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The determination of α_s by the ALPHA collaboration

ALPHA Collaboration

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Abstract

We review the ALPHA collaboration strategy for obtaining the QCD coupling at high scale. In the three-flavor effective theory it avoids the use of perturbation theory at $\alpha \gtrsim 0.2$ and at the same time has the physical scales small compared to the cutoff $1/a$ in all stages of the computation. The result $\Lambda^{(3)}_{\overline{\rm MS}} = 332(14)\,$MeV is translated to $\alpha_{\overline{\rm MS}}(m_Z) = 0.1179(10)(2)$ by use of (high order) perturbative relations between the effective theory couplings at the charm and beauty quark "thresholds". The error of this perturbative step is discussed and estimated as 0.0002.

Keywords: QCD, Perturbation theory, Strong coupling constant, Λ-parameter, Lattice QCD

1. Introduction

This talk discusses the non-perturbative determination of α_s, using lattice QCD as the non-perturbative definition of the theory and for its evaluation. One wants to relate $\alpha_s(\mu) = \bar{g}_s^2(\mu)/(4\pi)$ (conventionally at $\mu = m_Z$ and in the $s = \overline{\rm MS}$ scheme) to experimental observables with negligible truncation errors from the use of perturbation theory (PT) at intermediate scales. This is a very relevant task, since $\alpha_{\overline{\rm MS}}(\mu)$ enters many important theory predictions, whether in LHC- or in flavor-physics. But it seems that it is sometimes overlooked that it is also a true challenge to achieve a good systematic precision. Both the Particle Data Group [1] and the Flavour Lattice Averaging Group [2] are therefore not just taking weighted averages of the individual determinations to arrive at their world averages.

We start with a note on definitions of the QCD coupling and Λ-parameters which is needed in order to understand what can be said non-perturbatively and what is intrinsically perturbative.

The standard is to use the $\overline{\rm MS}$ renormalization scheme for QCD. Order by order in the coupling it defines the relation between the bare coupling and the renormalized $\overline{\rm MS}$ coupling. There is no general definition of this relation beyond this series, i.e. beyond PT. Therefore, it is also hard to make firm statements about non-perturbative "contributions" or "corrections". However, it is not hard to get around this conceptual and practical problem. One may start from some short-distance QCD observable with a perturbative expansion

$$O_s(\mu) = k\,\bar{g}^2_{\overline{\rm MS}}(\mu)\,[1 + c_1^s \bar{g}^2_{\overline{\rm MS}}(\mu) + \ldots] \qquad (1)$$

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and define the coupling in the associated physical scheme via

$$\bar{g}_s^2(\mu) \equiv O_s(\mu)/k = \bar{g}^2_{\overline{\rm MS}}(\mu) + c_1^s \bar{g}^4_{\overline{\rm MS}}(\mu) + \dots . \quad (2)$$

Short-distance means that $O_s(\mu)$ is defined in terms of fields concentrated within a 4-d region of linear size $R = 1/\mu$. In this way, μ is the only energy scale that enters and the coupling runs with μ. *Observable* simply means that all c_i^s (or more precisely O_s itself) are finite; no renormalization beyond the one of the coupling and quark masses is needed.

While it is not easy to start from experimentally accessible cross sections and directly relate them to such quantities, sufficient inclusiveness / smearing over energy makes it possible to approximately define physical couplings in terms of experimental data. However, a direct relation to experimental numbers is not really necessary, rather it is sufficient that the same theory and bare coupling uniquely predict the physical coupling and experimental quantities such as the mass of the proton or decay constants of pion and kaon, $f_\pi, f_{\rm K}$. For a lattice computation this means that there is a great opportunity to choose coupling definitions which can be handled well technically. We will choose two different (families of) couplings, for reasons to be mentioned below. Note also that – rather exceptionally – it is a true advantage that lattice gauge theory works in Euclidean space-time.

We turn to the Λ-parameters. In massless renormalization schemes, which we assume throughout,[1] the integration of the Callan–Symanzik equation

$$\mu\partial_\mu \bar{g}_s(\mu) = \beta_s(\bar{g}(\mu)) \quad (3)$$

yields the exact relation (at any μ)

$$\Lambda_s \;=\; \varphi_s(\bar{g}_s(\mu)) \times \mu\,, \quad (4)$$

with

$$\varphi_s(\bar{g}_s) \;=\; (b_0\bar{g}_s^2)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}_s^2)} \quad (5)$$
$$\times\;\; \exp\Big\{-\int_0^{\bar{g}_s} \mathrm{d}x \Big[\frac{1}{\beta_s(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0^2 x}\Big]\Big\}.$$

