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ON THE CHARACTERIZATION OF THE HIGGS PHASE IN LATTICE GAUGE THEORIES

by

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On the characterization of the Higgs phase in lattice gauge theories

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Abstract: We consider Higgs models on a lattice in 3 or 4 dimensions. Higgs scalars are assumed to transform trivially under a finite subgroup Γ of the compact gauge group G. We adopt 't Hooft's definition of the Higgs phase, it is characterized by a nonvanishing free energy per unit length (area) of a vortex in 3 (4) dimensions. By using a Peierls argument we show that the models are in the Higgs phase in this sense for suitable coupling constants.

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1. Introduction and summary of results.

Since the invention of lattice gauge theories [1,2] many attempts have been made to determine their phase structure (see for example refs. [3,4,5]). The question, how to characterize the different phases, is more difficult to answer for lattice gauge models than for spin models and was investigated by several people.

Recent work of Mack and Petkova [6] and of 't Hooft [7] suggests to classify the phases of pure lattice gauge theories by the dependence of the free energy of a system in a box or vortex container on certain boundary conditions. This is reviewed in sect.3 of this paper, including fundamental scalar fields (Higgs fields) coupled to the gauge field in a gauge invariant way. We consider gauge fields on a lattice in 3 or 4 dimensions. The gauge group G is a compact semi-simple group with nontrivial center. If a Higgs field is included, it is assumed to transform trivially under a nontrivial finite subgroup Γ of the center of G. The theory then possesses Γ-vortices. The system is considered on a finite lattice Λ with some boundary conditions. Λ may either be a vortex container of ref.[6] with fixed boundary conditions on $\partial\Lambda$, or a torus as in ref.[7] , where the boundary conditions for the gauge field specify transition functions of a fibre bundle. Then a change of the boundary conditions is defined by certain singular gauge transformations with elements of Γ . It has the effect to change the vortex content (the vorticity) of the system by a definite amount. Now one asks how the free energy of the system changes under the singular gauge transformations. This change gives a quantitative measure for the free energy of a vortex, which is used to characterize different phases. According to 't Hooft's definition the system is in the Higgs phase if the free energy of a vortex per unit length resp. area does not tend to zero in the limit of infinite width.

In sect.4 we investigate models with Higgs fields. We show that the Higgs phase exists at low temperatures by using a Peierls argument. On the other hand it can be shown [8] that the high temperature phase is not a Higgs phase. These results support the idea that above is a useful definition of a Higgs phase.

The reader should be warned not to confuse the Higgs phase in 't Hooft's sense with the Higgs mechanism. What is popularly called Higgs mechanism does not imply that the system is in a Higgs phase. In particular, one speaks of a Higgs mechanism even for models where the stability group Γ of the scalar fields is trivial-such as for instance a SU(2) gauge theory with a Higgs doublet. On the other hand, if Γ is trivial the model with the scalars has no Higgs phase in 't Hooft's sense by definition. In Higgs models of this type the high temperature region is linked continuously to a region of coupling parameters, where the Higgs mechanism takes place according to conventional wisdom. See the discussion in ref.[9]. Nevertheless in such a model there can be different phases. Their existence has been established for the Z_2-Higgs model [10] . The second phase is obtained from the Higgs phase of the pure Z_2 gauge theory without the scalars by convergent perturbation theory in the effect of the scalar field.

In sect.5 we discuss the relationship between the Higgs phase of ν-dimensional lattice gauge theory and spontaneous breakdown of a global Γ-symmetry in ν-1 -dimensional Heisenberg models with fluctuating couplings.

2. 2-dimensional Ising model and 3-dimensional $\mathbb{Z}_2$ lattice gauge theories as introductory examples.

a) The 2-dimensional Ising model.

Consider a 2-dimensional square lattice Λ with spin variables $\sigma(x) = \pm 1$ at the lattice points $x \in \Lambda$. The Hamiltonian is

$$H = -J \sum_{\langle xy \rangle} \sigma(x)\, \sigma(y) \qquad , \qquad J > 0 \tag{2.1}$$

$\langle xy \rangle$: pair of nearest neighbour lattice points

and describes ferromagnetic coupling. In the statistical mechanics of this model the probability measure on the space of configurations is given by the Boltzmann factor

$$\frac{1}{Z}\, e^{-\beta H} \tag{2.2}$$

where $Z = \sum e^{-\beta H}$, $\beta = \frac{1}{kT}$ the inverse temperature.

The correlation function is

$$\langle \sigma(x)\, \sigma(y) \rangle = \frac{1}{Z} \sum \sigma(x)\, \sigma(y)\, e^{-\beta H} \tag{2.3}$$

It is well known that the Ising model has a critical point β_c. At small temperatures β^{-1} we have an ordered phase with spontaneous magnetization

$$\lim_{|x-y| \to \infty} \langle \sigma(x)\, \sigma(y) \rangle = m^2 > 0 \tag{2.4}$$

whereas at high temperatures there is a disordered phase without spontaneous magnetization.

Peierls [11] was the first to give an argument for long range order at low temperatures and we will use his idea of contours in the following discussion. A Peierls contour is defined to be a set of links $b = \langle xy \rangle$ such that $\sigma(b) := \sigma(x)\sigma(y) = -1$. On the dual lattice a Peierls contour is closed (Fig.1), because of

$$\prod_{b \in \partial P} \sigma(b) = 1 \quad , \text{ P a plaquette of four links} \tag{2.5}$$

Peierls contours can be chosen nonintersecting.

For two distant points x and y, linked by a path $\mathcal{C}$, we have

$$\sigma(x)\, \sigma(y) = \prod_{b \in \mathcal{C}} \sigma(b) = (-1)^N \tag{2.6}$$

where N is the number of Peierls contours intersecting $\mathcal{C}$ an odd number of times, and therefore

$$\langle \sigma(x)\, \sigma(y) \rangle = \sum_N (-1)^N p_N \quad . \tag{2.7}$$

p_N is the probability that N contours wind around x or y (not around both). Obviously contours which do not wind around x or y are of no interest (Fig.1). Formula (2.7) expresses the correlation function through the probability distribution of contours. If there were no contours, $\langle \sigma(x)\sigma(y)\rangle$ would be 1 identically. A suppression of long contours would lead to the result that the correlation function tends to a non-zero constant for large $|x-y|$. On the other hand, if contours of arbitrary length are abundant, the correlation function goes to zero exponentially.

Peierls showed that long contours are suppressed at low temperatures and that there is spontaneous magnetization. A quantitative measure of this suppression is the free energy of a contour per unit length. It is defined in the following way. Consider a finite lattice of length ℓ and width t : $x = (x_1, x_2)$, $0 \le x_1 \le \ell$, $0 \le x_2 \le t$. Impose periodic boundary conditions in the x_1-direction, whereas in the x_2-direction the boundary conditions are

$$\sigma(x_1, 0) = \sigma(x_1, t)\, \gamma \, , \qquad \gamma = \pm 1 \quad . \tag{2.8}$$

The system now lives on a torus T^2 and the boundary conditions fix transition functions for a fiber bundle over T^2, which is nontrivial if $\gamma = -1$. In this case every closed curve which winds around the torus in the x_2-direction contains an odd number of links with $\sigma(b) = -1$ From this follows the existence of an odd number of Peierls contours winding around the torus in the x_1-direction. In the case $\gamma = +1$ the number has to be even. The corresponding partition functions are denoted by Z_γ and the free energy F_γ is given by

$$\beta F_\gamma = - \ln Z_\gamma \quad . \tag{2.9}$$

Then the free energy of a Peierls contour per unit length is defined by

$$f(t) = \lim_{\ell \to \infty} \frac{1}{\ell} \beta (F_- - F_+) = - \lim_{\ell \to \infty} \frac{1}{\ell} \ln \frac{Z_-}{Z_+} \tag{2.10}$$

and we consider $\qquad f = \lim_{t \to \infty} f(t) \quad ,$ (2.11)

so that the effects of the finite lattice width are ignored.

If $f > 0$, the probability of finding a contour of length l decreases exponentially with l. This happens at low temperatures, as can be shown by the Peierls argument.

