

Four-loop results on anomalous dimensions and splitting functions in QCD

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We report on recent progress on the flavour non-singlet splitting functions in perturbative QCD. The exact four-loop ($N^3\text{LO}$) contribution to these functions has been obtained in the planar limit of a large number of colours. Phenomenologically sufficient approximate expressions have been obtained for the parts not exactly known so far. Both cases include results for the four-loop cusp and virtual anomalous dimensions which are relevant well beyond the evolution of non-singlet quark distributions, for which an accuracy of (well) below 1% has now been reached.

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1. Introduction

Up to power corrections, observables in ep and pp hard scattering can be schematically expressed as

$$O^{ep} = f_i \otimes c_i^0, \quad O^{pp} = f_i \otimes f_k \otimes c_{ik}^0 \quad (1.1)$$

in terms of the respective partonic cross sections (coefficient functions) c^0 and the universal parton distribution functions (PDFs) $f_i(x, \mu^2)$ of the proton at a (renormalization and factorization) scale μ of the order of a physical hard scale, e.g., M_H for the total cross section for the production of the Higgs boson. The dependence of the PDFs on the momentum fraction x is not calculable in perturbative QCD; their scale dependence is governed by the renormalization-group evolution equations

$$\frac{\partial}{\partial \ln \mu^2} f_i(x, \mu^2) = [P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)](x) \quad (1.2)$$

where $\otimes$ denotes the Mellin convolution. The splitting functions, which are closely related to the anomalous dimensions of twist-2 operators in the light-cone operator-product expansion (OPE), and the coefficient functions can be expanded in powers of the strong coupling $a_s \equiv \alpha_s(\mu^2)/(4\pi)$,

$$P = a_s P^{(0)} + a_s^2 P^{(1)} + a_s^3 P^{(2)} + a_s^4 P^{(3)} + \dots, \quad (1.3)$$

$$c_a^0 = a_s^{n_0} [c_0^{(0)} + a_s c_0^{(1)} + a_s^2 c_0^{(2)} + a_s^3 c_0^{(3)} + \dots]. \quad (1.4)$$

Together the first three terms in eqs. (1.3) and (1.4) provide the next-to-next-to-leading order (N²LO) of perturbative QCD for the observables (1.1). This is now the standard approximation for many hard processes; see refs. [1–4] for the corresponding splitting functions.

Corrections beyond N²LO are of phenomenological interest where high precision is required, such as in determinations of α_s from deep-inelastic scattering (DIS) (see refs. [5, 6] for the N³LO corrections to the most important structure functions), and where the perturbation series shows a slow convergence, such as for Higgs production via gluon-gluon fusion calculated in ref. [7] at N³LO. The size and structure of the corrections beyond N²LO are also of theoretical interest.

Here we briefly report about considerable recent progress on the three four-loop (N³LO) non-singlet splitting functions. We focus on the quantities $P_{ns}^{\pm(3)}(x)$ for the evolution of flavour-differences $q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k)$ of quark and antiquark distributions; for more details see ref. [8].

2. Diagram calculations of fixed- N moments

Two methods have been applied for obtaining Mellin moments of the quantities $P^{(3)}$ in eq. (1.3). Depending on the function, both can be used to determine the same even- N or the odd- N moments.

In the first one calculates, via the optical theorem and a dispersion relation in x , the unfactorized structure functions in DIS, as done at two and three loops in refs. [9–12]. The construction of the FORCER program [13] has facilitated the extension of those computations (which also provide moments of the coefficient functions) to four loops. For the hardest diagrams, the complexity of these computations rises quickly with N , hence only $N \leq 6$ has been covered completely so far [14]. Much higher N can be accessed for simpler cases, e.g., values up to $N > 40$ have been reached for high- n_f parts. These were sufficient to determine the complete n_f^2 and n_f^3 parts of the non-singlet splitting functions $P_{ns}^{(3)}(x)$ and the n_f^3 parts of the corresponding flavour-singlet quantities [15].

The increase of the complexity of the Feynman integrals with N is more benign for the second method based on the OPE which was applied to the present non-singlet cases at NLO in ref. [16], see also ref. [17]. FORCER calculations in this framework have reached $N = 16$ for all contributions to the functions $P_{\text{ns}}^{(3)}$, $N = 18$ for their n_f parts and $N = 20$ for the complete limit of a large number of colours n_c [8]. See refs. [18–21] for earlier calculations of $P_{\text{ns}}^{\pm(3)}$ at $N \leq 4$.

