac{71}{27}\zeta_2-\frac{575}{18}L\zeta_2-\frac{11}{2}L^2\zeta_2+\frac{4}{3}L^3\zeta_2\\
&+12\ln(2)\zeta_2+\frac{637}{5}\zeta_2^2+\frac{88}{5}L\zeta_2^2+\frac{1577}{9}\zeta_3+\frac{260}{3}L\zeta_3+26L^2\zeta_3-2\zeta_2\zeta_3-157\zeta_5\\
&+x\Bigg(-\frac{10577}{27}+24c_1+\frac{3799L}{18}+\frac{112L^2}{3}-9L^3+\frac{L^4}{6}-\frac{L^5}{10}+\frac{1744}{3}\zeta_2+153L\zeta_2+42L^2\zeta_2+\frac{32}{3}L^3\zeta_2\\
&-864\ln(2)\zeta_2-\frac{2944}{5}\zeta_2^2-\frac{606}{5}L\zeta_2^2+184\zeta_3+20L\zeta_3-10L^2\zeta_3-136\zeta_2\zeta_3+576\zeta_5\Bigg)\\
&+x^2\Bigg(-\frac{12007}{27}-48c_1-\frac{19681L}{162}-\frac{22517L^2}{54}+\frac{956L^3}{27}-\frac{37L^4}{6}-\frac{L^5}{15}-\frac{25085}{54}\zeta_2-\frac{4418}{9}L\zeta_2+43L^2\zeta_2-36L^3\zeta_2\\
&+1344\ln(2)\zeta_2+\frac{8004}{5}\zeta_2^2+548L\zeta_2^2+\frac{4042}{9}\zeta_3-\frac{2720}{3}L\zeta_3+164L^2\zeta_3+628\zeta_2\zeta_3-2994\zeta_5\Bigg)\Bigg]\Bigg\}\\
&+C_Fn_lT_F\Bigg\{\frac{1}{\varepsilon^2}\left[-\frac{4}{3}-\frac{4L}{3}-\frac{8Lx^2}{3}\right]+\frac{1}{\varepsilon}\left[\frac{20}{9}+\frac{20L}{9}+\frac{40Lx^2}{9}\right]+\Bigg[\frac{424}{27}+\frac{418L}{27}+\frac{38L^2}{9}+\frac{4L^3}{9}-\frac{28}{9}\zeta_2\\
&+\frac{8}{3}L\zeta_2-\frac{16}{3}\zeta_3+x\left(\frac{200}{9}-\frac{68L}{3}-4L^2+8\zeta_2\right)+x^2\left(-\frac{164}{9}+\frac{1352L}{27}+\frac{88L^2}{9}+\frac{8L^3}{9}-\frac{176}{9}\zeta_2+\frac{16}{3}L\zeta_2-\frac{32}{3}\zeta_3\right)\Bigg]\\
&+\varepsilon\Bigg[\frac{5204}{81}+\frac{5813L}{81}+\frac{562L^2}{27}+\frac{94L^3}{27}+\frac{L^4}{3}-\frac{176}{27}\zeta_2+\frac{74}{9}L\zeta_2+2L^2\zeta_2-\frac{96}{5}\zeta_2^2-\frac{280}{9}\zeta_3-\frac{16}{3}L\zeta_3\\
&+x\left(\frac{3808}{27}-\frac{862L}{9}-\frac{80L^2}{3}-4L^3+\frac{232}{3}\zeta_2-12L\zeta_2+48\zeta_3\right)\\
&+x^2\left(-\frac{4234}{27}+\frac{20254L}{81}+\frac{1838L^2}{27}+\frac{224L^3}{27}+\frac{2L^4}{3}-\frac{3838}{27}\zeta_2+\frac{184}{9}L\zeta_2+4L^2\zeta_2-\frac{192}{5}\zeta_2^2-\frac{896}{9}\zeta_3-\frac{32}{3}L\zeta_3\right)\Bigg]\Bigg\}\\
&+C_FT_F\Bigg\{\Bigg[\frac{1532}{27}+\frac{530L}{27}+\frac{38L^2}{9}+\frac{4L^3}{9}-\frac{8}{3}\zeta_2+4L\zeta_2+x\left(-\frac{160}{9}-\frac{164L}{3}-20L^2-136\zeta_2\right)\\
&+x^2\left(\frac{320}{9}+\frac{2980L}{27}+\frac{520L^2}{9}-\frac{16L^3}{9}+\frac{1136}{3}\zeta_2-8L\zeta_2\right)\Bigg]\\
&+\varepsilon\Bigg[\frac{4138}{27}+\frac{191L}{3}+\frac{562L^2}{27}+\frac{94L^3}{27}+\frac{L^4}{3}-\frac{1616}{27}\zeta_2+6L\zeta_2+2L^2\zeta_2+\frac{224}{3}\ln(2)\zeta_2-\frac{8}{5}\zeta_2^2-\frac{184}{3}\zeta_3-\frac{56}{9}L\zeta_3\\
&+x\left(\frac{3628}{27}-\frac{1726L}{9}-\frac{176L^2}{3}-\frac{44L^3}{3}-400\zeta_2-12L\zeta_2+576\ln(2)\zeta_2-256\zeta_3\right)\\
&+x^2\Bigg(-\frac{10537}{27}+\frac{10904L}{27}+\frac{2972L^2}{27}+\frac{1196L^3}{27}-\frac{4L^4}{3}\\
&+\frac{25118}{27}\zeta_2+40L\zeta_2-4L^2\zeta_2-\frac{4736}{3}\ln(2)\zeta_2-\frac{8}{5}\zeta_2^2+\frac{6208}{9}\zeta_3+\frac{176}{9}L\zeta_3\Bigg)\Bigg]\Bigg\}. \qquad (4.26)
\end{aligned}
$$

Here we observe that for $x=0$, the electric vector and axial-vector form factors are the same, as expected. The bare singlet piece for the electric axial-vector form factor is given by

$$\begin{aligned}\hat{\mathcal{F}}_{A,1}^{(2),\text{“s”}} = C_F T_F\bigg\{&-\frac{6}{\varepsilon}+\bigg[-29-12L+8\zeta_2+x(8+24L+8L^2-48\zeta_2-32L\zeta_2+64\zeta_3)\\&+x^2\bigg(-4-16L-56L^2-\frac{16L^3}{3}+32\zeta_2+32L\zeta_2+32L^2\zeta_2-\frac{64}{5}\zeta_2^2-128\zeta_3-128L\zeta_3\bigg)\bigg]\\&+\varepsilon\bigg[-\frac{199}{2}-58L-12L^2+26\zeta_2+16L\zeta_2+8\zeta_3+x\bigg(-4+76L+56L^2+8L^3-176\zeta_2\\&-48L\zeta_2-48L^2\zeta_2+384\ln(2)\zeta_2+\frac{224}{5}\zeta_2^2-80\zeta_3+32L\zeta_3\bigg)\\&+x^2\bigg(-274+88L-180L^2-\frac{136L^3}{3}-\frac{10L^4}{3}+632\zeta_2-16L\zeta_2+64L^2\zeta_2+32L^3\zeta_2-1536\ln(2)\zeta_2\\&-48\zeta_2^2-\frac{448}{5}L\zeta_2^2+400\zeta_3+224L\zeta_3-32L^2\zeta_3+160\zeta_2\zeta_3-144\zeta_5\bigg)\bigg]\bigg\}.\end{aligned}\tag{4.27}$$

The magnetic parts of the axial-vector form factor read

$$\begin{aligned}\mathcal{F}_{A,2}^{(1),\text{“ns”}} = C_F\bigg\{&-4(2+3L)x-16(1+L)x^2+\varepsilon[x(-16-24L-6L^2+12\zeta_2)+x^2(-56-8L-8L^2+16\zeta_2)]\\&+\varepsilon^2\bigg[x(-32-48L-12L^2-2L^3+20\zeta_2+6L\zeta_2+24\zeta_3)\\&+x^2\bigg(-136-16L-4L^2-\frac{8L^3}{3}+8L\zeta_2+32\zeta_3\bigg)\bigg]\bigg\}.\end{aligned}\tag{4.28}$$

$$\begin{aligned}\mathcal{F}_{A,2}^{(2),\text{“ns”}} = C_F^2\bigg\{&\frac{1}{\varepsilon}[8(1+L)(2+3L)x+32(1+L)^2x^2]+\bigg[x(68+122L+86L^2+24L^3+132\zeta_2-48L\zeta_2-192\ln(2)\zeta_2+48\zeta_3)\\&+x^2\bigg(-80+416L+48L^2+32L^3+\frac{4L^4}{3}+16L\zeta_2+384\ln(2)\zeta_2+\frac{32}{5}\zeta_2^2-160\zeta_3-64L\zeta_3\bigg)\bigg]\\&+\varepsilon\bigg[x\bigg(242+32c_1+335L+250L^2+\frac{218L^3}{3}+14L^4+812\zeta_2-28L\zeta_2-48L^2\zeta_2-1152\ln(2)\zeta_2\\&-\frac{1944}{5}\zeta_2^2+176\zeta_3-96L\zeta_3\bigg)+x^2\bigg(316-64c_1+548L+620L^2+16L^3+\frac{56L^4}{3}+\frac{4L^5}{5}-616\zeta_2+384L\zeta_2-24L^2\zeta_2\\&-\frac{64}{3}L^3\zeta_2+3072\ln(2)\zeta_2+\frac{96}{5}\zeta_2^2+\frac{496}{5}L\zeta_2^2-2000\zeta_3-176L\zeta_3-112L^2\zeta_3+576\zeta_2\zeta_3+768\zeta_5\bigg)\bigg]\bigg\}\\&+C_FC_A\bigg\{\bigg[x\bigg(-\frac{968}{9}-\frac{458L}{3}-22L^2-84\zeta_2+96\ln(2)\zeta_2+48\zeta_3\bigg)\\&+x^2\bigg(-\frac{2800}{9}-\frac{1936L}{9}-\frac{304L^2}{3}-\frac{2L^4}{3}+\frac{1184}{3}\zeta_2+144L\zeta_2+32L^2\zeta_2-192\ln(2)\zeta_2+\frac{528}{5}\zeta_2^2-560\zeta_3-224L\zeta_3\bigg)\bigg]\\&+\varepsilon\bigg[x\bigg(-\frac{14872}{27}-16c_1-\frac{6853L}{9}-\frac{590L^2}{3}-22L^3-\frac{908}{3}\zeta_2-82L\zeta_2+576\ln(2)\zeta_2+\frac{1248}{5}\zeta_2^2+232\zeta_3+144L\zeta_3\bigg)\\&+x^2\bigg(-\frac{61028}{27}+32c_1-\frac{18980L}{27}-\frac{5092L^2}{9}-96L^3-\frac{2L^5}{5}+\frac{12992}{9}\zeta_2+568L\zeta_2+176L^2\zeta_2\\&+\frac{128}{3}L^3\zeta_2-1536\ln(2)\zeta_2-\frac{7296}{5}\zeta_2^2-\frac{2424}{5}L\zeta_2^2+1312\zeta_3+304L\zeta_3-40L^2\zeta_3-544\zeta_2\zeta_3+2304\zeta_5\bigg)\bigg]\bigg\}\end{aligned}$$

$$
+ C_F n_l T_F \left\{ \left[x\left(\frac{304}{9} + \frac{136L}{3} + 8L^2 - 16\zeta_2\right) + x^2\left(\frac{896}{9} + \frac{320L}{9} + \frac{32L^2}{3} - \frac{64}{3}\zeta_2\right)\right]\right.
$$

$$
+ \varepsilon\left[x\left(\frac{4664}{27} + \frac{2036L}{9} + \frac{184L^2}{3} + 8L^3 - \frac{224}{3}\zeta_2 + 24L\zeta_2 - 96\zeta_3\right)\right.
$$

$$
\left.\left. + x^2\left(\frac{18544}{27} + \frac{2704L}{27} + \frac{368L^2}{9} + \frac{32L^3}{3} + \frac{128}{9}\zeta_2 + 32L\zeta_2 - 128\zeta_3\right)\right]\right\}
$$

$$
+ C_F T_F \left\{ \left[x\left(-\frac{128}{9} + \frac{136L}{3} + 8L^2 + 80\zeta_2\right) + x^2\left(-\frac{1024}{9} - \frac{1408L}{9} - \frac{160L^2}{3} - 320\zeta_2\right)\right]\right.
$$

$$
+ \varepsilon\left[x\left(-\frac{304}{27} + \frac{2036L}{9} + \frac{184L^2}{3} + 8L^3 + \frac{1072}{3}\zeta_2 + 24L\zeta_2 - 384\ln(2)\zeta_2 + 192\zeta_3\right)\right.
$$

$$
\left.\left. + x^2\left(\frac{5200}{27} - \frac{5936L}{27} - \frac{1360L^2}{9} - 32L^3 - \frac{6080}{9}\zeta_2 + 32L\zeta_2 + 1280\ln(2)\zeta_2 - \frac{1600}{3}\zeta_3\right)\right]\right\}. \tag{4.29}
$$

The bare singlet piece for the magnetic axial-vector form factor is given by

$$
\hat{\mathcal{F}}_{A,2}^{(2),\text{“s”}} = C_F T_F \left\{ \left[x(-80 - 48L - 8L^2 - 32\zeta_2) + x^2\left(-160 + \frac{4L^4}{3} - 256\zeta_2 - 128L\zeta_2 + 32L^2\zeta_2 + \frac{256}{5}\zeta_2^2 + 256\zeta_3 + 192L\zeta_3\right)\right]\right.
$$

$$
+ \varepsilon\left[x\left(-360 - 264L - 72L^2 - \frac{32L^3}{3} + 64\zeta_2 + 16L\zeta_2 - 48\zeta_3\right)\right.
$$

$$
+ x^2\left(-768 - 288L - 32L^2 - \frac{16L^3}{3} - \frac{8L^4}{3} + \frac{16L^5}{15} - 896\zeta_2 - 192L\zeta_2 - 192L^2\zeta_2 + 16L^3\zeta_2 + 1536\ln(2)\zeta_2\right.
$$

$$
\left.\left.\left. + \frac{256}{5}\zeta_2^2 - 32L\zeta_2^2 - 448\zeta_3 - 512L\zeta_3 + 208L^2\zeta_3 + 64\zeta_2\zeta_3 - 704\zeta_5\right)\right]\right\}. \tag{4.30}
$$

Similar to the vector form factor, the magnetic part of the axial-vector form factor also vanishes for $x = 0$.