The parameters Λ_s are renormalization group invariant, i.e. independent of μ and, together with β_s, give the coupling at any μ. Starting from the above equations it is a simple exercise to derive the *exact* relation of Λ-parameters $\Lambda_s/\Lambda_{\overline{\rm MS}} = \exp(c_1^s/(2b_0))$. Here one-loop PT yields the non-perturbative result. This is one reason, why we aim for the Λ-parameter. The second is that once we thus have converted from our scheme to $\Lambda_{\overline{\rm MS}}$, the coupling can be computed by inserting the perturbative approximation

$$\beta_s^{\rm pert}(g) = -g^3 b_0 - g^5 b_1 - g^3 \sum_{n=2}^{l_{\rm b}-1} b_{n,s} g^{2n}$$

into eq. (4) and eq. (5) for $s = \overline{\rm MS}$. There, $l_{\rm b} = 5$ loops are known [3, 4, 5], making the correction term

$$\Delta\Lambda_s/\Lambda_s = \Delta\varphi_s/\varphi_s = c_{l_{\rm b}}\alpha^{l_{\rm b}-1}(\mu) + \dots \quad (6)$$

due to the difference $\beta_s^{\rm pert} - \beta_s$ very small *in the region where PT applies at all*.[2]

As we will see, the strategy of the ALPHA collaboration allows to reach $\mu = {\rm O}(100\,{\rm GeV})$ non-perturbatively and only there uses PT. The perturbative error eq. (6) is then around $[\alpha_{\rm SF}(100\,{\rm GeV})]^2 = 10^{-2}$ since in the used SF scheme, the function $\beta_{\rm SF}^{\rm pert}$ is known to $l_{\rm b} = 3$ loops.

2. Non-perturbative α_s: meeting the challenge

2.1. Challenge

As said, we have great freedom in our choice for $O_s(\mu)$, defining the coupling, but it is a challenge to reach large μ (small error term eq. (6)) in a lattice computation. The reason is that numerical computations involve both a discretization length, the lattice spacing, a, and a total size of the system, L that is simulated. For standard observables, e.g. the potential of static quarks at short distance, there are finite L effects of order $\exp(-m_\pi L)$ requiring L to be several fm. At the same time, one needs to suppress discretization errors of order $a^2\mu^2$ and should extrapolate to $a^2\mu^2 \to 0$. The resulting inequalities

$$L \gg 1/m_\pi\,, \quad 1/a \gg \mu \quad \to\; L/a \ggg \mu/m_\pi \quad (7)$$

lead to the need of very large lattices. To get a feeling for numbers, we show a semiquantitative plot of the region in $\alpha(\mu)$ which enters eq. (6) vs. $a^2\mu^2$, which determines the size of the (minimal) discretization errors for the range of lattice spacings $a > 0.04$ fm reached in the simulations that dominate the present estimates of α_s by the PDG and FLAG [1, 2]. The desired $(0, 0)$ point in that plot can only be reached by large extrapolations.

[1] The issue of effective theories with different $N_{\rm f}$ and quark mass thresholds will be discussed below.

[2] The coefficients $c_{l_{\rm b}}$, are, for $l_{\rm b} \le 5$, of order one in the $\overline{\rm MS}$ scheme and expected to be so in "good" schemes in general.

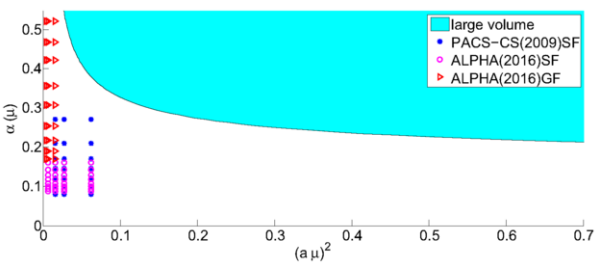

Figure 1: The shaded area shows the region of lattice spacings $a > 0.04$fm of present day large volume simulations with $\alpha_{\overline{\rm MS}}(\mu)$ evaluated to two loops and $\Lambda^{(3)} = 332\,\mathrm{MeV}$. The data points on the left are simulation points in the finite size scaling computations.

2.2. Finite size schemes

However, it has been proposed long ago [6], that one may identify the scales $R = L = 1/\mu$ by introducing an observable O_s which depends only on the scale L, not on any other ones. Finite size effects are part of the observable rather than one of its errors. Instead of eq. (7) the only restriction is

$$L/a \gg 1\,, \tag{8}$$

such that $L/a = 10 - 50$ lattices are sufficient. Apart from the definition of such observables, it remains to clarify how one connects the perturbative region (large μ where one can use eq. (4), eq. (5) with perturbative $\beta_s = \beta_s^{\rm pert}$) with the hadronic region (large L, where $\exp(-m_\pi L)$ effects are negligible and one can connect the theory parameters to $f_\pi, f_{\rm K}$). This is achieved by

2.3. Step scaling

One replaces the derivative of the coupling with respect to the scale, i.e. the β-function, by the change of the coupling when the scale is varied by a factor of two [6].

