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STABILITY PROPERTIES OF EQUILIBRIUM STATES

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Abstract

The significance of stability of an equilibrium state under local perturbations of the dynamics (as defined in [1]) and the different degree of stability with respect to extended perturbations of states at phase transition points are discussed. The general conclusions are tested and illustrated in the example of the free Bose gas. A more transparent proof of the relation between local stability and the Kubo-Martin-Schwinger relation is given.

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Introduction

It was argued in [1] that thermodynamic equilibrium states of an infinitely extended medium are distinguished among (possibly other) stationary states by a certain stability with respect to small changes of the dynamical law. In fact, this stability should be considered as the defining property of an equilibrium state. Specifically we consider the quantum physics of an infinitely extended system. The system is described by the algebra $\mathfrak{A}$ of its quasi local observables, the states by expectation functionals on $\mathfrak{A}$ and the dynamics by the 1-parameter automorphism group α_t .[1)]

The conceptual definition of stability is then the following:
Consider a small change of the dynamical law to the automorphism group α_t^h which results from α_t by the "addition of a perturbation Hamiltonian h ". Then if ω is a stationary state with respect to α_t it is called stable under this perturbation if there exists a state ω^h , stationary with respect to α_t^h , which is close to ω . In particular, in [1] we took h to be an element of $\mathfrak{A}$, which means physically that we consider essentially local, bounded perturbations. If λ is a coupling constant which we let tend to zero ultimately then the stability requirement is that $\|\omega^{\lambda h} - \omega\| \to 0$ as $\lambda \to 0$ for all such perturbations i.e. that $\omega^{\lambda h}$ be continuous in the norm topology of state space at $\lambda = 0$. Let us call this specific form of the stability requirements S_1 . More generally a stability criterion will involve on the one hand the specification of a class of perturbations and on the other hand the specification of the type of continuity of the state which is demanded.

Requiring S_1 i.e. norm continuity of ω under local perturbations ($h \in \mathfrak{A}$) it was shown[2)] in [1] that this leads to the condition

1) Notation: $\omega(A)$ denotes the expectation value of the element $A \in \mathfrak{A}$ in the state ω . $\alpha_t(A)$ is the time translated element A.

2) We had to assume there in addition that ω possesses sufficiently strong clusting properties. This may be replaced however by a requirement of sufficiently strong asymptotic abelianness of the dynamical system i.e. by a property of $\mathfrak{A}, \alpha_t$ which does not refer to the particular state. See Section IV and appendix.

$$\int_{-\infty}^{\infty} \omega\left([h, \alpha_t(A)]\right) dt = 0 \qquad \text{for all } A \in \mathfrak{A} \tag{1.1}$$

Secondly it was shown[2] that if an extremal stationary state ω satisfies condition (1.1) for a dense set of $h \in \mathfrak{A}$ then ω satisfies the Kubo-Martin-Schwinger (KMS)-condition for some value of the inverse temperature β. Thirdly, in the appendix of [1] the condition (1.1) was used directly to compute all the extremal stationary stable states of a free Fermi gas and verify that they are a 2-parametric set of quasifree states with a 1-particle momentum distribution of the form

$$\rho(\underset{\sim}{p}) = \left(1 + \exp(\alpha + \beta\varepsilon_p)\right)^{-1} . \tag{1.2}$$

It is instructive to do the analogous computation for the free Bose gas. In that case the condition (1.1) selects two branches of extremal stationary states: the normal one (labelled by α,β) and the "superfluid" one labelled by β and some constants c, c_i ... describing the state of the superfluid component. This will be done in Section II.

One may note that while the derivation of (1.1) from the conceptual definition of stability in Ref.[1] took $\mathfrak{A}$ to be a C^*-algebra (with A and h belonging to $\mathfrak{A}$) it appears that condition (1.1) itself remains meaningful under wider circumstances. Thus in Section III we shall apply (1.1) to an algebra of unbounded observables. This is, in the Bose case, a much simpler procedure. One verifies then afterwards that the resulting states are indeed KMS-states of a certain C^*-algebra (the uniform closure of the associated Weyl algebra) but this algebra itself depends sensitively on the class of wave functions, which one allows.

One expects that an equilibrium state which is away from a phase transition point is stable not only under local but also under homogeneous perturbations and that the stability class decreases at a phase transition point. Possibly the remaining stability class depends on the nature of the phase transition ("hierarchy of stability"). Heuristically a test of this idea would be to see whether (1.1) remains satisfied for extended perturbations of the form

$$h = \int \alpha_{\underset{\sim}{x}}(v) \left(1 + |\underset{\sim}{x}|\right)^{-n} d^3x \tag{1.3}$$

where $v \in \mathfrak{N}$ is local, $\alpha_{\underset{\sim}{x}}$ denotes space translations. For $n = 0$ we have homogeneous perturbations; $n = 1$ is roughly equivalent to surface perturbations which should be the beginning of instability in the case of first order phase transitions; $n = 3 + \varepsilon$ are quasilocal perturbations. The question is then for which range of n the integration over $\underset{\sim}{x}$ and that over t can still be interchanged if one inserts (1.3) into (1.1). We shall consider this question in Section III.