3. Scalar form factor

Next, we present the scalar form factor in the asymptotic limit.

$$
\mathcal{F}_S^{(1)} = C_F \left\{ \frac{1}{\varepsilon}[-2 - 2L - 4Lx^2] + [-2 - L^2 + 2\zeta_2 + 4(-1 + 3L)x + x^2(1 - 2L - 2L^2 + 4\zeta_2)]\right.
$$

$$
+ \varepsilon\left[-4 - 2L - \frac{L^3}{3} - \zeta_2 + L\zeta_2 + 4\zeta_3 + x(-4 + 12L + 6L^2 - 12\zeta_2)\right.
$$

$$
\left. + x^2\left(\frac{45}{2} - 24L - L^2 - \frac{2L^3}{3} + 2\zeta_2 + 2L\zeta_2 + 8\zeta_3\right)\right]
$$

$$
+ \varepsilon^2\left[-8 - 4L - L^2 - \frac{L^4}{12} + \frac{1}{2}L^2\zeta_2 + \frac{14}{5}\zeta_2^2 + \frac{2}{3}\zeta_3 + \frac{14}{3}L\zeta_3 + \zeta_2 + x(-8 + 28L + 6L^2 + 2L^3 - 14\zeta_2 - 6L\zeta_2 - 24\zeta_3)\right.
$$

$$
\left.\left. + x^2\left(\frac{193}{4} - 34L - 12L^2 - \frac{L^3}{3} - \frac{L^4}{6} + \frac{49}{2}\zeta_2 + L\zeta_2 + L^2\zeta_2 + \frac{28}{5}\zeta_2^2 + 4\zeta_3 + \frac{28}{3}L\zeta_3\right)\right]\right\}. \tag{4.31}
$$

$$
\begin{aligned}
\mathcal{F}_S^{(2)} = C_F^2 \Bigg\{ & \frac{1}{\varepsilon^2}[2+4L+2L^2+8L(1+L)x^2] + \frac{1}{\varepsilon}[4+4L+2L^2+2L^3-4\zeta_2-4L\zeta_2 \\
& -8(1+L)(-1+3L)x + x^2(-2+10L+8L^2+8L^3-8\zeta_2-16L\zeta_2)] \\
& + \bigg[29+12L+6L^2+\frac{2L^3}{3}+\frac{7L^4}{6}+6\zeta_2+12L\zeta_2-4L^2\zeta_2-\frac{118}{5}\zeta_2^2-56\zeta_3-32L\zeta_3 \\
& + x\left(32-72L+4L^2-24L^3-112\zeta_2+64L\zeta_2+8L^2\zeta_2+288\ln(2)\zeta_2+\frac{136}{5}\zeta_2^2-120\zeta_3-64L\zeta_3\right) \\
& + x^2\left(\frac{1069}{2}-380L+166L^2-\frac{10L^3}{3}+5L^4-36\zeta_2-132L\zeta_2-36L^2\zeta_2-576\ln(2)\zeta_2-\frac{724}{5}\zeta_2^2+72\zeta_3+224L\zeta_3\right)\bigg] \\
& + \varepsilon\bigg[-\frac{113}{2}+36L+14L^2+\frac{14L^3}{3}+\frac{L^4}{6}+\frac{L^5}{2}-40\zeta_2+24L\zeta_2+12L^2\zeta_2 \\
& -\frac{8}{3}L^3\zeta_2+264\ln(2)\zeta_2-\frac{314}{5}\zeta_2^2-\frac{106}{5}L\zeta_2^2-\frac{478}{3}\zeta_3-\frac{212}{3}L\zeta_3-\frac{112}{3}L^2\zeta_3-12\zeta_2\zeta_3-18\zeta_5 \\
& + x\bigg(232-48c_1-216L-40L^2+12L^3-\frac{41L^4}{3}-1016\zeta_2+24L\zeta_2+12L^2\zeta_2+8L^3\zeta_2 \\
& +1536\ln(2)\zeta_2+\frac{1552}{5}\zeta_2^2-\frac{544}{5}L\zeta_2^2-616\zeta_3+240L\zeta_3-24L^2\zeta_3-64\zeta_2\zeta_3+672\zeta_5\bigg) \\
& + x^2\bigg(-\frac{2149}{4}+96c_1+895L-221L^2+\frac{472L^3}{3}+\frac{37L^4}{6}+\frac{5L^5}{3}+2568\zeta_2-366L\zeta_2 \\
& +214L^2\zeta_2-52L^3\zeta_2-2304\ln(2)\zeta_2-\frac{306}{5}\zeta_2^2+100L\zeta_2^2+604\zeta_3-\frac{4240}{3}L\zeta_3+\frac{128}{3}L^2\zeta_3-520\zeta_2\zeta_3-2916\zeta_5\bigg)\bigg]\Bigg\} \\
& + C_F C_A\Bigg\{\frac{1}{\varepsilon^2}\left[\frac{11}{3}+\frac{11L}{3}+\frac{22Lx^2}{3}\right]+\frac{1}{\varepsilon}\left[-\frac{49}{9}-\frac{67L}{9}+2\zeta_2+2L\zeta_2-2\zeta_3+x^2\left(4-\frac{188L}{9}-4L^2-\frac{4L^3}{3}-4\zeta_2-8\zeta_3\right)\right] \\
& + \bigg[-\frac{869}{27}-\frac{242L}{27}-\frac{67L^2}{9}-\frac{11L^3}{9}+\frac{182}{9}\zeta_2-\frac{22}{3}L\zeta_2+2L^2\zeta_2-\frac{3}{5}\zeta_2^2+\frac{98}{3}\zeta_3+26L\zeta_3 \\
& + x\left(-\frac{580}{9}+\frac{416L}{3}+12L^2+92\zeta_2-144\ln(2)\zeta_2-44\zeta_3\right) \\
& + x^2\left(\frac{3463}{36}-\frac{5705L}{54}-\frac{37L^2}{18}-\frac{43L^3}{9}-\frac{L^4}{2}-\frac{611}{9}\zeta_2+\frac{58}{3}L\zeta_2+10L^2\zeta_2+288\ln(2)\zeta_2-\frac{122}{5}\zeta_2^2+\frac{364}{3}\zeta_3+96L\zeta_3\right)\bigg] \\
& + \varepsilon\bigg[-\frac{6437}{162}-\frac{2122L}{81}-\frac{341L^2}{27}-\frac{134L^3}{27}-\frac{11L^4}{12}+\frac{1972}{27}\zeta_2-\frac{103}{9}L\zeta_2-\frac{11}{2}L^2\zeta_2+\frac{4}{3}L^3\zeta_2 \\
& -132\ln(2)\zeta_2+65\zeta_2^2+\frac{88}{5}L\zeta_2^2+\frac{1055}{9}\zeta_3+\frac{152}{3}L\zeta_3+26L^2\zeta_3-2\zeta_2\zeta_3-157\zeta_5 \\
& + x\left(-\frac{2756}{27}+24c_1+\frac{3700L}{9}+\frac{476L^2}{3}+\frac{32L^3}{3}-\frac{L^4}{6}+\frac{746}{3}\zeta_2+14L\zeta_2+2L^2\zeta_2-768\ln(2)\zeta_2-360\zeta_2^2+220\zeta_3-16L\zeta_3\right) \\
& + x^2\bigg(\frac{40951}{216}-48c_1-\frac{85217L}{324}-\frac{24019L^2}{108}+\frac{553L^3}{54}-\frac{79L^4}{12}-\frac{L^5}{6}-\frac{13060}{27}\zeta_2-\frac{188}{9}L\zeta_2 \\
& +28L^2\zeta_2+\frac{2}{3}L^3\zeta_2+1152\ln(2)\zeta_2+841\zeta_2^2+\frac{798}{5}L\zeta_2^2-\frac{1916}{9}\zeta_3-\frac{908}{3}L\zeta_3+92L^2\zeta_3+388\zeta_2\zeta_3-942\zeta_5\bigg)\bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+C_F n_l T_F\left\{\frac{1}{\varepsilon^2}\left[-\frac{4}{3}-\frac{4L}{3}-\frac{8Lx^2}{3}\right]+\frac{1}{\varepsilon}\left[\frac{20}{9}+\frac{20L}{9}+\frac{40Lx^2}{9}\right]\right.\\
&+\left[\frac{196}{27}+\frac{112L}{27}+\frac{20L^2}{9}+\frac{4L^3}{9}+\frac{8}{9}\zeta_2+\frac{8}{3}L\zeta_2-\frac{16}{3}\zeta_3+x\left(\frac{128}{9}-\frac{112L}{3}-8L^2+16\zeta_2\right)\right.\\
&\left.+x^2\left(-\frac{290}{9}+\frac{1064L}{27}+\frac{52L^2}{9}+\frac{8L^3}{9}-\frac{104}{9}\zeta_2+\frac{16}{3}L\zeta_2-\frac{32}{3}\zeta_3\right)\right]\\
&+\varepsilon\left[\frac{1706}{81}+\frac{1232L}{81}+\frac{148L^2}{27}+\frac{40L^3}{27}+\frac{L^4}{3}+\frac{328}{27}\zeta_2+\frac{20}{9}L\zeta_2+2L^2\zeta_2-\frac{96}{5}\zeta_2^2-\frac{64}{9}\zeta_3-\frac{16}{3}L\zeta_3\right.\\
&+x\left(\frac{1504}{27}-\frac{1328L}{9}-\frac{136L^2}{3}-8L^3+\frac{344}{3}\zeta_2-24L\zeta_2+96\zeta_3\right)\\
&\left.\left.+x^2\left(-\frac{7375}{27}+\frac{16600L}{81}+\frac{1496L^2}{27}+\frac{116L^3}{27}+\frac{2L^4}{3}-\frac{3154}{27}\zeta_2+\frac{76}{9}L\zeta_2+4L^2\zeta_2-\frac{192}{5}\zeta_2^2-\frac{464}{9}\zeta_3-\frac{32}{3}L\zeta_3\right)\right]\right\}\\
&+C_F T_F\left\{\left[\frac{1628}{27}+\frac{224L}{27}+\frac{20L^2}{9}+\frac{4L^3}{9}-\frac{68}{3}\zeta_2+4L\zeta_2\right.\right.\\
&+x\left(-\frac{160}{9}+\frac{128L}{3}+8L^2-\frac{L^4}{3}+16\zeta_2-8L^2\zeta_2-\frac{64}{5}\zeta_2^2-48L\zeta_3\right)\\
&\left.+x^2\left(\frac{2192}{9}-\frac{1088L}{27}-\frac{416L^2}{9}-\frac{136L^3}{9}-\frac{4L^4}{3}+\frac{512}{3}\zeta_2+40L\zeta_2-32L^2\zeta_2-\frac{256}{5}\zeta_2^2-256\zeta_3-192L\zeta_3\right)\right]\\
&+\varepsilon\left[\frac{4214}{27}+\frac{64L}{9}+\frac{148L^2}{27}+\frac{40L^3}{27}+\frac{L^4}{3}-\frac{4028}{27}\zeta_2+2L^2\zeta_2+\frac{512}{3}\ln(2)\zeta_2-\frac{8}{5}\zeta_2^2-\frac{328}{3}\zeta_3-\frac{56}{9}L\zeta_3\right.\\
&+x\left(\frac{2368}{27}+\frac{1696L}{9}+\frac{344L^2}{3}+\frac{32L^3}{3}+\frac{L^4}{3}-\frac{4L^5}{15}-\frac{520}{3}\zeta_2-88L\zeta_2-8L^2\zeta_2-4L^3\zeta_2+192\ln(2)\zeta_2\right.\\
&\left.+\frac{96}{5}\zeta_2^2+8L\zeta_2^2+240\zeta_3+112L\zeta_3-52L^2\zeta_3-16\zeta_2\zeta_3+176\zeta_5\right)\\
&+x^2\left(\frac{2621}{27}+\frac{10172L}{27}-\frac{10276L^2}{27}-\frac{2116L^3}{27}-14L^4-\frac{16L^5}{15}+\frac{29510}{27}\zeta_2+148L\zeta_2+148L^2\zeta_2-16L^3\zeta_2\right.\\
&\left.\left.\left.-\frac{7040}{3}\ln(2)\zeta_2-\frac{1504}{5}\zeta_2^2+32L\zeta_2^2+\frac{3904}{9}\zeta_3-\frac{5296}{9}L\zeta_3-208L^2\zeta_3-64\zeta_2\zeta_3+704\zeta_5\right)\right]\right\}. \qquad (4.32)
\end{aligned}
$$

4. Pseudoscalar form factor

The nonsinglet part of the pseudoscalar form factor can be obtained in this limit using Eq. (2.13) as

$$
\mathcal{F}_P^{(n),\text{“ns”}}=\mathcal{F}_{A,1}^{(n),\text{“ns”}}+\left(-\frac{(1-x)^2}{4x}\right)\mathcal{F}_{A,2}^{(n),\text{“ns”}}\quad \text{for } n=1,2. \qquad (4.33)
$$

The unrenormalized singlet piece is given by

$$
\begin{aligned}
\hat{\mathcal{F}}_P^{(2),\text{“s”}}=C_F T_F&\left\{\left[x\left(8L^2-\frac{L^4}{3}+32\zeta_2-8L^2\zeta_2-\frac{64}{5}\zeta_2^2-48L\zeta_3\right)+x^2(112-64L+16L^2+16L^3+32\zeta_2+96L\zeta_2)\right]\right.\\
&+\varepsilon\left[x\left(16L^2+8L^3+L^4-\frac{4L^5}{15}+16L\zeta_2+8L^2\zeta_2-4L^3\zeta_2+\frac{224}{5}\zeta_2^2+8L\zeta_2^2+48\zeta_3+208L\zeta_3-52L^2\zeta_3\right.\right.\\
&\left.-16\zeta_2\zeta_3+176\zeta_5\right)+x^2\left(-400+480L-80L^2-\frac{56L^3}{3}+\frac{32L^4}{3}-64\zeta_2-160L\zeta_2-48L^2\zeta_2\right.\\
&\left.\left.\left.-\frac{176}{5}\zeta_2^2+32\zeta_3-128L\zeta_3\right)\right]\right\}. \qquad (4.34)
\end{aligned}
$$

C. Threshold region $q^2 \sim 4m^2$

Now we provide the expansion of the form factors in the threshold region $q^2 \sim 4m^2$ or $x \to -1$. We expand the form factors around $\beta = 0$ up to $\mathcal{O}(\beta^2)$ and denote the form factors by $\tilde{F}$ in this limit.