$$\sigma(u) \equiv \bar{g}^2(\tfrac{1}{2L})\big|_{\bar{g}^2(\frac{1}{L})=u,m=0} = u + 2b_0\log(2)\,u^2 + \ldots$$

is called the step scaling function. Non-perturbatively it is computed as the continuum limit

$$\sigma(u) = \lim_{a/L\to 0} \Sigma(u, a/L) \tag{9}$$

of its lattice approximants Σ. At finite lattice spacing the conditions $\bar{g}^2(1/L) = u$ and $m = 0$ refer to a $(L/a)^4$ lattice and fix the bare coupling and bare quark mass of the theory. $\bar{g}^2(1/(2L))$ is evaluated for the same bare parameters on a $(2L/a)^4$ lattice, cf. figure 2. Setting $m = 0$ ensures the quark mass independence of the scheme [7]. A recursion

$$u_k = \sigma(u_{k+1}), \tag{10}$$

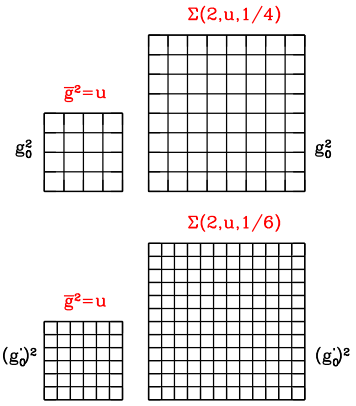

Figure 2: Illustration of the computation of the continuum step scaling function from finer and finer lattices. $\Sigma(2, u, a/L)$ in the illustration corresponds to our lattice step scaling function $\Sigma(u, a/L)$. The two different lattice spacings mean two different values g_0^2 and $(g_0')^2$ of the bare coupling.

then provides us with $\bar{g}^2$ at discrete points along the energy axis,

$$\bar{g}^2(\mu_k = 2^k/L_0) = u_k\,,\ k = 1, 2, \ldots\,. \tag{11}$$

Ten such steps cover three orders of magnitude in μ.

In the 90's and 00's a suitable definition of the coupling was developed [8, 9] and the above programme was carried out for $N_{\rm f} = 0, 2$ by the ALPHA collaboration. PACS-CS applied the same strategy for $N_{\rm f} = 3$ [10] and partial results are available for $N_{\rm f} = 4$ [11]. The review [12] contains more references. In the following, we report on our new results for $N_{\rm f} = 3$ which achieve a precision which far exceeds previous ones and leads to a determination of $\alpha_{\overline{\rm MS}}(m_Z)$ as precise as the current world average and – as we would argue – with much improved systematic control over perturbative errors and discretization effects.

3. An optimized strategy

One reason for the enhanced precision is that at approximately 4 GeV we switch to a new scheme [13, 14] which has much better statistical accuracy for small μ. Unfortunately, the computation now has an increased number of steps to be explained. Here we can just give an overview following the sketch in figure 3.

4. High energy region: Schrödinger functional coupling

Our scheme [9] for the high energy region, reviewed in [16, 17, 12], is based on the so-called Schrödinger

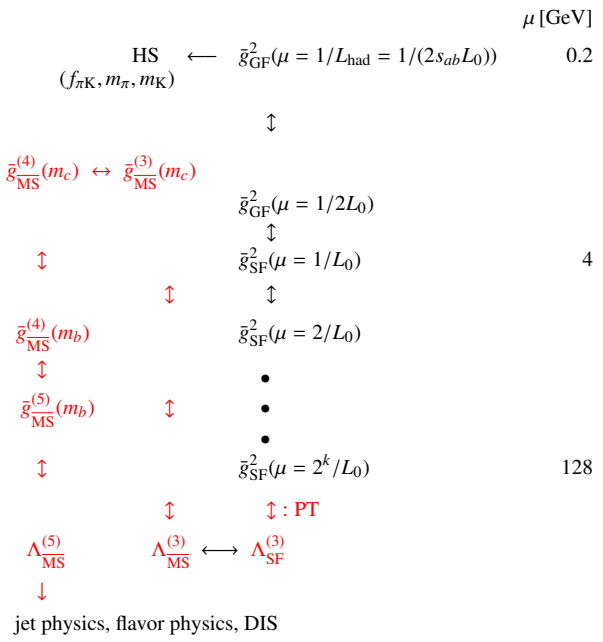

Figure 3: Our strategy for the computation of the three-flavor $\Lambda^{(3)}$, followed by the exact translation to the $\overline{\rm MS}$-scheme (right half). On the left we sketch the perturbative connection of the 3-flavor effective theory to the 5-flavor effective theory by standard matching relations at the quark thresholds. HS refers to the hadronic scheme used at low energy. Red parts involve PT with corresponding uncertainties.

