

# On Multiple Zeta Values of Even Arguments

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## Abstract

For  $k \leq n$ , let  $E(2n, k)$  be the sum of all multiple zeta values with even arguments whose weight is  $2n$  and whose depth is  $k$ . Of course  $E(2n, 1)$  is the value  $\zeta(2n)$  of the Riemann zeta function at  $2n$ , and it is well known that  $E(2n, 2) = \frac{3}{4}\zeta(2n)$ . Recently Z. Shen and T. Cai gave formulas for  $E(2n, 3)$  and  $E(2n, 4)$  in terms of  $\zeta(2n)$  and  $\zeta(2)\zeta(2n-2)$ . We give two formulas for  $E(2n, k)$ , both valid for arbitrary  $k \leq n$ , one of which generalizes the Shen-Cai results; by comparing the two we obtain a Bernoulli-number identity. We also give explicit generating functions for the numbers  $E(2n, k)$  and for the analogous numbers  $E^*(2n, k)$  defined using multiple zeta-star values of even arguments.

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# 1 Introduction and Statement of Results

For positive integers  $i_1, \dots, i_k$  with  $i_1 > 1$ , we define the multiple zeta value  $\zeta(i_1, \dots, i_k)$  by

$$\zeta(i_1, \dots, i_k) = \sum_{n_1 > \dots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}. \quad (1)$$

The multiple zeta value (1) is said to have weight  $i_1 + \dots + i_k$  and depth  $k$ . Many remarkable identities have been proved about these numbers, but in this note we will concentrate on the case where the  $i_j$  are even integers. Let  $E(2n, k)$  be the sum of all the multiple zeta values of even-integer arguments having weight  $2n$  and depth  $k$ , i.e.,

$$E(2n, k) = \sum_{\substack{i_1, \dots, i_k \text{ even} \\ i_1 + \dots + i_k = 2n}} \zeta(i_1, \dots, i_k).$$

Of course

$$E(2n, 1) = \zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}, \quad (2)$$

where  $B_{2n}$  is the  $2n$ th Bernoulli number, by the classical formula of Euler. Euler also studied double zeta values (i.e., multiple zeta values of depth 2) and in his paper [3] gave two identities which read

$$\begin{aligned} \sum_{i=2}^{2n-1} (-1)^i \zeta(i, 2n-i) &= \frac{1}{2} \zeta(2n) \\ \sum_{i=2}^{2n-1} \zeta(i, 2n-i) &= \zeta(2n) \end{aligned}$$

in modern notation. From these it follows that

$$E(2n, 2) = \frac{3}{4} \zeta(2n),$$

though Gangl, Kaneko and Zagier [4] seem to be the first to have pointed it out in print. Recently Shen and Cai [11] proved the formulas

$$E(2n, 3) = \frac{5}{8} \zeta(2n) - \frac{1}{4} \zeta(2) \zeta(2n-2), \quad n \geq 3 \quad (3)$$

$$E(2n, 4) = \frac{35}{64} \zeta(2n) - \frac{5}{16} \zeta(2) \zeta(2n-2), \quad n \geq 4. \quad (4)$$

Identity (3) was also proved by Machide [10] using a different method.

This begs the question whether there is a general formula of this type for  $E(2n, k)$ . The pattern

$$\frac{3}{4}, \quad \frac{3}{4} \cdot \frac{5}{6} = \frac{5}{8}, \quad \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} = \frac{35}{64}$$

of the leading coefficients makes one curious. In fact, the general result is as follows.