1. Vector form factor

First, we present the electric component of the vector form factor

$$
\begin{aligned}
\tilde{F}^{(1)}_{V,1} = C_F \Bigg\{ & \frac{1}{\varepsilon}\left[\frac{8\beta^2}{3} + i\pi\left(-\frac{1}{\beta} - \beta\right)\right] \\
& + \left[\frac{6\zeta_2}{\beta} - 6 + 6\beta\zeta_2 - \frac{10\beta^2}{9} + i\pi\left(\frac{1}{\beta}(-1 + 2\ln(2) + 2\log(\beta)) + \beta(-1 + 2\ln(2) + 2\log(\beta))\right)\right] \\
& + \varepsilon\Bigg[\frac{1}{\beta}(6\zeta_2 - 12\zeta_2\ln(2) - 12\zeta_2\log(\beta)) + 4 + \beta(6\zeta_2 - 12\zeta_2\ln(2) - 12\zeta_2\log(\beta)) + \beta^2\left(\frac{344}{27} + \frac{4\zeta_2}{3}\right) \\
& + i\pi\Bigg(\frac{1}{\beta}\left(-4 + \frac{3}{2}\zeta_2 + 2\ln(2) - 2\ln^2(2) + 2\log(\beta) - 4\ln(2)\log(\beta) - 2\log^2(\beta)\right) \\
& + \beta\left(-3 + \frac{3}{2}\zeta_2 + 2\ln(2) - 2\ln^2(2) + 2\log(\beta) - 4\ln(2)\log(\beta) - 2\log^2(\beta)\right)\Bigg)\Bigg] \\
& + \varepsilon^2\Bigg[\frac{1}{\beta}(24\zeta_2 + 3\zeta_2^2 - 12\zeta_2\ln(2) + 12\zeta_2\ln^2(2) - 12\zeta_2\log(\beta) + 24\zeta_2\ln(2)\log(\beta) + 12\zeta_2\log^2(\beta)) \\
& - 24 - 3\zeta_2 + \beta(18\zeta_2 + 3\zeta_2^2 - 12\zeta_2\ln(2) + 12\zeta_2\ln^2(2) - 12\zeta_2\log(\beta) + 24\zeta_2\ln(2)\log(\beta) + 12\zeta_2\log^2(\beta)) \\
& + \beta^2\left(-\frac{472}{81} - \frac{5\zeta_2}{9} - \frac{8\zeta_3}{9}\right) + i\pi\Bigg(\frac{1}{\beta}\Bigg(-8 + \frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3 + 8\ln(2) - 3\zeta_2\ln(2) - 2\ln^2(2) + \frac{4\ln^3(2)}{3} + 8\log(\beta) \\
& - 3\zeta_2\log(\beta) - 4\ln(2)\log(\beta) + 4\ln^2(2)\log(\beta) - 2\log^2(\beta) + 4\ln(2)\log^2(\beta) + \frac{4\log^3(\beta)}{3}\Bigg) \\
& + \beta\Bigg(-4 + \frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3 + 6\ln(2) - 3\zeta_2\ln(2) - 2\ln^2(2) + \frac{4\ln^3(2)}{3} + 6\log(\beta) - 3\zeta_2\log(\beta) \\
& - 4\ln(2)\log(\beta) + 4\ln^2(2)\log(\beta) - 2\log^2(\beta) + 4\ln(2)\log^2(\beta) + \frac{4\log^3(\beta)}{3}\Bigg)\Bigg)\Bigg]\Bigg\},
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
\tilde{F}^{(2)}_{V,1} = C_F^2 \Bigg\{ & \frac{1}{\varepsilon^2}\left[-\frac{3\zeta_2}{\beta^2} - 6\zeta_2 - 3\beta^2\zeta_2 + i\pi\left(-\frac{8\beta}{3}\right)\right] + \frac{1}{\varepsilon}\Bigg[\frac{1}{\beta^2}(-6\zeta_2 + 12\zeta_2\ln(2) + 12\zeta_2\log(\beta)) - 12\zeta_2 + 16\beta\zeta_2 \\
& + 24\zeta_2\ln(2) + 24\zeta_2\log(\beta) + \beta^2(-16 + 3\zeta_2 + 12\zeta_2\ln(2) + 12\zeta_2\log(\beta)) \\
& + i\pi\left(-\frac{6\zeta_2}{\beta^2} + \frac{6}{\beta} - 12\zeta_2 + \frac{8}{9}\beta(5 + 6\ln(2) + 6\log(\beta)) - 6\beta^2\zeta_2\right)\Bigg] + \Bigg[\frac{1}{\beta^2}(-28\zeta_2 + 15\zeta_2^2 + 24\zeta_2\ln(2) - 24\zeta_2\ln^2(2) \\
& + 24\zeta_2\log(\beta) - 48\zeta_2\ln(2)\log(\beta) - 24\zeta_2\log^2(\beta)) - \frac{36\zeta_2}{\beta} + \frac{421}{15} - \frac{1926}{25}\zeta_2 + 30\zeta_2^2 - \frac{81}{5}\zeta_3 + \frac{156}{5}\zeta_2\ln(2) - 48\zeta_2\ln^2(2) \\
& + \frac{56}{5}\zeta_2\log(\beta) - 96\zeta_2\ln(2)\log(\beta) - 48\zeta_2\log^2(\beta) + \beta\left(\frac{256}{3}\zeta_2 - 32\zeta_2\ln(2) - 32\zeta_2\log(\beta)\right) \\
& + \beta^2\left(\frac{691}{35} - \frac{51302\zeta_2}{3675} + 15\zeta_2^2 + \frac{1192}{35}\zeta_3 - \frac{108}{35}\zeta_2\ln(2) - 24\zeta_2\ln^2(2) + \frac{412}{35}\zeta_2\log(\beta) - 48\zeta_2\ln(2)\log(\beta) - 24\zeta_2\log^2(\beta)\right) \\
& + i\pi\Bigg(\frac{1}{\beta^2}(-12\zeta_2 + 24\zeta_2\ln(2) + 24\zeta_2\log(\beta)) - \frac{1}{\beta}(12(\ln(2) + \log(\beta))) - \frac{28}{5}\zeta_2 + 48\zeta_2\ln(2) + 48\zeta_2\log(\beta) \\
& + \beta\left(-\frac{2498}{27} + \frac{8}{3}\zeta_2 + \frac{448\ln(2)}{9} - \frac{16}{3}\ln^2(2) + \frac{256\log(\beta)}{9} - \frac{32}{3}\ln(2)\log(\beta) - \frac{16}{3}\log^2(\beta)\right) \\
& + \beta^2\left(-\frac{281}{35}\zeta_2 + 24\zeta_2\ln(2) + 24\zeta_2\log(\beta)\right)\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\bigg[\frac{1}{\beta^2}(-28\zeta_2+30\zeta_2^2-106\zeta_2\zeta_3+112\zeta_2\ln(2)-60\zeta_2^2\ln(2)-48\zeta_2\ln^2(2)+32\zeta_2\ln^3(2)\\
&+112\zeta_2\log(\beta)-60\zeta_2^2\log(\beta)-96\zeta_2\ln(2)\log(\beta)+96\zeta_2\ln^2(2)\log(\beta)-48\zeta_2\log^2(\beta)\\
&+96\zeta_2\ln(2)\log^2(\beta)+32\zeta_2\log^3(\beta))+\frac{1}{\beta}(72\ln(2)\zeta_2+72\log(\beta)\zeta_2)-\frac{36163}{450}-\frac{10c_1}{3}+\frac{157876}{375}\zeta_2\\
&-\frac{306}{5}\zeta_2^2-\frac{43756}{75}\zeta_3-212\zeta_2\zeta_3-\frac{32552}{75}\zeta_2\ln(2)-120\zeta_2^2\ln(2)+\frac{32}{5}\zeta_2\ln^2(2)+64\zeta_2\ln^3(2)\\
&-\frac{7144}{25}\zeta_2\log(\beta)-120\zeta_2^2\log(\beta)-\frac{224}{5}\zeta_2\ln(2)\log(\beta)+192\zeta_2\ln^2(2)\log(\beta)-\frac{112}{5}\zeta_2\log^2(\beta)\\
&+192\zeta_2\ln(2)\log^2(\beta)+64\zeta_2\log^3(\beta)+\beta\bigg(\frac{14416}{9}\zeta_2+16\zeta_2^2-\frac{3776}{3}\zeta_2\ln(2)+32\zeta_2\ln^2(2)\\
&-\frac{2624}{3}\zeta_2\log(\beta)+64\zeta_2\ln(2)\log(\beta)+32\zeta_2\log^2(\beta)\bigg)\\
&+\beta^2\bigg(\frac{1880267}{22050}+\frac{16c_1}{21}+\frac{61140199\zeta_2}{385875}+\frac{26498}{175}\zeta_2^2+\frac{286234\zeta_3}{3675}-106\zeta_2\zeta_3+\frac{800012\zeta_2\ln(2)}{3675}\\
&-60\zeta_2^2\ln(2)+\frac{208}{7}\zeta_2\ln^2(2)+32\zeta_2\ln^3(2)+\frac{175684\zeta_2\log(\beta)}{1225}-60\zeta_2^2\log(\beta)-\frac{2368}{35}\zeta_2\ln(2)\log(\beta)\\
&+96\zeta_2\ln^2(2)\log(\beta)-\frac{824}{35}\zeta_2\log^2(\beta)+96\zeta_2\ln(2)\log^2(\beta)+32\zeta_2\log^3(\beta)\bigg)\\
&+i\pi\bigg(\frac{1}{\beta^2}(-56\zeta_2-18\zeta_2^2+48\zeta_2\ln(2)-48\zeta_2\ln^2(2)+48\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta))\\
&+\frac{1}{\beta}(36-6\zeta_2+12\ln^2(2)+24\ln(2)\log(\beta)+12\log^2(\beta))+\frac{3572}{25}\zeta_2-36\zeta_2^2+\frac{112}{5}\zeta_2\ln(2)\\
&-96\zeta_2\ln^2(2)+\frac{112}{5}\zeta_2\log(\beta)-192\zeta_2\ln(2)\log(\beta)-96\zeta_2\log^2(\beta)\\
&+\beta\bigg(-\frac{49163}{81}+\frac{1064}{9}\zeta_2+\frac{64}{9}\zeta_3+\frac{19408\ln(2)}{27}-\frac{16}{3}\zeta_2\ln(2)-\frac{2656}{9}\ln^2(2)+\frac{32\ln^3(2)}{9}+\frac{14416\log(\beta)}{27}\\
&-\frac{16}{3}\zeta_2\log(\beta)-\frac{3776}{9}\ln(2)\log(\beta)+\frac{32}{3}\ln^2(2)\log(\beta)-\frac{1312}{9}\log^2(\beta)+\frac{32}{3}\ln(2)\log^2(\beta)+\frac{32\log^3(\beta)}{9}\bigg)\\
&+\beta^2\bigg(-\frac{62992\zeta_2}{1225}-18\zeta_2^2-\frac{76}{35}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{674}{35}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\bigg)\bigg)\bigg]\bigg\}\\
&+C_FC_A\bigg\{\frac{1}{\varepsilon^2}\bigg[-\frac{44\beta^2}{9}+i\pi\bigg(\frac{11}{6\beta}+\frac{11\beta}{6}\bigg)\bigg]+\frac{1}{\varepsilon}\bigg[-8\beta\zeta_2+\beta^2\bigg(\frac{376}{27}+\frac{32\zeta_2}{3}\bigg)+i\pi\bigg(-\frac{31}{18\beta}+\frac{1}{6}\beta(-13-32\ln(2)-16\log(\beta))\bigg)\bigg]\\
&+\bigg[\frac{1}{\beta}\bigg(\frac{146}{3}\zeta_2-44\zeta_2\ln(2)-44\zeta_2\log(\beta)\bigg)-\frac{379}{15}+\frac{5482}{75}\zeta_2-\frac{166}{5}\zeta_3-\frac{312}{5}\zeta_2\ln(2)\\
&-\frac{144}{5}\zeta_2\log(\beta)+\beta\bigg(-\frac{16}{3}\zeta_2+84\zeta_2\ln(2)+36\zeta_2\log(\beta)\bigg)+\beta^2\bigg(\frac{26779}{2835}+88\zeta_2-\frac{346}{105}\zeta_3-\frac{1952}{35}\zeta_2\ln(2)-48\zeta_2\log(\beta)\bigg)\\
&+i\pi\bigg(\frac{1}{\beta}\bigg(-\frac{394}{27}+\frac{146\ln(2)}{9}-\frac{22\ln^2(2)}{3}+\frac{146\log(\beta)}{9}-\frac{44}{3}\ln(2)\log(\beta)-\frac{22\log^2(\beta)}{3}\bigg)+\frac{72}{5}\zeta_2\\
&+\beta\bigg(-\frac{139}{9}-\frac{40}{3}\zeta_2-\frac{152\ln(2)}{9}+\frac{74\ln^2(2)}{3}-\frac{16\log(\beta)}{9}+28\ln(2)\log(\beta)+6\log^2(\beta)\bigg)+\frac{351}{14}\beta^2\zeta_2\bigg)\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\left[\frac{1}{\beta}\left(\frac{2422}{9}\zeta_2+99\zeta_2^2-\frac{752}{3}\zeta_2\ln(2)+132\zeta_2\ln^2(2)-\frac{752}{3}\zeta_2\log(\beta)+264\zeta_2\ln(2)\log(\beta)+132\zeta_2\log^2(\beta)\right)\right.\\
&-\frac{28301}{150}+\frac{28c_1}{5}+\frac{228613\zeta_2}{1125}-\frac{7748}{25}\zeta_2^2+\frac{8539}{75}\zeta_3-\frac{1076}{5}\zeta_2\ln(2)-\frac{256}{5}\zeta_2\ln^2(2)\\
&-\frac{4144}{25}\zeta_2\log(\beta)+\frac{576}{5}\zeta_2\ln(2)\log(\beta)+\frac{288}{5}\zeta_2\log^2(\beta)\\
&+\beta\left(\frac{976}{9}\zeta_2+59\zeta_2^2+\frac{1612}{3}\zeta_2\ln(2)-508\zeta_2\ln^2(2)+\frac{796}{3}\zeta_2\log(\beta)-632\zeta_2\ln(2)\log(\beta)-172\zeta_2\log^2(\beta)\right)\\
&+\beta^2\left(\frac{39802157}{198450}-\frac{284c_1}{105}+\frac{5048597\zeta_2}{9450}-\frac{209161\zeta_2^2}{1050}-\frac{244703\zeta_3}{7350}-\frac{323804\zeta_2\ln(2)}{3675}+\frac{264}{7}\zeta_2\ln^2(2)\right.\\
&\left.-\frac{1274}{5}\zeta_2\log(\beta)+144\zeta_2\ln(2)\log(\beta)+96\zeta_2\log^2(\beta)\right)\\
&+i\pi\left(\frac{1}{\beta}\left(-\frac{13177}{162}+\frac{38}{3}\zeta_2+\frac{22}{3}\zeta_3+\frac{2422\ln(2)}{27}-11\zeta_2\ln(2)-\frac{376}{9}\ln^2(2)+\frac{44\ln^3(2)}{3}+\frac{2422\log(\beta)}{27}-11\zeta_2\log(\beta)\right.\right.\\
&\left.-\frac{752}{9}\ln(2)\log(\beta)+44\ln^2(2)\log(\beta)-\frac{376}{9}\log^2(\beta)+44\ln(2)\log^2(\beta)+\frac{44\log^3(\beta)}{3}\right)\\
&+\frac{2072}{25}\zeta_2-\frac{288}{5}\zeta_2\ln(2)-\frac{288}{5}\zeta_2\log(\beta)+\beta\left(-\frac{2950}{27}-\frac{596}{9}\zeta_2-\frac{194}{3}\zeta_3-\frac{256\ln(2)}{27}+\frac{335}{3}\zeta_2\ln(2)\right.\\
&+150\ln^2(2)-\frac{764}{9}\ln^3(2)+\frac{976\log(\beta)}{27}+77\zeta_2\log(\beta)+\frac{1612}{9}\ln(2)\log(\beta)-\frac{508}{3}\ln^2(2)\log(\beta)\\
&\left.\left.\left.\left.+\frac{398\log^2(\beta)}{9}-\frac{316}{3}\ln(2)\log^2(\beta)-\frac{172}{9}\log^3(\beta)\right)+\beta^2\left(\frac{792}{5}\zeta_2-\frac{786}{7}\zeta_2\ln(2)-\frac{657}{7}\zeta_2\log(\beta)\right)\right)\right]\right\}\\
&+C_FT_Fn_l\left\{\frac{1}{\varepsilon^2}\left[\frac{16\beta^2}{9}+i\pi\left(-\frac{2}{3\beta}-\frac{2\beta}{3}\right)\right]+\frac{1}{\varepsilon}\left[-\frac{80\beta^2}{27}+i\pi\left(\frac{10}{9\beta}+\frac{10\beta}{9}\right)\right]+\left[\frac{1}{\beta}\left(-\frac{64}{3}\zeta_2+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\right)\right.\right.\\
&+4+\beta\left(-\frac{64}{3}\zeta_2+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\right)+\beta^2\left(-\frac{868}{81}-\frac{64\zeta_2}{9}\right)\\
&+i\pi\left(\frac{1}{\beta}\left(\frac{206}{27}-\frac{64\ln(2)}{9}+\frac{8\ln^2(2)}{3}-\frac{64\log(\beta)}{9}+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\right)\right.\\
&\left.\left.+\beta\left(\frac{116}{27}-\frac{64\ln(2)}{9}+\frac{8\ln^2(2)}{3}-\frac{64\log(\beta)}{9}+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\right)\right)\right]\\
&+\varepsilon\left[\frac{1}{\beta}\left(-\frac{1112}{9}\zeta_2-36\zeta_2^2+\frac{304}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{304}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\right)\right.\\
&+\frac{574}{9}+36\zeta_2+\beta\left(-\frac{680}{9}\zeta_2-36\zeta_2^2+\frac{304}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{304}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\right)\\
&+\beta^2\left(-\frac{518}{81}-\frac{220\zeta_2}{27}-\frac{128\zeta_3}{9}\right)+i\pi\left(\frac{1}{\beta}\left(\frac{3211}{81}-\frac{16}{3}\zeta_2-\frac{8}{3}\zeta_3-\frac{1112\ln(2)}{27}\right.\right.\\
&+4\zeta_2\ln(2)+\frac{152\ln^2(2)}{9}-\frac{16}{3}\ln^3(2)-\frac{1112\log(\beta)}{27}\\
&\left.+4\zeta_2\log(\beta)+\frac{304}{9}\ln(2)\log(\beta)-16\ln^2(2)\log(\beta)+\frac{152\log^2(\beta)}{9}-16\ln(2)\log^2(\beta)-\frac{16}{3}\log^3(\beta)\right)\\
&+\beta\left(\frac{934}{81}-\frac{16}{3}\zeta_2-\frac{8}{3}\zeta_3-\frac{680\ln(2)}{27}+4\zeta_2\ln(2)+\frac{152\ln^2(2)}{9}-\frac{16}{3}\ln^3(2)-\frac{680\log(\beta)}{27}+4\zeta_2\log(\beta)\right.\\
&\left.\left.\left.\left.+\frac{304}{9}\ln(2)\log(\beta)-16\ln^2(2)\log(\beta)+\frac{152\log^2(\beta)}{9}-16\ln(2)\log^2(\beta)-\frac{16}{3}\log^3(\beta)\right)\right)\right]\right\}
\end{aligned}
$$

