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ASYMPTOTIC FREEDOM CONSTRAINTS ON GRAND UNIFIED GAUGE THEORIES

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Asymptotic Freedom Constraints on Grand Unified Gauge Theories

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Abstract

Constraints on the fermion and Higgs scalar content of grand unified gauge theories, imposed by the requirement of asymptotic freedom for the gauge couplings, are derived for models which have fermion representations with only color singlets and color triplets. The constraint $n_f \leqslant 16$ on the number n_f of flavors of color triplet quarks in pure QCD is removed. Definitive limits are placed on the representation content of theories based on the exceptional groups.

I. INTRODUCTION

Gauge theories of the weak and electromagnetic interactions [1,2] and of the strong interactions [3,4] have become the theoretical foundation of contemporary elementary particle physics. Not only have these theories passed many phenomenological tests (for recent reviews, see [5,6]), but they possess general properties, such as renormalizability [7,8] and asymptotic freedom [9,10], which many theorists find attractive. Thus, it is not surprising that there have been many attempts [11-26] to construct a grand unified gauge theory of elementary particle interactions, in which only the role of gravitation is at present uncertain (for attempts to include gravitation, see [27] and the works cited therein).

Gell-Mann, Ramond, and Slansky [27] have recently enumerated all possible simple gauge groups which contain a product $G_{f\ell} \otimes SU^c(3)$ of a flavor group $G_{f\ell}$ which contains the weak and electromagnetic gauge group $G_W \supseteq SU(2) \otimes U(1)$ and the color gauge group $SU^c(3)$ of quantum chromodynamics, and which in addition admits a fermion representation with only color singlets and color triplets (3^c and perhaps $\bar{3}^c$). They find that such representations can be divided into four classes, based on the structure of the flavor group $G_{f\ell}$:

<u>Class I.</u> $G_{f\ell} = G_\ell \otimes G_q \otimes U(1)$. The leptons transform under G_ℓ, the quarks under G_q, and the U(1) factor distinguishes leptons and quarks. In this class, the fermions transform under the defining representation of one of the classical groups SU(n), SO(n), Sp(2n).

<u>Class II.</u> $G_{f\ell} = SU(n_\ell) \otimes SU(n_q) \otimes SU(n_r) \otimes U(1) \otimes U(1)$. The fermions

transform under the fundamental representation of SU(n), with $n = n_{\ell} + 3n_q + 3n_r$, and consist of n_{ℓ} color singlet leptons which transform under $SU(n_{\ell})$, n_q q-type quarks (3^c) which transform under $SU(n_q)$, and n_r r-type quarks ($\bar{3}^c$) which transform under $SU(n_r)$.

Class III. $G_{f\ell} = G_{\ell+q} \otimes U(1)$. There are two representations which lead to this structure:

1. If the fundamental representation of SU(n) is expressed as

$$\underset{\sim}{n} = (1,\ 3^c) + (n - 3,\ 1^c) \tag{1}$$

under $SU(n-3) \otimes SU^c(3)$, then the fermions can transform as one or more antisymmetric tensor representations; the flavor group is $SU(n-3) \otimes U(1)$.

2. The fermions transform as one or more spinor representations of SO(n), with flavor group $SO(n-6) \times U(1)$.

Class IV. $G_{f\ell} = G_{\ell+q}$. The fermions transform as one or more fundamental representations of one of the exceptional groups F_4, E_6, E_7.

Only theories of class III and class IV provide a natural connection between the weak and electromagnetic interactions of leptons and quarks, although the connection could arise as a necessary consequence of the symmetry breaking in theories of class I and class II.

The present work examines the constraints placed on grand unified gauge theories by requiring asymptotic freedom as a whole. It should be noted that such an extrapolation of asymptotic freedom may not be required by present experiment, in view of the Appelquist-Carrazone decoupling theorem [28], which states that heavy fields decouple from a theory at low energy, but there is recent work [29,30] which suggests that there may be exceptions to this theorem. There is a conjecture by Fradkin and Kalashnikov [31], based on an old observation of Landau [32], that asymptotic freedom is necessary for a consistent field theory, but that conjecture is by no means proven (F1). On the other hand, attempts to resolve the Landau problem either by an ultraviolet fixed point [33-35] or by gravitation [36,37] have not been completely successful, so the question of overall asymptotic freedom may be of more than academic interest.