The purpose of this section is to illustrate a concept which will be used for the study of lattice gauge fields. The important points are

i) objects which disorder the system,

ii) the study of these objects by considering nontrivial boundary conditions for some containers,

iii) the characterization of the phases by the change of the free energy for different boundary conditions.

b) <u>The Z_2 lattice gauge theory in 3 dimensions.</u>

This is a gauge theory analogue of the Ising model [12]. The gauge field variables $\sigma(b) = \pm 1 \in Z_2$ sit on the links b of a 3-dimensional cubic lattice. For each plaquette P define

$$\sigma(\dot{P}) = \prod_{b \in \partial P} \sigma(b) \qquad . \tag{2.12}$$

The action is $$L = \beta \sum_{P} \sigma(\dot{P}) \qquad . \tag{2.13}$$

It appears in the measure of path integrals

$$d\mu = \frac{1}{Z} e^{L} \prod_{b} d\sigma(b) \quad , \qquad Z = \int e^{L} \prod_{b} d\sigma(b) \tag{2.14}$$

$d\sigma$ the discrete Haar measure on Z_2 .

A gauge invariant correlation function is the Wilson loop integral [12, 1]

$$\mathcal{W}(\mathcal{C}) = \left\langle \prod_{b \in \mathcal{C}} \sigma(b) \right\rangle = \left\langle \sigma(\mathcal{C}) \right\rangle \tag{2.15}$$

for closed paths $\mathcal{C} = \partial \Xi$ on the lattice. We have the "second Maxwell equation" for cubes c

$$\prod_{P \in \partial c} \sigma(\dot{P}) = 1 \tag{2.16}$$

and $$\sigma(\mathcal{C}) = \prod_{P \in \Xi} \sigma(\dot{P}) \tag{2.17}$$

Now a vortex is defined to be a set of plaquettes P with $\sigma(\dot{P}) = -1$. Because of (2.16) they form closed paths on the dual lattice (Fig.2) and we consider separate paths as different vortices.

If there are N vortices winding around $\mathcal{C}$ and thereby going through Ξ , we find

$$\sigma(\mathcal{C}) = (-1)^N \tag{2.18}$$

and

$$\langle \sigma(\mathcal{C}) \rangle = \sum_N (-1)^N p_N \tag{2.19}$$

where p_n is the probability that N vortices wind around $\mathcal{C}$. If vortices were totally absent, $\langle \sigma(\mathcal{C}) \rangle$ would be 1 identically. We see that vortices are the analogues of Peierls contours. At high temperatures β^{-1} long vortices are abundant and $W(\mathcal{C})$ falls off exponentially with the area of Ξ , which means confinement of static quarks. On the other hand they are suppressed at low temperatures and $W(\mathcal{C})$ falls off only exponentially with the perimeter of Ξ . The importance of vortices for the confinement problem has been emphasized by 't Hooft, Glimm and Jaffe, Yoneya, Förster, Mack and Petkova [13, 14, 15, 16, 17, 6] and others. Yoneya pointed out that in the Z_n gauge theory vortices are the relevant objects for confinement.

The suppression of long vortices at low temperatures can be described quantitatively by their free energy per unit length. Consider a finite lattice Λ as a vortex container (Fig.3). Fix boundary conditions $\sigma(b)$ on $\partial\Lambda$ and denote the partition Z_Λ . Now change the boundary conditions by a singular gauge transformation

$$\sigma(b) \longrightarrow \begin{cases} \sigma(b) \, , & b \notin T \\ -\sigma(b) \, , & b \in T \end{cases} \quad , \quad b \in \partial\Lambda \tag{2.20}$$

and call the partition function $Z_{\Lambda,-}$. It is easy to see that this change of the boundary conditions changes the number of vortices which wind once through the container by one (mod 2). The free energy of a vortex is defined by

$$\begin{aligned} f(t) &= - \lim_{\ell \to \infty} \frac{1}{\ell} \ln \frac{Z_{\Lambda,-}}{Z_\Lambda} \\ f &= \lim_{t \to \infty} f(t) \end{aligned} \tag{2.21}$$

where ℓ is the length and t the width of the container. By a Peierls argument one can show that $f > 0$ at low temperatures [18] .

As in the case of the Ising model one might also use cyclic and anticyclic boundary conditions, if one wants to preserve translation invariance. Then one has a trivial respectively nontrivial Z_2 fiber bundle over the torus T^3. We will come back to this point later.

3. Definition of the Higgs phase.

In this section we consider lattice gauge theories with or without charged Higgs fields. Let the gauge group G be a compact group with nontrivial center Z(G). The gauge field $\mathcal{U}(b)\in G$ sits on the links b of a hypercubical lattice Λ in $\nu = 3$ or 4 dimensions. For plaquettes P one defines

$$\mathcal{U}(\dot{P}) = \prod_{b\in\partial P} \mathcal{U}(b) \qquad \text{(path-ordered product).} \qquad (3.1)$$

The action of the gauge field is

$$L_G = \beta_P \sum_P \left(Re\, \chi(\mathcal{U}(\dot{P})) - \delta \right) \leq 0 \;, \qquad \beta_P > 0 \;, \; \delta = \chi(\mathbb{1}), \qquad (3.2)$$

where χ is the character of a faithful representation of G. The Higgs field, if it is included, lives on the lattice points x and takes values $\phi(x)$ in a linear space Φ. Gauge transformations by functions $V(x)\in G$ are defined by

$$\begin{aligned} \mathcal{U}(x,y) &\longrightarrow V(x)\, \mathcal{U}(x,y)\, V(y)^{-1} \\ \phi(x) &\longrightarrow D\big(V(x)\big)\, \phi(x) \end{aligned} \qquad , \qquad (3.3)$$

where D is a representation of G on Φ with kernel Γ. We assume that Γ is a discrete subgroup of G and therefore also of Z(G). D may be looked at as a faithful representation of G/Γ. Physically this means that the Higgs field does not bear the fundamental charge in the theory if Γ is nontrivial. In the case of a pure gauge field theory we set $\Gamma = Z(G)$. Let (,) be an inner product on Φ, unitary with respect to D, and define the gauge invariant link variables

$$W(x,y) = \big(\phi(x) ,\; D(\mathcal{U}(x,y))\, \phi(y) \big) \quad . \qquad (3.4)$$

The Higgs potential is a gauge invariant continuous function

$$\mathcal{V} : \Phi \to \mathbb{R} \qquad (3.5)$$

with zero as its minimum value. The usual action of the Higgs field is

$$L_\phi = \beta_\ell \sum_{\langle xy\rangle} \left\{ 2\, Re\, W(x,y) - |\phi(x)|^2 - |\phi(y)|^2 \right\} - \beta_x \sum_x \mathcal{V}(\phi(x)) \leq 0 \qquad (3.6)$$
$$\beta_\ell ,\, \beta_x \geq 0$$

The total action $L = L_G + L_\phi$ appears in the measure of path integrals

$$d\mu = \frac{1}{Z} e^L d\hat{\mu} \;, \quad Z = \int e^L d\hat{\mu} \;, \quad d\hat{\mu} = \prod_b d\mathcal{U}(b) \;\; \text{resp.} \; \prod_b d\mathcal{U}(b) \prod_x d\phi(x) \qquad (3.7)$$

dU is the Haar measure on G and $d\phi$ the usual invariant Lebesgue measure on Φ .

From the work of 't Hooft and others [13 - 16] we know that also in the case of a continuous gauge group G there are topological nontrivial objects, called vortices, which are important for the phase structure of the theory. They are (ν-2)-dimensional objects and are characterized by elements of the finite subgroup Γ . Mack and Petkova [17] showed that thin vortices produce confinement of static quarks in a high temperature SU(2) model and a confining Coulomb potential in the 3-dimensional U(1) model at all temperatures[6].

In general vortices cannot be localized as in the case of Z_n gauge theories. Due to the continuity of G they spread out and are to be defined through the boundary values of the gauge field on some vortex containers. We shall consider two approaches to study the free energy of vortices. First we follow ref. [6] and take for the lattice Λ a vortex container as defined there. They are constructed in such a way that the homotopy group $\pi_1(\Lambda^c)$ of their complement Λ^c is that of a circle. See figs. 3, 4 for the case $\nu = 3$. If one takes a loop in Λ^c with winding number one and shrinks it without touching the interior of Λ, one finally gets a closed curve $\mathcal{C}$ in $\partial\Lambda$.