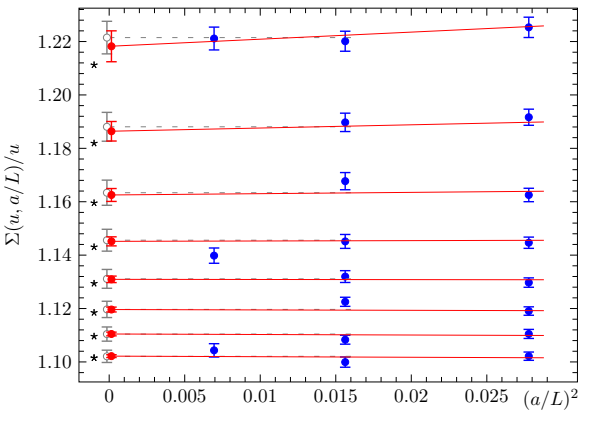

Figure 4: Continuum limit of step scaling function in the SF scheme after subtraction of cutoff effects to 2 loops [15]. The $\star$-symbols show the perturbative σ computed from the three-loop $\beta^{\rm pert}$.

functional (SF) [8, 18]. Here, we just summarize what is needed to judge our findings below. Dirichlet boundary conditions are imposed in Euclidean time,

$$A_k(x)|_{x_0=0} = C_k\,, \quad A_k(x)|_{x_0=L} = C'_k\,, \quad k = 1, 2, 3, \tag{12}$$

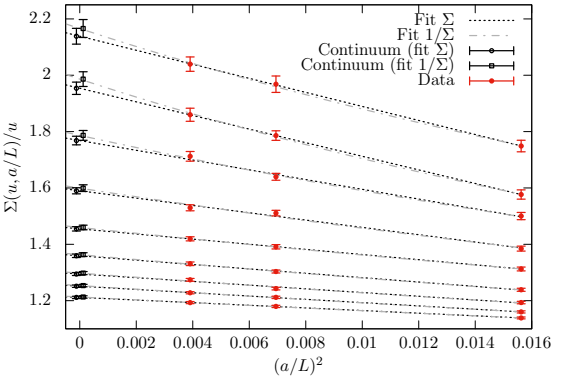

Figure 5: Continuum extrapolation of Σ in our GF scheme [14] using the discretized flow of [20].

and the gauge potentials A_μ are taken periodic in space with period L. The six dimensionless matrices

$$LC_k = i\,\mathrm{diag}(\eta - \tfrac{\pi}{3}, \eta(\nu - \tfrac{1}{2}), -\eta(\nu + \tfrac{1}{2}) + \tfrac{\pi}{3})\,,$$
$$LC'_k = i\,\mathrm{diag}(-(\eta + \pi), \eta(\nu + \tfrac{1}{2}) + \tfrac{\pi}{3}, -\eta(\nu - \tfrac{1}{2}) + \tfrac{2\pi}{3})\,,$$

depend on the two real parameters η, ν.

With these boundary conditions the field which minimizes the action is unique up to gauge equivalence [9] and denoted by $A_\mu = B^{\rm class}_\mu$. It is a constant Abelian color electric field, given in the temporal gauge, $B_0 = 0$, by $B^{\rm class}_k(x) = C_k + (C'_k - C_k)x_0/L$. A family of couplings [19], $\bar{g}_\nu$, is then obtained by taking $1/O_\nu$ in eq. (2) to be the η-derivative of the effective action. This yields a simple path integral expectation value,

$$\langle \partial_\eta S \rangle_{\eta=0} = \frac{12\pi}{\bar{g}^2_\nu}\,, \tag{13}$$

well suited for a Monte Carlo evaluation in the latticised theory. Small fluctuations around the background field generate the non-trivial orders in PT. The whole one-parameter family of couplings can be obtained from numerical simulations at $\nu = 0$, since we have

$$\frac{1}{\bar{g}^2_\nu} = \frac{1}{\bar{g}^2} - \nu\,\bar{\nu}\,, \tag{14}$$

with $\bar{g}^2 \equiv \bar{g}^2_{\nu=0}$ and $12\pi\bar{\nu} = -\langle \partial_\nu \partial_\eta S \rangle_{\eta=\nu=0}$.

Advantageous properties of these couplings are: 1. at large μ the statistical error decreases proportional to $\bar{g}^4$. 2. The typical $\sim \mu^{-1}, \mu^{-2}$ renormalon contributionsare absent due the finite volume infrared momentum cutoff. Instead, the leading known non-perturbative contribution is of order $(\Lambda/\mu)^{3.8}$ [15]. 3. The β-function is known including its three-loop term. It is well behaved. 4. As shown in figure 4 and discussed in [15] discretisation effects are very small. At tree-level of perturbation

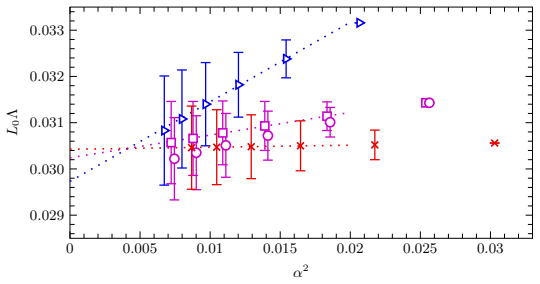

Figure 6: The dependence of the Λ-parameter on the coupling, α. From right to left, $k = 0, 1, \ldots, 5$ steps of non-perturbative step-scaling are performed to arrive at $\alpha(\mu)$ at $\mu = 2^k/L_0$, before using perturbative running. From top to bottom the different symbols correspond to $\nu = -0.5, 0, 0.3$. For $\nu = 0$ two different ways of performing the continuum limit are shown. Dotted straight lines guide the eye.