Finally, in Section IV we discuss again the relation between the conceptual formulation of stability and condition (1.1). Apart from a proof of this relation under somewhat modified assumptions (as compared to [1]) we are interested in the qualitative physical picture of the perturbed state in the case when ω is not S_1-stable. It appears that in this case the local perturbation causes the expectation value of almost every local current, to decrease only like $|\underset{\sim}{x}|^{-2}$ with distance so that the flux through each solid angle becomes asymptotically constant (independent of the distance from the origin) whereas in the stable case the asymptotic outward flux vanishes. This is the mechanism by which $\|\omega^{\lambda h} - \omega\|$ becomes small in the stable case and remains equal to 2 independent of the size of λ in the unstable case.

In the appendix we sketch the derivation of the KMS-condition from (1.1) for a primary state without the use of additional asumptions on the rate of decrease of correlation functions.

II. The Extremal, Stable, Stationary States of the Free Bose Gas

Consider an infinite system of non-interacting identical Bose particles and let

$$a^*(f) = \int a^*(\underset{\sim}{p}) f(\underset{\sim}{p}) d^3p , \qquad a(f) = \int a(\underset{\sim}{p}) \bar{f}(\underset{\sim}{p}) d^3p \tag{2}$$

be the creation- (respectively annihilation) operators of a particle with momentum space wave function f. The commutation relations are

$$[a(f), a^*(g)] = (f, g) = \int \bar{f}(\underset{\sim}{p}) g(\underset{\sim}{p}) d^3p , \tag{2.1}$$

$$[a(f), a(g)] = 0$$

Let $\mathcal{F}$ denote the (not closed, nor normed) algebra of polynomials of the $a^+(f)$, $a(g)$ where f, g are smooth wave functions[3] and $\mathfrak{A}$ the subalgebra of "gauge invariant" elements of $\mathcal{F}$ i.e. those polynomials which have in each term an equal number of creation and annihilation operators (conservation of particle number). $\mathfrak{A}$ will be considered as the observable algebra. The dynamical law (in $\mathcal{F}$) is given by

$$\alpha_t(a^*p) = e^{i\varepsilon_p t} a^*(p) \; ; \qquad \varepsilon_p = p^2/2m \; . \tag{2.2}$$

We shall analyse the consequences of the "stability condition" (1.1) for a state ω, allowing for A and h arbitrary elements of $\mathfrak{A}$. The state ω may be described by the set of "Wightman distributions"

$$W^{(2n)}(p_1 \cdots p_n ; p_1', \cdots p_n') = \omega(a^*(p_1) \cdots a^*(p_n)\, a(p_1') \cdots a(p_n')) \tag{2.3}$$

which are Laurent Schwartz distributions over $\mathcal{D}^{(6n)}$.

First take

$$A = a^*(f_1) \cdots a^*(f_n)\, a(f_1') \cdots a(f_n')$$
$$h = a^*(g)\, a(g') \tag{2.4}$$

and denote the support of the respective wave functions by

$$\mathrm{supp}\, f_i = K_i \; ; \quad \mathrm{supp}\, f_i' = K_i' \; ; \quad \mathrm{supp}\, g = K \; ; \quad \mathrm{supp}\, g' = K' .$$

If we choose

$$\begin{aligned} &K' \text{ disjoint from } K_i \text{ for } i = 2,\ldots n \\ &K \text{ disjoint from } K_j' \text{ for } j = 1,\ldots n \end{aligned} \tag{2.5}$$

then condition (1.1) gives (for stationary ω)

3) Specifically we take $f \in \mathcal{D}$, the infinitely often differentiable functions with compact support.

$$\int W^{(2n)}(p_1, \cdots p_n ; p_1' \cdots p_n')\, F(p_1) f_2(p_2) \cdots f_n(p_n) \bar{f}_1'(p_1') \cdots \bar{f}_n'(p_n') \prod d^3p_i\, d^3p_j' = 0 \tag{2.6}$$

with

$$F(p) = g(p) \int \delta(\varepsilon_p - \varepsilon_{p'})\, f_1(p') \bar{g}'(p')\, d^3p' \tag{2.7}$$