$$+C_F T_F\left\{\left[\frac{148}{3}-\frac{416}{15}\zeta_2+\beta^2\left(-\frac{1372}{81}+\frac{52\zeta_2}{5}\right)+i\pi\left(\frac{2\zeta_2}{3\beta}+\beta\left(\frac{16}{15}+\frac{2\zeta_2}{3}\right)\right)\right]\right.$$
$$+\varepsilon\left[-\frac{4\zeta_2^2}{\beta}+\frac{2102}{15}-\frac{28616}{225}\zeta_2-\frac{1456}{15}\zeta_3+\frac{832}{5}\zeta_2\ln(2)+\beta\left(-\frac{32\zeta_2}{5}-4\zeta_2^2\right)\right.$$
$$+\beta^2\left(-\frac{47902}{1215}+\frac{475514\zeta_2}{11025}+\frac{5104}{135}\zeta_3-\frac{6232}{105}\zeta_2\ln(2)+\frac{24}{7}\zeta_2\log(\beta)\right)+i\pi\left(\frac{1}{\beta}\left(\frac{2}{3}\zeta_2-\frac{4}{9}\zeta_3-\frac{4}{3}\zeta_2\ln(2)-\frac{4}{3}\zeta_2\log(\beta)\right)\right.$$
$$\left.\left.\left.+\beta\left(\frac{16}{15}+\frac{2}{3}\zeta_2-\frac{4}{9}\zeta_3-\frac{32\ln(2)}{15}-\frac{4}{3}\zeta_2\ln(2)-\frac{32\log(\beta)}{15}-\frac{4}{3}\zeta_2\log(\beta)\right)\right)\right]\right\}. \quad (4.36)$$

Next, we present the magnetic component of the vector form factor

$$\tilde{F}_{V,2}^{(1)}=C_F\left\{-2+\frac{4\beta^2}{3}+i\pi\left(\frac{1}{\beta}-\beta\right)+\varepsilon\left[-\frac{6\zeta_2}{\beta}-4+6\beta\zeta_2+\frac{28\beta^2}{9}\right.\right.$$
$$\left.+i\pi\left(-\frac{1}{\beta}(2(-2+\ln(2)+\log(\beta)))+\beta(-5+2\ln(2)+2\log(\beta))\right)\right]$$
$$+\varepsilon^2\left[\frac{1}{\beta}(-24\zeta_2+12\zeta_2\ln(2)+12\zeta_2\log(\beta))-8-\zeta_2+\beta(30\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta))+\beta^2\left(\frac{160}{27}+\frac{2\zeta_2}{3}\right)\right.$$
$$+i\pi\left(\frac{1}{\beta}\left(8-\frac{3}{2}\zeta_2-8\ln(2)+2\ln^2(2)-8\log(\beta)+4\ln(2)\log(\beta)+2\log^2(\beta)\right)\right.$$
$$\left.\left.\left.+\beta\left(-12+\frac{3}{2}\zeta_2+10\ln(2)-2\ln^2(2)+10\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\right)\right)\right]\right\}. \quad (4.37)$$

$$\tilde{F}_{V,2}^{(2)}=C_F^2\left\{\frac{1}{\varepsilon}\left[\frac{6\zeta_2}{\beta^2}+\beta^2\left(-\frac{16}{3}-6\zeta_2\right)+i\pi\left(\frac{2}{\beta}+\frac{10\beta}{3}\right)\right]+\left[\frac{1}{\beta^2}(28\zeta_2-24\zeta_2\ln(2)-24\zeta_2\log(\beta))+\frac{269}{15}-\frac{4922}{75}\zeta_2\right.\right.$$
$$-\frac{12\zeta_2}{\beta}-60\beta\zeta_2+\frac{41}{5}\zeta_3+\frac{404}{5}\zeta_2\ln(2)+\frac{24}{5}\zeta_2\log(\beta)+\beta^2\left(\frac{3658}{315}+\frac{103928\zeta_2}{3675}-\frac{471}{35}\zeta_3-\frac{1628}{35}\zeta_2\ln(2)+\frac{512}{35}\zeta_2\log(\beta)\right)$$
$$\left.+i\pi\left(\frac{12\zeta_2}{\beta^2}-\frac{1}{\beta}(4(\ln(2)+\log(\beta)))-\frac{12}{5}\zeta_2-\frac{2}{9}\beta(-73+90\ln(2)+90\log(\beta))-\frac{181}{35}\beta^2\zeta_2\right)\right]$$
$$+\varepsilon\left[\frac{1}{\beta^2}(28\zeta_2-30\zeta_2^2-112\zeta_2\ln(2)+48\zeta_2\ln^2(2)-112\zeta_2\log(\beta)+96\zeta_2\ln(2)\log(\beta)+48\zeta_2\log^2(\beta))\right.$$
$$+\frac{1}{\beta}(24\ln(2)\zeta_2+24\log(\beta)\zeta_2)+\frac{20713}{450}-\frac{38c_1}{3}-\frac{529628\zeta_2}{1125}+\frac{1342}{5}\zeta_2^2-\frac{15944}{75}\zeta_3+\frac{27752}{75}\zeta_2\ln(2)+\frac{448}{5}\zeta_2\ln^2(2)$$
$$+\frac{344}{25}\zeta_2\log(\beta)-\frac{96}{5}\zeta_2\ln(2)\log(\beta)-\frac{48}{5}\zeta_2\log^2(\beta)+\beta\left(-\frac{200}{3}\zeta_2+280\zeta_2\ln(2)+280\zeta_2\log(\beta)\right)$$
$$+\beta^2\left(-\frac{4863529}{33075}+\frac{250c_1}{21}+\frac{115201589\zeta_2}{385875}-\frac{42283}{175}\zeta_2^2+\frac{330121\zeta_3}{1225}-\frac{749248\zeta_2\ln(2)}{3675}-\frac{3672}{35}\zeta_2\ln^2(2)\right.$$
$$\left.+\frac{245324\zeta_2\log(\beta)}{1225}-\frac{1328}{35}\zeta_2\ln(2)\log(\beta)-\frac{1024}{35}\zeta_2\log^2(\beta)\right)+i\pi\left(\frac{1}{\beta^2}(56\zeta_2-48\zeta_2\ln(2)-48\zeta_2\log(\beta))\right.$$
$$+\frac{1}{\beta}(-4-2\zeta_2+4\ln^2(2)+8\ln(2)\log(\beta)+4\log^2(\beta))-\frac{172}{25}\zeta_2+\frac{48}{5}\zeta_2\ln(2)+\frac{48}{5}\zeta_2\log(\beta)$$
$$+\beta\left(-\frac{241}{9}-\frac{50}{3}\zeta_2-\frac{8\ln(2)}{9}+\frac{140\ln^2(2)}{3}-\frac{200\log(\beta)}{9}+\frac{280}{3}\ln(2)\log(\beta)+\frac{140\log^2(\beta)}{3}\right)$$
$$\left.\left.\left.+\beta^2\left(-\frac{147512\zeta_2}{1225}+\frac{1924}{35}\zeta_2\ln(2)+\frac{1174}{35}\zeta_2\log(\beta)\right)\right)\right]\right\}$$

$$
\begin{aligned}
&+C_FC_A\left\{\left[-\frac{28\zeta_2}{\beta}-\frac{373}{45}+\frac{1156}{25}\zeta_2-\frac{94}{5}\zeta_3-\frac{328}{5}\zeta_2\ln(2)-\frac{96}{5}\zeta_2\log(\beta)+60\beta\zeta_2\right.\right.\\
&+\beta^2\left(-\frac{1234}{315}-\frac{1577}{25}\zeta_2+\frac{288}{35}\zeta_3+\frac{2344}{35}\zeta_2\ln(2)+\frac{32}{5}\zeta_2\log(\beta)\right)+i\pi\left(-\frac{1}{9\beta}(4(-25+21\ln(2)+21\log(\beta)))+\frac{48}{5}\zeta_2\right.\\
&\left.\left.+\frac{2}{3}\beta(-49+38\ln(2)+30\log(\beta))-\frac{299}{70}\beta^2\zeta_2\right)\right]+\varepsilon\left[\frac{1}{\beta}\left(-\frac{682}{3}\zeta_2+168\zeta_2\ln(2)+168\zeta_2\log(\beta)\right)\right.\\
&-\frac{181841}{1350}+\frac{116c_1}{15}+\frac{104129}{375}\zeta_2-\frac{6652}{25}\zeta_2^2+\frac{6236}{75}\zeta_3-\frac{4112}{15}\zeta_2\ln(2)-\frac{224}{5}\zeta_2\ln^2(2)-\frac{1856}{25}\zeta_2\log(\beta)\\
&+\frac{384}{5}\zeta_2\ln(2)\log(\beta)+\frac{192}{5}\zeta_2\log^2(\beta)+\beta\left(\frac{1822}{3}\zeta_2-520\zeta_2\ln(2)-424\zeta_2\log(\beta)\right)\\
&+\beta^2\left(\frac{16374871}{99225}-\frac{128c_1}{21}-\frac{975547\zeta_2}{5250}+\frac{82843}{350}\zeta_2^2-\frac{458029\zeta_3}{3675}-\frac{71626\zeta_2\ln(2)}{3675}\right.\\
&\left.+\frac{4056}{35}\zeta_2\ln^2(2)-\frac{638}{25}\zeta_2\log(\beta)+\frac{112}{5}\zeta_2\ln(2)\log(\beta)-\frac{64}{5}\zeta_2\log^2(\beta)\right)\\
&+i\pi\left(\frac{1}{\beta}\left(\frac{4015}{54}-\frac{15}{2}\zeta_2-\frac{682\ln(2)}{9}+28\ln^2(2)-\frac{682\log(\beta)}{9}+56\ln(2)\log(\beta)+28\log^2(\beta)\right)+\frac{928}{25}\zeta_2-\frac{192}{5}\zeta_2\ln(2)\right.\\
&-\frac{192}{5}\zeta_2\log(\beta)+\beta\left(-\frac{12113}{54}+\frac{79}{2}\zeta_2+\frac{2270\ln(2)}{9}-108\ln^2(2)+\frac{1822\log(\beta)}{9}-\frac{520}{3}\ln(2)\log(\beta)-\frac{212}{3}\log^2(\beta)\right)\\
&\left.\left.\left.+\beta^2\left(-\frac{456}{25}\zeta_2+\frac{1018}{35}\zeta_2\ln(2)+\frac{373}{35}\zeta_2\log(\beta)\right)\right)\right]\right\}\\
&+C_FT_Fn_l\left\{\left[\frac{8\zeta_2}{\beta}+\frac{52}{9}-8\beta\zeta_2-\frac{40\beta^2}{9}+i\pi\left(\frac{1}{9\beta}(2(-25+12\ln(2)+12\log(\beta)))-\frac{2}{9}\beta(-31+12\ln(2)+12\log(\beta))\right)\right]\right.\\
&+\varepsilon\left[\frac{1}{\beta}\left(\frac{296}{3}\zeta_2-48\zeta_2\ln(2)-48\zeta_2\log(\beta)\right)+\frac{1010}{27}+12\zeta_2+\beta\left(-\frac{368}{3}\zeta_2+48\zeta_2\ln(2)+48\zeta_2\log(\beta)\right)\right.\\
&+\beta^2\left(-\frac{2228}{81}-8\zeta_2\right)+i\pi\left(\frac{1}{\beta}\left(-\frac{961}{27}+2\zeta_2+\frac{296\ln(2)}{9}-8\ln^2(2)+\frac{296\log(\beta)}{9}-16\ln(2)\log(\beta)-8\log^2(\beta)\right)\right.\\
&\left.\left.\left.+\beta\left(\frac{1405}{27}-2\zeta_2-\frac{368\ln(2)}{9}+8\ln^2(2)-\frac{368\log(\beta)}{9}+16\ln(2)\log(\beta)+8\log^2(\beta)\right)\right)\right]\right\}\\
&+C_FT_F\left\{\left[-\frac{92}{9}+\frac{32}{5}\zeta_2+\beta^2\left(\frac{248}{27}-\frac{36\zeta_2}{5}\right)\right]+\varepsilon\left[\frac{82}{135}+\frac{324}{25}\zeta_2+\frac{112}{5}\zeta_3-\frac{192}{5}\zeta_2\ln(2)\right.\right.\\
&\left.\left.+\beta^2\left(\frac{772}{405}-\frac{171322\zeta_2}{11025}-\frac{96}{5}\zeta_3+\frac{1032}{35}\zeta_2\ln(2)-\frac{24}{7}\zeta_2\log(\beta)\right)+i\pi\left(-\frac{2\zeta_2}{3\beta}+\beta\left(\frac{16}{15}+\frac{2\zeta_2}{3}\right)\right)\right]\right\}. \qquad (4.38)
\end{aligned}
$$