**Theorem 1.** *For  $k \leq n$ ,*

$$\begin{aligned} E(2n, k) &= \frac{1}{2^{2(k-1)}} \binom{2k-1}{k} \zeta(2n) \\ &\quad - \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{2^{2k-3}(2j+1)B_{2j}} \binom{2k-2j-1}{k} \zeta(2j) \zeta(2n-2j). \end{aligned}$$

The next two cases after (4) are

$$\begin{aligned} E(2n, 5) &= \frac{63}{128} \zeta(2n) - \frac{21}{64} \zeta(2) \zeta(2n-2) + \frac{3}{64} \zeta(4) \zeta(2n-4) \\ E(2n, 6) &= \frac{231}{512} \zeta(2n) - \frac{21}{64} \zeta(2) \zeta(2n-2) + \frac{21}{256} \zeta(4) \zeta(2n-4). \end{aligned}$$

We prove Theorem 1 in §3 below, using the generating function

$$F(t, s) = 1 + \sum_{n \geq k \geq 1} E(2n, k) t^n s^k.$$

In §2 we establish the following explicit formula.

**Theorem 2.**

$$F(t, s) = \frac{\sin(\pi\sqrt{1-s}\sqrt{t})}{\sqrt{1-s} \sin(\pi\sqrt{t})}.$$

Our proof uses symmetric functions. We define a homomorphism  $\mathfrak{Z} : \text{Sym} \rightarrow \mathbf{R}$ , where Sym is the algebra of symmetric functions, and a family  $N_{n,k} \in \text{Sym}$  such that  $\mathfrak{Z}$  sends  $N_{n,k}$  to  $E(2n, k)$ . We then obtain a formula for the generating functions

$$\mathcal{F}(t, s) = 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k \in \text{Sym}[[t, s]]$$

and apply  $\mathfrak{Z}$  to get Theorem 2.

Along with the multiple zeta values there are the multiple zeta-star values

$$\zeta^*(i_1, \dots, i_k) = \sum_{n_1 \geq \dots \geq n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}},$$

where the strict inequalities in equation (1) are replaced by  $\geq$ . These coincide with multiple zeta values for depth 1, and for greater depths they are simply sums of multiple zeta values, e.g.,

$$\zeta^*(6, 2, 4) = \zeta(6, 2, 4) + \zeta(8, 4) + \zeta(6, 6) + \zeta(12).$$

Conversely, multiple zeta values are sums of multiple zeta-star values with signs alternating by depth, e.g.,

$$\zeta(6, 2, 4) = \zeta^*(6, 2, 4) - \zeta^*(8, 4) - \zeta^*(6, 6) + \zeta^*(12).$$

For  $k \leq n$  we can define  $E^*(2n, k)$  as the sum of all multiple zeta-star values with even arguments having weight  $2n$  and depth  $k$ . The generating function

$$F^*(t, s) = 1 + \sum_{n \geq k \geq 1} E^*(2n, k) t^n s^k$$

turns out to have a remarkably simple relation to  $F(t, s)$ , as we show in §2.

**Corollary 1.** *The generating functions  $F(t, s)$  and  $F^*(t, s)$  are related by  $F^*(t, s) = F(t, -s)^{-1}$ .*

From the form of  $\mathcal{F}(t, s)$  we show that it satisfies a partial differential equation (Proposition 1 below), which is equivalent to a recurrence for the  $N_{n,k}$ . From the latter we obtain a formula for the  $N_{n,k}$  in terms of complete and elementary symmetric functions, to which  $\mathfrak{Z}$  can be applied to give the following alternative formula for  $E(2n, k)$ .

**Theorem 3.** *For  $k \leq n$ ,*

$$E(2n, k) = \frac{(-1)^{n-k-1} \pi^{2n}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} 2(2^{2i-1} - 1) B_{2i}.$$

Note that the sum given by Theorem 3 has  $n-k+1$  terms, while that given by Theorem 1 has  $\lfloor \frac{k-1}{2} \rfloor + 1$  terms. Yet another explicit formula for  $E(2n, k)$  can be obtained by setting  $d = 1$  in Theorem 7.1 of Komori, Matsumoto and Tsumura [8]. That formula expresses  $E(2n, k)$  as a sum over partitions of  $k$ , and it is not immediately clear how it relates to our two formulas.

Comparison of Theorems 1 and 3 establishes the following Bernoulli-number identity.