Fix boundary conditions $\mathcal{U} = \{\mathcal{U}(b)\}, \phi = \{\phi(x)\}$ on $\partial\Lambda$ and denote the partition function $Z_\Lambda(\mathcal{U},\phi)$. Now a change of vorticity is defined to be a singular gauge transformation $\mathcal{U} \rightarrow \mathcal{U}_\gamma$ of the boundary conditions by an element $\gamma \in \Gamma$.

$$\mathcal{U}(b) \longrightarrow \begin{cases} \mathcal{U}(b) \, , & b \notin T \\ \mathcal{U}(b)\gamma \, , & b \in T \end{cases} \quad , \quad b \in \partial\Lambda \tag{3.8}$$

T is a set of links in $\partial\Lambda$ as defined in ref.[6] (see fig.3). The Higgs field remains unaffected. Call the partition function $Z_\Lambda(\mathcal{U}_\gamma,\phi)$. If you take a closed curve $\mathcal{C}$ in $\partial\Lambda$ as described above and compute

$$\mathcal{U}(\mathcal{C}) = \prod_{b\in\mathcal{C}} \mathcal{U}(b) \qquad \text{(path-ordered product)} \quad , \tag{3.9}$$

the transformation (3.8) just changes it to $\mathcal{U}(\mathcal{C})\gamma$. Here it is crucial that γ is in the center of G. This explains the notion of a change of vorticity. The Higgs field, transforming trivially under Γ , is not able to compensate the effect of this change. Now the free energy of a vortex per unit length ($\nu = 3$) respectively unit area ($\nu = 4$) is naturally defined as

$$f_{\Lambda,\gamma}(\mathcal{U},\phi) = -\frac{1}{\ell}\ln\frac{Z_\Lambda(\mathcal{U}_\gamma,\phi)}{Z_\Lambda(\mathcal{U},\phi)} \quad , \tag{3.10}$$

where ℓ is the length resp. area of the vortex container.

In 't Hooft's [7] approach Λ is taken to be a torus T^ν in order to have translation invariance. With periodic boundary conditions in all directions one gets a partition function Z_Λ . Then singular gauge transformations characterized by elements γ of Γ are performed, which preserve the periodicity for gauge invariant variables. These transformations introduce magnetic vortex flux in Λ in different directions. Choose a specific direction and call the partition function $Z_{\Lambda,\gamma}$ for this case. Define the free energy of a vortex per unit length resp. area

$$f_\gamma(d_i) = \lim_{\ell\to\infty} f_{\Lambda,\gamma} := -\lim_{\ell\to\infty}\frac{1}{\ell}\ln\frac{Z_{\Lambda,\gamma}}{Z_\Lambda} \quad , \tag{3.11}$$

where d_i are the widths of the torus and ℓ the length resp. area as in (3.10).

As in the examples of sect.2 the phases of the model can now be characterized by these quantities. In ref.[6] the following criterion for confinement is proven. Let $\tilde{f}_{\Lambda,\gamma}$ be the maximum of the $f_{\Lambda,\gamma}(\mathcal{U})$ for all boundary conditions $\mathcal{U}$. If $\tilde{f}_\gamma(d_i) = \lim_{\ell\to\infty}\tilde{f}_{\Lambda,\gamma}$ goes to zero exponentially with d_i, static quarks are confined by an approximately linearly rising potential.

On the other hand the Higgs phase is defined according to 't Hooft [7] by

$$f_\gamma := \lim_{d_i\to\infty} f_\gamma(d_i) \neq 0 \quad . \tag{3.12}$$

Physically this means that long vortices are suppressed. Their thermodynamic probability decreases exponentially with their extension.

The two approaches are related by the fact that the partition functions on the torus can be obtained by integrating the partition functions $Z_\Lambda(\mathcal{U},\phi)$ of the vortex container over a certain class of boundary conditions $\mathcal{U},\phi$ in $\nu = 3$ dimensions. (In $\nu = 4$ dimensions the topologies are different.) If the confinement condition of Mack and Petkova is fulfilled, the system is surely not in the Higgs phase.

It should be noted that the socalled Higgs mechanism in the old terminology does not imply that one is in the Higgs phase in the above sense. Consider the case where the Higgs field carries the fundamental charge and Γ is trivial. There cannot be a Higgs phase in the sense of (3.12). But nevertheless one speaks of a Higgs mechanism, which is really a kind of screening mechanism. See the discussion in ref. [9].

4. The Peierls argument for the Higgs model.

In this section we shall give some results which support the idea that the above definition of the Higgs phase makes physical sense. The pure Z_n lattice gauge theory can be shown to be in the Higgs phase at low temperatures by using a Peierls argument, as we mentioned in sect.2 [18]. This is in agreement with the conventional wisdom about the phase diagram of this theory.

In the following we shall prove rigorously the existence of the Higgs phase in the low temperature Higgs model by a Peierls argument. For the sake of intuition we shall deal first with the 3-dimensional case and turn to 4 dimensions later. The general definition of the model has been given in sect.3. We specify the Higgs field further by the following requirement. Consider the set of non-generic vectors in Φ, namely

$$\Phi_0 = \{ \phi \in \Phi \mid H_\phi \neq \Gamma \} \tag{4.1}$$

where $H_\phi = \{ g \in G \mid D(g)\phi = \phi \}$ is the stability group of ϕ.
We demand that Φ_0 forms a subset of measure zero. Generic vectors have Γ as their stability group. The orbits

$$B(\phi) = \{ D(g)\phi \mid g \in G \} \tag{4.2}$$

consist either solely of generic or solely of non-generic vectors. The Higgs potential $\mathcal{V}$ is required to be such that

$$\min_{\Phi_0} \mathcal{V} = c > 0 \tag{4.3}$$

so that $\mathcal{V}$ does not assume its minimum for non-generic vectors. For simplicity we demand further that $\mathcal{V}$ assumes its minimum value on one orbit only, and we rescale ϕ to be of length 1 there. For $|\phi| \to \infty$ $\mathcal{V}$ should grow like $|\phi|^4$.

Example 1 : Abelian Higgs model

$G = U(1)$, $\Phi = \mathbb{C}$, $D(e^{i\varphi}) = e^{iq\varphi}$, q is the Higgs charge, $q \in \mathbb{N}$

$$\Gamma = \{ e^{i\frac{k}{q}2\pi} \mid k = 0, 1, \ldots, q-1 \} \cong Z_q$$

$$\Phi_0 = \{ 0 \}$$

$$W(x,y) = \overline{\phi(x)}\, \mathcal{U}^q(x,y)\, \phi(y) \quad , \quad \mathcal{V}(\phi) = (\, |\phi|^2 - 1\,)^2$$

Example 2 : Non-Abelian Higgs model

$$G = SU(2) \ , \quad Z(G) = \{+1 \ , \ -1\} = Z_2$$

$$\Phi = \mathbb{R}^3 \oplus \mathbb{R}^3 \ , \quad \phi = (\phi_1, \phi_2) \ , \quad \phi_i \in \mathbb{R}^3$$

This is a pair of Higgs fields which transform according to the 3-dimensional adjoint representation $D^{(1)}$ of $SU(2)$.

$$D = D^{(1)} \oplus D^{(1)} \qquad , \quad \Gamma = Z_2$$

$$\Phi_0 = \{ (\phi_1, \phi_2) \mid \phi_1 \text{ and } \phi_2 \text{ are linearly dependent} \}$$

$$(\phi, \phi') = (\phi_1, \phi_1')_{\mathbb{R}^3} + (\phi_2, \phi_2')_{\mathbb{R}^3} \qquad \text{the inner product on } \Phi \ .$$

$$U(\phi) = \left(|\phi_1|^2 - \tfrac{1}{2} \right)^2 + \left(|\phi_2|^2 - \tfrac{1}{2} \right)^2 + (\phi_1, \phi_2)_{\mathbb{R}^3} \ .$$

In the limit of an infinite sharp potential, $\beta_* \to \infty$, we get the Higgs model with fixed length of ϕ .