2. Axial-vector form factor

In the following, we provide the nonsinglet part of the axial-vector form factors in the threshold region

$$
\begin{aligned}
\tilde{F}_{A,1}^{(1),\text{“ns”}}=C_F&\left\{\frac{1}{\varepsilon}\left[\frac{8\beta^2}{3}+i\pi\left(-\frac{1}{\beta}-\beta\right)\right]+\left[\frac{6\zeta_2}{\beta}-4+6\beta\zeta_2-\frac{22\beta^2}{9}+i\pi\left(\frac{1}{\beta}(2(-1+\ln(2)+\log(\beta)))+2\beta(\ln(2)+\log(\beta))\right)\right]\right.\\
&+\varepsilon\left[\frac{1}{\beta}(12\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta))+\beta(-12\ln(2)\zeta_2-12\log(\beta)\zeta_2)+\beta^2\left(\frac{404}{27}+\frac{4\zeta_2}{3}\right)\right.\\
&+i\pi\left(\frac{1}{\beta}\left(-4+\frac{3}{2}\zeta_2+4\ln(2)-2\ln^2(2)+4\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\right)\right.\\
&\left.\left.+\beta\left(-2+\frac{3}{2}\zeta_2-2\ln^2(2)-4\ln(2)\log(\beta)-2\log^2(\beta)\right)\right)\right]
\end{aligned}
$$

$$+\varepsilon^2\Bigg[\frac{1}{\beta}(24\zeta_2+3\zeta_2^2-24\zeta_2\ln(2)+12\zeta_2\ln^2(2)-24\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta))-16-2\zeta_2+\beta(12\zeta_2+3\zeta_2^2$$
$$+12\zeta_2\ln^2(2)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta))+\beta^2\left(-\frac{808}{81}-\frac{11\zeta_2}{9}-\frac{8\zeta_3}{9}\right)$$
$$+i\pi\Bigg(\frac{1}{\beta}\Bigg(-8+3\zeta_2+\frac{7}{3}\zeta_3+8\ln(2)-3\zeta_2\ln(2)-4\ln^2(2)+\frac{4\ln^3(2)}{3}+8\log(\beta)$$
$$-3\zeta_2\log(\beta)-8\ln(2)\log(\beta)+4\ln^2(2)\log(\beta)-4\log^2(\beta)+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\Bigg)$$
$$+\beta\Bigg(-4+\frac{7}{3}\zeta_3+4\ln(2)-3\zeta_2\ln(2)+\frac{4\ln^3(2)}{3}+4\log(\beta)-3\zeta_2\log(\beta)+4\ln^2(2)\log(\beta)$$
$$+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\Bigg)\Bigg)\Bigg]\Bigg\}. \tag{4.39}$$

$$\tilde{F}_{A,1}^{(2),\text{“ns”}}=C_F^2\Bigg\{\frac{1}{\varepsilon^2}\left[-\frac{3\zeta_2}{\beta^2}-6\zeta_2-3\beta^2\zeta_2+i\pi\left(-\frac{8\beta}{3}\right)\right]+\frac{1}{\varepsilon}\Bigg[\frac{1}{\beta^2}(-12\zeta_2+12\zeta_2\ln(2)+12\zeta_2\log(\beta))-12\zeta_2$$
$$+16\beta\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta)+\beta^2\left(-\frac{32}{3}+9\zeta_2+12\zeta_2\ln(2)+12\zeta_2\log(\beta)\right)$$
$$+i\pi\left(-\frac{6\zeta_2}{\beta^2}+\frac{4}{\beta}-12\zeta_2+\frac{2}{9}\beta(5+24\ln(2)+24\log(\beta))-6\beta^2\zeta_2\right)\Bigg]$$
$$+\Bigg[\frac{1}{\beta^2}(-24\zeta_2+15\zeta_2^2+48\zeta_2\ln(2)-24\zeta_2\ln^2(2)+48\zeta_2\log(\beta)-48\zeta_2\ln(2)\log(\beta)$$
$$-24\zeta_2\log^2(\beta))+\frac{46}{3}-32\zeta_2-\frac{24\zeta_2}{\beta}+30\zeta_2^2-27\zeta_3+84\zeta_2\ln(2)-48\zeta_2\ln^2(2)+8\zeta_2\log(\beta)$$
$$-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)+\beta\left(-\frac{20}{3}\zeta_2-32\zeta_2\ln(2)-32\zeta_2\log(\beta)\right)$$
$$+\beta^2\left(\frac{247}{9}-\frac{1508}{75}\zeta_2+15\zeta_2^2+50\zeta_3-\frac{172}{5}\zeta_2\ln(2)-24\zeta_2\ln^2(2)-\frac{4}{5}\zeta_2\log(\beta)-48\zeta_2\ln(2)\log(\beta)-24\zeta_2\log^2(\beta)\right)$$
$$+i\pi\Bigg(\frac{1}{\beta^2}(-24\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta))-\frac{1}{\beta}(8(-1+\ln(2)+\log(\beta)))-4\zeta_2+48\zeta_2\ln(2)$$
$$+48\zeta_2\log(\beta)+\beta\left(-\frac{956}{27}+\frac{8}{3}\zeta_2-\frac{20\ln(2)}{9}-\frac{16}{3}\ln^2(2)-\frac{20\log(\beta)}{9}-\frac{32}{3}\ln(2)\log(\beta)-\frac{16}{3}\log^2(\beta)\right)$$
$$+\beta^2\left(\frac{2}{5}\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta)\right)\Bigg)\Bigg]$$
$$+\varepsilon\Bigg[\frac{1}{\beta^2}(48\zeta_2+60\zeta_2^2-106\zeta_2\zeta_3+96\zeta_2\ln(2)-60\zeta_2^2\ln(2)-96\zeta_2\ln^2(2)+32\zeta_2\ln^3(2)+96\zeta_2\log(\beta)$$
$$-60\zeta_2^2\log(\beta)-192\zeta_2\ln(2)\log(\beta)+96\zeta_2\ln^2(2)\log(\beta)-96\zeta_2\log^2(\beta)+96\zeta_2\ln(2)\log^2(\beta)+32\zeta_2\log^3(\beta))$$
$$+\frac{1}{\beta}(-48\zeta_2+48\zeta_2\ln(2)+48\zeta_2\log(\beta))-\frac{313}{9}-\frac{38c_1}{3}+\frac{940}{3}\zeta_2+\frac{282}{5}\zeta_2^2-\frac{1202}{3}\zeta_3-212\zeta_2\zeta_3$$
$$-\frac{424}{3}\zeta_2\ln(2)-120\zeta_2^2\ln(2)+32\zeta_2\ln^2(2)+64\zeta_2\ln^3(2)-304\zeta_2\log(\beta)-120\zeta_2^2\log(\beta)$$
$$-32\zeta_2\ln(2)\log(\beta)+192\zeta_2\ln^2(2)\log(\beta)-16\zeta_2\log^2(\beta)+192\zeta_2\ln(2)\log^2(\beta)+64\zeta_2\log^3(\beta)$$
$$+\beta\left(\frac{3964}{9}\zeta_2+16\zeta_2^2+\frac{40}{3}\zeta_2\ln(2)+32\zeta_2\ln^2(2)+\frac{40}{3}\zeta_2\log(\beta)+64\zeta_2\ln(2)\log(\beta)+32\zeta_2\log^2(\beta)\right)$$

$$+\beta^2\left(-\frac{35237}{270}+\frac{28c_1}{5}-\frac{117812\zeta_2}{1125}+\frac{3639}{25}\zeta_2^2+\frac{724}{15}\zeta_3-106\zeta_2\zeta_3-\frac{624}{5}\zeta_2\ln(2)-60\zeta_2^2\ln(2)+\frac{488}{5}\zeta_2\ln^2(2)\right.$$
$$+32\zeta_2\ln^3(2)+\frac{3696}{25}\zeta_2\log(\beta)-60\zeta_2^2\log(\beta)+\frac{16}{5}\zeta_2\ln(2)\log(\beta)+96\zeta_2\ln^2(2)\log(\beta)$$
$$\left.+\frac{8}{5}\zeta_2\log^2(\beta)+96\zeta_2\ln(2)\log^2(\beta)+32\zeta_2\log^3(\beta)\right)$$
$$+i\pi\left(\frac{1}{\beta^2}(-48\zeta_2-18\zeta_2^2+96\zeta_2\ln(2)-48\zeta_2\ln^2(2)+96\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta))\right.$$
$$+\frac{1}{\beta}(32-4\zeta_2-16\ln(2)+8\ln^2(2)-16\log(\beta)+16\ln(2)\log(\beta)+8\log^2(\beta))+152\zeta_2-36\zeta_2^2+16\zeta_2\ln(2)$$
$$-96\zeta_2\ln^2(2)+16\zeta_2\log(\beta)-192\zeta_2\ln(2)\log(\beta)-96\zeta_2\log^2(\beta)$$
$$+\beta\left(-\frac{18635}{81}-\frac{10}{9}\zeta_2+\frac{64}{9}\zeta_3+\frac{5116\ln(2)}{27}-\frac{16}{3}\zeta_2\ln(2)+\frac{20\ln^2(2)}{9}+\frac{32\ln^3(2)}{9}+\frac{3964\log(\beta)}{27}-\frac{16}{3}\zeta_2\log(\beta)\right.$$
$$\left.+\frac{40}{9}\ln(2)\log(\beta)+\frac{32}{3}\ln^2(2)\log(\beta)+\frac{20\log^2(\beta)}{9}+\frac{32}{3}\ln(2)\log^2(\beta)+\frac{32\log^3(\beta)}{9}\right)$$
$$\left.\left.\left.+\beta^2\left(-\frac{1848}{25}\zeta_2-18\zeta_2^2-\frac{8}{5}\zeta_2\ln(2)-48\zeta_2\ln^2(2)-\frac{8}{5}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\right)\right)\right]\right\}$$
$$+C_FC_A\left\{\frac{1}{\varepsilon^2}\left[-\frac{44\beta^2}{9}+i\pi\left(\frac{11}{6\beta}+\frac{11\beta}{6}\right)\right]+\frac{1}{\varepsilon}\left[-8\beta\zeta_2+\beta^2\left(\frac{376}{27}+\frac{32\zeta_2}{3}\right)\right.\right.$$
$$\left.+i\pi\left(-\frac{31}{18\beta}+\frac{1}{6}\beta(-13-32\ln(2)-16\log(\beta))\right)\right]+\left[\frac{1}{\beta}\left(\frac{194}{3}\zeta_2-44\zeta_2\ln(2)-44\zeta_2\log(\beta)\right)\right.$$
$$-\frac{202}{9}+\frac{178}{3}\zeta_2-18\zeta_3-72\zeta_2\ln(2)-16\zeta_2\log(\beta)+\beta\left(\frac{8}{3}\zeta_2+84\zeta_2\ln(2)+36\zeta_2\log(\beta)\right)$$
$$+\beta^2\left(\frac{1159}{405}+\frac{6263}{75}\zeta_2-\frac{346}{15}\zeta_3-\frac{216}{5}\zeta_2\ln(2)-\frac{336}{5}\zeta_2\log(\beta)\right)$$
$$+i\pi\left(\frac{1}{\beta}\left(-\frac{478}{27}+\frac{194\ln(2)}{9}-\frac{22\ln^2(2)}{3}+\frac{194\log(\beta)}{9}-\frac{44}{3}\ln(2)\log(\beta)-\frac{22\log^2(\beta)}{3}\right)\right.$$
$$\left.\left.+8\zeta_2+\beta\left(-\frac{143}{9}-\frac{40}{3}\zeta_2-\frac{80\ln(2)}{9}+\frac{74\ln^2(2)}{3}+\frac{8\log(\beta)}{9}+28\ln(2)\log(\beta)+6\log^2(\beta)\right)+\frac{168}{5}\beta^2\zeta_2\right)\right]$$
$$+\varepsilon\left[\frac{1}{\beta}\left(\frac{2704}{9}\zeta_2+99\zeta_2^2-\frac{1040}{3}\zeta_2\ln(2)+132\zeta_2\ln^2(2)-\frac{1040}{3}\zeta_2\log(\beta)+264\zeta_2\ln(2)\log(\beta)+132\zeta_2\log^2(\beta)\right)\right.$$
$$-\frac{4103}{27}+\frac{28c_1}{3}+\frac{2294}{9}\zeta_2-\frac{1404}{5}\zeta_2^2+73\zeta_3-\frac{940}{3}\zeta_2\ln(2)-64\zeta_2\ln^2(2)-144\zeta_2\log(\beta)+64\zeta_2\ln(2)\log(\beta)$$
$$+32\zeta_2\log^2(\beta)+\beta\left(\frac{1702}{9}\zeta_2+59\zeta_2^2+\frac{988}{3}\zeta_2\ln(2)-508\zeta_2\ln^2(2)+\frac{460}{3}\zeta_2\log(\beta)-632\zeta_2\ln(2)\log(\beta)-172\zeta_2\log^2(\beta)\right)$$
$$+\beta^2\left(\frac{361877}{1350}-4c_1+\frac{2123531\zeta_2}{3375}-\frac{17228}{75}\zeta_2^2-\frac{8419}{150}\zeta_3-\frac{6966}{25}\zeta_2\ln(2)+\frac{576}{5}\zeta_2\ln^2(2)\right.$$
$$\left.-\frac{9536}{25}\zeta_2\log(\beta)+\frac{1344}{5}\zeta_2\ln(2)\log(\beta)+\frac{672}{5}\zeta_2\log^2(\beta)\right)$$
$$+i\pi\left(\frac{1}{\beta}\left(-\frac{6338}{81}+\frac{97}{6}\zeta_2+\frac{22}{3}\zeta_3+\frac{2704\ln(2)}{27}-11\zeta_2\ln(2)-\frac{520}{9}\ln^2(2)+\frac{44\ln^3(2)}{3}+\frac{2704\log(\beta)}{27}-11\zeta_2\log(\beta)\right.\right.$$
$$\left.-\frac{1040}{9}\ln(2)\log(\beta)+44\ln^2(2)\log(\beta)-\frac{520}{9}\log^2(\beta)+44\ln(2)\log^2(\beta)+\frac{44\log^3(\beta)}{3}\right)$$

$$+72\zeta_2-32\zeta_2\ln(2)-32\zeta_2\log(\beta)+\beta\Bigg(-\frac{9149}{54}-\frac{727}{18}\zeta_2-\frac{194}{3}\zeta_3+\frac{1238\ln(2)}{27}+\frac{335}{3}\zeta_2\ln(2)+94\ln^2(2)$$
$$-\frac{764}{9}\ln^3(2)+\frac{1702\log(\beta)}{27}+77\zeta_2\log(\beta)+\frac{988}{9}\ln(2)\log(\beta)-\frac{508}{3}\ln^2(2)\log(\beta)$$
$$+\frac{230\log^2(\beta)}{9}-\frac{316}{3}\ln(2)\log^2(\beta)-\frac{172}{9}\log^3(\beta)\Bigg)+\beta^2\left(\frac{4768}{25}\zeta_2-\frac{672}{5}\zeta_2\ln(2)-\frac{672}{5}\zeta_2\log(\beta)\right)\Bigg)\Bigg]\Bigg\}$$
$$+C_FT_Fn_l\Bigg\{\frac{1}{\varepsilon^2}\left[\frac{16\beta^2}{9}+i\pi\left(-\frac{2}{3\beta}-\frac{2\beta}{3}\right)\right]+\frac{1}{\varepsilon}\left[-\frac{80\beta^2}{27}+i\pi\left(\frac{10}{9\beta}+\frac{10\beta}{9}\right)\right]+\Bigg[\frac{1}{\beta}\left(-\frac{88}{3}\zeta_2+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\right)$$
$$+\frac{56}{9}+\beta\left(-\frac{40}{3}\zeta_2+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\right)+\beta^2\left(-\frac{940}{81}-\frac{64\zeta_2}{9}\right)$$
$$+i\pi\Bigg(\frac{1}{\beta}\left(\frac{248}{27}-\frac{88\ln(2)}{9}+\frac{8\ln^2(2)}{3}-\frac{88\log(\beta)}{9}+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\right)$$
$$+\beta\left(\frac{38}{27}-\frac{40\ln(2)}{9}+\frac{8\ln^2(2)}{3}-\frac{40\log(\beta)}{9}+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\right)\Bigg)\Bigg]$$
$$+\varepsilon\Bigg[\frac{1}{\beta}\left(-\frac{1280}{9}\zeta_2-36\zeta_2^2+\frac{448}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{448}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\right)$$
$$+\frac{1228}{27}+24\zeta_2+\beta\left(-\frac{296}{9}\zeta_2-36\zeta_2^2+\frac{160}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{160}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\right)$$
$$+\beta^2\left(\frac{22}{9}-\frac{4\zeta_2}{27}-\frac{128\zeta_3}{9}\right)+i\pi\Bigg(\frac{1}{\beta}\Bigg(\frac{3160}{81}-\frac{22}{3}\zeta_2-\frac{8}{3}\zeta_3-\frac{1280\ln(2)}{27}+4\zeta_2\ln(2)+\frac{224\ln^2(2)}{9}-\frac{16}{3}\ln^3(2)$$
$$-\frac{1280\log(\beta)}{27}+4\zeta_2\log(\beta)+\frac{448}{9}\ln(2)\log(\beta)-16\ln^2(2)\log(\beta)+\frac{224\log^2(\beta)}{9}-16\ln(2)\log^2(\beta)-\frac{16}{3}\log^3(\beta)\Bigg)$$
$$+\beta\Bigg(\frac{733}{81}-\frac{10}{3}\zeta_2-\frac{8}{3}\zeta_3-\frac{296\ln(2)}{27}+4\zeta_2\ln(2)+\frac{80\ln^2(2)}{9}-\frac{16}{3}\ln^3(2)-\frac{296\log(\beta)}{27}$$
$$+4\zeta_2\log(\beta)+\frac{160}{9}\ln(2)\log(\beta)-16\ln^2(2)\log(\beta)+\frac{80\log^2(\beta)}{9}-16\ln(2)\log^2(\beta)-\frac{16}{3}\log^3(\beta)\Bigg)\Bigg)\Bigg]\Bigg\}$$
$$+C_FT_F\Bigg\{\left[\frac{320}{9}-\frac{64}{3}\zeta_2+\beta^2\left(-\frac{388}{81}+\frac{16\zeta_2}{5}\right)+i\pi\left(\frac{2\zeta_2}{3\beta}+\frac{2}{3}\beta\zeta_2\right)\right]$$
$$+\varepsilon\Bigg[-\frac{4\zeta_2^2}{\beta}+\frac{2944}{27}-\frac{884}{9}\zeta_2-\frac{224}{3}\zeta_3+128\zeta_2\ln(2)-4\beta\zeta_2^2+\beta^2\left(-\frac{18106}{1215}+\frac{1256}{75}\zeta_2+\frac{2512}{135}\zeta_3-\frac{448}{15}\zeta_2\ln(2)\right)$$
$$+i\pi\left(\frac{1}{\beta}\left(\frac{4}{3}\zeta_2-\frac{4}{9}\zeta_3-\frac{4}{3}\zeta_2\ln(2)-\frac{4}{3}\zeta_2\log(\beta)\right)+\beta\left(-\frac{4}{9}\zeta_3-\frac{4}{3}\zeta_2\ln(2)-\frac{4}{3}\zeta_2\log(\beta)\right)\right)\Bigg]\Bigg\}. \quad (4.40)$$