whenever (2.5) is satisfied. If we choose $2n$ points $p_1, \ldots p_n$, $p_1', \ldots p_n'$ such that $p_1 \neq p_j'$ for any j and $p_1 \neq 0$ then there exists a neighborhood K of p_1 and neighborhoods K_i of $p_i (i = 2,\ldots n)$, K_j' of $p_j' (j = 1,\ldots n)$ such that for arbitrary F, $f_i (i = 2,\ldots n)$, $f_j' (j = 1,\ldots n)$ from class $\mathcal{D}$ with supports respectively in K, K_i, K_j' we can choose test functions f_1 g, g' with supports respectively in K_1, K, K' so that (2.5) and (2.7) is satisfied[4]. This means that $W^{(n)}(p_1, \ldots p_n', p_n', \ldots p_1')$ vanishes unless either $p_1 = 0$ or $p_1 = p_j'$ for some j. Since the argument may be repeated, interchanging all the primed quantities with the unprimed ones in the choice of the supports and since the Index 1 is not preferred ($W^{(2n)}$ being symmetric under permutations of the p_i), this means that W^{2n} is a sum of terms each one having point support at the origin in some of the p_i, p_j' and point support at $(p_k - p_\ell') = 0$ for some pairing of the remaining momenta.

In particular, the 2-point function is of the form

$$W^{(2)}(p, p') = c(p)\, \bar{c}(p') + \rho^{(2)}(p, p') \tag{2.8}$$

where

$$c(p) = \left(c + \sum_i c_i \frac{\partial}{\partial p_i} + \sum_{ij} c_{ij} \frac{\partial^2}{\partial p_i \partial p_j} + \cdots \right) \delta^3(p) \tag{2.9}$$

4) If $p_1 \neq p_j'$ we can choose the neighborhoods of the points $p_1, \ldots p_n'$ small enough so that (2.5) is satisfied with a K' which covers the whole energy spread of K. Then for $f_1 = g'$ the function $\wp(p) = \int \delta(\varepsilon - \varepsilon') f_1(p') \bar{g}'(p')\, d^3p'$ will be infinitely differentiable and nonvanishing in K if K excludes the point 0 (Note that $\wp(0) = 0$). Thus we may choose $g(p) = F(p)/\wp(p)$.

$$\rho^{(2)}(\underset{\sim}{p},\underset{\sim}{p}') = \left(\rho(\underset{\sim}{p}) + \sum_i \rho_i(p)\frac{\partial}{\partial p_i} + \cdots\right)\delta^3(\underset{\sim}{p}-\underset{\sim}{p}') . \tag{2.10}$$

Here c, c_i etc. are constants, the indices i, j referring to the 3 components of the vector $\underset{\sim}{p}$. The stationarity and positivity of the state ω implies that in $\rho^{(2)}$ no derivatives may occur[5]. Thus

$$\rho^{(2)}(\underset{\sim}{p},\underset{\sim}{p}') = \rho(p)\,\delta^3(\underset{\sim}{p}-\underset{\sim}{p}') \tag{2.11}$$

It also demands that

$$\sum_i c_{ii} = \sum_i c_{iij} = \cdots = 0 . \tag{2.12}$$

So far the argument is completely analoguous to the Fermi case. But there the annihilators a(f) are bounded by the $\mathcal{L}^{(2)}$-norm of f and this implies immediately that in the Fermi case $C(\underset{\sim}{p}) = 0$. In the Bose case we have to carry $C(\underset{\sim}{p})$ along.

The next argument shall show that if the state ω is an extremal stationary state then the $W^{(2n)}$ for arbitrary n are determined by $W^{(2)}$. In fact

$$W^{(2n)}(\underset{\sim}{p}_1\cdots p_n; p_1'\cdots p_n') = \sum\prod c(p_i)\,\bar{c}(p_j')\,\rho^{(2)}(p_k, p_\ell') \tag{2.13}$$

where the sum runs over all partitions of the 2n arguments into clusters of one or two arguments and for the latter we can pair only a primed with an unprimed momentum. An extremal stationary state has the mean clustering property

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\omega(A\,\alpha_t(B)) - \omega(A)\,\omega(B)\right)dt = 0$$

[5]If we go e.g. in (2.10) up to the second derivatives then the positivity demands that $\rho(p)$, $\rho_{ij}(\underset{\sim}{p})$ are real, $\rho_i(\underset{\sim}{p})$ purely imaginary and at every $\underset{\sim}{p}$ the 4 × 4-matrix $\begin{pmatrix}\rho & \rho_i\\ -\rho_j & \rho_{ij}\end{pmatrix}$ shall be positive. Stationarity demands $\sum\rho_i p_i = 0$ $\sum\rho_{ij}p_j = 0$, $\sum\rho_{ii} = 0$. The latter two conditions are incompatible with positivity.

Thus we should have, e.g.