**Theorem 4.** *For  $k \leq n$ ,*

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} = (-1)^k 2^{2k-2n} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} (2^{2i-1} - 1) B_{2i}.$$

It is interesting to contrast this result with the Gessel-Viennot identity (see [2, Theorem 4.2]) valid on the complementary range:

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} = \frac{2n+1}{2} \binom{2k-2n}{k}, \quad k > n. \quad (5)$$

Note that the right-hand side of equation (5) is zero unless  $k \geq 2n$ .

## 2 Symmetric Functions

We think of  $\text{Sym}$  as the subring of  $\mathbf{Q}[[x_1, x_2, \dots]]$  consisting of those formal power series of bounded degree that are invariant under permutations of the  $x_i$ . A useful reference is the first chapter of Macdonald [9]. We denote the elementary, complete, and power-sum symmetric functions of degree  $i$  by  $e_i$ ,

$h_i$ , and  $p_i$  respectively. They have associated generating functions

$$\begin{aligned} E(t) &= \sum_{j=0}^{\infty} e_j t^j = \prod_{i=1}^{\infty} (1 + tx_i) \\ H(t) &= \sum_{j=0}^{\infty} h_j t^j = \prod_{i=1}^{\infty} \frac{1}{1 - tx_i} = E(-t)^{-1} \\ P(t) &= \sum_{j=1}^{\infty} p_j t^{j-1} = \sum_{i=1}^{\infty} \frac{x_i}{1 - tx_i} = \frac{H'(t)}{H(t)}. \end{aligned}$$

As explained in [6] and in greater detail in [7], there is a homomorphism  $\zeta : \text{Sym}^0 \rightarrow \mathbf{R}$ , where  $\text{Sym}^0$  is the subalgebra of  $\text{Sym}$  generated by  $p_2, p_3, p_4, \dots$ , such that  $\zeta(p_i)$  is the value  $\zeta(i)$  of the Riemann zeta function at  $i$ , for  $i \geq 2$  (in [6, 7] this homomorphism is extended to all of  $\text{Sym}$ , but we do not need the extension here). Let  $\mathcal{D} : \text{Sym} \rightarrow \text{Sym}$  be the degree-doubling map that sends  $x_i$  to  $x_i^2$ . Then  $\mathcal{D}(\text{Sym}) \subset \text{Sym}^0$ , so the composition  $\mathfrak{Z} = \zeta \mathcal{D}$  is defined on all of  $\text{Sym}$ . (Alternatively, we can simply think of  $\mathfrak{Z}$  as sending  $x_i$  to  $1/i^2$ : see [9, Ch. I, §2, ex. 21].) Note that  $\mathfrak{Z}(p_i) = \zeta(2i) \in \mathbf{R}$ . Further,  $\mathfrak{Z}$  sends the monomial symmetric function  $m_{i_1, \dots, i_k}$  to the symmetrized sum of multiple zeta values

$$\frac{1}{|\text{Iso}(i_1, \dots, i_k)|} \sum_{\sigma \in S_k} \zeta(2i_{\sigma(1)}, 2i_{\sigma(2)}, \dots, 2i_{\sigma(k)}),$$

where  $S_k$  is the symmetric group on  $k$  letters and  $\text{Iso}(i_1, \dots, i_k)$  is the subgroup of  $S_k$  that fixes  $(i_1, \dots, i_k)$  under the obvious action.

Now let  $N_{n,k}$  be the sum of all the monomial symmetric functions corresponding to partitions of  $n$  having length  $k$ . Of course  $N_{n,k} = 0$  unless  $k \leq n$ , and  $N_{k,k} = e_k$ . Then  $\mathfrak{Z}$  sends  $N_{n,k}$  to  $E(2n, k)$ . Also, if we define (as in the introduction)

$$\mathcal{F}(t, s) = 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k,$$

then  $\mathfrak{Z}$  sends  $\mathcal{F}(t, s)$  to the generating function  $F(t, s)$ . We have the following simple description of  $\mathcal{F}(t, s)$ .