Next we introduce variables which will be useful in the later discussion. Let ϕ be generic, so that G/Γ acts freely on the orbit $B(\phi)$. Choose a standard representative ϕ_0 of the orbit. To each $\phi' \in B(\phi)$ there exists exactly one $\dot{u} \in G/\Gamma$, which fulfills

$$D(u)\,\phi' = \phi_0 \ , \qquad u \in G \quad \text{a representative of } \dot{u} \tag{4.4}$$

We use this fact to associate with each $\phi(x)$ two variables

$$\varsigma(x) \in \{\text{orbits}\} \ , \qquad \dot{V}(x) \in G/\Gamma \ . \tag{4.5}$$

$\varsigma(x)$ denotes the orbit of $\phi(x)$. Let $\phi_0(\varsigma(x))$ be a standard representative vector in this orbit. Then $\dot{V}(x)$ is chosen such that

$$D(V(x))\,\phi(x) = \phi_0(\varsigma(x)) \qquad \text{for} \quad \dot{V}(x) = V(x)\,\Gamma \ . \tag{4.6}$$

If $\phi(x)$ is generic, $\dot{V}(x)$ is defined uniquely by (4.6). $\phi_0(\varsigma)$ can be chosen continuous almost everywhere. For the volume elements we have

$$d\phi(x) = d\dot{V}(x) \ \ d\nu(\varsigma(x)) \tag{4.7}$$

where $d\nu$ is a certain measure on the orbit space.

Now choose a sheet $G' \subset G$ of the covering $G \to G/\Gamma$ with $\mathbb{1}$ in its interior. To every $\dot{V}(x) \in G/\Gamma$ we find an unique group element $V(x) \in G'$ with

$$\dot{V}(x) = V(x)\,\Gamma \quad , \qquad V(x) \in G' \quad . \tag{4.8}$$

Define
$$R(x,y) = V(x)\,\mathcal{U}(x,y)\,V(y)^{-1} \quad . \tag{4.9}$$

It transforms nontrivially under gauge transformations in Γ . We have

$$d\mathcal{U}(b) = dR(b) \tag{4.10}$$

$$W(x,y) = \left(\phi_0(\varrho(x))\ ,\ D(R(x,y))\,\phi_0(\varrho(y)) \right) \quad . \tag{4.11}$$

We introduce variables $H(b) \in G'$, $\gamma(b) \in \Gamma$ .They are uniquely determined by

$$R(b) = H(b)\,\gamma(b) \quad , \qquad H(b) \in G' \quad . \tag{4.12}$$

H(b) is gauge invariant, while $\gamma(b)$ transforms nontrivially under Γ . For the measure we write

$$dR(b) = dH(b)\ d\gamma(b) \quad . \tag{4.13}$$

dH is the restriction of the Haar measure dR onto G' and

$$d\gamma = \sum_{\gamma \in \Gamma} \delta(g^{-1}\gamma)\,dg \tag{4.14}$$

the discrete measure on Γ . For gauge invariant integrable functions $f(\{u(b),\ \phi(x)\})$ one has

$$\int f\,d\hat{\mu} = \int \prod_b dH(b) \prod_b d\gamma(b) \prod_x d\nu(\varrho(x))\ f\left(\left\{H(b)\gamma(b)\ ,\ \phi_0(\varrho(x))\right\}\right) \ . \tag{4.15}$$

Therefore the model is well described by the variables $\varrho(x)$, H(b) and $\gamma(b)$. Because $\varrho(x)$ and H(b) are gauge invariant and $\gamma(b) \in \Gamma$ we recognize Γ as the effective gauge group.

Example 1 : (see above)

$$\phi(x) = \varrho(x)\,e^{i\Theta(x)} \quad , \qquad \varrho(x) \geq 0 \ , \qquad -\pi < \Theta(x) \leq \pi$$

$$\dot{V}(x) = e^{-i\frac{\Theta(x)}{q}}\,\Gamma \quad ,$$

$$G' = \left\{ e^{i\varphi} \ \middle|\ -\frac{\pi}{q} < \varphi \leq \frac{\pi}{q} \right\} \quad , \qquad V(x) = e^{-i\frac{\Theta(x)}{q}}$$

$$H(x,y) = \left(e^{-i\Theta(x)}\,\mathcal{U}^q(x,y)\,e^{i\Theta(y)} \right)^{\frac{1}{q}}$$, the q^{th} root is defined to be in G'.

The action is easily expressed in the new variables.

$$L = \sum_P L_P \tag{4.16}$$

with the plaquette action

$$L_P = \beta\Big[\ \hat\beta_P\big\{\mathrm{Re}\,\chi(H(\dot P)\gamma(\dot P)) - \delta\big\} - \frac{1}{12}\hat\beta_x \sum_{x\in\partial P} \mathcal{V}(\varsigma(x))$$

$$+\frac{1}{4}\hat\beta_\ell \sum_{\langle xy\rangle\in\partial P}\Big\{2\,\mathrm{Re}\,W(x,y) - |\phi_o(\varsigma(x))|^2 - |\phi_o(\varsigma(y))|^2\Big\}\Big] \quad (4.17)$$

where $H(\dot P) = \prod_{b\in\partial P} H(b)$, $\gamma(\dot P) = \prod_{b\in\partial P}\gamma(b)$ (4.18)

$$W(x,y) = \big(\phi_o(\varsigma(x)),\ D(H(x,y))\,\phi_o(\varsigma(y))\big) \quad (4.19)$$

$$\mathcal{V}(\varsigma(x)) := \mathcal{V}(\phi_o(\varsigma(x))) \quad . \quad (4.20)$$

We introduced an overall temperature $\beta = \frac{\beta_i}{\hat\beta_i}$ by renormalizing $\min \hat\beta_i = 1$.

We shall now investigate the vortices in the Higgs model. If $\gamma(\dot P) = \gamma \neq 1$ on a plaquette P, the variable $\mathcal{U}(\dot P)$ is near γ. Because of

$$\prod_{P\in\partial c}\gamma(\dot P) = 1 \quad (4.21)$$

the plaquettes with $\gamma(\dot P)\neq 1$ form closed lines on the dual lattice. These lines may split or join according to the multiplication law of Γ (Fig.5). A vortex is defined to be a set of plaquettes with $\gamma(\dot P)\neq 1$ which is connected on the dual lattice. We define the vortex flux through a surface Ξ by

$$\gamma(\dot\Xi) = \prod_{P\in\Xi}\gamma(\dot P) \quad . \quad (4.22)$$

Now the action L_p of a plaquette P in a vortex will be studied, where $\gamma(\dot P) = \gamma \neq 1$. χ has the property

$$\mathrm{Re}\,\chi(g) < \delta \qquad \text{for all} \quad g \neq 1 \quad . \quad (4.23)$$

Suppose $H(\dot P)\in G'$, so that $H(\dot P)\gamma(\dot P)\notin G'$. G' contains an open neighbourhood $\mathcal{O}$ of 1 and we conclude by compactness of $G\setminus\mathcal{O}$

$$\mathrm{Re}\,\chi(H(\dot P)\gamma(\dot P)) - \delta \leq -c_1 < 0 \quad (4.24)$$

so that

$$L_P \leq -c_1\beta_P \quad (4.25)$$

where c_1 is a numerical constant.

The other possibility is $H(\dot P) = \prod_{b\in\partial P} H(b)\notin G'$. There exists an open neighbourhood $G'' \subset G$ of 1, with the property

$$g_i \in G'', \; i=1,2,3,4 \quad \Rightarrow \prod_{i=1}^{4} g_i \in G' \tag{4.26}$$

and we conclude that there is at least one link $b_0 \in \partial P$ with $H(b_0) \notin G''$. Let $b_0 = \langle xy \rangle$ and consider

$$f = \frac{\hat{\beta}_\ell}{4}\left\{ 2\,\mathrm{Re}\left(\phi_0(\varsigma(x)),\, D(H(x,y))\,\phi_0(\varsigma(y))\right) - \left|\phi_0(\varsigma(x))\right|^2 - \left|\phi_0(\varsigma(y))\right|^2 \right\}$$

$$- \frac{\hat{\beta}_x}{12}\left\{ \mathcal{V}(\varsigma(x)) + \mathcal{V}(\varsigma(y)) \right\} \quad \geqslant \beta^{-1} L_P \tag{4.27}$$

The maximum of f is at $H(b_0) = 1$ and some orbit $\varsigma(x) = \varsigma(y) = \varsigma_0$, where $f = 0$. By a continuity argument, one obtains a bound

$$f \leqslant -K < 0 \qquad \text{if} \quad H(b_0) \notin G'' \tag{4.28}$$

For the example 1 (Abelian Higgs model) K is of the form $\min(c_2 \hat{\beta}_\ell, c_3 \hat{\beta}_x)$. So we have the result

$$L_P \leqslant -\beta\varkappa \qquad \text{if} \quad \gamma(\dot{P}) \neq 1, \quad \varkappa > 0. \tag{4.29}$$

$\varkappa$ depends on the $\hat{\beta}_i$.