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(W^{(4)}(\underset{\sim}{p}_1,\underset{\sim}{p}_2;\underset{\sim}{p}_1',\underset{\sim}{p}_2') - W^{(2)}(\underset{\sim}{p}_1,\underset{\sim}{p}_1')\,W^{(2)}(\underset{\sim}{p}_2,\underset{\sim}{p}_2')\right)e^{i(\varepsilon_2-\varepsilon_2')t}\,\bar{f}(p_1)\cdots f(\underset{\sim}{p}_2')\,dp_1\cdots dp_2' = 0 \tag{2.14}$$

Due to the singular support of $W^{(4)}$ the integrand will, for suitable choice of the f_i, f_i' always contain a time independent term which violates (2.14) unless the part of $W^{(4)}$, having support at $\underset{\sim}{p}_2 - \underset{\sim}{p}_2' = 0$ factors, and cancels the corresponding term in $W^{(2)}(p_1,p_1')\,W^{(2)}(p_2,p_2')$ and the same holds for the part of W^4 with support at $\underset{\sim}{p}_2 = 0$, $\underset{\sim}{p}_2' = 0$. Using the symmetry of W^4 under interchange of $\underset{\sim}{p}_1$ with $\underset{\sim}{p}_2$ one gets expression (2.13) for $W^{(4)}$.

To get further restrictions on the function $\rho(p)$ and the constants c, c_i... we apply the stability condition (1.1) to

$$h = a^*(g_1)\,a^*(g_2)\,a(g_1')\,a(g_2') ; \quad A = a^*(f_1)\,a^*(f_2)\,a(f_1')\,a(f_2') .$$

The condition (1.1) gives then with (2.8), (2.11), 2.13)

$$\left\{\rho(p_1)\rho(p_2)(1+\rho(p_1'))(1+\rho(p_2')) - (1+\rho(p_1))(1+\rho(p_2))\rho(p_1')\rho(p_2')\right\}\delta(\varepsilon_1+\varepsilon_2-\varepsilon_1'-\varepsilon_2') = 0 \tag{2.15}$$

and in addition either

$$c(\underset{\sim}{p}) = 0 \tag{2.16}$$

or

$$\left\{\rho(p_2)(1+\rho(p_1'))(1+\rho(p_2')) - (1+\rho(p_2))\rho(p_1')\rho(p_2')\right\}\delta(\varepsilon_2-\varepsilon_1'-\varepsilon_2') = 0 \tag{2.17}$$

Condition (2.15) implies that $\log \frac{1+\rho}{\rho}$ shall be a linear function of the energy. Thus

$$\rho(\underset{\sim}{p}) = \left(e^{\alpha+\beta\varepsilon_p} - 1\right)^{-1} . \tag{2.18}$$

The remaining condition gives the alternative between (2.16) (absence of the "superfluid" part) and

$$c(\underset{\sim}{p}) \neq 0 \quad ; \quad \alpha = 0 \tag{2.19}$$

Remarks:

i) The possibility (2.19) arises formally from the fact that in 3 (or more) dimensions $\delta(\varepsilon) = 0$ since

$$\int f(\underset{\sim}{p})\, \delta(\varepsilon)\, d^3p = \int f(p)\, \delta(\varepsilon) \sqrt{\varepsilon}\, d\varepsilon\, d\Omega = 0$$

if f is finite at $\underset{\sim}{p} = 0$. In 2 dimensions a solution with $c(p) \neq 0$ is not allowed. The superfluid state is not stable under perturbations involving $a(f)$ for functions $f(\underset{\sim}{p})$ which are singular at the origin like $\varepsilon^{-1/2}$ though still square integrable.

ii) The "1-point function" $\underset{\sim}{c}(p)$ is the wave function of the macroscopically occupied 1-particle state. The conditions (2.12) read in x-space $\Delta\tilde{c}(\underset{\sim}{x}) = 0$ which is the Schroedinger equation for zero energy[6]. If one approaches the problem from a finite system then the solutions where c_i, c_{ij} etc. are not zero correspond to boundary conditions which increase as the box becomes larger. The superfluid density increases in them as one moves away from the origin and one has a circulating superfluid flow in this case.

iii) The expectation functional over $\mathfrak{A}$ defined above can most simply be summarized by saying that ω may be extended to a quasifree state over the Weyl algebra generated by the unitaries $\exp i\,(a^*(f) + a(f))$

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[6] This significance of the conditions (2.12) was pointed out to us by D. Ruelle.

with $f \in \mathcal{D}$. The truncated 2-point function is given by (2.11), (2.18), the 1-point function by (2.9), (2.12). One checks then that ω has the right analyticity properties for a KMS-state with respect to the appropriate combination of time translations and gauge transformations corresponding to the parameters α, β.

III. Degree of Stability

Suppose ω is an extremal, stationary state satisfying (1.1) for local, bounded perturbation h (i.e. $h \in \mathfrak{A}$). We may ask whether it will still satisfy (1.1) for extended perturbations of the form (1.3). The left hand side is of the form

$$\int (1+|\underset{\sim}{x}|)^{-n}\, \phi(\underset{\sim}{x},t)\, d^3x\, dt \tag{3.1}$$

where

$$\phi(\underset{\sim}{x},t) = \omega\left([\alpha_{\underset{\sim}{x},t}(h), A]\right) \tag{3.2}$$

and the x-integration shall be performed first.
We know that if we integrate first over t then (3.1) will indeed vanish. If the expression is absolutely integrable we can interchange the order of the $\underset{\sim}{x}$ and t-integration and hence get zero for (3.1). A very crude criterion for extended stability is then obtained by the following power counting argument.