3. Towards all- N expressions

If the anomalous dimensions $\gamma_{\text{ns}}(N) = -P_{\text{ns}}(N)$ at $N^{n>2}\text{LO}$ are analogous to the lower orders, then they can be expressed in terms of harmonic sums $S_{\vec{w}}$ [22, 23] and denominators $D_a^k \equiv (N+a)^{-k}$ as

$$\gamma_{\text{ns}}^{(n)}(N) = \sum_{w=0}^{2n+1} c_{00\vec{w}} S_{\vec{w}}(N) + \sum_a \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{ak\vec{w}} D_a^k S_{\vec{w}}(N). \quad (3.1)$$

The denominators at the calculated values of N indicate $a = 0, 1$ for $\gamma_{\text{ns}}^{\pm}$, with coefficients $c_{00\vec{w}}$, $c_{ak\vec{w}}$ that are integer modulo low powers of $1/2$ and $1/3$. Sums up to weight $w = 2n+1$ occur at $N^n\text{LO}$.

Based on a conformal symmetry of QCD at an unphysical number of space-time dimensions D , it has been conjectured that the $\overline{\text{MS}}$ functions $\gamma_{\text{ns}}(N)$ are constrained by ‘self-tuning’ [24, 25],

$$\gamma_{\text{ns}}(N) = \gamma_{\text{u}}(N + \sigma \gamma_{\text{ns}}(N) - \beta(a_s)/a_s) \quad (3.2)$$

where $\beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \dots$ is the beta function, for its present status see refs. [26, 27]. The initial-state (PDF) and final-state (fragmentation-function) anomalous dimensions are obtained for $\sigma = -1$ and $\sigma = 1$, respectively, and the universal kernel γ_{u} is reciprocity respecting (RR), i.e., invariant under replacement $N \rightarrow (1-N)$. Eq. (3.2) implies that the non-RR parts and the spacelike/timelike difference are inherited from lower orders. Hence ‘only’ γ_{u} , which includes 2^{w-1} RR (combinations of) harmonic sums of weight w , needs to be determined at four loops.

Present information, given by the even- N (odd- N) values $N \leq 16$ (15) of $\gamma_{\text{ns}}^{+(3)}(N)$ ($\gamma_{\text{ns}}^{-(3)}(N)$) and endpoint constraints (see below), is insufficient to determine the $n = 3$ coefficients in eq. (3.1). However, $\gamma_{\text{ns}}^+ = \gamma_{\text{ns}}^-$ in the large- n_c limit, hence the known even- N and odd- N values can be used. Moreover, alternating sums do not contribute to $\gamma_{\text{ns}}^{\pm}$ in this limit, leaving 1, 1, 2, 3, 5, 8, 13 = Fibonacci(w) RR sums at weight $w = 1, \dots, 7$ and a total of 87 basis functions for $n = 3$ in eq. (3.1).

Large- N and small- x limits provide more than 40 constraints on their coefficients. At large- N , the non-singlet anomalous dimensions have the form [33–35]

$$\gamma_{\text{ns}}^{(n-1)}(N) = A_n \ln \tilde{N} - B_n + N^{-1} \{C_n \ln \tilde{N} - \tilde{D}_n + \frac{1}{2} A_n\} + O(N^{-2}) \quad (3.3)$$

with $\ln \tilde{N} \equiv \ln N + \gamma_e$, where γ_e denotes the Euler-Mascheroni constant. C_n and $\tilde{D}_n$ are given by

$$C(a_s) = (A(a_s))^2, \quad \tilde{D}(a_s) = A(a_s) \cdot (B(a_s) - \beta(a_s)/a_s), \quad (3.4)$$

in terms of lower-order information on the cusp anomalous dimension $A(a_s) = A_1 a_s + A_2 a_s^2 + \dots$ and the quantity $B(a_s) = B_1 a_s + B_2 a_s^2 + \dots$ sometimes called the virtual anomalous dimension.

The resummation of small- x double logarithms [28–31] provides the four-loop coefficients of $x^a \ln^b x$ at $4 \leq b \leq 6$ and all a in the large- n_c limit (in full QCD, this holds only at even a for $P_{\text{ns}}^+(x)$ and odd a for $P_{\text{ns}}^-(x)$). Moreover, a relation leading to a single-logarithmic resummation at $a = 0$,

$$\gamma_{\text{ns}}^+(N) \cdot (\gamma_{\text{ns}}^+(N) + N - \beta(a_s)/a_s) = O(1), \quad (3.5)$$

has been conjectured in ref. [32]. As far as it can be checked so far, this relation is found to be correct except for terms with $\zeta_2 = \pi^2/6$ that vanish in the large- n_c limit.