We proceed with the Peierls argument by specifying the boundary conditions. The lattice Λ is chosen to be a torus T^3 of extension $\ell \times d \times t$ and cyclic boundary conditions are required for all gauge invariant variables. Consider a closed cross-section of Λ, for example the surface $\Xi_1 : x_1 = \text{const.}$ (Fig.4). In order to have nonvanishing vortex flux through Ξ_1 we have to impose anticyclic boundary conditions on the variables $\gamma(b)$, for instance

$$\gamma(b) \;\; \text{cyclic in} \;\; x_1, x_2$$

$$\gamma(b) = \gamma(b + t\vec{e}_3)\cdot\delta(b), \qquad \delta(b) = \begin{cases} 1, & b \notin T \\ \delta \in \Gamma, & b \in T \end{cases} \tag{4.30}$$

See fig.4 for T. In this case we find $\prod_{P \in \Xi_1} \gamma(\dot{P}) = \delta$. The translation invariance for gauge invariant quantities is not violated by (4.30). Instead of the variables $\gamma(b)$ it will be advantageous to use the plaquette variables $\gamma(\dot{P})$, which are cyclic in all directions. They obey the restriction (4.21), and therefore it is

$$\prod_{P \in \Xi} \gamma(\dot{P}) = 1 \qquad \text{for } \Xi \text{ a closed surface of the form } \Xi = \partial\Omega. \tag{4.31}$$

But on T^3 we have three homology classes of closed surfaces, $\partial\Xi = 0$, which are not boundaries of any volumes Ω. Representatives are $\Xi_i : x_i = \text{const.}$, $i = 1, 2, 3$. For any surface Ξ in the homology class of Ξ_i, i.e. $\Xi - \Xi_i = \partial\Omega$, one has

$$\prod_{P \in \Xi} \gamma(\dot{P}) = \prod_{P \in \Xi_i} \gamma(\dot{P}) =: \sigma_i \in \Gamma \qquad (4.32)$$

So the variables $\gamma(\dot{P})$ obey three further constraints (4.32) in addition to (4.21). Summarizing we can write down the following partition functions

$$Z_{\sigma_1,\sigma_2,\sigma_3} = \int \prod_x d\nu(g(x)) \prod_b dH(b) \prod_P d\gamma(\dot{P})\, e^L \prod_c \delta\Big(\prod_{P \in \partial c} \gamma(\dot{P})\Big) \prod_{i=1}^{3} \delta\big(\gamma(\dot{\Xi}_i)\sigma_i^{-1}\big)$$

$$=: \int d\tilde{\mu}\; e^L \prod_{i=1}^{3} \delta\big(\gamma(\dot{\Xi}_i)\sigma_i^{-1}\big) \qquad (4.33)$$

The product over c goes over all cubes. With the definitions

$$d\mu = \frac{1}{Z} e^L d\tilde{\mu} \quad , \quad Z = \int e^L d\tilde{\mu} = \sum_{\{\sigma_i\}} Z_{\sigma_1,\sigma_2,\sigma_3} \qquad (4.34)$$

we write

$$Z_{\sigma_1,\sigma_2,\sigma_3} = Z \left\langle \prod_{i=1}^{3} \delta\big(\gamma(\dot{\Xi}_i)\sigma_i^{-1}\big) \right\rangle . \qquad (4.35)$$

Expectation values are performed with the measure $d\mu$. Making use of translation invariance we can find an upper bound for $Z_{\sigma,1,1}/Z_{1,1,1}$. Our argument uses the chessboard estimates of ref.[17]*. They can be proven for our model in the form

$$\left| \left\langle \prod_{P \in P_{2h}} F_P(\gamma(\dot{P})) \right\rangle \right| \leq \prod_{P' \in P_{2h}} \left\langle \prod_{P \in P_{2h}} F_{P'}(\gamma(\dot{P})) \right\rangle^{\frac{1}{|P_{2h}|}} \qquad (4.36)$$

P_{2h} is a set of even parallel plaquettes (see [17]) and F_P some observables. First we estimate

$$Z_{\sigma,1,1} \leq \sum_{\sigma_2,\sigma_3} Z_{\sigma,\sigma_2,\sigma_3} := Z_\sigma = Z \left\langle \delta\big(\gamma(\dot{\Xi}_1)\sigma^{-1}\big) \right\rangle . \qquad (4.37)$$

Every configuration with $\gamma(\dot{\Xi}_1) = \sigma \neq 1$ has a vortex running through Λ in the x_1-direction. Let Ψ be the set of all these vortices.

$$\left\langle \delta\big(\gamma(\dot{\Xi}_1)\sigma^{-1}\big) \right\rangle \leq \sum_{\mathcal{C} \in \Psi} \left\langle \prod_{P \in \mathcal{C}} \delta\big(\gamma(\dot{P})\sigma_P^{-1}\big) \right\rangle \qquad (4.38)$$

$\sigma_P \neq 1$ is the value of $\gamma(\dot{P})$ in the vortex $\mathcal{C}$. If $s = |\mathcal{C}|$ is the total length of $\mathcal{C}$, there exists a set P_{2h} containing at least $\frac{s}{6}$ plaquettes of $\mathcal{C}$. The chessboard estimates give us

*Chessboard estimates require translation invariance, which we achieved by going onto a torus. Furthermore they demand the three δ-functions for $\gamma(\dot{\Xi}_i)$ to be treated as observables and not as part of the measure.

$$\left\langle \prod_{P\in \mathcal{C}} \delta(\gamma(\dot{P})\,\zeta_P^{-1}) \right\rangle \leqslant \left\langle \prod_{P\in \mathcal{C}\cap P_{2h}} \delta(\gamma(\dot{P})\,\zeta_P^{-1}) \right\rangle$$

$$\leqslant \left\langle \prod_{P\in P_{2h}} \delta(\gamma(\dot{P})\,\tilde{\zeta}^{-1}) \right\rangle^{\frac{s}{3|\Lambda|}} \quad , \ |\Lambda| = \ell\cdot d\cdot t \qquad (4.39)$$

$\tilde{\zeta}$ is the value of $\zeta \neq 1$ for which $\left\langle \prod_{P\in P_{2h}} \delta(\gamma(\dot{P})\,\zeta^{-1}) \right\rangle$ is maximal. Every configuration which contributes to the expectation value on the r.h.s. of (4.39) carries an action $L_P \leqslant -\beta\varkappa$ at all plaquettes on P_{2h}. This allows one to estimate the expectation value. We give only the result and put the calculational details in an appendix.

$$\left\langle \prod_{P\in P_{2h}} \delta(\gamma(\dot{P})\,\tilde{\zeta}^{-1}) \right\rangle \leqslant D(\beta)^{3|\Lambda|} \qquad (4.40)$$

where $D(\beta)$ goes to zero exponentially, when β goes to infinity. From this we get

$$Z_\zeta \leqslant Z \sum_{s=\ell}^{\infty} N(s)\, D(\beta)^s \qquad (4.41)$$

N(s) is the number of vortices in Υ of length s . The bound

$$N(s) \leqslant d\cdot t\cdot 5^s \qquad (4.42)$$

is easily derived. Therefore

$$Z_{\zeta,1,1} \leqslant Z\, d t \sum_{s=\ell}^{\infty} (5\,D(\beta))^s \qquad (4.43)$$

This bound goes to zero exponentially for $\beta \to \infty$. For the other partition functions $Z_{\zeta_1,\zeta_2,\zeta_3}$ exept $Z_{1,1,1}$ we get similar bounds in the same way and find

$$Z_{1,1,1} = Z - \sum_{(\zeta_1,\zeta_2,\zeta_3)\neq(1,1,1)} Z_{\zeta_1,\zeta_2,\zeta_3} \geqslant \frac{Z}{2} \qquad \text{for large enough } \beta . \qquad (4.44)$$

If we choose β so large that $5\,D(\beta) < \frac{1}{2}$, then

$$\frac{Z_{\zeta,1,1}}{Z_{1,1,1}} \leqslant 4\, d\cdot t\, (5\,D(\beta))^{\ell} \qquad (4.45)$$

and

$$-f_\zeta = \lim_{d,t\to\infty} \lim_{\ell\to\infty} \frac{1}{\ell} \ln \frac{Z_{\zeta,1,1}}{Z_{1,1,1}} \leqslant \ln(5\,D(\beta)) < 0 . \qquad (4.46)$$

The result is : there exists a value β_c with

$$-f_\zeta < 0 \quad \text{for} \quad \beta > \beta_c . \qquad (4.47)$$

Thus the model is in the Higgs phase at low temperatures.