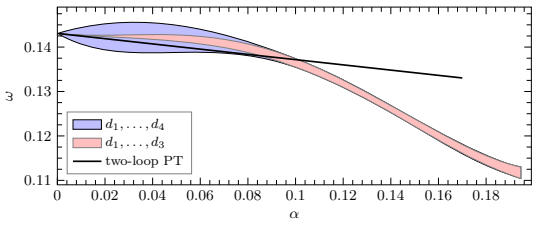

Figure 7: The function $\omega(\bar{g}^2)$ after continuum extrapolation, covering the $\pm 1\sigma$ band of two fits described in the text.

theory they are $O((a/L)^4)$. We subtract [21] the known perturbative pieces including two-loop order [22].

The main downside of the SF scheme (see [15] for details) is that at larger couplings the precision deteriorates. This is avoided by the switch to the GF scheme.

Results: We performed a careful tuning of the bare parameters to have $m \approx 0$ within sharp limitsand to have 8 fixed values of $\bar{g}^2(1/L) = u$ on $L/a = 4, 6, 8, 12$ lattices. We computed $\Sigma(u, a/L) = \bar{g}^2(1/2L)$ and extrapolated to the continuum limit as sketched in figure 4. Also $\bar{v}(L), \bar{v}(2L)$ were computed and the function $\omega(u) = \lim_{a/L\to 0} \bar{v}(L)|_{\bar{g}(L)=u}$ was obtained. These results allow to start at $\bar{g}^2_\nu(1/L_0)$ where

$$\bar{g}^2_{\rm SF}(1/L_0) = 2.012 \quad \text{at } \nu = 0 \tag{15}$$

defines L_0, and construct the coupling at $\mu = 2^k/L_0$ for $k \le 5$ and for different ν. We can then compute effective values for $L_0\Lambda_{{\rm SF},\nu}$ using the 3-loop $\beta^{\rm pert}$. These are changed to the $\nu = 0$ default scheme via $L_0\Lambda_{\rm SF} = \frac{\Lambda_{{\rm SF},0}}{\Lambda_{{\rm SF},\nu}} L_0\Lambda_{{\rm SF},\nu}$ with the (exact) ratio $\frac{\Lambda_{{\rm SF},0}}{\Lambda_{{\rm SF},\nu}}$. These numbers (points with error bars in figure 6) have to converge to the true $L_0\Lambda_{\rm SF}$ with a rate proportional to $\alpha^2(2^k/L_0)$. The numerical results strongly support this. Given that all data points in the graph use PT for $\alpha < 0.2$ only, the magnitude of differences at finite α is surprisingly big. In order to exclude that this is a statistical fluctuation, we show the function ω in figure 7. It differs by many standard deviations from the two-loop (linear) function, shown in the graph.

These **inaccuracies of PT** do not pose a problem to us because we can (and do) simply go to $\alpha \lesssim 0.1$ but they are a warning about estimating uncertainties of perturbative predictions.

In figure 3 we have now connected $\bar{g}^2_{\rm SF}(\mu = 1/L_0)$ to $\Lambda^{(3)}_{\rm SF}$, obtaining

$$L_0\Lambda^{(3)}_{\rm SF} = 0.0303(8) \;\to\; L_0\Lambda^{(3)}_{\overline{\rm MS}} = 0.0791(21) \tag{16}$$

and move on to lower energy.

5. From 4 GeV to 200 MeV: Gradient Flow coupling

The first step combines the change of scale with the change to the new scheme derived from the Gradient Flow [23, 13, 20, 14] into

$$\bar{g}^2_{\rm GF}(1/2L_0) = 2.6723(64)\,. \tag{17}$$

The continuum limit is understood. Step scaling functions in the GF scheme are then computed and extrapolated to the continuum limit in a similar manner as before; only extra care has to be taken about higher than a^2/L^2 discretization effects – a glance at figure 5 shows that a^2/L^2 terms are significant. We fit the continuum σ to a parameterization of the β-function,

$$\beta(g) = \frac{-g^3}{P(g^2)}\,, \quad P(g^2) = p_0 + p_1 g^2 + p_2 g^4 + \ldots\,.$$

using the relation

$$\log(2) = -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{{\rm d}x}{\beta(x)}\,. \tag{18}$$

The parametrization allows us to directly obtain scale factors corresponding to the change of couplings,

$$\log(s_{ab}) = \int_{g_a}^{g_b} {\rm d}x \frac{P(x^2)}{x^3}\,. \tag{19}$$

In particular a careful analysis yields

$$s_{ab} = 10.93(20) \quad \text{for } g_a^2 = 2.6723\,,\; g_b^2 = 11.31\,, \tag{20}$$

or combined with eq. (17) we get $L_{\rm had}/L_0 = 21.86(42)$ where $\bar{g}^2(1/L_{\rm had}) = 11.31$.