Here only the restrictions on the fermion representations and Higgs scalar representations imposed by asymptotic freedom for the gauge couplings alone are considered. The theory as a whole can be asymptotically free, of course, only if the Yukawa couplings and scalar quartic couplings are also asymptotically free. It has been emphasized by Cheng, Eichten and Li [38] that it is a non-trivial task to construct asymptotically free scalar quartic couplings, particularly when enough scalars are included to break a gauge symmetry down to U(1). In the examples they considered (see also [39]), asymptotic freedom and symmetry breaking to U(1) were, in fact, incompatible.

Apart from the fact that for grand unified theories, the requisite unbroken symmetry is $SU^c(3) \otimes U(1)$, and the representations of Higgs scalars can be more complicated than those previously considered, there is another interesting possibility: the Yukawa coupling constants and scalar quartic coupling constants may be functions of the gauge coupling constants [31,40-43], corresponding to an ultraviolet unstable fixed point of the renormalization group equations. This may correspond to a supersymmetric theory [41] , or to a non-supersymmetric theory in which dynamical symmetry breakdown has been described by an equivalent set of Higgs scalars [44]. These issues will be studied in detail elsewhere; however, asymptotic freedom of the gauge couplings is a prerequisite for any of these prospects to materialize.

The constraints on grand unified theories of class I and class II are minimal, as explained in Sec. II. The restriction

$$n_f \leqslant 16$$

on the number of flavors n_f of color triplet quarks in pure QCD [9] is removed; it is replaced by a restriction on the number of times the fundamental representation can be repeated. For sufficiently large groups. Higgs scalars are forbidden to transform as tensors of rank higher than two.

The constraints on class III theories are somewhat stronger, but not of immediate consequence: for the SO(n) spinor representations, no more than 64 quark flavors and 64 leptons are allowed, while for the antisymmetric tensor representations of SU(n), tensors of rank higher than two are forbidden for $n > 15$ with four-component fermions; limits are also obtained for the two-component assignments to anomaly free reducible representations, which are generalizations of the $\underset{\sim}{5} + \underset{\sim}{10}^*$ in the SU(5) model of Georgi and Glashow [13].

For theories based on the exceptional groups (class IV), definitive limits are obtained on the allowed multiplets of Higgs scalars.

The reason that the present constraints are weaker than those of pure QCD is that there are massive colored gauge fields in the grand unification models which tend to further stabilize the gauge couplings. Although this additional stabilization does not occur until the grand unification mass, which is superheavy in models which do not conserve baryon number (F2), it is not possible to obtain stronger constraints on the mass spectrum of fermions and scalars (F3).

II. ASYMPTOTIC FREEDOM OF GAUGE COUPLINGS

The standard renormalization group equation for the running gauge coupling constant has the one-loop approximation

$$16\pi^2 \frac{dg}{dt} = - b_0 g^3 \tag{2}$$

where $t = \ln \lambda^2$, with λ a parameter which sets the momentum scale. The coefficient b_0 is given in terms of the gauge group representation content of the theory by [9,10,38]

$$b_0 = \frac{11}{3} S_2(G) - \frac{2}{3} S_2(F) - \frac{1}{6} S_2(S) \tag{3}$$

where G is the adjoint representation of the gauge group, F is the representation of the fermions expressed as two-component (Weyl) spinors, and S is the representation of the scalars expressed as real fields (F4). $S_2(R)$ is a quadratic invariant associated with the (possibly reducible) representation

R of the gauge group; it is defined in terms of the representation matrices $\underset{\sim}{t}_A$ of the generators of the group by

$$\mathrm{Tr}(\underset{\sim}{t}_A \underset{\sim}{t}_B) = S_2(R)\delta_{AB} \qquad (4)$$

The theory can be asymptotically free only if

$$b_o > 0$$

(further necessary conditions on the Yukawa couplings, scalar quartic couplings, and requirements that the initial values lie in the domain of attraction of the origin, are not considered in the present study). The restrictions on the representation content of the fermions and scalars in the theory due to this condition are outlined here for the grand unified theories classified in [27].

The quadratic invariants $S_2(R)$ for the relevant representations of the classical groups are collected in the Appendix. These lead to the results given below.