In the case of the fixed length Higgs field ($\beta_x \to \infty$) the calculation can be made along the same line. Large β means that β_p as well as β_ℓ is large. This is just the region where one expects the Higgs phase from other arguments [9] .

In four dimensions the calculations are similar. We take a torus T^4 as the lattice and the relevant homology group $H_2(T^4)$ is generated by six elements Ξ_{ij} , $1 \leq i < j \leq 4$. The partition functions are $Z_{\{\sigma_{ij}\}}$ and we get a bound for large β .

$$-f_\sigma = \lim_{d_3, d_4 \to \infty} \lim_{d_1, d_2 \to \infty} \frac{1}{d_1 d_2} \ln \frac{Z_{\sigma,1,1,1,1,1}}{Z_{1,1,1,1,1,1}} \leq \ln \left(5 \tilde{D}(\beta)\right) \tag{4.48}$$

$$\tilde{D}(\beta) \to 0 \quad \text{as} \quad \beta \to \infty \quad , \qquad \sigma \neq 1$$

d_i are the lattice widths.

For the Z_n gauge theory a Peierls argument can be carried through along the same line without the need of chessboard estimates. This case might be regarded as the limit $\beta_\ell \to \infty$ of the fixed length Higgs model for $\Gamma = Z_n$.

We already noted the importance of the group Γ for the characterization of the Higgs phase. If Γ is trivial, there are obviously no vortices of the kind we discussed and a Higgs phase is not defined. This, for example, is the case in the Abelian Higgs model if $q = 1$. For this model analyticity has been shown in a large region of the coupling parameters [9] . There one has a kind of screening phase. In contrast, if $q \neq 1$ one finds two well separated phases in this region, a high temperature phase (confinement) and a Higgs phase in the sense of 't Hooft.

5. Symmetry breaking aspects.

We ask ourselves wether the Higgs phase can be characterized by a broken symmetry. The Higgs phase is sometimes brought into connection with the breaking of a gauge symmetry. But a local gauge symmetry, i.e. a symmetry under gauge transformations which are unity outside a compact set, cannot be broken spontaneously. This is a trivial consequence of the local structure of the theory, as described by the Dobrushin - Lanford - Ruelle equations [5] . One may consider the behaviour under more general gauge transformations. However there is also a simpler possibility : in this section we shall argue that the Higgs phase in ν -dimensional lattice gauge theory is accompanied by the spontaneous breaking of a global discrete symmetry in a $\nu-1$ -dimensional spin model.

We consider lattice gauge theory as defined in sect.3 . For definiteness we take $G = SU(2)$ as our gauge group and $\nu = 3$. The lattice Λ^{ν} is a box of size $\ell \times d \times t$. It contains a sublattice $\Lambda^{\nu-1}$: $x_2 = 0$ (see fig. 6), which is chosen in such a way that the endpoints of the links $b \in T$ touch $\Lambda^{\nu-1}$. With $\Lambda^{\nu-1}$ we shall associate a spin model à la Fröhlich and Durhuus [19] . Define

$$\mathcal{U}(x) := \mathcal{U}(b) \in SU(2) \qquad \text{for} \qquad x \in \Lambda^{\nu-1} \tag{5.1}$$

where $b = \langle (x,0)(x,1) \rangle$ is the link pointing from x in the positive x_2-direction. The total action L contains a part

$$L' = \beta_P \sum_P{}' \operatorname{Tr} \mathcal{U}(\dot{P}) + \beta_L \sum_{\langle xy \rangle}{}' 2 \operatorname{Re} W(x,y) \tag{5.2}$$

which includes only those plaquettes resp. links which involve an element $\mathcal{U}(x), x \in \Lambda^{\nu-1}$. If $x,y \in \Lambda^{\nu-1}$ are nearest neighbours, the corresponding variables $\mathcal{U}(x), \mathcal{U}(y)$ are coupled through the term

$$\operatorname{Tr}(\mathcal{U}(\dot{P})) = \operatorname{Tr}\left(\mathcal{U}(b_1)\mathcal{U}(b_2)\mathcal{U}(b_3)\mathcal{U}(b_4)\right) = \operatorname{Tr}\left(\mathcal{U}(x)\mathcal{U}(b_2)\mathcal{U}^{-1}(y)\mathcal{U}(b_4)\right) \tag{5.3}$$

See fig.7 . Now we associate 4-dimensional unit spin vectors with the variables $\mathcal{U}(x)$ by the formula

$$\mathcal{U}(x) = s^{\mu}(x)\,\sigma_\mu \quad , \qquad (\sigma_\mu) = (i\vec{\sigma}, 1) \quad , \mu = 1,2,3,4 \tag{5.4}$$

$$\sum_\mu (s^\mu(x))^2 = 1 \quad , \quad \vec{\sigma} \text{ the Pauli matrices}$$

With the help of

$$\operatorname{Tr}(\mathcal{U}_1 \mathcal{U}_2^{-1}) = 2 \sum_\mu s_1^\mu s_2^\mu = 2\, s_1 s_2 \quad , \qquad \mathcal{U}_i = s_i^\mu \sigma_\mu \tag{5.5}$$

we find

$$\mathrm{Tr}\,(\mathcal{U}(\dot{p})) = 2\, s(x)\, J(x,y)\, s(y) \quad . \tag{5.6}$$

The matrix $J(x,y) \in SO(4) \cong {SU(2) \times SU(2)}/{Z_2}$ is given by

$$\mathcal{U}^{-1}(b_4)\,\mathcal{U}(y)\,\mathcal{U}^{-1}(b_2) = \left[\, J(x,y)\, s(y)\right]^{\mu} \sigma_{\mu} \tag{5.7 a}$$

or $$J_{\mu\nu}(x,y) = \frac{1}{2}\,\mathrm{Tr}\left(\sigma_\mu^+\ \mathcal{U}^{-1}(b_4)\ \sigma_\nu\ \mathcal{U}^{-1}(b_2)\right) \tag{5.7 b}$$

The Higgs field is chosen as in example 2 of sect.3 and we get for $\mathcal{U}(b) = \mathcal{U}(x),\ x \in \Lambda^{\nu-1}$

$$\mathrm{Re}\ W(b) = s(x)\, K(x)\, s(x) \tag{5.8}$$

with some real matrix K(x) depending on the Higgs field at the endpoints of b . So we have

$$L' = 2\beta_P \sum_{\langle xy\rangle} s(x) J(x,y) s(y) \ + \beta_e \sum_x s(x) K(x) s(x) \ , \quad x,y \in \Lambda^{\nu-1} \tag{5.9}$$

$$s^2(x) = s^2(y) = 1$$

which is the action of a SO(4) Heisenberg model with variable coupling matrices J(x,y), K(x) . These are given by the external field $F_{ext} = \{\mathcal{U}(b),\, b \notin \Lambda^{\nu-1};\ \phi(x)\}$. With

$$Z_{\Lambda^{\nu-1}}(F_{ext}) = \int \prod_{x\in\Lambda^{\nu-1}} d\mathcal{U}(x)\ e^{L'} \tag{5.10}$$

we write

$$Z_{\Lambda^\nu} = \int \prod_{b\in\Lambda^\nu\setminus\Lambda^{\nu-1}} d\mathcal{U}(b) \prod_{x\in\Lambda^\nu} d\phi(x)\ \ e^{L''(F_{ext})}\, Z_{\Lambda^{\nu-1}}(F_{ext}) \quad . \tag{5.11}$$

We recognize this as the partition function of a $\nu-1$ -dimensional Heisenberg model with fluctuating coupling matrices, immersed in a ν-dimensional heat bath. The Heisenberg model on $\Lambda^{\nu-1}$ has for generic F_{ext} only the global symmetry

$$s(x) \rightarrow -s(x) \tag{5.12}$$

and the symmetry group is $\Gamma = Z_2$.