Introducing x/t and t was variables it is reasonable to assume that the correlation function ϕ will depend smoothly on $\underset{\sim}{x}/t$ and decrease fast for $\underset{\sim}{x}/t \to \infty$ (for fixed t). If

$$\sigma(\underset{\sim}{x}/t) = \lim_{t\to\infty} t^r \phi \qquad (\text{for fixed } \underset{\sim}{x}/t) \tag{3.3}$$

is finite then (3.1) will be absolutely integrable as long as

$$n + r > 4 . \tag{3.4}$$

Let us consider the example of the free Bose gas! Here one may compute the function $\phi(\underset{\sim}{x}, t)$ for bounded observables v, A from the Weyl-algebra mentioned at the end of Section II (Remark iii) when ω is taken as one of the equilibrium states described. It turns out[7] that the generic asymptotic behaviour of ϕ for large arguments is the same as the one obtained when one takes for A and v the simplest unbounded observables

$$A = a^*(f)\, a(f)\ ; \qquad v = a^*(g)\, a(g)\,.$$

Then, for an equilibrium state $\omega_{\rho,c}$ we get

$$\phi = \phi_\rho + \phi_c$$

with

$$\phi_\rho = \int \exp i\{(\underset{\sim}{p}' - \underset{\sim}{p})\underset{\sim}{x} - (\varepsilon' - \varepsilon)t\}\,(\rho(\underset{\sim}{p}') - \rho(\underset{\sim}{p}))\,\bar f(\underset{\sim}{p})\, f(\underset{\sim}{p}')\,\bar g(\underset{\sim}{p}')\, g(\underset{\sim}{p})\, d^3p\, d^3p'$$

$$\phi_c = 2\,\mathrm{Im} \int \exp i\{(\underset{\sim}{p}' - \underset{\sim}{p})\underset{\sim}{x} - (\varepsilon' - \varepsilon)t\}\,(c, f)\,\bar f(\underset{\sim}{p})\, g(\underset{\sim}{p})\,\bar g(\underset{\sim}{p}')\, c(\underset{\sim}{p}')\, d^3p\, d^3p'.$$

Consider first the branch where ρ is given by (2.18) and $\alpha \neq 0$, $c = 0$. Then the asymptotic evaluation of ϕ_ρ by the stationary phase method gives

$$\phi = \phi_\rho \to t^{-4} F(\underset{\sim}{x}/t) \qquad \text{i.e.} \quad r = 4\,.$$

The power 4 arises because the stationary point in both the $\underset{\sim}{p}'$- and the $\underset{\sim}{p}$-integration is at $\underset{\sim}{p} = \underset{\sim}{p}' = \underset{\sim}{x}/t$ and the dominant contribution which would have a decrease like t^{-3} vanishes because of the factor $(\rho(\underset{\sim}{p}) - \rho(\underset{\sim}{p}'))$; so one has to carry the stationary phase approximation as in [2; Eq.(1.12)] up to terms of order $t^{-5/2}$. The lowest non-vanishing term comes then from the cross terms which are or order $t^{-5/2}, t^{-3/2} = t^{-4}$.
Next consider the boundary point between the two branches where $\alpha = 0$ and still $c = 0$. Here the function ρ becomes singular like $|p|^{-2}$ at the origin so that now

$$\phi = \phi_\rho \to t^{-3} F(\underset{\sim}{x}/t) \qquad \text{i.e.} \quad r = 3\,.$$

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[7] We are very much indebted to K. Fredenhagen for this observation.

Finally, when $\alpha = 0$, $c \neq 0$ the slowest decreasing part comes from ϕ_c and $\phi_c \approx t^{-3/2} F(\underset{\sim}{x}/t)$.

Thus we have $r = 4$ in the normal case, indicating stability up to $n = 0$. At the boundary of the two branches $(\alpha = 0,\ c = 0)$ r decreases to $r = 3$ and ultimately we have $r = 3/2$ in the superfluid case.

IV. Discussion of the Stability Criterion (1.1)

A dynamical system possesses in general many different types of equilibrium states (e.g. phase transition points, critical points). It is therefore desirable to separate as fas as possible the properties of the dynamical system as such (algebraic properties) from those of a particular state (e.g. behavior of correlation functions). One relevant algebraic property for our purposes is the degree of asymptotic Abelianness in time and space-time.