Taking into account all the above information, it is possible to set up systems of Diophantine equations for the coefficients $c_{00\bar{w}}, c_{ak\bar{w}}$ of $\gamma_{\text{ns}}^{\pm(3)}(N)$ in the large- n_c limit that can be solved using the moments $1 \leq N \leq 18$, leaving the results of the diagram calculation at $N = 19, 20$ as checks.

4. All- N anomalous dimension in the large- n_c limit

The exact expressions for the new n_f^0 and n_f^1 parts cannot be shown here due to their length, they can be found in eq. (3.6) and (3.7) of ref. [8]. For the n_f^2 and n_f^3 terms see ref. [15]. The resulting large- N coefficients $A_{L,4}$ and $B_{L,4}$ – the subscript L indicates the large- n_c limit – are found to be

$$\begin{aligned} A_{L,4} = & C_F n_c^3 \left(\frac{84278}{81} - \frac{88832}{81} \zeta_2 + \frac{20992}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_2 \zeta_3 - 352 \zeta_5 - 32 \zeta_3^2 - 876 \zeta_6 \right) \\ & - C_F n_c^2 n_f \left(\frac{39883}{81} - \frac{26692}{81} \zeta_2 + \frac{16252}{27} \zeta_3 + \frac{440}{3} \zeta_4 - \frac{256}{3} \zeta_2 \zeta_3 - 224 \zeta_5 \right) \\ & + C_F n_c n_f^2 \left(\frac{2119}{81} - \frac{608}{81} \zeta_2 + \frac{1280}{27} \zeta_3 - \frac{64}{3} \zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27} \zeta_3 \right) \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} B_{L,4} = & C_F n_c^3 \left(-\frac{1379569}{5184} + \frac{24211}{27} \zeta_2 - \frac{9803}{162} \zeta_3 - \frac{9382}{9} \zeta_4 + \frac{838}{9} \zeta_2 \zeta_3 + 1002 \zeta_5 + \frac{16}{3} \zeta_3^2 \right. \\ & \left. + 135 \zeta_6 - 80 \zeta_2 \zeta_5 + 32 \zeta_3 \zeta_4 - 560 \zeta_7 \right) \\ & + C_F n_c^2 n_f \left(\frac{353}{3} - \frac{85175}{162} \zeta_2 - \frac{137}{9} \zeta_3 + \frac{16186}{27} \zeta_4 - \frac{584}{9} \zeta_2 \zeta_3 - \frac{248}{3} \zeta_5 - \frac{16}{3} \zeta_3^2 - 144 \zeta_6 \right) \\ & - C_F n_c n_f^2 \left(\frac{127}{18} - \frac{5036}{81} \zeta_2 + \frac{932}{27} \zeta_3 + \frac{1292}{27} \zeta_4 - \frac{160}{9} \zeta_2 \zeta_3 - \frac{32}{3} \zeta_5 \right) \\ & - C_F n_f^3 \left(\frac{131}{81} - \frac{32}{81} \zeta_2 - \frac{304}{81} \zeta_3 + \frac{32}{27} \zeta_4 \right). \end{aligned} \quad (4.2)$$

The agreement of the four-loop cusp anomalous dimension (4.1) with the result obtained from the large- n_c photon-quark form factor [36,37] provides a further non-trivial check of the determination of the all- N expressions from the moments at $N \leq 18$, and hence also of the relations (3.1) – (3.5).

The maximum-weight ζ_3^2 and ζ_6 parts of eq. (4.1) also agree with the result obtained in planar $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory (MSYM) obtained before in ref. [38]. There is no such direct connection between the four-loop virtual anomalous dimension (4.2) and its counterparts in planar $\mathcal{N} = 4$ MSYM; see ref. [39] where the maximum-weight part of eq. (4.2) has been employed to derive the four-loop collinear anomalous dimension in planar $\mathcal{N} = 4$ MSYM.

The all- N large- n_c limit of $\gamma_{\text{ns}}^{\pm(3)}(N)$ is compared in fig. 1 with the integer- N QCD results at $N \leq 16$. As illustrated in the left panel, the former are a decent approximation to the latter for the individual n_f^k contributions. However, as shown in the right panel, there are considerable cancellations between the these contributions. These cancellations are most pronounced for the physically relevant number of $n_f = 5$ light quark flavours outside the large- N /large- x region. Hence the large- n_c suppressed contributions – indicated by the subscript N below – need to be taken into account in phenomenological N³LO analyses.