Now we come to the boundary conditions. First we choose periodic boundary conditions in all directions. Λ^ν has to be considered as a torus T^ν and $\Lambda^{\nu-1}$ is a torus $T^{\nu-1}$ with periodic boundary conditions too. Next we perform the singular gauge transformation (3.8) , so that we get anticyclic boundary conditions in x_3

$$\mathcal{U}(b) = \mathcal{U}(b + t\vec{e}_3) \cdot \delta(b) \quad , \qquad \delta(b) = \begin{cases} +1 , & b \notin T \\ -1 , & b \in T \end{cases} \tag{5.13}$$

and call the partition function $Z_{\Lambda^\nu,-}$. The singular gauge transformation affects only variables $\mathcal{U}(x), x \in \Lambda^{\nu-1}$ and we write for the partition function in $\Lambda^{\nu-1}$ $Z_{\Lambda^{\nu-1},-}(F_{ext})$, so that

$$Z_{\Lambda^\nu,-} = \int \prod_{b \in \Lambda^\nu \setminus \Lambda^{\nu-1}} d\mathcal{U}(b) \prod_{x \in \Lambda^\nu} d\phi(x) \; e^{L''} \, Z_{\Lambda^{\nu-1},-}(F_{ext}) \quad . \tag{5.14}$$

One recognizes easily that the singular gauge transformation introduced a Bloch wall in the SO(4) Heisenberg spin field. It is characterized by a spin flip between two distant lattice boundaries. In contrast to the Ising model, where we had a thin Bloch wall, namely the Peierls contour, the spins here may vary slowly between the boundaries, forming a thick Bloch wall. As it was explained in sect.2 for the Ising model, the log of the ratio $Z_{\Lambda^{\nu-1},-}/Z_{\Lambda^{\nu-1}}$ is a qualitative measure for the free energy of a Bloch wall. If the free energy of a Bloch wall per unit length is not zero, long Bloch walls are suppressed and we expect an ordered phase, where the global Z_2-symmetry is broken spontaneously.

Now we see from (5.14) that this just happens, when the gauge field is in the Higgs phase. Thus the Higgs phase in ν dimensions is accompanied by a spontaneous breakdown of the global Z_2 symmetry of an associated Heisenberg model in $\nu-1$ dimensions with fluctuating coupling matrices. This result can be derived in the same way for $\nu = 4, 5, \ldots$ and for $G = SU(n)$, where we find a $SU(n) \times SU(n)$ Heisenberg model with variable couplings and global symmetry group Γ .

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Appendix: Inequality (4.40)

To estimate the expectation value

$$E = \Big\langle \prod_{p \in P_{2h}} \delta\big(\gamma(\dot{p})\, \tilde{\gamma}^{-1}\big) \Big\rangle \tag{A.1}$$

we modify the measure $d\nu$.

$$e^{\beta_x \mathcal{V}(g)}\, d\nu(g) = e^{\beta_x \mathcal{V}_1(g)}\, e^{\beta_0 \mathcal{V}_2(g)}\, d\nu(g) =: e^{\beta_x \mathcal{V}_1(g)}\, d\hat{\nu}(g) \,, \tag{A.2}$$

where

$$\mathcal{V}(g) = \mathcal{V}_1(g, \beta_x) + \frac{\beta_0}{\beta_x}\, \mathcal{V}_2(g) \quad , \quad \beta_0 < \beta_x \tag{A.3}$$

$$\mathcal{V}_2(g) = \begin{cases} 0 & , \quad |\phi_0(g)| < a \\ \mathcal{V}(g) & , \quad |\phi_0(g)| > a \end{cases} \tag{A.4}$$

The constant $a > 1$ can be chosen so large that the modified action $\hat{L}$, which contains $\mathcal{V}_1$ instead of $\mathcal{V}$, has the property (4.29) :

$$\hat{L}_p \leq -\beta\varkappa \qquad \text{if} \quad \gamma(\dot{p}) \neq 1 \tag{A.5}$$

Define the new measure $d\hat{\mu}$ by

$$\frac{1}{Z}\, e^{L}\, d\tilde{\mu} = \frac{1}{Z}\, e^{\hat{L}} \prod_x e^{\beta_0 \mathcal{V}_2(g(x))}\, d\tilde{\mu} = \frac{1}{\hat{Z}}\, e^{\hat{L}}\, d\hat{\mu} \tag{A.6}$$

$$d\hat{\mu} = \omega^{-|\Lambda|} \prod_x e^{\beta_0 \mathcal{V}_2(g(x))}\, d\tilde{\mu} \tag{A.7}$$

with $$\omega^{|\Lambda|} = \int \prod_x e^{\beta_0 \mathcal{V}_2(g(x))}\, d\tilde{\mu} = \Big(\int e^{\beta_0 \mathcal{V}_2(g)}\, d\nu(g) \Big)^{|\Lambda|} \tag{A.8}$$

so that $$\int d\hat{\mu} = 1 \tag{A.9}$$

$$\hat{Z} = \int e^{\hat{L}}\, d\hat{\mu} = Z\, \omega^{-|\Lambda|} \,. \tag{A.10}$$

The purpose of these manipulations is to get a normalizable measure $d\hat{\mu}$ while maintaining property (4.29). Now we have

$$E = \frac{1}{\hat{Z}} \int e^{\hat{L}} \prod_{p \in P_{2h}} \delta\big(\gamma(\dot{p})\, \tilde{\gamma}^{-1}\big)\, d\hat{\mu} \;\leq\; \frac{1}{\hat{Z}}\, e^{-\beta\varkappa \frac{|\Lambda|}{2}} \tag{A.11}$$

because in all contributing configurations the action obeys

$$\hat{L} \leq -\beta\varkappa\, \frac{|\Lambda|}{2} \,. \tag{A.12}$$

We estimate

$$\hat{Z} \geq \int_{\Omega^{\varepsilon}} e^{\hat{L}} \, d\hat{\mu} \geq e^{-\beta\varepsilon|\Lambda|} \int_{\Omega_{\varepsilon}} d\hat{\mu} =: e^{-\beta\varepsilon|\Lambda|} \, \tau(\varepsilon)^{|\Lambda|} \tag{A.13}$$

where Ω_{ε} is an integration volume around the minimum of $\hat{L}$ with $\hat{L}_p \geq -\frac{\varepsilon}{3}$. $\tau(\varepsilon)$ can be seen to go like a power of ε. We obtain

$$E \leq \left(\tau(\varepsilon)^{-1} \, e^{-\beta(\frac{\varkappa}{2} - \varepsilon)} \right)^{|\Lambda|} =: D_{\varepsilon}(\beta)^{|\Lambda|} \qquad \text{for all } \varepsilon > 0 . \tag{A.14}$$

With
$$D^3(\beta) = \min_{\varepsilon} D_{\varepsilon}(\beta) \tag{A.15}$$

we get the result

$$E \leq D(\beta)^{3|\Lambda|} . \tag{A.16}$$

$D(\beta)$ decreases like $e^{-\beta\frac{\varkappa}{6}}$ for $\beta \to \infty$. The factor of $\frac{1}{6}$ is due to the fact that we catch only a sixth of a vortex with our chessboard estimates (4.36).

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Figure captions.

Fig. 1. Peierls contours (dotted lines) in 2-dimensional Ising model. Lattice points x and y are linked by a path $\mathcal{C}$.

Fig. 2. Vortex in 3-dimensional Z_2 lattice gauge theory, winding around a path $\mathcal{C} = \partial\Xi$.

Fig. 3. Vortex container Λ in 3 dimensions, which winds around the path $\mathcal{C}$. On the boundary of Λ the set T is marked.

Fig. 4. Vortex container in 3 dimensions, equivalent to that of fig.3. The righthand face and the lefthand face (hatched) are to be identified. Ξ_1 is a cross-section of the container.
If one identifies the other faces pairwise, one gets a torus T^3.

Fig. 5. Part of a Γ -vortex (for example $\Gamma = Z_3$), which splits.

Fig. 6. Vortex container Λ^3 with sublattice Λ^2 . The container is as in fig. 4 .

Fig. 7. Illustration to equation (5.3).