**Lemma 1.**  $\mathcal{F}(t, s) = E((s-1)t)H(t)$ .

*Proof.* Evidently  $\mathcal{F}(t, s)$  has the formal factorization

$$\prod_{i=1}^{\infty} (1 + stx_i + st^2x_i^2 + \dots) = \prod_{i=1}^{\infty} \frac{1 + (s-1)tx_i}{1 - tx_i} = E((s-1)t)H(t).$$

□

*Proof of Theorem 2.* Using the well-known formula for  $\zeta(2, 2, \dots, 2)$  [5, Cor. 2.3],

$$\mathfrak{Z}(e_n) = \zeta(\underbrace{2, 2, \dots, 2}_n) = \frac{\pi^{2n}}{(2n+1)!}. \quad (6)$$

Hence

$$\mathfrak{Z}(E(t)) = \frac{\sinh(\pi\sqrt{t})}{\pi\sqrt{t}},$$

and the image of  $H(t) = E(-t)^{-1}$  is

$$\mathfrak{Z}(H(t)) = \frac{\pi\sqrt{-t}}{\sinh(\pi\sqrt{-t})} = \frac{\pi\sqrt{t}}{\sin(\pi\sqrt{t})}.$$

Thus from Lemma 1  $F(t, s) = \mathfrak{Z}(\mathcal{F}(t, s))$  is

$$\mathfrak{Z}(E((s-1)t)H(t)) = \frac{\sinh(\pi\sqrt{(s-1)t})}{\pi\sqrt{(s-1)t}} \frac{\pi\sqrt{t}}{\sin(\pi\sqrt{t})} = \frac{\sin(\pi\sqrt{(1-s)t})}{\sqrt{1-s}\sin(\pi\sqrt{t})}.$$

□

Taking limits as  $s \rightarrow 1$  in Theorem 2, we obtain

$$F(t, 1) = \frac{\pi\sqrt{t}}{\sin \pi\sqrt{t}}$$

and so, taking the coefficient of  $t^n$ , the following result.

**Corollary 2.** *For all  $n \geq 1$ ,*

$$\sum_{k=1}^n E(2n, k) = \frac{2(2^{2n-1} - 1)(-1)^{n-1} B_{2n} \pi^{2n}}{(2n)!} = 2(1 - 2^{1-2n}) \zeta(2n).$$

**Remark.** This result was obtained previously by Aoki, Kombu and Ohno [1], who stated it in the language of multiple zeta-star values; since

$$\zeta^*(\underbrace{2, \dots, 2}_n) = E(2n, n) + E(2n, n-1) + \dots + E(2n, 1), \quad (7)$$

Corollary 2 can be recognized as [1, equation (4.6)]. In fact, using Euler's infinite product for sine, one sees that

$$F(z^2, 1) = \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right)^{-1} = 1 + \sum_{n \geq 1} \zeta^*(\underbrace{2, \dots, 2}_n) z^{2n}.$$

Recall that we defined  $E^*(2n, k)$  as the sum of all multiple zeta-star values with even arguments having depth  $k$  and weight  $2n$ . The left-hand side of equation (7) is  $E^*(2n, n)$ , and that equation generalizes as follows.

**Lemma 2.** For  $n \geq k \geq 1$ ,

$$E^*(2n, k) = \sum_{j=1}^k \binom{n-j}{k-j} E(2n, j).$$

*Proof.* Let  $I = (i_1, \dots, i_k)$  be a composition (i.e., ordered partition) of  $n$ . We can think of  $I$  as specified by placing  $k-1$  dividers within a row of  $n$  dots, which makes it clear that there are  $\binom{n-1}{k-1}$  compositions of  $n$  with  $k$  parts. If we associate to  $I$  the multiple zeta and multiple star zeta values

$$\zeta(2I) = \zeta(2i_1, \dots, 2i_k), \quad \zeta^*(2I) = \zeta^*(2i_1, \dots, 2i_k),$$

of even values, then  $\zeta^*(2I) = \sum_{I \succeq J} \zeta(2J)$ , where  $\succeq$  is the partial order on compositions given by refinement, i.e.,  $I \succeq J$  if  $J$  can be obtained by combining adjacent parts of  $I$ ; in terms of the dividers-in-row-of-dots picture,  $J$  is obtained by removing some dividers from  $I$ .