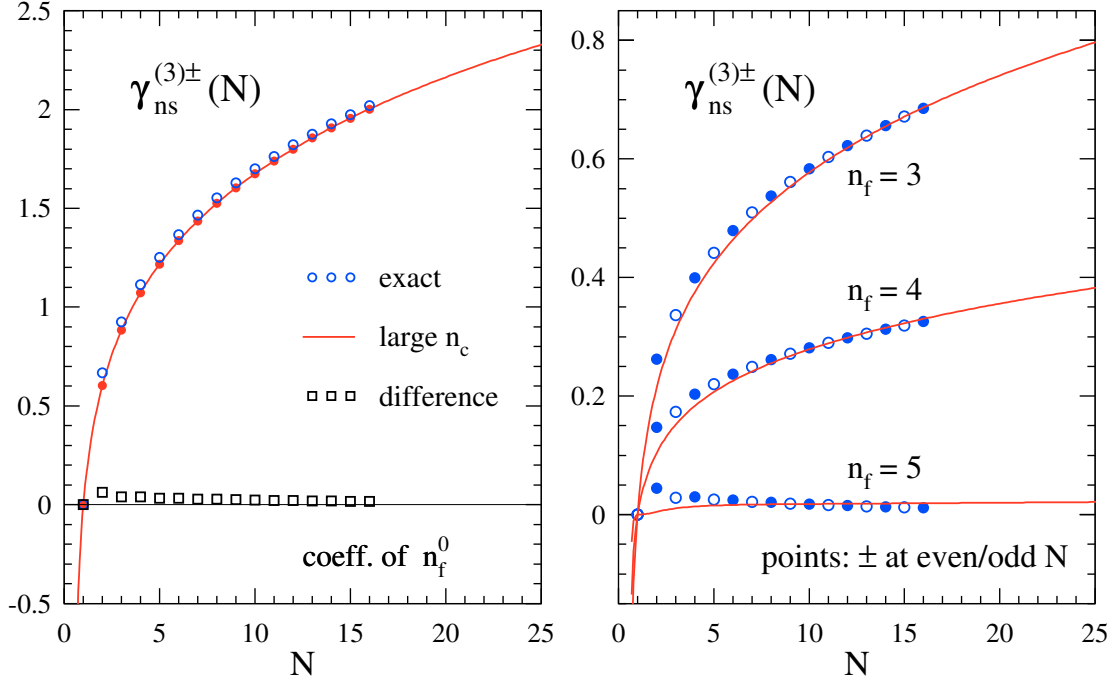


Figure 1: The large- n_c limit of the four-loop anomalous dimensions $\gamma_{\text{ns}}^{\pm(3)}(N)$ (lines) compared to the QCD results for $\gamma_{\text{ns}}^{+(3)}(N)$ at even N and $\gamma_{\text{ns}}^{-(3)}(N)$ at odd N (points). Left: the n_f -independent contributions. Right: the results for physically relevant values of n_f . The values have been converted to an expansion in α_s .

5. x -space approximations of the large- n_c suppressed parts

With eight integer- N moments known for both $P_{\text{ns}}^{+(3)}(x)$ and $P_{\text{ns}}^{-(3)}(x)$ and the large- x and small- x knowledge discussed in section 2, it is possible to construct approximate x -space expressions which are analogous to (but more accurate than) those used before 2004 at N²LO, see refs. [40–43]. For this purpose an ansatz consisting of

- the two large- x parameters A_4 and B_4 in eq. (3.3),
- two of three suppressed large- x logs $(1-x)\ln^k(1-x)$, $k = 1, 2, 3$,
- one of ten two-parameter polynomials in x that vanish for $x \rightarrow 1$,
- two of the three unknown small- x logarithms $\ln^k x$, $k = 1, 2, 3$

is built for the large- n_c suppressed n_f^0 and n_f^1 parts $P_{\text{N},0/1}^{+(3)}$ of $P_{\text{ns}}^{+(3)}(x)$. This results in 90 trial functions, the parameters of which can be fixed from the eight available moments. Of these functions, two representatives A and B are then chosen that indicate the remaining uncertainty, see fig. 2.

This non-rigorous procedure can be checked by comparing the same treatment for the large- n_c parts to our exact results. Moreover, the trial functions lead to very similar values for the next moment, e.g., $N = 18$ for $P_{\text{ns}}^{+(3)}$. The residual uncertainty at this N -value is a consequence of the width of the band at large x , which in turn is correlated with the uncertainties at smaller x . If the spread of the result A and B would underestimate the true remaining uncertainties, then a comparison with an additional analytic result at this next value of N should reveal a discrepancy.