The unrenormalized singlet part is given by

$$\tilde{\tilde{F}}_{A,1}^{(2),\text{"s"}}=C_FT_F\Bigg\{-\frac{6}{\varepsilon}-\frac{6}{\beta^2}-\frac{53}{3}-\frac{76}{3}\zeta_2+64\zeta_2\ln(2)+\beta^2\left(\frac{32}{3}+\frac{44}{15}\zeta_2-\frac{256}{5}\zeta_2\ln(2)\right)$$
$$+\varepsilon\Bigg[\frac{1}{\beta^2}(-15+20\zeta_2-42\zeta_3+16\ln(2)+48\zeta_2\ln(2))-\frac{48\zeta_2}{\beta}-\frac{1169}{18}-\frac{32c_1}{3}-\frac{1424}{9}\zeta_2$$
$$+\frac{944}{5}\zeta_2^2-\frac{98}{3}\zeta_3+\frac{32\ln(2)}{3}-\frac{88}{3}\zeta_2\ln(2)+64\zeta_2\ln^2(2)+48\beta\zeta_2$$
$$+\beta^2\left(\frac{2528}{45}+\frac{128c_1}{15}+\frac{15374}{225}\zeta_2-\frac{3776}{25}\zeta_2^2-\frac{14}{15}\zeta_3-\frac{176\ln(2)}{15}-\frac{4936}{75}\zeta_2\ln(2)-\frac{256}{5}\zeta_2\ln^2(2)\right)$$
$$+i\pi\left(\frac{1}{\beta^2}(-8+12\zeta_2)-\frac{1}{\beta}(16(-1+\ln(2)))-\frac{16}{3}-16\zeta_2+\beta\left(-\frac{64}{3}+16\ln(2)\right)+\beta^2\left(\frac{88}{15}+\frac{16\zeta_2}{5}\right)\right)\Bigg]\Bigg\}, \quad (4.41)$$

$$\tilde{F}_{A,2}^{(1),\text{“ns”}} = C_F\Big\{-2-\frac{4\beta^2}{3}+i\pi\left(\frac{2}{\beta}-\beta\right)+\varepsilon\Big[-\frac{12\zeta_2}{\beta}+4+6\beta\zeta_2-\frac{28\beta^2}{9}$$
$$+i\pi\left(-\frac{1}{\beta}(4(-1+\ln(2)+\log(\beta)))+2\beta(-2+\ln(2)+\log(\beta))\right)\Big]$$
$$+\varepsilon^2\Big[\frac{1}{\beta}(-24\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta))-8-\zeta_2+\beta(24\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta))+\beta^2\left(-\frac{160}{27}-\frac{2\zeta_2}{3}\right)$$
$$+i\pi\Big(\frac{1}{\beta}(8-3\zeta_2-8\ln(2)+4\ln^2(2)-8\log(\beta)+8\ln(2)\log(\beta)+4\log^2(\beta))$$
$$+\beta\left(-8+\frac{3}{2}\zeta_2+8\ln(2)-2\ln^2(2)+8\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\right)\Big)\Big]\Big\}. \tag{4.42}$$

$$\tilde{F}_{A,2}^{\text{“2”(ns)}} = C_F^2\Big\{\frac{1}{\varepsilon}\left[\frac{12\zeta_2}{\beta^2}+6\zeta_2+\beta^2\left(-\frac{16}{3}-12\zeta_2\right)+i\pi\left(\frac{2}{\beta}+\frac{26\beta}{3}\right)\right]$$
$$+\Big[\frac{1}{\beta^2}(24\zeta_2-48\zeta_2\ln(2)-48\zeta_2\log(\beta))-\frac{12\zeta_2}{\beta}+\frac{41}{3}-18\zeta_2-45\zeta_3-60\zeta_2\ln(2)-80\zeta_2\log(\beta)+20\beta\zeta_2$$
$$+\beta^2\left(\frac{218}{9}-\frac{3292}{75}\zeta_2-7\zeta_3-\frac{268}{5}\zeta_2\ln(2)+\frac{264}{5}\zeta_2\log(\beta)\right)$$
$$+i\pi\left(\frac{24\zeta_2}{\beta^2}-\frac{1}{\beta}(4(3+\ln(2)+\log(\beta)))+40\zeta_2+\frac{2}{9}\beta(-265+126\ln(2)+30\log(\beta))-\frac{132}{5}\beta^2\zeta_2\right)\Big]$$
$$+\varepsilon\Big[\frac{1}{\beta^2}(-48\zeta_2-60\zeta_2^2-96\zeta_2\ln(2)+96\zeta_2\ln^2(2)-96\zeta_2\log(\beta)+192\zeta_2\ln(2)\log(\beta)+96\zeta_2\log^2(\beta))$$
$$+\frac{1}{\beta}(72\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta))-\frac{823}{18}-\frac{10c_1}{3}+\frac{776}{3}\zeta_2-\frac{1572}{5}\zeta_2^2-\frac{673}{3}\zeta_3-\frac{1052}{3}\zeta_2\ln(2)$$
$$+112\zeta_2\ln^2(2)-128\zeta_2\log(\beta)+320\zeta_2\ln(2)\log(\beta)+160\zeta_2\log^2(\beta)$$
$$+\beta\left(\frac{3608}{3}\zeta_2-968\zeta_2\ln(2)-584\zeta_2\log(\beta)\right)+\beta^2\Big(\frac{6661}{135}+\frac{266c_1}{15}-\frac{193438\zeta_2}{1125}-\frac{6534}{25}\zeta_2^2+\frac{3811}{15}\zeta_3$$
$$+\frac{9292}{15}\zeta_2\ln(2)-\frac{1248}{5}\zeta_2\ln^2(2)+\frac{9704}{25}\zeta_2\log(\beta)-\frac{1056}{5}\zeta_2\ln(2)\log(\beta)-\frac{528}{5}\zeta_2\log^2(\beta)\Big)$$
$$+i\pi\Big(\frac{1}{\beta^2}(48\zeta_2-96\zeta_2\ln(2)-96\zeta_2\log(\beta))+\frac{1}{\beta}(-8-2\zeta_2+24\ln(2)+4\ln^2(2)+24\log(\beta)$$
$$+8\ln(2)\log(\beta)+4\log^2(\beta))+64\zeta_2-160\zeta_2\ln(2)-160\zeta_2\log(\beta)$$
$$+\beta\left(-\frac{1031}{3}+\frac{302}{3}\zeta_2+\frac{5080\ln(2)}{9}-\frac{740}{3}\ln^2(2)+\frac{3608\log(\beta)}{9}-\frac{968}{3}\ln(2)\log(\beta)-\frac{292}{3}\log^2(\beta)\right)$$
$$+\beta^2\left(-\frac{4852}{25}\zeta_2+\frac{528}{5}\zeta_2\ln(2)+\frac{528}{5}\zeta_2\log(\beta)\right)\Big)\Big]\Big\}$$
$$+C_FC_A\Big\{\Big[-\frac{44\zeta_2}{\beta}+\frac{151}{9}+\frac{104}{3}\zeta_2-30\zeta_3-72\zeta_2\ln(2)-32\zeta_2\log(\beta)+84\beta\zeta_2$$
$$+\beta^2\left(-\frac{2666}{45}-\frac{5138}{75}\zeta_2+\frac{252}{5}\zeta_3+\frac{656}{5}\zeta_2\ln(2)+\frac{256}{5}\zeta_2\log(\beta)\right)$$
$$+i\pi\left(-\frac{1}{9\beta}(4(-32+33\ln(2)+33\log(\beta)))+16\zeta_2+\frac{14}{9}\beta(-23+18\ln(2)+18\log(\beta))-\frac{128}{5}\beta^2\zeta_2\right)\Big]$$
$$+\varepsilon\Big[\frac{1}{\beta}\left(-\frac{776}{3}\zeta_2+264\zeta_2\ln(2)+264\zeta_2\log(\beta)\right)-\frac{1433}{54}+\frac{20c_1}{3}+\frac{415}{9}\zeta_2-\frac{1644}{5}\zeta_2^2+46\zeta_3-\frac{104}{3}\zeta_2\ln(2)$$
$$-32\zeta_2\ln^2(2)-96\zeta_2\log(\beta)+128\zeta_2\ln(2)\log(\beta)+64\zeta_2\log^2(\beta)$$

$$+\beta\left(\frac{1990}{3}\zeta_2-504\zeta_2\ln(2)-504\zeta_2\log(\beta)\right)+\beta^2\left(-\frac{145603}{2025}-\frac{40c_1}{3}-\frac{202052\zeta_2}{1125}+\frac{14536}{25}\zeta_2^2\right.$$
$$\left.-\frac{2001}{25}\zeta_3+\frac{9148}{75}\zeta_2\ln(2)+\frac{384}{5}\zeta_2\ln^2(2)+\frac{4736}{25}\zeta_2\log(\beta)-\frac{1024}{5}\zeta_2\ln(2)\log(\beta)-\frac{512}{5}\zeta_2\log^2(\beta)\right)$$
$$+i\pi\left(\frac{1}{\beta}\left(\frac{1924}{27}-11\zeta_2-\frac{776\ln(2)}{9}+44\ln^2(2)-\frac{776\log(\beta)}{9}+88\ln(2)\log(\beta)+44\log^2(\beta)\right)+48\zeta_2-64\zeta_2\ln(2)\right.$$
$$-64\zeta_2\log(\beta)+\beta\left(-\frac{10195}{54}+\frac{157}{6}\zeta_2+\frac{2182\ln(2)}{9}-84\ln^2(2)+\frac{1990\log(\beta)}{9}-168\ln(2)\log(\beta)-84\log^2(\beta)\right)$$
$$\left.\left.\left.+\beta^2\left(-\frac{2368}{25}\zeta_2+\frac{512}{5}\zeta_2\ln(2)+\frac{512}{5}\zeta_2\log(\beta)\right)\right)\right]\right\}$$
$$+C_FT_Fn_l\left\{\left[\frac{16\zeta_2}{\beta}-\frac{44}{9}-8\beta\zeta_2+8\beta^2+i\pi\left(\frac{1}{9\beta}(16(-4+3\ln(2)+3\log(\beta)))-\frac{2}{9}\beta(-25+12\ln(2)+12\log(\beta))\right)\right]\right.$$
$$+\varepsilon\left[\frac{1}{\beta}\left(\frac{352}{3}\zeta_2-96\zeta_2\ln(2)-96\zeta_2\log(\beta)\right)+\frac{338}{27}+12\zeta_2+\beta\left(-\frac{296}{3}\zeta_2+48\zeta_2\ln(2)+48\zeta_2\log(\beta)\right)\right.$$
$$+\beta^2\left(\frac{3044}{81}+8\zeta_2\right)+i\pi\left(\frac{1}{\beta}\left(-\frac{944}{27}+4\zeta_2+\frac{352\ln(2)}{9}-16\ln^2(2)+\frac{352\log(\beta)}{9}-32\ln(2)\log(\beta)-16\log^2(\beta)\right)\right.$$
$$\left.\left.\left.+\beta\left(\frac{889}{27}-2\zeta_2-\frac{296\ln(2)}{9}+8\ln^2(2)-\frac{296\log(\beta)}{9}+16\ln(2)\log(\beta)+8\log^2(\beta)\right)\right)\right]\right\}$$
$$+C_FT_F\left\{\left[\frac{196}{9}-\frac{32}{3}\zeta_2+\beta^2\left(-\frac{632}{27}+\frac{64\zeta_2}{5}\right)+i\pi\left(\frac{16\beta}{15}\right)\right]\right.$$
$$+\varepsilon\left[\frac{1322}{27}-\frac{412}{9}\zeta_2-\frac{112}{3}\zeta_3+64\zeta_2\ln(2)-\frac{32}{5}\beta\zeta_2+\beta^2\left(-\frac{13348}{405}+\frac{9232}{225}\zeta_2+\frac{224}{5}\zeta_3-\frac{384}{5}\zeta_2\ln(2)\right)\right.$$
$$\left.\left.+i\pi\left(-\frac{4\zeta_2}{3\beta}+\beta\left(\frac{32}{15}+\frac{2}{3}\zeta_2-\frac{32\ln(2)}{15}-\frac{32\log(\beta)}{15}\right)\right)\right]\right\}. \tag{4.43}$$

The unrenormalized singlet contribution is

$$\tilde{\tilde{F}}_{A,2}^{(2),\text{“s”}}=C_FT_F\left\{\left[\frac{6}{\beta^2}+\frac{8}{3}+\frac{136}{3}\zeta_2-42\zeta_3+16\ln(2)-16\zeta_2\ln(2)-48\beta\zeta_2\right.\right.$$
$$\left.+\beta^2\left(-30-\frac{904}{15}\zeta_2+70\zeta_3+\frac{16\ln(2)}{3}+\frac{176}{5}\zeta_2\ln(2)\right)+i\pi\left(-8+12\zeta_2+\beta(16-16\ln(2))+\beta^2\left(-\frac{8}{3}-20\zeta_2\right)\right)\right]$$
$$+\varepsilon\left[\frac{1}{\beta^2}(15-20\zeta_2+42\zeta_3-16\ln(2)-48\zeta_2\ln(2))+\frac{48\zeta_2}{\beta}+\frac{364}{9}-4c_1+\frac{2684}{9}\zeta_2-\frac{976}{5}\zeta_2^2+\frac{182}{3}\zeta_3\right.$$
$$+\frac{112\ln(2)}{3}-\frac{416}{3}\zeta_2\ln(2)-56\zeta_3\ln(2)-16\ln^2(2)+200\zeta_2\ln^2(2)+\beta(-240\zeta_2+288\zeta_2\ln(2)+96\zeta_2\log(\beta))$$
$$+\beta^2\left(-\frac{427}{5}+\frac{236c_1}{45}-\frac{101524}{225}\zeta_2+\frac{26288}{75}\zeta_2^2-\frac{1666}{15}\zeta_3-\frac{3392\ln(2)}{45}+\frac{33536}{75}\zeta_2\ln(2)+\frac{280}{3}\zeta_3\ln(2)\right.$$
$$\left.-\frac{16}{3}\ln^2(2)-\frac{1624}{5}\zeta_2\ln^2(2)\right)+i\pi\left(\frac{1}{\beta^2}(8-12\zeta_2)+\frac{1}{\beta}(16(-1+\ln(2)))-\frac{56}{3}+28\zeta_2+28\zeta_3+16\ln(2)\right.$$
$$-24\zeta_2\ln(2)+\beta\left(\frac{160}{3}-80\zeta_2-80\ln(2)+48\ln^2(2)-32\log(\beta)+32\ln(2)\log(\beta)\right)$$
$$\left.\left.\left.+\beta^2\left(\frac{1696}{45}-\frac{428}{15}\zeta_2-\frac{140}{3}\zeta_3+\frac{16\ln(2)}{3}+40\zeta_2\ln(2)\right)\right)\right]\right\}. \tag{4.44}$$