Now  $E^*(2n, k)$  is the sum of all  $\zeta^*(2I)$  with  $I$  having  $k$  parts. Write each of these as a sum of multiple zeta values. Then the coefficient of  $\zeta(2J)$ , where  $J$  has  $j \leq k$  parts, is the number of distinct compositions  $I$  with  $k$  parts such that  $J \preceq I$ ; this corresponds to the number of ways to insert  $k-j$  additional dividers into  $J$ . Since there are  $n-1-(j-1)=n-j$  places to put them, this number is  $\binom{n-j}{k-j}$ .  $\square$

*Proof of Corollary 1.* Using Lemma 2 and Theorem 2,

$$\begin{aligned}
F^*(t, s) &= 1 + \sum_{n \geq k \geq 1} E^*(2n, k) t^n s^k = 1 + \sum_{n \geq k \geq 1} \sum_{j=1}^k \binom{n-j}{k-j} E(2n, j) t^n s^k \\
&= 1 + \sum_{n \geq j \geq 1} E(2n, j) \sum_{i=0}^{n-j} \binom{n-j}{i} t^n s^{j+i} = 1 + \sum_{n \geq j \geq 1} E(2n, j) t^n s^j (1+s)^{n-j} \\
&= F\left(t(1+s), \frac{s}{1+s}\right) = \frac{\sin\left(\pi\sqrt{1-\frac{s}{1+s}}\sqrt{t(1+s)}\right)}{\sqrt{1-\frac{s}{1+s}}\sin(\pi\sqrt{t(1+s)})} = \frac{\sqrt{1+s}\sin(\pi\sqrt{t})}{\sin(\pi\sqrt{(1+s)t})} \\
&= \frac{1}{F(t, -s)}.
\end{aligned}$$

□

Another consequence of Lemma 1 is the following partial differential equation.

**Proposition 1.**

$$t \frac{\partial \mathcal{F}}{\partial t}(t, s) + (1-s) \frac{\partial \mathcal{F}}{\partial s}(t, s) = tP(t)\mathcal{F}(t, s).$$

*Proof.* From Lemma 1 we have

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial t}(t, s) &= (s-1)E'((s-1)t)H(t) + E((s-1)t)H'(t) \\
\frac{\partial \mathcal{F}}{\partial s}(t, s) &= tE'((s-1)t)H(t)
\end{aligned}$$

from which the conclusion follows. □

Now examine the coefficient of  $t^n s^k$  in Proposition 1 to get the following.

**Proposition 2.** For  $n \geq k+1$ ,

$$p_1 N_{n-1, k} + p_2 N_{n-2, k} + \cdots + p_{n-k} N_{k, k} = (n-k)N_{n, k} + (k+1)N_{n, k+1}.$$

It is also possible to prove this result directly via a counting argument like that used to prove the lemma of [7, p. 16].

The preceding result allows us to write  $N_{n, k}$  explicitly in terms of complete and elementary symmetric functions as follows.

**Lemma 3.** For  $r \geq 0$ ,

$$N_{k+r,k} = \sum_{i=0}^r (-1)^i \binom{k+i}{i} h_{r-i} e_{k+i}.$$

*Proof.* We use induction on  $r$ , the result being evident for  $r = 0$ . Proposition 2 gives

$$\sum_{i=1}^{r+1} p_i N_{k+r+1-i,k} = (r+1)N_{k+r+1,k} + (k+1)N_{k+r+1,k+1},$$

which after application of the induction hypothesis becomes

$$\begin{aligned} \sum_{i=1}^{r+1} \sum_{j=0}^{r+1-i} (-1)^j p_i \binom{k+j}{j} h_{r+1-i-j} N_{k+j,k+j} = \\ (r+1)N_{k+r+1,k} + (k+1) \sum_{j=0}^r \binom{k+1+j}{j} h_{r-j} N_{k+1+j,k+1+j}. \end{aligned}$$