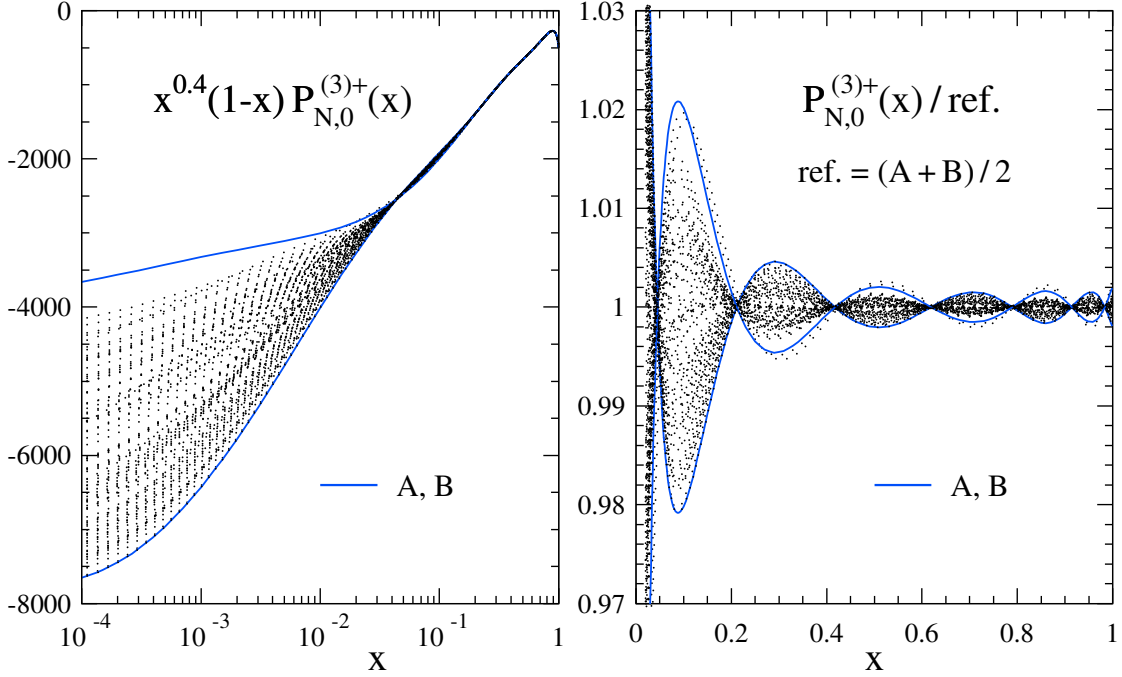


Figure 2: About 90 trial functions for the n_f -independent contribution to the large- n_c suppressed part of splitting function $P_{\text{ns}}^{+(3)}(x)$, multiplied by $x^{0.4}(1-x)$. The two functions chosen to represent the remaining uncertainty are denoted by A and B and shown by solid (blue) lines. Due to the factor $(1-x)$ the contribution $A_{N,4}$ to the four-loop cusp anomalous dimension can be read off at $x = 1$.

We were able to extend the diagram computations of the n_f^1 parts of $P_{\text{ns}}^{+(3)}(x)$ to $N = 18$ and find

$$P_{N,1}^{+(3)}(N=18) = 195.8888792_B < 195.8888857 \dots_{\text{exact}} < 195.8888968_A . \quad (5.1)$$

A similar check for $P_{N,0}^{+(3)}$ has been carried out by deriving a less accurate approximation using only seven moments and comparing the results to the now unused value at $N = 16$.

The case of $P_{\text{ns}}^{-(3)}(x)$ has been treated in the same manner, but taking into account that only its leading small- x logarithm is known up to now [29]. See ref. [8] for the (large- N suppressed) additional $d^{abc}d_{abc}$ contribution $P_{\text{ns}}^{s(3)}(x)$ to the splitting function for the total valence quark PDF.

6. Numerical results for the cusp and virtual anomalous dimensions

Combining the exact large- n_c results, the approximations for the remaining n_f^0 and n_f^1 contributions and the complete high- n_f contributions of ref. [15], the four-loop cusp anomalous dimension for QCD with n_f quark flavours are given by

$$A_4 = 20702(2) - 5171.9(2)n_f + 195.5772n_f^2 + 3.272344n_f^3 , \quad (6.1)$$

where the numbers in brackets represent a conservative estimate of the remaining uncertainty. The conversion of this result to an expansion in powers of α_s leads to

$$\begin{aligned} A_q(\alpha_s, n_f=3) &= 0.42441 \alpha_s (1 + 0.72657 \alpha_s + 0.73405 a_s^2 + 0.6647(2) a_s^3 + \dots) , \\ A_q(\alpha_s, n_f=4) &= 0.42441 \alpha_s (1 + 0.63815 \alpha_s + 0.50998 a_s^2 + 0.3168(2) a_s^3 + \dots) , \\ A_q(\alpha_s, n_f=5) &= 0.42441 \alpha_s (1 + 0.54973 \alpha_s + 0.28403 a_s^2 + 0.0133(2) a_s^3 + \dots) . \end{aligned} \quad (6.2)$$