We shall proceed here from the assumption that $\|[A, \alpha_t(B)]\|$ is an absolutely integrable function of t if A and B lie in a domain $\mathcal{D}$ which is norm dense in $\mathfrak{A}$, and that the same holds for the perturbed dynamics $\alpha_t^{\lambda h}$ for sufficiently small λ when $h \in \mathcal{D}$.
This assumption will be called $\mathcal{L}^{(1)}$-asymptotic Abelianness.

Consider the automorphisms

$$\beta_t^{\lambda h} = \alpha_t^{-1}\, \alpha_t^{\lambda h}\,; \tag{4.1}$$

One has

$$i\,\frac{\partial}{\partial t}\, \beta_t^{\lambda h}(A) = -\lambda\, \alpha_t^{-1}\left(\left[h, \alpha_t^{\lambda h}(A)\right]\right) \tag{4.2}$$

$$i\,\frac{\partial}{\partial t}\, \left(\beta_t^{\lambda h}\right)^{-1}(A) = +\lambda \left(\alpha_t^{\lambda h}\right)^{-1}\left(\left[h, \alpha_t(A)\right]\right). \tag{4.3}$$

In the case of $\mathcal{L}^{(1)}$-asymptotic Abelianness the norm of the right hand side of (4.3) is absolutely integrable provided $h, A \in \mathcal{D}$ and the same holds for the r.h.s. of (4.2) if λ is sufficiently small. This means that for $h \in \mathcal{D}$ the limits

$$\beta_{\pm}^{\lambda h} = \lim_{t \to \pm\infty} \beta_t^{\lambda h} \tag{4.4}$$

exist and are automorphisms on $\mathfrak{A}$ (compare [3]). In analogy to scattering theory one may call $\beta_{\pm}^{\lambda h}$ the "Möller automorphisms".

Given a primary state ω which is stationary under α_t we get then immediately two primary states $\omega_{\pm}^{\lambda h}$ which are stationary under $\alpha_t^{\lambda h}$ namely

$$\omega_{\pm}^{\lambda h}(A) = \omega\left(\beta_{\pm}^{\lambda h}(A)\right) \tag{4.5}$$

If we now require as the stability requirement for ω that

$$\left\| \omega_{\pm}^{\lambda h} - \omega \right\| = c_{\pm}(\lambda) \to 0 \quad \text{as} \quad \lambda \to 0 \tag{4.6}$$

(norm continuity) then for sufficiently small λ, say $\lambda < \lambda_0$ (i.e. as soon as $c_{\pm}(\lambda) < 2$) $\omega_{+}^{\lambda h}$, $\omega_{-}^{\lambda h}$ and ω must all lie in the same folium[8]. But a primary folium can contain at most one stationary state for the asymptotically Abelian dynamics $\alpha_t^{\lambda h}$. Therefore

$$\omega_{+}^{\lambda h} = \omega_{-}^{\lambda h} \qquad \text{for any} \qquad \lambda < \lambda_0 . \tag{4.7}$$

From (4.3) we obtain

$$\omega_{\pm}^{\lambda h}(A) - \omega(A) = i\lambda \int_0^{\pm\infty} \omega_{\pm}^{\lambda h}\left(\left[h, \alpha_t(A)\right]\right) dt . \tag{4.8}$$

Hence by (4.7)

$$\int_{-\infty}^{0} \omega_{-}^{\lambda h}\left(\left[h, \alpha_t(A)\right]\right) dt + \int_0^{\infty} \omega_{+}^{\lambda h}\left(\left[h, \alpha_t(A)\right]\right) dt = 0 \text{ for } 0 < \lambda < \lambda_0 \tag{4.9}$$

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8) By the folium of ω we mean the set of states which correspond to density matrices in the Hilbert space representation of $\mathfrak{A}$ arising by the GNS-construction from the state ω.

Then, if we choose $A \in \mathcal{D}$ due to (4.6) and $\mathcal{L}^{(1)}$-asymptotic Abelianness we get

$$\int_{-\infty}^{\infty} \omega\left(\left[h, \alpha_t(A)\right]\right) dt < M_+ c_+(\lambda) + M_- c_-(\lambda) \tag{4.10}$$

where M_+, M_- are finite constants (depending on A and h but not on λ) namely the time integrals of the norms $\| [h, \alpha_t(A)] \|$. Since the left hand side of (4.10) is independent of λ and the right hand side goes to zero as $\lambda \to 0$ we have (1.1) for $A, h \in \mathcal{D}$.

We shall sketch in the appendix how one gets from this result without additional assumptions to the KMS-condition which implies also that (1.1) must hold for all pairs h, A and not only for those in the dense set $\mathcal{D}$.