The latter equation can be rewritten

$$\begin{aligned} \sum_{j=0}^r (-1)^j \binom{k+j}{j} N_{k+j,k+j} \sum_{i=1}^{r+1-j} p_i h_{r+1-i-j} = \\ (r+1)N_{k+r+1,k} - (k+1) \sum_{j=1}^{r+1} (-1)^j \binom{k+j}{j-1} h_{r+1-j} N_{k+j,k+j}. \end{aligned}$$

Now the inner sum on the left-hand side is  $(r+1-j)h_{r+1-j}$  by the recurrence relating the complete and power-sum symmetric functions, so we have

$$\begin{aligned} (r+1)N_{k+r+1,k} - (r+1)N_{k,k} h_{r+1} = \\ \sum_{j=1}^{r+1} (-1)^j h_{r+1-j} N_{k+j,k+j} \left( (r+1-j) \binom{k+j}{j} + (k+1) \binom{k+j}{j-1} \right), \end{aligned}$$

and the conclusion follows after the observation that  $(k+1) \binom{k+j}{j-1} = j \binom{k+j}{j}$ .  $\square$

*Proof of Theorem 3.* Rewrite Lemma 3 in the form

$$N_{n,k} = \sum_{i=0}^{n-k} \binom{n-i}{k} (-1)^{n-k-i} h_i e_{n-i}$$

and apply the homomorphism  $\mathfrak{Z}$ , using equation (6) and

$$\mathfrak{Z}(h_i) = \frac{2(2^{2i-1} - 1)(-1)^{i-1} B_{2i} \pi^{2i}}{(2i)!}.$$

□

### 3 Proof of Theorems 1 and 4

From the introduction we recall the statement of Theorem 1:

$$\begin{aligned} E(2n, k) &= \frac{1}{2^{2(k-1)}} \binom{2k-1}{k} \zeta(2n) \\ &\quad - \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} \frac{1}{2^{2k-3} (2j+1) B_{2j}} \binom{2k-2j-1}{k} \zeta(2j) \zeta(2n-2j). \end{aligned}$$

We note that Euler's formula (2) can be used to write the result in the alternative form

$$E(2n, k) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^j \pi^{2j} \zeta(2n-2j)}{2^{2k-2j-2} (2j+1)!} \binom{2k-2j-1}{k} \quad (8)$$

which avoids mention of Bernoulli numbers.

We now expand out the generating function  $F(t, s)$ . We have

$$\begin{aligned} F(t, s) &= \frac{1}{\sqrt{1-s} \sin \pi \sqrt{t}} \sin(\pi \sqrt{t} \sqrt{1-s}) \\ &= \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \sum_{j=0}^{\infty} \frac{(-1)^j \pi^{2j} t^j (1-s)^j}{(2j+1)!} = \sum_{k=0}^{\infty} s^k G_k(t), \end{aligned}$$

where

$$G_k(t) = (-1)^k \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \sum_{j \geq k} \frac{(-1)^j \pi^{2j} t^j}{(2j+1)!} \binom{j}{k}. \quad (9)$$

Then Theorem 1 is equivalent to the statement that

$$G_k(t) = \sum_{n \geq k} t^n \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^j \pi^{2j} \zeta(2n-2j)}{2^{2k-2j-2} (2j+1)!} \binom{2k-2j-1}{k}$$

for all  $k$ . We can write the latter sum as

$$\begin{aligned} & \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2} (2j+1)!} \binom{2k-2j-1}{k} \sum_{n \geq j+1} \zeta(2n-2j) t^{n-j} - \\ & \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2} (2j+1)!} \binom{2k-2j-1}{k} \sum_{n=j+1}^{k-1} \zeta(2n-2j) t^{n-j} = \\ & \frac{1}{2} (1 - \pi\sqrt{t} \cot \pi\sqrt{t}) \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2} (2j+1)!} \binom{2k-2j-1}{k} - \\ & \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2} (2j+1)!} \binom{2k-2j-1}{k} \sum_{n=j+1}^{k-1} \zeta(2n-2j) t^{n-j}, \quad (10) \end{aligned}$$

where we have used the generating function

$$\frac{1}{2} (1 - \pi\sqrt{t} \cot \pi\sqrt{t}) = \sum_{i=1}^{\infty} \zeta(2i) t^i.$$