The corresponding results for the virtual anomalous dimension, i.e., the coefficient of $\delta(1-x)$ show a similarly benign expansion with

$$B_4 = 23393(10) - 5551(1)n_f + 193.8554n_f^2 + 3.014982n_f^3 \quad (6.3)$$

and

$$\begin{aligned} B_q(\alpha_s, n_f=3) &= 0.31831 \alpha_s (1 + 0.99712 \alpha_s + 1.24116 a_s^2 + 1.0791(13) a_s^3 + \dots), \\ B_q(\alpha_s, n_f=4) &= 0.31831 \alpha_s (1 + 0.87192 \alpha_s + 0.97833 a_s^2 + 0.5649(13) a_s^3 + \dots), \\ B_q(\alpha_s, n_f=5) &= 0.31831 \alpha_s (1 + 1.74672 \alpha_s + 0.71907 a_s^2 + 0.1085(13) a_s^3 + \dots). \end{aligned} \quad (6.4)$$

Due to constraints by large- N moments, the errors of A_4 and B_4 are fully correlated. The accuracy in eqs. (6.2) and (6.4) should be amply sufficient for phenomenological applications.

By repeating the approximation procedure in section 5 for individual colour factors, it is possible to obtain corresponding approximate coefficients for A_4 and B_4 which can be summarized as (for a table of the relevant group invariants see, e.g., appendix C of ref. [44])

	A_4	B_4
C_F^4	0	$197. \pm 3.$
$C_F^3 C_A$	0	$-687. \pm 10.$
$C_F^2 C_A^2$	0	$1219. \pm 12.$
$C_F C_A^3$	610.3 ± 0.3	295.6 ± 2.4
$d_R^{abcd} d_A^{abcd} / N_R$	-507.5 ± 6.0	$-996. \pm 45.$
$n_f C_F^3$	-31.00 ± 0.4	81.4 ± 2.2
$n_f C_F^2 C_A$	38.75 ± 0.2	-455.7 ± 1.1
$n_f C_F C_A^2$	-440.65 ± 0.2	-274.4 ± 1.1
$n_f d_R^{abcd} d_R^{abcd} / N_R$	-123.90 ± 0.2	-143.5 ± 1.2
$n_f^2 C_F^2$	-21.31439	-5.775288
$n_f^2 C_F C_A$	58.36737	51.03056
$n_f^3 C_F$	2.454258	2.261237

where the exactly known n_f^2 and n_f^3 coefficients have been included for completeness. Due to the constraint provided by the exact large- n_c limit, the errors in this table are highly correlated; for numerical applications in QCD eqs. (6.2) and (6.4) should be used instead. The above results show that both quartic group invariants definitely contribute to the four-loop cusp anomalous dimension, for this issue see also refs. [45–48] and references therein. This implies that the so-called Casimir scaling between the quark and gluon cases, $A_q = C_F/C_A A_g$, does not hold beyond three loops.

7. N³LO corrections to the evolution of non-singlet PDFs

The effect of the fourth-order contributions on the evolution of the non-singlet PDFs can be illustrated by considering the logarithmic derivatives of the respective combinations of quark PDFs with respect to the factorization scale, $\dot{q}_{\text{ns}}^i \equiv d \ln q_{\text{ns}}^i / d \ln \mu_f^2$, at a suitably chosen reference point.

As in ref. [1], we choose the schematic, order-independent initial conditions

$$xq_{\text{ns}}^{\pm,v}(x, \mu_0^2) = x^{0.5}(1-x)^3 \quad \text{and} \quad \alpha_s(\mu_0^2) = 0.2. \quad (7.1)$$

For $\alpha_s(M_Z^2) = 0.114 \dots 0.120$ this value for α_s corresponds to $\mu_0^2 \simeq 25 \dots 50 \text{ GeV}^2$ beyond the leading order, a scale range typical for DIS at fixed-target experiments and at the ep collider HERA.

The new N³LO corrections to $\dot{q}_{\text{ns}}^i$ are generally small, hence they are illustrated in fig. 3 by comparing their relative effect to that of the N²LO contributions for the standard identification $\mu_r = \mu_f \equiv \mu$ of the renormalization scale with the factorization scale. Except close to the sign change of the scaling violations at $x \simeq 0.07$, the relative N³LO effects are (well) below 1% for the flavour-differences q_{ns}^+ and q_{ns}^- (left and middle panel). The N²LO and N³LO corrections are larger for the valence distribution q_{ns}^v at $x < 0.07$ due to the effect of the $d^{abc}d_{abc}$ ‘sea’ contribution $P_{\text{ns}}^s(x)$, note the different scale of the right panel in fig. 3. Also in this case the N³LO evolution represents a clear improvement, and the relative four-loop corrections are below 2%.