3. Scalar form factor

The scalar form factor is in this limit given by

$$\begin{aligned}\tilde{F}_S^{(1)} = C_F\Bigg\{&\frac{1}{\varepsilon}\left[\frac{8\beta^2}{3}+i\pi\left(-\frac{1}{\beta}-\beta\right)\right]+\left[\frac{6\zeta_2}{\beta}-2+6\beta\zeta_2-\frac{76\beta^2}{9}\right.\\
&+i\pi\left(\frac{1}{\beta}(2(-1+\ln(2)+\log(\beta)))+\beta(3+2\ln(2)+2\log(\beta))\right)\Bigg]\\
&+\varepsilon\left[\frac{1}{\beta}(12\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta))+4+\beta(-18\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta))+\beta^2\left(\frac{404}{27}+\frac{4\zeta_2}{3}\right)\right.\\
&+i\pi\left(\frac{1}{\beta}\left(-4+\frac{3}{2}\zeta_2+4\ln(2)-2\ln^2(2)+4\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\right)\right.\\
&\left.\left.+\beta\left(4+\frac{3}{2}\zeta_2-6\ln(2)-2\ln^2(2)-6\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\right)\right)\right]\\
&+\varepsilon^2\left[\frac{1}{\beta}(24\zeta_2+3\zeta_2^2-24\zeta_2\ln(2)+12\zeta_2\ln^2(2)-24\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta))\right.\\
&-8-\zeta_2+\beta(-24\zeta_2+3\zeta_2^2+36\zeta_2\ln(2)+12\zeta_2\ln^2(2)+36\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta))\\
&+\beta^2\left(-\frac{2752}{81}-\frac{38\zeta_2}{9}-\frac{8\zeta_3}{9}\right)+i\pi\left(\frac{1}{\beta}\left(-8+3\zeta_2+\frac{7}{3}\zeta_3+8\ln(2)-3\zeta_2\ln(2)-4\ln^2(2)+\frac{4\ln^3(2)}{3}\right.\right.\\
&\left.+8\log(\beta)-3\zeta_2\log(\beta)-8\ln(2)\log(\beta)+4\ln^2(2)\log(\beta)-4\log^2(\beta)+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\right)\\
&+\beta\left(8-\frac{9}{2}\zeta_2+\frac{7}{3}\zeta_3-8\ln(2)-3\zeta_2\ln(2)+6\ln^2(2)+\frac{4\ln^3(2)}{3}-8\log(\beta)-3\zeta_2\log(\beta)+12\ln(2)\log(\beta)\right.\\
&\left.\left.\left.+4\ln^2(2)\log(\beta)+6\log^2(\beta)+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\right)\right)\right]\Bigg\}.\end{aligned}\tag{4.45}$$

$$\begin{aligned}\tilde{F}_S^{(2)} = C_F^2\Bigg\{&\frac{1}{\varepsilon^2}\left[-\frac{3\zeta_2}{\beta^2}-6\zeta_2-3\beta^2\zeta_2+i\pi\left(-\frac{8\beta}{3}\right)\right]+\frac{1}{\varepsilon}\left[\frac{1}{\beta^2}(-12\zeta_2+12\zeta_2\ln(2)+12\zeta_2\log(\beta))+6\zeta_2\right.\\
&+24\zeta_2\ln(2)+24\zeta_2\log(\beta)+16\beta\zeta_2+\beta^2\left(-\frac{16}{3}+27\zeta_2+12\zeta_2\ln(2)+12\zeta_2\log(\beta)\right)\\
&\left.+i\pi\left(-\frac{6\zeta_2}{\beta^2}+\frac{2}{\beta}-12\zeta_2+\frac{2}{9}\beta(23+24\ln(2)+24\log(\beta))-6\beta^2\zeta_2\right)\right]\\
&+\left[\frac{1}{\beta^2}(-24\zeta_2+15\zeta_2^2+48\zeta_2\ln(2)-24\zeta_2\ln^2(2)+48\zeta_2\log(\beta)-48\zeta_2\ln(2)\log(\beta)-24\zeta_2\log^2(\beta))-\frac{12\zeta_2}{\beta}\right.\\
&+5+98\zeta_2+30\zeta_2^2-44\zeta_3-40\zeta_2\ln(2)-48\zeta_2\ln^2(2)-88\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\\
&+\beta\left(-\frac{92}{3}\zeta_2-32\zeta_2\ln(2)-32\zeta_2\log(\beta)\right)\\
&+\beta^2\left(\frac{1684}{45}-\frac{2446}{25}\zeta_2+15\zeta_2^2-\frac{58}{5}\zeta_3-\frac{964}{5}\zeta_2\ln(2)-24\zeta_2\ln^2(2)-\frac{684}{5}\zeta_2\log(\beta)-48\zeta_2\ln(2)\log(\beta)-24\zeta_2\log^2(\beta)\right)\\
&+i\pi\left(\frac{1}{\beta^2}(-24\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta))-\frac{4(\ln(2)+\log(\beta))}{\beta}+44\zeta_2+48\zeta_2\ln(2)+48\zeta_2\log(\beta)\right.\\
&+\beta\left(-\frac{632}{27}+\frac{8}{3}\zeta_2-\frac{92\ln(2)}{9}-\frac{16}{3}\ln^2(2)-\frac{92\log(\beta)}{9}-\frac{32}{3}\ln(2)\log(\beta)-\frac{16}{3}\log^2(\beta)\right)\\
&\left.\left.+\beta^2\left(\frac{342}{5}\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta)\right)\right)\right]\end{aligned}$$

$$+\varepsilon\Bigg[\frac{1}{\beta^2}(48\zeta_2+60\zeta_2^2-106\zeta_2\zeta_3+96\zeta_2\ln(2)-60\zeta_2^2\ln(2)-96\zeta_2\ln^2(2)+32\zeta_2\ln^3(2)+96\zeta_2\log(\beta)-60\zeta_2^2\log(\beta)$$
$$-192\zeta_2\ln(2)\log(\beta)+96\zeta_2\ln^2(2)\log(\beta)-96\zeta_2\log^2(\beta)+96\zeta_2\ln(2)\log^2(\beta)+32\zeta_2\log^3(\beta))$$
$$+\frac{1}{\beta}(24\ln(2)\zeta_2+24\log(\beta)\zeta_2)-\frac{451}{6}-8c_1+\frac{2368}{3}\zeta_2-\frac{1278}{5}\zeta_2^2-291\zeta_3-212\zeta_2\zeta_3$$
$$-740\zeta_2\ln(2)-120\zeta_2^2\ln(2)+176\zeta_2\ln^2(2)+64\zeta_2\ln^3(2)-688\zeta_2\log(\beta)-120\zeta_2^2\log(\beta)$$
$$+352\zeta_2\ln(2)\log(\beta)+192\zeta_2\ln^2(2)\log(\beta)+176\zeta_2\log^2(\beta)+192\zeta_2\ln(2)\log^2(\beta)+64\zeta_2\log^3(\beta)$$
$$+\beta\left(\frac{3316}{9}\zeta_2+16\zeta_2^2+\frac{184}{3}\zeta_2\ln(2)+32\zeta_2\ln^2(2)+\frac{184}{3}\zeta_2\log(\beta)+64\zeta_2\ln(2)\log(\beta)+32\zeta_2\log^2(\beta)\right)$$
$$+\beta^2\Bigg(-\frac{100966}{675}+\frac{28c_1}{3}-\frac{61918}{125}\zeta_2-\frac{10939}{25}\zeta_2^2-\frac{15298}{75}\zeta_3-106\zeta_2\zeta_3-\frac{5392}{75}\zeta_2\ln(2)-60\zeta_2^2\ln(2)$$
$$+\frac{1144}{5}\zeta_2\ln^2(2)+32\zeta_2\ln^3(2)+\frac{4776}{25}\zeta_2\log(\beta)-60\zeta_2^2\log(\beta)+\frac{2736}{5}\zeta_2\ln(2)\log(\beta)$$
$$+96\zeta_2\ln^2(2)\log(\beta)+\frac{1368}{5}\zeta_2\log^2(\beta)+96\zeta_2\ln(2)\log^2(\beta)+32\zeta_2\log^3(\beta)\Bigg)$$
$$+i\pi\Bigg(\frac{1}{\beta^2}(-48\zeta_2-18\zeta_2^2+96\zeta_2\ln(2)-48\zeta_2\ln^2(2)+96\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta))$$
$$+\frac{1}{\beta}(8-2\zeta_2+4\ln^2(2)+8\ln(2)\log(\beta)+4\log^2(\beta))+344\zeta_2-36\zeta_2^2-176\zeta_2\ln(2)-96\zeta_2\ln^2(2)$$
$$-176\zeta_2\log(\beta)-192\zeta_2\ln(2)\log(\beta)-96\zeta_2\log^2(\beta)+\beta\Bigg(-\frac{11021}{81}-\frac{46}{9}\zeta_2+\frac{64}{9}\zeta_3+\frac{4468\ln(2)}{27}-\frac{16}{3}\zeta_2\ln(2)$$
$$+\frac{92\ln^2(2)}{9}+\frac{32\ln^3(2)}{9}+\frac{3316\log(\beta)}{27}-\frac{16}{3}\zeta_2\log(\beta)+\frac{184}{9}\ln(2)\log(\beta)+\frac{32}{3}\ln^2(2)\log(\beta)+\frac{92\log^2(\beta)}{9}$$
$$+\frac{32}{3}\ln(2)\log^2(\beta)+\frac{32\log^3(\beta)}{9}\Bigg)+\beta^2\Bigg(-\frac{2388}{25}\zeta_2-18\zeta_2^2-\frac{1368}{5}\zeta_2\ln(2)-48\zeta_2\ln^2(2)$$
$$-\frac{1368}{5}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\Bigg)\Bigg)\Bigg]\Bigg\}$$
$$+C_FC_A\Bigg\{\frac{1}{\varepsilon^2}\left[-\frac{44\beta^2}{9}+i\pi\left(\frac{11}{6\beta}+\frac{11\beta}{6}\right)\right]+\frac{1}{\varepsilon}\left[-8\beta\zeta_2+\beta^2\left(\frac{376}{27}+\frac{32\zeta_2}{3}\right)+i\pi\left(-\frac{31}{18\beta}+\frac{1}{6}\beta(-13-32\ln(2)-16\log(\beta))\right)\right]$$
$$+\Bigg[\frac{1}{\beta}\left(\frac{194}{3}\zeta_2-44\zeta_2\ln(2)-44\zeta_2\log(\beta)\right)+\frac{49}{9}+38\zeta_2-20\zeta_3-64\zeta_2\ln(2)$$
$$-16\zeta_2\log(\beta)+\beta\left(-\frac{100}{3}\zeta_2+84\zeta_2\ln(2)+36\zeta_2\log(\beta)\right)+\beta^2\left(-\frac{16472}{405}+\frac{1679}{15}\zeta_2-\frac{472}{15}\zeta_3-\frac{336}{5}\zeta_2\ln(2)-80\zeta_2\log(\beta)\right)$$
$$+i\pi\Bigg(\frac{1}{\beta}\left(-\frac{478}{27}+\frac{194\ln(2)}{9}-\frac{22\ln^2(2)}{3}+\frac{194\log(\beta)}{9}-\frac{44}{3}\ln(2)\log(\beta)-\frac{22\log^2(\beta)}{3}\right)+8\zeta_2$$
$$+\beta\left(\frac{43}{9}-\frac{40}{3}\zeta_2-\frac{188\ln(2)}{9}+\frac{74\ln^2(2)}{3}-\frac{100\log(\beta)}{9}+28\ln(2)\log(\beta)+6\log^2(\beta)\right)+40\beta^2\zeta_2\Bigg)\Bigg]$$
$$+\varepsilon\Bigg[\frac{1}{\beta}\left(\frac{2704}{9}\zeta_2+99\zeta_2^2-\frac{1040}{3}\zeta_2\ln(2)+132\zeta_2\ln^2(2)-\frac{1040}{3}\zeta_2\log(\beta)+264\zeta_2\ln(2)\log(\beta)+132\zeta_2\log^2(\beta)\right)$$
$$-\frac{383}{54}+8c_1+\frac{409}{3}\zeta_2-264\zeta_2^2-\zeta_3-220\zeta_2\ln(2)-64\zeta_2\ln^2(2)-144\zeta_2\log(\beta)+64\zeta_2\ln(2)\log(\beta)+32\zeta_2\log^2(\beta)$$