Let us now ask the converse question. Suppose ω is a primary, stationary state which does not satisfy the stability condition (1.1). How does the "instability" of ω manifest itself? Such states exist for instance in the free Fermi gas. (More generally one expects them if the dynamics is "non-ergodic" in some sense). As an example we may take in the case of the free Fermi gas any quasifree state over the algebra of Fermi creation and destruction operators (CAR-algebra) with a 2-point function

$$w^{(2)}(\underline{p}, \underline{p}') = \rho(\underline{p})\, \delta^3(\underline{p} - \underline{p}') \tag{4.12}$$

where ρ is not of the form $(1+\exp(\alpha+\beta\varepsilon))^{-1}$. If we take ρ direction dependent then the stability condition (1.1) is already violated for h and A of the form

$$h = \int a^*(\underline{p})\, a(\underline{p}')\, g(\underline{p}, \underline{p}')\, d^3p\, d^3p' ; \quad A = \int a^*(\underline{p})\, a(\underline{p}')\, f(\underline{p}, \underline{p}')\, d^3p\, d^3p' \tag{4.13}$$

The physical picture of the consequence of this violation emerges if we compute $\omega_{\pm}^{\lambda h}(A)$ up to the second order in λ. With

$$\omega_{+}^{\lambda h} = \omega + \lambda\, \omega^{(1)} + \lambda^2 \omega^{(2)} + \cdots$$

we have from (4.5), (4.4), (4.2)

$$\omega_+^{(2)}(A) = - \int\limits_{\infty > t_2 > t_1 > 0} \omega\left(\left[\alpha_{-t_2}(h), \left[\alpha_{-t_1}(h), A\right]\right]\right) dt_1 \, dt_2 \, . \tag{4.14}$$

For h, A as in (4.13) and ω having the 2--point function (4.12)

$$\omega_+^{(2)}(A) = - \int\limits_{\infty > t_2 > t_1 > 0} \left(\rho(p_1) - \rho(p_2) \right) g(p_1, p_1') \, g(p_1', p_2') \, \hat{f}(p_2', p_1) \, e^{i(\varepsilon_1' - \varepsilon_1) t_2 + i(\varepsilon_2' - \varepsilon_1') t} d^3p_1 \, d^3p_1' \, d^3p_2 \times dt_1 \, dt_2 + * \tag{4.15}$$

where $*$ means the complex conjugate if h and A are chosen self adjoint. We evaluate this for an observable which is localized at a large distance from the origin i.e. from the approximate localization region of h . Thus we put $\alpha_x(A)$ instead of A in (4.14) and consider the asymptotic region of large $|x|$. This changes the integrand of (4.15) by a factor $e^{i(p_1 - p_2')x}$ We may then use the method of stationary phase for the integrations over p_1 and p_2' and subsequently over t_1, t_2 . The stationary phase occurs at

$$p_1 = p_2' = \frac{x}{|x|} \, |p_1'| \tag{4.16}$$

$$t_2 = t_1 = m \frac{x}{|p_1'|} \tag{4.17}$$

and we obtain up to a numerical factor

$$\omega_+^{(2)}(\alpha_x(A)) \rightarrow \frac{1}{|x|^2} \int f(p, p) \, |g(p, p')|^2 \left(\rho(p) - \rho(p') \right) d^3 p' \tag{4.18}$$

with

$$p = |p'| \, \frac{x}{|x|} \, . \tag{4.19}$$

This is the (first order) expression for the effect produced by the difference between the number of particles scattered per unit time from an arbitrary momentum p to p' and that of the inverse process. In the stable case, where $\rho(p) = \rho(p')$ on the energy shell (4.18) vanishes and $\omega^{(2)}(\alpha_x(A))$ decreases faster than $\frac{1}{|x|^2}$ with the distance. If we perform the same

calculation with $h = \int g(p_1, p_2; p_1', p_2') \, a^*(p_1) \, a^*(p_2) \, a(p_1') \, a(p_2') \, d^3p_1 \cdots d^3p_2'$ describing two-body interparticle forces we get as the dominant term proportional to $\frac{1}{|x|^2}$ an expression similar to (4.18) with $\rho(p) - \rho(p')$ replaced by the 2-particle collision factor $\rho(p_1)\rho(p_2)(1-\rho(p_1'))(1-\rho(p_2')) - (1-\rho(p_1))(1-\rho(p_2))\rho(p_1')\rho(p_2')$ which vanishes (on the energy shell) if and only if ρ is a Fermi distribution. Thus, a primary, stationary state which does not satisfy the stability condition (1.1) will change under a local perturbation h into a stationary state $\omega_+^{\lambda h}$ which has a steady radial flux, asymptotically constant in each solid angle. In $\omega_-^{\lambda h}$ this flux has the opposite sign. In a stable state $\omega_+ = \omega_-$ and the perturbation does not produce an outward flux. Its effect is then absorbed essentially in a finite region.