The remaining uncertainty due to the approximate character of the four-loop splitting functions beyond the large- n_c limit is indicated by the difference between the solid and dotted (red) curves in fig. 3 and fig. 4 below. Due to the small size of the four-loop contributions and the ‘ x -averaging’ effect of the Mellin convolution,

$$[P_{\text{ns}} \otimes q_{\text{ns}}](x) = \int_x^1 \frac{dy}{y} P_{\text{ns}}(y) q_{\text{ns}}\left(\frac{x}{y}\right), \quad (7.2)$$

the results of section 4 are safely applicable to lower values of x than one might expect from fig. 2.

The stability of the NLO, N²LO and N³LO results under variation of the renormalization scale over the range $\frac{1}{8}\mu_f^2 \leq \mu_r^2 \leq 8\mu_f^2$ is illustrated in fig. 4 at typical values of x . Except close to the sign change of $\dot{q}_{\text{ns}}^+$, the variation is well below 1% for the conventional interval $\frac{1}{2}\mu_f \leq \mu_r \leq 2\mu_f$.

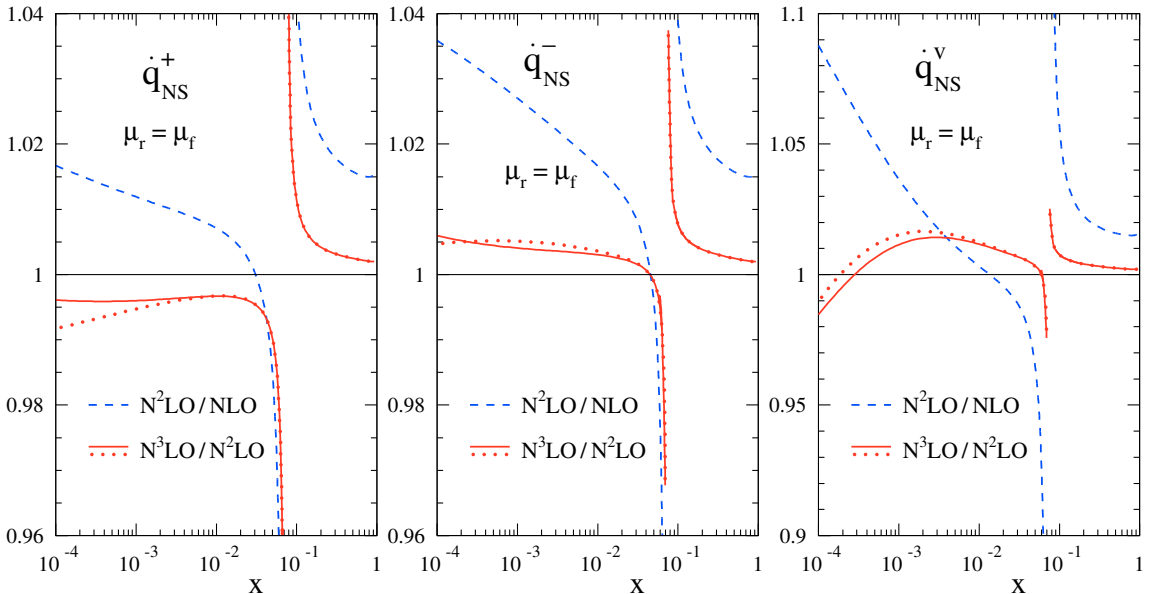


Figure 3: The relative N²LO and N³LO corrections to the logarithmic scale derivative of the non-singlet combinations q_{ns}^a of quark PDFs for the schematic order-independent input (7.1) for $n_f = 4$ at $\mu_r = \mu_f$.