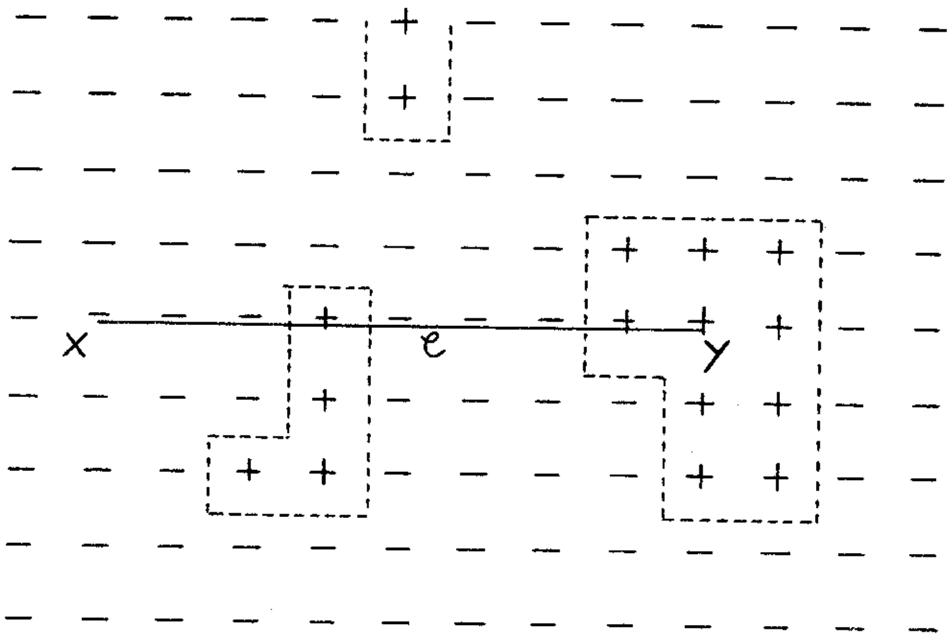

Fig. 1

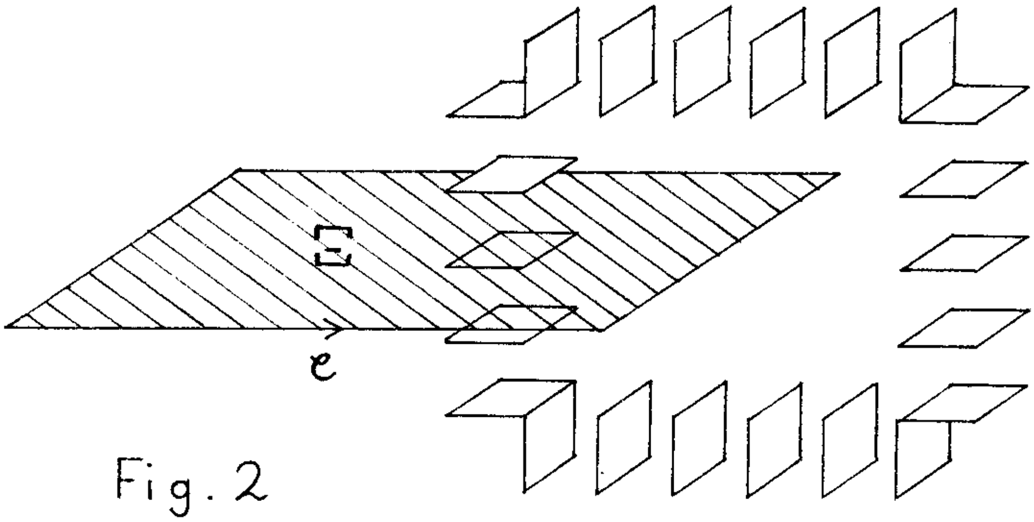

Fig. 2

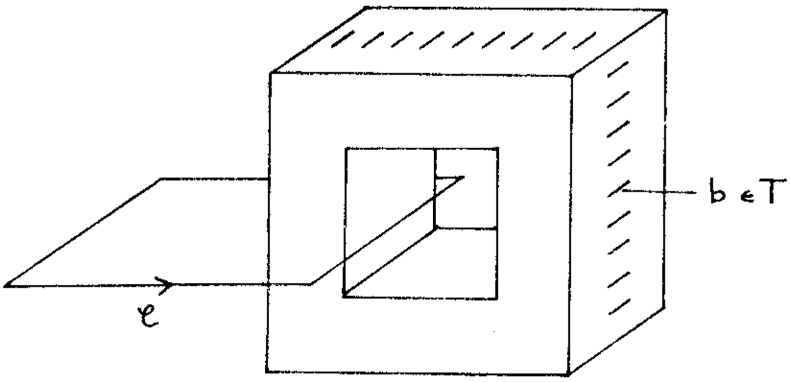

Fig. 3

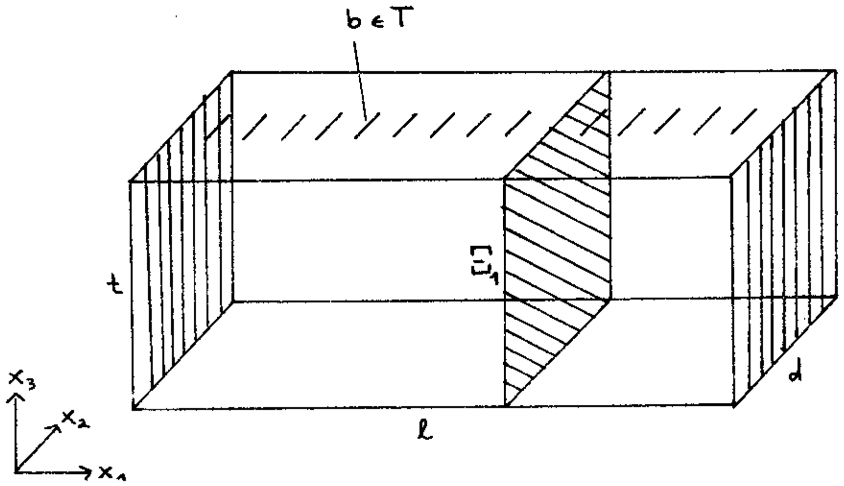

Fig. 4

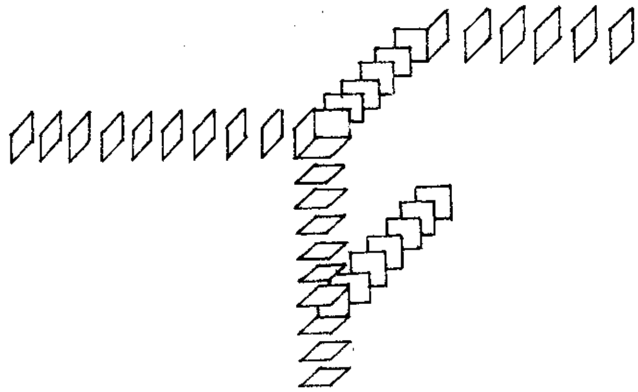

Fig. 5

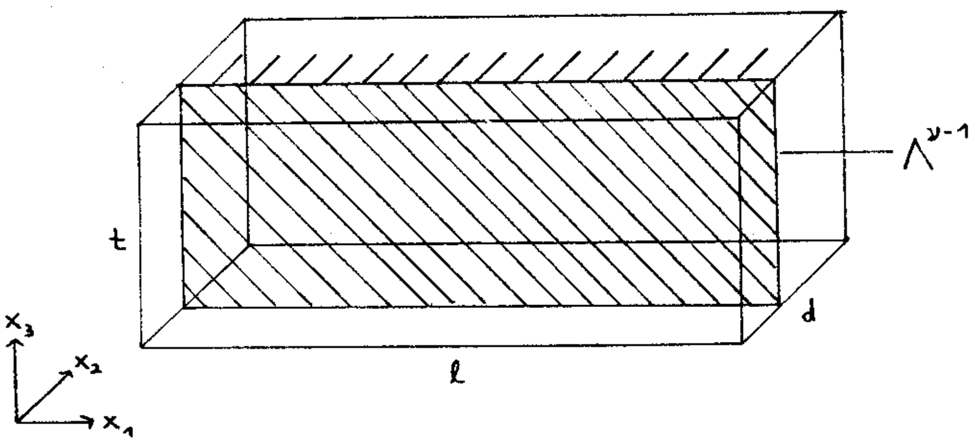

Fig. 6

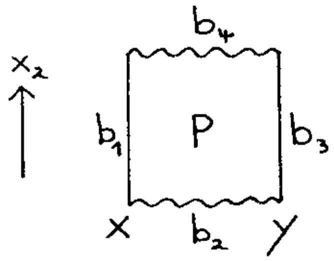

Fig. 7