From this analysis together with the one in the previous section, we also obtained the non-perturbative β-functions in the two schemes, in the respective energy ranges considered. A nice graph is found in [14].

6. Hadronic scales

We have to fix $L_{\rm had}$ in physical units from $L_{\rm had} = (L_{\rm had}m_{\rm had})^{(3)}/m_{\rm had}^{\rm exp}$ where $m_{\rm had}$ is an experimentally accessible low energy mass (scale) and $(L_{\rm had}m_{\rm had})^{(3)}$ is the dimensionless number computed in QCD with three quark flavors. While it is most natural to use the proton mass, m_p, technical limitations explained in detail in [24] lead us to choose the leptonic decay constant of pion and kaon, even though their phenomenological values $f_\pi = 130.4(2)\,\text{MeV}$ and $f_{\rm K} = 156.2(7)\,\text{MeV}$ depend on the knowledge of $V_{\rm ud}$ and $V_{\rm us}$ [2].

Our computation of hadronic scales is based on the CLS large volume simulations with two degenerate light quarks, $m_u = m_d$ and one additional strange quark [25]. In these simulations the trace, $m_u + m_d + m_s$, of the quark mass matrix is held constant [26] while varying $m_u = m_d$ in approaching the physical point defined by physical values for $m_\pi/f_{\pi{\rm K}}$, $m_{\rm K}/f_{\pi{\rm K}}$. Along this trajectory in the quark mass plane the linear combination

$$f_{\pi{\rm K}} = (2f_{\rm K} + f_\pi)/3 \tag{21}$$

has a particularly simple dependence on m_u. Thus it can be extrapolated well from the simulation points to the physical point. Using this feature, the physical $f_{\pi{\rm K}}$ was related [27] to t_0^*, the Gradient Flow scale, t_0 introduced by M. Lüscher [23] at the particular reference mass point

$$12m_\pi^2 t_0^* = 1.12 \quad \text{and} \quad m_u = m_d = m_s\,. \tag{22}$$

Inserting the phenomenological numbers for f_π and $f_{\rm K}$ yielded

$$(8t_0^*)^{1/2} = 0.413(5)(1)\,\text{fm}\,. \tag{23}$$

Just like our running couplings, it is irrelevant that t_0^* can't be measured directly in experiment. What matters is that we control the relation to Nature through $f_{\pi{\rm K}}$.

The scale t_0^* is convenient to finally determine $L_{\rm had}$ in physical units because it is defined in the mass-degenerate theory with quark masses far heavier than the physical up and down quark masses. Thus there are only two parameters and, since m_π is larger than in Nature, simulations are easier and finite size effects are smaller.

These properties enable determinations of t_0^*/a^2 and $L_{\rm had}/a$ at five common values of a (or bare coupling g_0) followed by a continuum extrapolation

$$(t_0^*)^{-1/2}L_{\rm max} = \left[(t_0^*)^{-1/2}L_{\rm max}\right]_{\rm cont} + B\,\frac{a^2}{t_0^*} \tag{24}$$

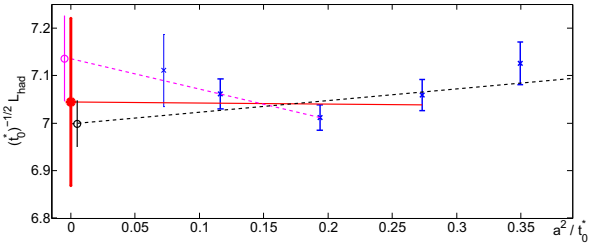

Figure 8: Preliminary continuum extrapolation of $(t_0^*)^{-1/2}L_{\rm max}$. The large volume simulation with the smallest lattice spacing is unfinished and therfore only included as an illustration. Extrapolations with 4,3 and 2 data points are shown together with a range for the continuum value covering all of them. It is taken as our preliminary result.

shown in figure 8. With eq. (23) we then find the preliminary values

$$L_{\rm max} = 1.03(3)\,\text{fm}\,, \quad \Lambda_{\overline{\rm MS}}^{(3)} = 332(14)\,\text{MeV}\,. \tag{25}$$

7. Connection to the 5-flavor theory and $\alpha_{\overline{\rm MS}}(m_Z)$

There is little doubt that 3-flavor QCD describes the low energy (E) phenomena including $L_{\rm had}f_{\pi{\rm K}}$ with high precision [28, 2]. In other words, the $(E/m_c)^2$ corrections in the effective theory expansion are small. However, $\Lambda^{(3)}$ needs to be related to $\Lambda^{(5)}$ because physical processes at high energies need $N_{\rm f} \geq 5$-flavor QCD and the standard $\alpha_{\overline{\rm MS}}(m_Z)$ is defined in the $N_{\rm f} = 5$ theory.