SU(n)

For the class I and class II embeddings of $SU^c(3)$ in SU(n), with four-component fermions transforming according to the fundamental representation,

$$b_0 = \frac{11}{3} - \frac{2}{3} p - \frac{1}{6} S_2 (S) \qquad (5)$$

(four-component fermions are required here to avoid anomalies). The Higgs scalar content is not severely restricted. If $p = 1$, rank two tensors are always allowed; for $n = 24$ (6 color triplet quarks and 6 leptons), one third-rank tensor (symmetric or antisymmetric) is allowed.

For the class III embedding of $SU^c(3)$ in SU(n), the vector-like assignment of the fermions to a single irreducible tensor of rank k is consistent with asymptotic freedom for $n \lesssim 9$ if $k = 4$, for $n \lesssim 15$ if $k = 3$, and for any n if $k = 2$. Also, the two-component assignment to the self-conjugate antisymmetric tensor of rank m in SU(2m) is allowed for $m \lesssim 5$.

An interesting type of fermion representation in this embedding is the anomaly-free reducible representation, which is a generalization of the "Woolworth representation" ($\underset{\sim}{5} + \underset{\sim}{10}^*$) of the SU(5) model of Georgi and Glashow [13]. These representations allow a two-component fermion assignment with no fermion mass term, and have an unequal number of left-handed and right-handed neutral leptons (thus leading naturally to massless neutrinos). Here only a brief characterization is given of the representations of this type which are consistent with asymptotic freedom; for full details, see [49].

The representation

$$(n - 4) \left\{1\right\} + \left\{1^2\right\}^*$$

containing the fundamental representation n-4 times, and the conjugate antisymmetric tensor of rank 2 once, is anomaly-free for $n \geq 4$. For $n = 5$ it is the $\underline{5} + \underline{10}^*$ of SU(5); for $n = 6$, it is the $\underline{6} + \underline{6} + \underline{15}^*$ of SU(6) obtained from the $\underline{27}$ of E_6 under the reduction $E_6 \rightarrow SU(6) \otimes SU(2)$. There are in general $\frac{1}{2}(n-3)(n-4)$ massless neutrinos in the representation, if only neutral and singly charged leptons are present.

The number p of copies ("generations") of the representation is restricted by

$$p < \frac{11n}{2(n-3)}$$

For SU(5), this gives $p < 13$, which is certainly consistent with the phenomenological arguments [46,47] which suggest p = 3 (or possibly 4).

There are, in addition, anomaly-free reducible representations of groups as large as SU(16) consistent with asymptotic freedom; twelve of these are consistent with the cosmological bound [50] of 7 on the number of massless neutrinos.

<u>SO(n)</u>

For the class I embedding of $SU^c(3)$ in SO(n), with p two-component fermions transforming according to the fundamental representation,

$$b_o = \frac{22}{3}(n-2) - \frac{4}{3}p - \frac{1}{6} S_2(S) \qquad (6)$$

The Higgs scalar content is here somewhat more restricted; for n = 24

(3 color triplet quarks and 3 leptons), one third-rank tensor (symmetric or antisymmetric) is allowed, as it is for n = 32; for n = 40, only one anti-symmetric third-rank tensor is allowed, while for n = 48, no third-rank tensors are allowed.

For the class III embedding of $SU^c(3)$ in SO(n), with fermions transforming according to one or more spinor representations, the constraint $b_o > 0$ allows 64 quark flavors and 64 leptons, but not more; if these are assigned as two-component fermions to a single irreducible spinor representation, the largest unified group of this type is SO(20).

Sp(2n)

For the class I embedding of $SU^c(3)$ in Sp(2n), with p two-component fermions transforming according to the fundamental representation,

$$b_o = \frac{22}{3}(n + 1) - \frac{2}{3}p - \frac{1}{6}S_2(S) \qquad (7)$$

Here the Higgs scalars are quite restricted; for $n \geq 16$, no third-rank tensors are allowed at all.

It remains to discuss the exceptional groups F_4, E_6, E_7, which have fermion representations with only color triplets and color singlets (F5). The constraints on the Higgs scalar multiplets for these groups, with minimal fermion representations, are summarized in Table I.