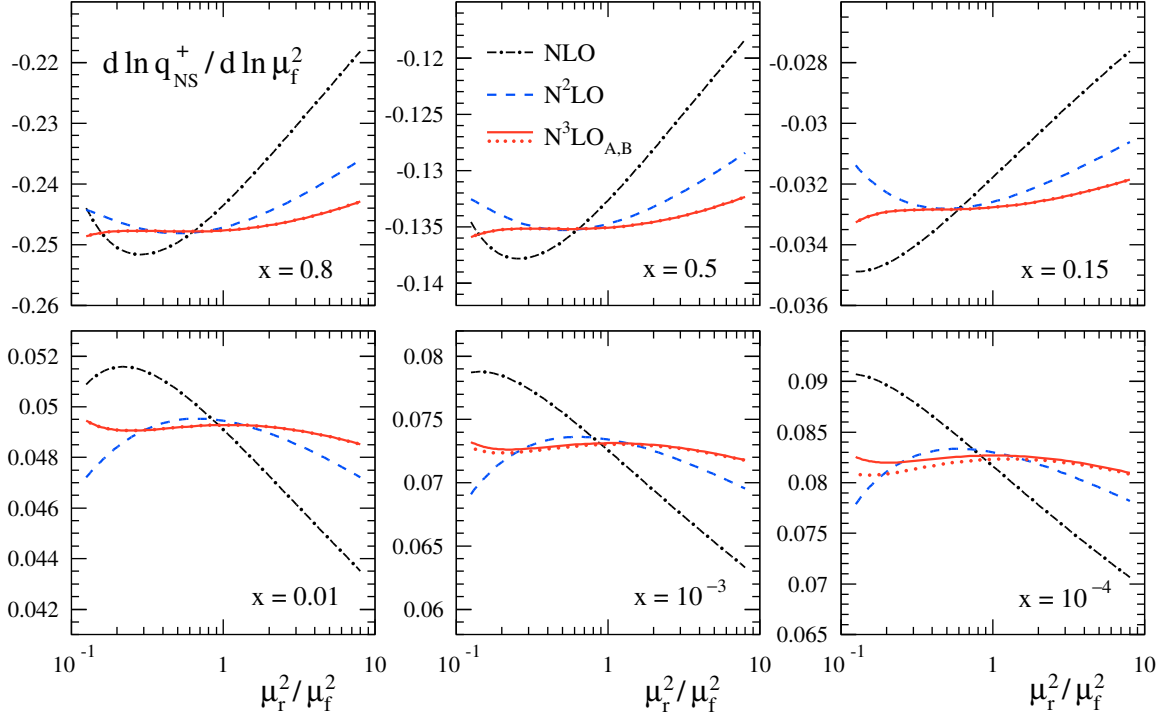


Figure 4: The dependence of the NLO, N²LO and N³LO results for $\dot{q}_{\text{ns}}^+ \equiv d \ln q_{\text{ns}}^+ / d \ln \mu_f^2$ on the renormalization scale μ_r at six typical values of x for the initial conditions (7.1) and $n_f = 4$ flavours. The remaining uncertainty of the four-loop splitting function $P_{\text{ns}}^{+(3)}(x)$ leads to the difference of the solid and dotted curves.

8. Summary and Outlook

The splitting functions for the non-singlet combinations of quark PDFs have been addressed at the fourth-order (N³LO) of perturbative QCD. The quantities $P_{\text{ns}}^{\pm(3)}$ are now known exactly in the limit of a large number of colours n_c . Present results for the large- n_c suppressed contributions with n_f^0 and n_f^1 are still approximate, but sufficiently accurate for phenomenological applications in deep-inelastic scattering and collider physics. FORM and FORTRAN files of these results can be obtained by downloading the source of ref. [8] from [arXiv.org](https://arxiv.org).

It would be desirable, mostly for theoretical purposes, to obtain also the analytic forms n_f^0 and n_f^1 parts of $P_{\text{ns}}^{\pm(3)}$. So far, only their contributions proportional to the values ζ_4 and ζ_5 of the Riemann ζ -function have been completely determined, together with the (unpublished) ζ_3 part of the n_f^1 contributions. The ζ_4 parts are particularly simple; in fact, it turns out that they (and other π^2 terms) can be predicted via physical evolution kernels from lower-order quantities, see refs. [49,50].

The ζ_5 part of $P_{\text{ns}}^{\pm(3)}$, presented in appendix D of ref. [8], includes a (non large- n_c) contribution

$$- \frac{128}{3} \left\{ 3C_F^2 C_A^2 - 2C_F C_A^3 + 12d_F^{abcd} d_A^{abcd} / N_R \right\} 5\zeta_5 [S_1(N)]^2. \quad (8.1)$$

The resulting $\ln^2 N$ large- N behaviour needs to be compensated by non- ζ_5 terms. Eq. (8.1) looks exactly like the ζ_5 -‘tail’ of the so-called wrapping correction in the anomalous dimensions in $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory, see refs. [51,52].

Phenomenologically, of course, one rather needs corresponding results for the flavour-singlet splitting functions $P_{ij}^{(3)}(x)$, $i, j = q, g$. At present, it appears computationally too hard to obtain moments of all four functions beyond $N = 6$ using the method of refs. [9–12]. Therefore one will need to resort to the OPE, which offers additional theoretical challenges in the massless flavour-singlet case, see refs. [53–55]. We hope to address this issue in a future publication.

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