It has long been known how to connect these theories perturbatively [29, 30] and we now have 4-loop precision [31, 32] in the relation

$$\bar{g}^{(N_{\rm f}-1)}(m_*) = \bar{g}^{(N_{\rm f})}(m_*)(1 + {\rm O}([\bar{g}^{(N_{\rm f})}(m_*)]^4)\,, \tag{26}$$

where $m_* = \bar{m}_{\overline{\rm MS}}(m_*)$ is the mass of the decoupled quark. Together with eq. (4) and $\beta \to \beta^{\rm pert}$, we obtain the ratio of the Λ-parameters. We illustrate this by the (red) steps on the left in figure 3.

With the available perturbative precision, we find

$$\Lambda_{\overline{\rm MS}}^{(5)} = 207(11)\,\text{MeV}\,, \tag{27}$$
$$\alpha_{\overline{\rm MS}}(m_Z) = 0.1179(10)(2)\,. \tag{28}$$

The first error in α is just propagated from the one in Λ, which in turn is obtained by standard error propagation of all previously discussed numbers which were put together. The second error represents our estimate of the uncertainty from using PT in the connection $\Lambda_{\overline{\rm MS}}^{(3)} \to \Lambda_{\overline{\rm MS}}^{(5)}$. We arrive at it as follows. The 2, 3, 4-loop terms in eq. (26) combined with the 3, 4, 5-loop running lead to

contributions 109, 15, 7 (in units of 10^{-5}) to $\alpha_{\overline{\rm MS}}(m_Z)$. We take the sum of the last two contributions as our error in eq. (28). *Within PT*, this represents a very conservative error estimate: the known terms of the series behave similar to a convergent series but we treat it like an asymptotic one. The possibility remains that PT is entirely misleading when we apply it at $\mu = m_c$, decoupling the charm quark. As long as we do not have a computation of all the above steps with $N_{\rm f} = 4$, we have to live with this – in our opinion unlikely [28] – possibility. It would mean that the second error estimate is far off due to a breakdown of PT for $\Lambda^{(3)}/\Lambda^{(4)}$.

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References

[1] K. A. Olive, et al., Review of Particle Physics, Chin. Phys. C38 (2014) 090001. doi:10.1088/1674-1137/38/9/090001.

[2] S. Aoki, et al., Review of lattice results concerning low-energy particle physics, arXiv:1607.00299.

[3] T. van Ritbergen, J. A. M. Vermaseren, S. A. Larin, The four loop beta function in quantum chromodynamics, Phys. Lett. B400 (1997) 379–384. arXiv:hep-ph/9701390.

[4] M. Czakon, The Four-loop QCD beta-function and anomalous dimensions, Nucl.Phys. B710 (2005) 485–498. arXiv:hep-ph/0411261, doi:10.1016/j.nuclphysb.2005.01.012.

[5] P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Five-Loop Running of the QCD coupling constant, arXiv:1606.08659.

[6] M. Lüscher, P. Weisz, U. Wolff, A Numerical method to compute the running coupling in asymptotically free theories, Nucl. Phys. B359 (1991) 221–243. doi:10.1016/0550-3213(91)90298-C.

[7] S. Weinberg, New approach to the renormalization group, Phys. Rev. D8 (1973) 3497–3509. doi:10.1103/PhysRevD.8.3497.

[8] M. Lüscher, R. Narayanan, P. Weisz, U. Wolff, The Schrödinger functional: a renormalizable probe for non-Abelian gauge theories, Nucl. Phys. B384 (1992) 168–228. arXiv:hep-lat/9207009, doi:10.1016/0550-3213(92)90466-O.

[9] M. Lüscher, R. Sommer, P. Weisz, U. Wolff, A Precise determination of the running coupling in the SU(3) Yang-Mills theory, Nucl. Phys. B413 (1994) 481–502. arXiv:hep-lat/9309005, doi:10.1016/0550-3213(94)90629-7.

[10] S. Aoki, et al., Precise determination of the strong coupling constant in $N_f = 2 + 1$ lattice QCD with the Schrödinger functional scheme, JHEP 0910 (2009) 053. arXiv:0906.3906, doi:10.1088/1126-6708/2009/10/053.

[11] F. Tekin, R. Sommer, U. Wolff, The Running coupling of QCD with four flavors, Nucl. Phys. B840 (2010) 114–128. arXiv:1006.0672, doi:10.1016/j.nuclphysb.2010.07.002.

[12] R. Sommer, U. Wolff, Non-perturbative computation of the strong coupling constant on the lattice, Nucl. Part. Phys. Proc. 261-262 (2015) 155–184. arXiv:1501.01861, doi:10.1016/j.nuclphysbps.2015.03.013.