For F_4, up to 16 two-component fermions transforming as $\underset{\sim}{26}$ are allowed (the Higgs scalar content must be reduced accordingly), corresponding to a maximum of 48 quarks and 144 leptons and antileptons.

For E_6 [18-21], the restrictions on the scalars are significant, especially in the flavor-chiral version of the theory currently favored [20,21]. Since

$$\underset{\sim}{27} \times \underset{\sim}{27} = (\underset{\sim}{27}^{*} + \underset{\sim}{351}') + \underset{\sim}{351}_A \,, \qquad (8)$$

it follows that only scalars transforming as $\underset{\sim}{78}$ or $\underset{\sim}{351}$ can be responsible for the superstrong symmetry breaking required to suppress proton decay, if no fermion masses are to be made superheavy. If a $\underset{\sim}{351}$ is used for this purpose (it may be possible with more than one $\underset{\sim}{78}$), then only $\underset{\sim}{27}$, $\underset{\sim}{27}^{*}$ Higgs scalars are available to grow fermion masses.

For E_7 [22-24], the restrictions are even more stringent. In order to provide superstrong breaking, it is necessary to use a representation of Higgs scalars larger than $\underset{\sim}{56}$, as explained in [24(b)]. But

$$\underset{\sim}{56} \times \underset{\sim}{56} = (\underset{\sim}{133} + \underset{\sim}{1463})_S + (\underset{\sim}{1} + \underset{\sim}{1539})_A \qquad (9)$$

so that use of $\underset{\sim}{133}$ or $\underset{\sim}{1463}$ for this purpose will make some fermions superheavy (F6). The representation $\underset{\sim}{912}$ is pseudoreal, not real, so that a Higgs scalar in this representation is necessarily complex; from Table I it can then be seen that there can be no scalars with Yukawa couplings to the fermions. This leaves $\underset{\sim}{1539}$, but then the fermion masses must be generated by at most two $\underset{\sim}{133}$ scalars, which is apparently not enough to reproduce the known pattern of quark and lepton masses [24(b)].

Thus an asymptotically free E_7 model with a single 56 of fermions is presently excluded. However, it might be possible to construct a theory with two 56-plets of fermions (then baryon number could also be conserved [27]), with symmetry breaking via Higgs scalars in 1463.

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APPENDIX. EVALUATION OF THE INVARIANT $S_2(R)$

Appearing in the coefficient b_o in the renormalization group equation (2) is the quadratic invariant $S_2(R)$ defined for a group representation R by

$$\mathrm{Tr}(\underset{\sim}{t}_A \underset{\sim}{t}_B) = S_2(R)\, \delta_{AB} \tag{A.1}$$

where the $\underset{\sim}{t}_A$ are the matrices representing the group generators in R. These invariants can be evaluated recursively using the relations

$$S_2\,(R_1 + R_2) = S_2\,(R_1) + S_2\,(R_2) \tag{A.2}$$

$$S_2\,(R_1 \times R_2) = d(R_1)\, S_2\,(R_2) + S_2\,(R_1)\, d\,(R_2) \tag{A.3}$$

where d(R) = dimension of R. The normalization can be defined by specifying the value of $S_2(R)$ in the fundamental representation.

Values of $S_2(R)$ for selected irreducible representations of the classical groups are given in Table A.1, with normalization according to the following conventions:

SU(n)

$$S_2(n) = \frac{1}{2} \tag{A.4}$$

corresponding to $\underset{\sim}{t}_A = \frac{1}{2}\underset{\sim}{\lambda}_A$ in the fundamental representation (the $\underset{\sim}{\lambda}_A$ are the generalized Pauli-Gell-Mann matrices).

SO(n)

$$S_2(n) = 2 \qquad (A.5)$$

chosen so that the generators of SO(3) coincide with those of SU(2).

Sp(2n)

$$S_2(2n) = 1 \qquad (A.6)$$

chosen so that the generators of Sp(4) coincide with those of SO(5).

The quadratic invariants for the representations of the exceptional groups shown in Table I are computed using the standard SU(3) normalization for the generators of the color group, and the known reduction of the representations under $G \rightarrow G_{fl} \otimes SU^c(3)$ (see [19,20,22-24,27]). Note also that $S_2(R)$ coincides (apart from normalization) with the Dynkin index computed by Wybourne and Bowick [52], who also give many additional useful properties of the exceptional group representations.