Note that the last sum in (10) is a polynomial that cancels exactly those terms in

$$\frac{1}{2} (1 - \pi\sqrt{t} \cot \pi\sqrt{t}) \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2} (2j+1)!} \binom{2k-2j-1}{k} \quad (11)$$

of degree less than  $k$ . Thus, to prove Theorem 1 it suffices to show that

$$G_k(t) = \text{terms of degree } \geq k \text{ in expression (11).}$$

From equation (9) it is evident that

$$G_k(t) = \frac{\pi\sqrt{t}}{\sin \pi\sqrt{t}} \cdot \frac{(-t)^k}{k!} \cdot \frac{d^k}{dt^k} \left( \frac{\sin \pi\sqrt{t}}{\pi\sqrt{t}} \right). \quad (12)$$

We use this to obtain an explicit formula for  $G_k(t)$ .

**Lemma 4.** For  $k \geq 0$ ,

$$G_k(t) = P_k(\pi^2 t) \pi \sqrt{t} \cot \pi \sqrt{t} + Q_k(\pi^2 t),$$

where  $P_k, Q_k$  are polynomials defined by

$$\begin{aligned} P_k(x) &= - \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4x)^j}{2^{2k-1}(2j+1)!} \binom{2k-2j-1}{k} \\ Q_k(x) &= \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-4x)^j}{2^{2k}(2j)!} \binom{2k-2j}{k}. \end{aligned}$$

*Proof.* In view of equation (12), the conclusion is equivalent to

$$f^{(k)}(t) = (-1)^k k! t^{-k} P_k(\pi^2 t) \cos \pi \sqrt{t} + (-1)^k k! t^{-k} Q_k(\pi^2 t) f(t),$$

where  $f(t) = \sin \pi \sqrt{t} / \pi \sqrt{t}$ . Differentiating, one sees that the polynomials  $P_k$  and  $Q_k$  are determined by the recurrence

$$\begin{aligned} (k+1)P_{k+1}(x) &= kP_k(x) - xP'_k(x) - \frac{1}{2}Q_k(x) \\ (k+1)Q_{k+1}(x) &= \frac{2k+1}{2}Q_k(x) - xQ'_k(x) + \frac{x}{2}P_k(x) \end{aligned}$$

together with the initial conditions  $P_0(x) = 0, Q_0(x) = 1$ . The recurrence and initial conditions are satisfied by the explicit formulas above.  $\square$

*Proof of Theorem 1.* Using Lemma 4, we have

$$\begin{aligned} G_k(t) &= - \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-1}(2j+1)!} \binom{2k-2j-1}{k} \pi \sqrt{t} \cot \pi \sqrt{t} \\ &\quad + \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k}(2j)!} \binom{2k-2j}{k} = \\ &\quad \frac{1}{2} (1 - \pi \sqrt{t} \cot \pi \sqrt{t}) \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!} \binom{2k-2j-1}{k} \\ &\quad + \text{terms of degree } < k, \end{aligned}$$

and this completes the proof.  $\square$

*Proof of Theorem 3.* Using Theorem 1 in the form of equation (8), eliminate  $\zeta(2n - 2j)$  using Euler's formula (2) and then compare with Theorem 3 to get

$$\sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^{n-1} \pi^{2n} B_{2n-2j}}{2^{2k-2n-1} (2n-2j)! (2j+1)!} \binom{2k-2j-1}{k} = \frac{(-1)^{n-k-1} \pi^{2n}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} 2(2^{2i-1} - 1) B_{2i}.$$

Now multiply both sides by  $(-1)^{n-1} 2^{2k-2n-1} \pi^{-2n} (2n+1)!$  and rewrite the factorials on the left-hand side as a binomial coefficient.  $\square$

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