[13] P. Fritzsch, A. Ramos, The gradient flow coupling in the Schrödinger Functional, JHEP 10 (2013) 008. arXiv:1301.4388, doi:10.1007/JHEP10(2013)008.

[14] M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Sint, R. Sommer, Slow running of the Gradient Flow coupling from 200 MeV to 4 GeV in $N_{\rm f} = 3$ QCD, arXiv:1607.06423.

[15] M. Dalla Brida, P. Fritzsch, T. Korzec, A. Ramos, S. Sint, R. Sommer, Determination of the QCD Λ-parameter and the accuracy of perturbation theory at high energies, Phys. Rev. Lett. 117 (18) (2016) 182001. arXiv:1604.06193, doi:10.1103/PhysRevLett.117.182001.

[16] M. Lüscher, Advanced lattice QCD, in: Probing the standard model of particle interactions. Proceedings, Summer School in Theoretical Physics, NATO Advanced Study Institute, 68th session, Les Houches, France, July 28-September 5, 1997. Pt. 1, 2, 1998, pp. 229–280. arXiv:hep-lat/9802029.

[17] R. Sommer, Non-perturbative QCD: Renormalization, O(a)-improvement and matching to Heavy Quark Effective Theory, in: Workshop on Perspectives in Lattice QCD Nara, Japan, October 31-November 11, 2005, 2006. arXiv:hep-lat/0611020.

[18] S. Sint, On the Schrödinger functional in QCD, Nucl. Phys. B421 (1994) 135–158. arXiv:hep-lat/9312079, doi:10.1016/0550-3213(94)90228-3.

[19] S. Sint, P. Vilaseca, Lattice artefacts in the Schrödinger Functional coupling for strongly interacting theories, PoS LATTICE2012 (2012) 031. arXiv:1211.0411.

[20] A. Ramos, S. Sint, Symanzik improvement of the gradient flow in lattice gauge theories, Eur. Phys. J. C76 (1) (2016) 15. arXiv:1508.05552, doi:10.1140/epjc/s10052-015-3831-9.

[21] G. de Divitiis, R. Frezzotti, M. Guagnelli, M. Lüscher, R. Petronzio, R. Sommer, P. Weisz, U. Wolff, Universality and the approach to the continuum limit in lattice gauge theory, Nucl. Phys. B437 (1995) 447–470. arXiv:hep-lat/9411017, doi:10.1016/0550-3213(94)00019-B.

[22] A. Bode, P. Weisz, U. Wolff, Two loop computation of the Schrödinger functional in lattice QCD, Nucl. Phys. B576 (2000) 517–539, [Erratum: Nucl. Phys.B608,481(2001)]. arXiv:hep-lat/9911018, doi:10.1016/S0550-3213(00)00187-5.

[23] M. Lüscher, Properties and uses of the Wilson flow in lattice QCD, JHEP 1008 (2010) 071. arXiv:1006.4518, doi:10.1007/JHEP08(2010)071.

[24] R. Sommer, Scale setting in lattice QCD, PoS LATTICE2013 (2014) 015. arXiv:1401.3270.

[25] M. Bruno, et al., Simulation of QCD with $N_f = 2 + 1$ flavors of non-perturbatively improved Wilson fermions, JHEP 02 (2015) 043. arXiv:1411.3982, doi:10.1007/JHEP02(2015)043.

[26] W. Bietenholz, et al., Tuning the strange quark mass in lattice simulations, Phys. Lett. B690 (2010) 436–441. arXiv:1003.1114, doi:10.1016/j.physletb.2010.05.067.

[27] M. Bruno, T. Korzec, S. Schaefer, Setting the scale for the CLS $2 + 1$ flavor ensembles, arXiv:1608.08900.

[28] M. Bruno, J. Finkenrath, F. Knechtli, B. Leder, R. Sommer, Effects of Heavy Sea Quarks at Low Energies, Phys. Rev. Lett. 114 (10) (2015) 102001. arXiv:1410.8374, doi:10.1103/PhysRevLett.114.102001.

[29] S. Weinberg, Effective gauge theories, Phys.Lett. B91 (1980) 51. doi:10.1016/0370-2693(80)90660-7.

[30] W. Bernreuther, W. Wetzel, Decoupling of Heavy Quarks in the Minimal Subtraction Scheme, Nucl. Phys. B197 (1982) 228–236, [Erratum: Nucl. Phys.B513,758(1998)]. doi:10.1016/0550-3213(82)90288-7, 10.1016/S0550-3213(97)00811-0.

[31] K. G. Chetyrkin, J. H. Kühn, C. Sturm, QCD decoupling at four loops, Nucl. Phys. B744 (2006) 121–135. arXiv:hep-ph/0512060, doi:10.1016/j.nuclphysb.2006.03.020.

[32] Y. Schröder, M. Steinhauser, Four-loop decoupling relations for the strong coupling, JHEP 01 (2006) 051. arXiv:hep-ph/0512058, doi:10.1088/1126-6708/2006/01/051.