APPENDIX

We sketch here how one obtains the KMS-condition from (1.1) without additional clustering assumptions for the correlation functions if the dynamics is $\mathcal{L}^{(1)}$-asymptotically Abelian and ω is primary. Noting that the domain $\mathcal{D}$ is by definition invariant under time translation and closed under the product operation we may, as in [1] put in (1.1)

$$h = h_1 \alpha_\tau(h_2) \; ; \quad A = A_1 \alpha_\tau(A_2)$$

and obtain

$$0 = \int_{-\infty}^{\infty} \omega\left(\left[h_1 \alpha_\tau(h_2), A_1 \alpha_t(A_2)\right]\right) dt = \int \omega\left(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\right) dt \tag{A.1}$$

$$\mathrm{I} = h_1 \left[\alpha_\tau(h_2), \alpha_t(A_1)\right] \alpha_{t+\tau}(A_2)$$

$$\mathrm{II} = h_1 \alpha_t(A_1) \left[\alpha_\tau(h_2), \alpha_{t+\tau}(A_2)\right]$$

$$\mathrm{III} = \left[h_1, \alpha_t(A_1)\right] \alpha_{t+\tau}(A_2) \alpha_\tau(h_2)$$

$$\mathrm{IV} = \alpha_t(A_1) \left[h_1, \alpha_{t+\tau}(A_2)\right] \alpha_\tau(h_2)$$

We consider the limit $\tau \to \infty$ of (A.1). The norms of II and III are bounded by $\mathcal{L}^{(1)}$-functions of t which are independent of τ. Thus we can interchange the τ-limit with the t-integration and obtain due to the clustering of a primary state

$$\lim_{\tau \to \infty} \int \omega(\mathrm{II} + \mathrm{III}) dt = \int \omega(h_1 \alpha_t(A_1))\, \omega[h_2, \alpha_t(A_2)] dt + \int \omega([h_1, \alpha_t(A_1)])\, \omega(\alpha_t(A_2) h_2)\, dt \tag{A.2}$$

$$= \int (F_1(t) F_2(t) - G_1(t) G_2(t))\, dt$$

with

$$F_i = \omega(h_i \alpha_t(A_i)) \; ; \quad G_i = \omega(\alpha_t(A_i) h_i)$$

Next one shows that $\lim\limits_{\tau\to\infty} \int \omega(\mathrm{I}) = \lim\limits_{\tau\to\infty} \int \omega(\mathrm{VI})\, dt = 0$. One has

$$\|\mathrm{I}\| \leq \|h_1\| \|A_2\| \, \|[h_2, \alpha_{t-\tau}(A_1)]\|$$

Since this is an $\mathcal{L}^{(1)}$-function and since ω satisfies (1.1) we can choose an arbitrarily small $\varepsilon > 0$ and find a T large enough so that

$$\int_{|t-\tau|>T} \omega(\mathrm{I}) dt < \varepsilon \qquad \text{and also} \left|\int_{-T}^{T} \omega([h_2, \alpha_{t'}(A_1)]) dt'\right| < \varepsilon \tag{A.3}$$

Then

$$\lim_{\tau\to\infty} \left|\int \omega(\mathrm{I}) dt\right| < \varepsilon + \lim_{\tau\to\infty} \int_{t-\tau=-T}^{t-\tau=T} \omega(\mathrm{I})\, dt$$

The second term is

$$\lim_{\tau\to\infty} \int_{-T}^{T} \omega(\alpha_{-\tau}(h_1)[h_2, \alpha_{t'}(A_1)]\, \alpha_{t'+\tau}(A_2)) dt'$$

Since the t'-integration is over a finite range we can perform the τ-limit under the integral and obtain for it[9)]

$$\omega(h_1)\, \omega(A_2) \int_{-T}^{T} \omega([h_2, \alpha_{t'}(A_1)])\, dt' \tag{A.4}$$

Thus by (A.3)

$$\left|\lim_{\tau\to\infty} \int \omega(\mathrm{I}) dt\right| < \varepsilon\left(1 + |\omega(h_1)\, \omega(A_2)|\right)$$

for any $\varepsilon > 0$. The corresponding estimate can be made with IV and we obtain finally by (A.1), (A.2)

$$\int_{-\infty}^{\infty} (F_1(t) F_2(t) - G_1(t) G_2(t)) dt = 0 \tag{A.5}$$

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9) The integrand becomes $\lim\limits_{\tau\to\infty} \omega([h_2, \alpha_{t'}(A_1)]\alpha_{-\tau}(h_1)\alpha_{t'+\tau}(A_2))$ and, since ω is primary and $\alpha_{-\tau}(h_1)\alpha_{t'+\tau}(A_2)$ moves into the center this equals $\lim\limits_{\tau\to\infty} \omega([h_2, \alpha_{t'}(A_1)])\, \omega(\alpha_{-\tau}(h_1)\, \alpha_{t'+\tau}(A_2)) = \omega[h_2, \alpha_{t'}(A_1)]\, \omega(h_1)\, \omega(A_2)$.

From (A.5) to the KMS-condition one may follow essentially the arguments of [1] using again the technique described above for the derivation of the triple relation ([1]; (4.3)].

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