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FOOTNOTES

(F1) The difficulty is that the singularity observed by Landau involves using the lowest order perturbation expansion for the renormalization group function $\beta(g)$ in the strong coupling region.

(F2) See [45-47] for a discussion of this point in SU(5).

(F3) Even with a perturbation theory approximation to $\beta(g)$, the recent work of Maiani, Parisi and Petronzio [48] suggests that even 16 quarks of relatively low mass, and a corresponding number of leptons, will not lead to strong coupling either in QCD, or in $SU(2) \otimes U(1)$, until the Planck mass or beyond.

(F4) The representations F, S are in general reducible, in which case S_2 is a sum over values of the quadratic invariant for irreducible representations. If the fermions can be expressed as four-component spinors, and F is the gauge group representation of the four-component fermions, then the coefficient of $S_2(F)$ is 4/3.

(F5) G_2 has no flavor, E_8 has a color octet in the fundamental representation **248**. It turns out that representations of E_8 larger than **248** (either for fermions or for scalars) are not consistent with asymptotic freedom.

(F6) This might not be a disaster if only the **175** of SU(6) contained in **1463** developed a large vacuum expectation value - then only some unwanted leptons would be made superheavy - but it is hard to see how to arrange this in a natural way.

Table I. Maximal multiplets of Higgs scalars consistent with asymptotic freedom for the exceptional groups F_4, E_6, E_7 with minimal fermion multiplets.

F_4 (fermions in 26 + 26)

R	26	52	273	324
$S_2(R)$	3	9	63	81
A	57	0	0	0
B	0	19	0	0
C	0	5	2	0
D	0	3	1	1
E	0	1	0	2

Note: One 52 is equivalent to three 26.

E_6 (fermions in 27 + 27)

R	27$^{(c)}$	78$^{(r)}$	351$^{(c)}$	351'$^{(c)}$
$S_2(R)$	3	12	75	84
A	39	0	0	0
B	1	19	0	0
C	0	7	1	0
D	1	5	0	1

Note: (r) = real representation, (c) = complex representation; one real 78 is equivalent to two complex 27.

E_7 (fermions in 56)

R	56	133	912	1463	1539
$S_2(R)$	6	18	180	330	324
A	61	0	0	0	0
B	1	20	0	0	0
C	1	0	2	0	0
D	0	2	0	1	0
E	1	2	0	0	1

Note: One real 133 is equivalent to three real 56; the representations 56, 912 are pseudoreal, so that for scalars these representations must occur in pairs.

Table A.1. The quadratic invariant $S_2(R)$ for selected irreducible representations of the classical groups. The representation R is denoted by the partition $\{p_1,\ldots,p_k\}$ with which it is associated (for a standard review, see [51]); $S^{\pm}_{2m}$, S_{2m+1} denote the spinor representations of SO(2m), SO(2m+1), respectively.

R	SU(n)	SO(n)	Sp(2n)
$\{1\}$	$\frac{1}{2}$	2	1
$\{2\}$	$\frac{1}{2}(n+2)$	$2(n+2)$	$2(n+1)$
$\{1^2\}$	$\frac{1}{2}(n-2)$	$2(n-2)$	$2(n-1)$
$\{3\}$	$\frac{1}{4}(n+2)(n+3)$	$(n+1)(n+4)$	$(n+1)(2n+3)$
$\{21\}$	$\frac{1}{2}(n^2-3)$	$2(n^2-4)$	$4(n^2-1)$
$\{1^3\}$	$\frac{1}{4}(n-2)(n-3)$	$(n-2)(n-3)$	$(n-2)(2n-1)$
$\{k\}$	$\frac{1}{2}\binom{n+k}{k-1}$	$2\,\frac{n+2k-2}{n+k-1}\binom{n+k-1}{n}$	$\binom{2n+k}{k-1}$
$\{1^k\}$	$\frac{1}{2}\binom{n-2}{k-1}$	$2\binom{n-2}{k-1}$	$\frac{n-k+1}{n}\binom{2n}{k-1}$
$S^{\pm}_{2m}$	-	2^{m-3}	-
S_{2m+1}	-	2^{m-2}	-