$$+\beta\left(-\frac{692}{9}\zeta_2+59\zeta_2^2+\frac{1636}{3}\zeta_2\ln(2)-508\zeta_2\ln^2(2)+\frac{1108}{3}\zeta_2\log(\beta)-632\zeta_2\ln(2)\log(\beta)-172\zeta_2\log^2(\beta)\right)$$
$$+\beta^2\left(\frac{17767}{675}-\frac{32c_1}{15}+\frac{450622}{675}\zeta_2-\frac{25112}{75}\zeta_2^2+\frac{2717}{150}\zeta_3-\frac{26306}{75}\zeta_2\ln(2)+\frac{608}{5}\zeta_2\ln^2(2)\right.$$
$$\left.-416\zeta_2\log(\beta)+320\zeta_2\ln(2)\log(\beta)+160\zeta_2\log^2(\beta)\right)$$
$$+i\pi\left(\frac{1}{\beta}\left(-\frac{6338}{81}+\frac{97}{6}\zeta_2+\frac{22}{3}\zeta_3+\frac{2704\ln(2)}{27}-11\zeta_2\ln(2)-\frac{520}{9}\ln^2(2)+\frac{44\ln^3(2)}{3}+\frac{2704\log(\beta)}{27}\right.\right.$$
$$\left.-11\zeta_2\log(\beta)-\frac{1040}{9}\ln(2)\log(\beta)+44\ln^2(2)\log(\beta)-\frac{520}{9}\log^2(\beta)+44\ln(2)\log^2(\beta)+\frac{44\log^3(\beta)}{3}\right)$$
$$+72\zeta_2-32\zeta_2\ln(2)-32\zeta_2\log(\beta)+\beta\left(-\frac{2317}{27}-\frac{422}{9}\zeta_2-\frac{194}{3}\zeta_3-\frac{1156\ln(2)}{27}+\frac{335}{3}\zeta_2\ln(2)+130\ln^2(2)\right.$$
$$-\frac{764}{9}\ln^3(2)-\frac{692\log(\beta)}{27}+77\zeta_2\log(\beta)+\frac{1636}{9}\ln(2)\log(\beta)-\frac{508}{3}\ln^2(2)\log(\beta)+\frac{554\log^2(\beta)}{9}$$
$$\left.\left.\left.\left.-\frac{316}{3}\ln(2)\log^2(\beta)-\frac{172}{9}\log^3(\beta)\right)+\beta^2(208\zeta_2-160\zeta_2\ln(2)-160\zeta_2\log(\beta))\right)\right]\right\}$$
$$+C_FT_Fn_l\left\{\frac{1}{\varepsilon^2}\left[\frac{16\beta^2}{9}+i\pi\left(-\frac{2}{3\beta}-\frac{2\beta}{3}\right)\right]+\frac{1}{\varepsilon}\left[-\frac{80\beta^2}{27}+i\pi\left(\frac{10}{9\beta}+\frac{10\beta}{9}\right)\right]\right.$$
$$+\left[\frac{1}{\beta}\left(-\frac{88}{3}\zeta_2+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\right)-\frac{20}{9}+\beta\left(\frac{32}{3}\zeta_2+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\right)\right.$$
$$+\beta^2\left(-\frac{400}{81}-\frac{64\zeta_2}{9}\right)+i\pi\left(\frac{1}{\beta}\left(\frac{248}{27}-\frac{88\ln(2)}{9}+\frac{8\ln^2(2)}{3}-\frac{88\log(\beta)}{9}+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\right)\right.$$
$$\left.\left.+\beta\left(-\frac{268}{27}+\frac{32\ln(2)}{9}+\frac{8\ln^2(2)}{3}+\frac{32\log(\beta)}{9}+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\right)\right)\right]$$
$$+\varepsilon\left[\frac{1}{\beta}\left(-\frac{1280}{9}\zeta_2-36\zeta_2^2+\frac{448}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{448}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\right)\right.$$
$$+\frac{62}{27}+12\zeta_2+\beta\left(\frac{1360}{9}\zeta_2-36\zeta_2^2-\frac{272}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)-\frac{272}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\right)$$
$$+\beta^2\left(\frac{800}{9}+\frac{968\zeta_2}{27}-\frac{128\zeta_3}{9}\right)+i\pi\left(\frac{1}{\beta}\left(\frac{3160}{81}-\frac{22}{3}\zeta_2-\frac{8}{3}\zeta_3-\frac{1280\ln(2)}{27}+4\zeta_2\ln(2)+\frac{224\ln^2(2)}{9}-\frac{16}{3}\ln^3(2)\right.\right.$$
$$\left.-\frac{1280\log(\beta)}{27}+4\zeta_2\log(\beta)+\frac{448}{9}\ln(2)\log(\beta)-16\ln^2(2)\log(\beta)+\frac{224\log^2(\beta)}{9}-16\ln(2)\log^2(\beta)-\frac{16}{3}\log^3(\beta)\right)$$
$$+\beta\left(-\frac{3848}{81}+\frac{8}{3}\zeta_2-\frac{8}{3}\zeta_3+\frac{1360\ln(2)}{27}+4\zeta_2\ln(2)-\frac{136}{9}\ln^2(2)-\frac{16}{3}\ln^3(2)+\frac{1360\log(\beta)}{27}+4\zeta_2\log(\beta)\right.$$
$$\left.\left.\left.\left.-\frac{272}{9}\ln(2)\log(\beta)-16\ln^2(2)\log(\beta)-\frac{136}{9}\log^2(\beta)-16\ln(2)\log^2(\beta)-\frac{16}{3}\log^3(\beta)\right)\right)\right]\right\}$$
$$+C_FT_F\left\{\left[\frac{580}{9}-\frac{212}{3}\zeta_2-16\ln(2)+64\zeta_2\ln(2)+\beta^2\left(-\frac{712}{81}+\frac{252}{5}\zeta_2-42\zeta_3+\frac{32\ln(2)}{3}-\frac{336}{5}\zeta_2\ln(2)\right)\right.\right.$$
$$\left.+i\pi\left(\frac{2\zeta_2}{3\beta}+8+\frac{2}{3}\beta\zeta_2+\beta^2\left(-\frac{16}{3}+12\zeta_2\right)\right)\right]+\varepsilon\left[-\frac{4\zeta_2^2}{\beta}+\frac{5186}{27}-\frac{32c_1}{3}-\frac{2800}{9}\zeta_2+\frac{944}{5}\zeta_2^2-\frac{616}{3}\zeta_3\right.$$
$$-\frac{224\ln(2)}{3}+\frac{584}{3}\zeta_2\ln(2)+16\ln^2(2)+64\zeta_2\ln^2(2)-4\beta\zeta_2^2$$
$$+\beta^2\left(-\frac{35116}{1215}+\frac{68c_1}{15}+\frac{18992}{75}\zeta_2-\frac{8656}{25}\zeta_2^2+\frac{5914}{135}\zeta_3+\frac{3584\ln(2)}{45}-\frac{8696}{75}\zeta_2\ln(2)-56\zeta_3\ln(2)-\frac{32}{3}\ln^2(2)+\frac{744}{5}\zeta_2\ln^2(2)\right)$$

$$+ i\pi\left(\frac{1}{\beta}\left(\frac{4}{3}\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2) - \frac{4}{3}\zeta_2\log(\beta)\right) + \frac{112}{3} - 12\zeta_2 - 16\ln(2)\right.$$
$$\left.+ \beta\left(-2\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2) - \frac{4}{3}\zeta_2\log(\beta)\right) + \beta^2\left(-\frac{1792}{45} + 40\zeta_2 + 28\zeta_3 + \frac{32\ln(2)}{3} - 24\zeta_2\ln(2)\right)\right)\Bigg]\Bigg\}. \quad (4.46)$$

4. Pseudoscalar form factor

The nonsinglet part of the pseudoscalar form factor can be obtained using the chiral Ward identity Eq. (2.13) as follows

$$\tilde{F}_P^{(n),\text{“ns”}} = \tilde{F}_{A,1}^{(n),\text{“ns”}} + (1 + \beta^2 + \beta^4)\tilde{F}_{A,2}^{(n),\text{“ns”}} + \mathcal{O}(\beta^3). \quad (4.47)$$

The bare singlet piece is given by

$$\tilde{\tilde{F}}_P^{(2),\text{“s”}} = C_F T_F\Bigg\{\Bigg[20\zeta_2 - 42\zeta_3 + 48\zeta_2\ln(2) - 48\beta\zeta_2 + \beta^2\left(-\frac{40}{3} - 12\zeta_2 + 28\zeta_3 + 16\ln(2) - 32\zeta_2\ln(2)\right)$$
$$+ i\pi(12\zeta_2 + \beta(16 - 16\ln(2)) + \beta^2(-8 - 8\zeta_2))\Bigg] + \varepsilon\Bigg[-\frac{44c_1}{3} + 88\zeta_2 - \frac{32}{5}\zeta_2^2 + 112\zeta_3 - 264\zeta_2\ln(2)$$
$$- 56\zeta_3\ln(2) + 264\zeta_2\ln^2(2) + \beta(-144\zeta_2 + 288\zeta_2\ln(2) + 96\zeta_2\log(\beta))$$
$$+ \beta^2\left(\frac{256}{9} + \frac{88c_1}{9} - 108\zeta_2 + \frac{64}{15}\zeta_2^2 - \frac{112}{3}\zeta_3 - 48\ln(2) + \frac{680}{3}\zeta_2\ln(2) + \frac{112}{3}\zeta_3\ln(2) - 16\ln^2(2) - 176\zeta_2\ln^2(2)\right)$$
$$+ i\pi\left(-12\zeta_2 + 28\zeta_3 - 24\zeta_2\ln(2) + \beta(16 - 80\zeta_2 - 48\ln(2) + 48\ln^2(2) - 32\log(\beta) + 32\ln(2)\log(\beta))\right.$$
$$\left.+ \beta^2\left(24 - \frac{4}{3}\zeta_2 - \frac{56}{3}\zeta_3 + 16\ln(2) + 16\zeta_2\ln(2)\right)\right)\Bigg]\Bigg\}. \quad (4.48)$$

V. CONCLUSION

The massive form factors are basic building blocks to many observables in heavy quark physics. The precision study of these objects will both shed light on the physical structure of the top quark itself and also on important aspects of the mechanism to create the fermion masses. A future electron-positron collider can achieve high precision and, hence, an equal or better theory prediction is indispensable. In a similar way, this also applies to the LHC for its high luminosity phase. In the present paper, we have computed the heavy quark form factors for vector, axial-vector, scalar and pseudoscalar currents at two-loop level up to the $\mathcal{O}(\varepsilon^2)$ contributions. These contributions constitute important ingredients to renormalize the three-loop form factors and do also contribute to potential future four-loop calculations. In addition, they serve as a cross-check of earlier results available in the literature. In the calculation, we have used both traditional techniques in solving the differential equations for the master integrals, as well as a more recent automated method, based on coupled difference equations. Both methods play a role in computing higher than second-order corrections to the different form factors.

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APPENDIX: RENORMALIZATION CONSTANTS

In this appendix, we present corresponding renormalization constants up to relevant order in ε. The wave function renormalization constants in the OS scheme up to two loop [35–37,40] are

$$Z^{(1)}_{2,OS} = C_F\left[-\frac{3}{\varepsilon} - 4 + \varepsilon\left\{-8 - \frac{3}{2}\zeta_2\right\} + \varepsilon^2\{-16 - 2\zeta_2 + \zeta_3\} + \varepsilon^3\left\{-32 - 4\zeta_2 - \frac{27}{40}\zeta_2^2 + \frac{4}{3}\zeta_3\right\}\right], \tag{A1}$$

$$\begin{aligned} Z^{(2)}_{2,OS} = C_F^2\Bigg[&\frac{1}{\varepsilon^2}\left\{\frac{9}{2}\right\} + \frac{1}{\varepsilon}\left\{\frac{51}{4}\right\} + \left\{\frac{433}{8} - \frac{147}{2}\zeta_2 - 24\zeta_3 + 96\zeta_2\ln(2)\right\} \\ &+ \varepsilon\left\{\frac{211}{16} - 16c_1 - \frac{1017}{4}\zeta_2 + \frac{1008}{5}\zeta_2^2 - 297\zeta_3 + 552\zeta_2\ln(2)\right\} \\ &+ \varepsilon^2\left\{\frac{4889}{32} - 92c_1 - \frac{96c_2}{5} - \frac{8851}{8}\zeta_2 + \frac{11703}{20}\zeta_2^2 - \frac{2069}{2}\zeta_3 + 264\zeta_2\zeta_3 + 2436\zeta_5 + 1968\zeta_2\ln(2)\right\}\Bigg] \\ &+ C_AC_F\Bigg[\frac{1}{\varepsilon^2}\left\{\frac{11}{2}\right\} + \frac{1}{\varepsilon}\left\{-\frac{127}{12}\right\} + \left\{-\frac{1705}{24} + 30\zeta_2 + 12\zeta_3 - 48\zeta_2\ln(2)\right\} \\ &+ \varepsilon\left\{-\frac{9907}{48} + 8c_1 + \frac{769}{12}\zeta_2 - \frac{504}{5}\zeta_2^2 + 129\zeta_3 - 276\zeta_2\ln(2)\right\} \\ &+ \varepsilon^2\left\{-\frac{79225}{96} + 46c_1 + \frac{48c_2}{5} + \frac{6367}{24}\zeta_2 - \frac{14359}{40}\zeta_2^2 + \frac{7595}{18}\zeta_3 - 132\zeta_2\zeta_3 - 1218\zeta_5 - 984\zeta_2\ln(2)\right\}\Bigg] \\ &+ C_FT_Fn_l\Bigg[\frac{1}{\varepsilon^2}\left\{-2\right\} + \frac{1}{\varepsilon}\left\{\frac{11}{3}\right\} + \left\{\frac{113}{6} + 8\zeta_2\right\} + \varepsilon\left\{\frac{851}{12} + \frac{127}{3}\zeta_2 + 16\zeta_3\right\} \\ &+ \varepsilon^2\left\{\frac{5753}{24} + \frac{853}{6}\zeta_2 + \frac{597}{10}\zeta_2^2 + \frac{610}{9}\zeta_3\right\}\Bigg] \\ &+ C_FT_F\Bigg[\frac{1}{\varepsilon} + \left\{\frac{947}{18} - 30\zeta_2\right\} + \varepsilon\left\{\frac{17971}{108} - \frac{445}{3}\zeta_2 - \frac{340}{3}\zeta_3 + 192\zeta_2\ln(2)\right\} \\ &+ \varepsilon^2\left\{\frac{422747}{648} - 32c_1 - \frac{8605}{18}\zeta_2 + \frac{1683}{10}\zeta_2^2 - \frac{4810}{9}\zeta_3 + 912\zeta_2\ln(2)\right\}\Bigg]. \end{aligned} \tag{A2}$$

The heavy quark mass renormalization constants in the OS scheme are [35–39]

$$Z^{(1)}_{m,OS} = C_F\left[-\frac{3}{\varepsilon} - 4 + \varepsilon\left\{-8 - \frac{3}{2}\zeta_2\right\} + \varepsilon^2\left\{-16 - 2\zeta_2 + \zeta_3\right\} + \varepsilon^3\left\{-32 - 4\zeta_2 - \frac{27}{40}\zeta_2^2 + \frac{4}{3}\zeta_3\right\}\right], \tag{A3}$$

$$\begin{aligned} Z^{(2)}_{m,OS} = C_F^2\Bigg[&\frac{1}{\varepsilon^2}\left\{\frac{9}{2}\right\} + \frac{1}{\varepsilon}\left\{\frac{45}{4}\right\} + \left\{\frac{199}{8} - \frac{51}{2}\zeta_2 - 12\zeta_3 + 48\zeta_2\ln(2)\right\} \\ &+ \varepsilon\left\{\frac{677}{16} - 8c_1 - \frac{615}{4}\zeta_2 + \frac{504}{5}\zeta_2^2 - 135\zeta_3 + 288\zeta_2\ln(2)\right\} \\ &+ \varepsilon^2\left\{\frac{1167}{32} - 48c_1 - \frac{48c_2}{5} - \frac{4821}{8}\zeta_2 + \frac{7719}{20}\zeta_2^2 - \frac{1203}{2}\zeta_3 + 132\zeta_2\zeta_3 + 1218\zeta_5 + 1056\zeta_2\ln(2)\right\}\Bigg] \\ &+ C_AC_F\Bigg[\frac{1}{\varepsilon^2}\left\{\frac{11}{2}\right\} + \frac{1}{\varepsilon}\left\{-\frac{97}{12}\right\} + \left\{-\frac{1111}{24} + 8\zeta_2 + 6\zeta_3 - 24\zeta_2\ln(2)\right\} \\ &+ \varepsilon\left\{-\frac{8581}{48} + 4c_1 + \frac{271}{12}\zeta_2 - \frac{252}{5}\zeta_2^2 + 52\zeta_3 - 144\zeta_2\ln(2)\right\} \\ &+ \varepsilon^2\left\{-\frac{58543}{96} + 24c_1 + \frac{24c_2}{5} + \frac{1537}{24}\zeta_2 - \frac{9783}{40}\zeta_2^2 + \frac{3929}{18}\zeta_3 - 66\zeta_2\zeta_3 - 609\zeta_5 - 528\zeta_2\ln(2)\right\}\Bigg] \\ &+ C_FT_Fn_l\Bigg[\frac{1}{\varepsilon^2}\left\{-2\right\} + \frac{1}{\varepsilon}\left\{\frac{5}{3}\right\} + \left\{\frac{71}{6} + 8\zeta_2\right\} + \varepsilon\left\{\frac{581}{12} + \frac{97}{3}\zeta_2 + 16\zeta_3\right\} + \varepsilon^2\left\{\frac{4079}{24} + \frac{643}{6}\zeta_2 + \frac{597}{10}\zeta_2^2 + \frac{478}{9}\zeta_3\right\}\Bigg] \\ &+ C_FT_F\Bigg[\frac{1}{\varepsilon^2}\left\{-2\right\} + \frac{1}{\varepsilon}\left\{\frac{5}{3}\right\} + \left\{\frac{143}{6} - 16\zeta_2\right\} + \varepsilon\left\{\frac{1133}{12} - \frac{227}{3}\zeta_2 - 56\zeta_3 + 96\zeta_2\ln(2)\right\} \\ &+ \varepsilon^2\left\{\frac{8135}{24} - 16c_1 - \frac{1553}{6}\zeta_2 + \frac{837}{10}\zeta_2^2 - \frac{2546}{9}\zeta_3 + 480\zeta_2\ln(2)\right\}\Bigg]. \end{aligned} \tag{A4}$$

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