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1. Problems and Results

An old problem in quantum field theory is to characterize all local fields leading to a given S-matrix [1] [2; Chapter 4.6], [3; Chapter 19.5]. Our interest in this question arose from the desire for a local criterion distinguishing field theories with interaction from non-interacting ones. For that purpose it would be sufficient to know all fields leading to a trivial S-matrix. However, even this simpler problem has not yet been solved.

In view of this situation we found it worth while to place the following analysis on record although it applies only to a rather special case. We consider in this paper the model of a scalar, massless particle which does not interact. It turns out that there each interpolating field ϕ of the (trivial) S-matrix is an element of the Borchers class [1] of the free, massless field A. All these elements are explicitly known (see [4] and the Appendix) and this solves the inverse scattering problem for this special model. Moreover, owing to the fact that the free, massless field commutes with itself at timelike separations one obtains a simple algebraic characterisation of interacting fields in the above framework: a local field ϕ, describing the dynamics of a scalar, massless particle leads to a non-trival S-matrix if and only if the commutator $[\phi(x), \phi(y)]$ does not vanish at timelike distances $(x-y)$.

Our argument is based on the following reasoning: if the asymptotic fields coincide, $\phi^{in} = \phi^{out} = A$, they have the same TCP-operator as the interpolating field ϕ; consequently, ϕ and A are weakly local with respect to each other [1]. Moreover, because of Huyghens' principle [5] the commutator between $\phi(x)$ and $A(y)$ vanishes at timelike distances $(x-y)$. Hence if B is any element of the Borchers class of A the support of the vacuum expectation value

$$K(x) = (\Omega, [B, \phi(x)]\Omega) \tag{1}$$

is confined to the lightcone $x^2 = 0$. It follows then from the spectrum condition and temperateness that

$$K(x) = \sum_{finite} c_{(m),n}\, x^{(m)}\, \varepsilon(x_0)\, \delta^{(n)}(x^2) \tag{2}$$

where (m) is a multiindex and $x^{(m)} = x_o^{m_o} x_1^{m_1} x_2^{m_2} x_3^{m_3}$. Since the right hand side of this equation is a sum of homogenous distributions we conclude that the unitary operators $D(\lambda), \lambda > 0$, which induce the dilations of the asymptotic field A act on ϕ according to

$$D(\lambda)\,\phi(\lambda^{-1}x)\,D(\lambda)^{-1} = \sum_{d=0}^{n} \lambda^{d}\,\phi_d(x) \qquad (3)$$

i.e. ϕ is a finite sum of fields carrying a dimension; in particular $\phi_o = (\Omega, \phi\Omega).1$ and $\phi_1 = A$. Locality of ϕ then leads to recursive relations for ϕ_d which imply that these fields are relatively local to A.

Admittedly, this method of proof is tailored to the massless case and cannot be generalized to massive models. Nevertheless we hope that this contribution will serve as a stimulant for further investigations on these problems.

2. Details

The assumption that we are dealing with a field ϕ which leads to a trivial S-matrix can be expressed as follows: let A be the free, massless, scalar field acting in Fock-space H. We identify A with the incoming field constructed from the real, local, temperate field ϕ, i.e. we require

a) ϕ transforms as a scalar field under the same unitary representation $(\Lambda, x) \longrightarrow U(x)U(\Lambda)$ of the Poincaré group as A.

b) $P_1\phi(x)\Omega = A(x)\Omega$ where P_1 is the projection onto the one-particle states in H.

c) $[A(x), \phi(y)] = 0$ for $(x-y) \in V_+$ (Huyghens' principle [5]).

These assumptions characterize A unambiguously as the incoming field constructed from ϕ.[1)] If A were the outgoing field the only change would

1) It is of course a more delicate problem whether, given a ϕ, one can always construct an incoming field A with properties specified above. This question has been answered affirmatively in the algebraic framework of field theory [5].

be in relation c) where we had to replace V_+ by the backward cone V_-. Hence, if there is no scattering in the model, relation c) holds for $(x-y) \in V_+ \cup V_-$. An alternative way of expressing the triviality of the S-matrix is [1]

d) ϕ is weakly local relative to A.

For completeness let us also specify the domains of A and ϕ: We suppose that we may apply arbitrary smeared polynomials in A and ϕ to the vacuum Ω. This rather strong assumption is not really necessary for the proof. However, it will allow us to neglect intricate domain questions in what follows.

Now let $D(\lambda), \lambda > 0$ be the dilation operators acting on A according to

$$D(\lambda)\,A(x)\,D(\lambda)^{-1} = \lambda \cdot A(\lambda x)\,. \qquad (4)$$

Since the Borchers class $\mathcal{B}(A)$ of A contains only Wick-polynomials of A and its derivatives (see the Appendix) each $B \in \mathcal{B}(A)$ can be decomposed into a finite sum of fields $B_d \in \mathcal{B}(A)$ with dimension $d \in (0, \mathbb{N})$. We take any such B_d and analyse the distribution

$$K_\lambda(x) = (\Omega, [B_d, \phi_\lambda(x)]\Omega) = \lambda^{-d}(\Omega, [B_d, \phi(\lambda^{-1}x)]\Omega) \qquad (5)$$

where $\phi_\lambda(x) = D(\lambda)\phi(\lambda^{-1}x)D(\lambda)^{-1}$. It follows from assumptions c) and d) that $(\Omega, [B_d, \phi(x)]\Omega)$ has support on the light cone $x^2 = 0$. The Fourier transform of this distribution vanishes for $p^2 < 0$ owing to the spectrum condition. We give the most general form of such a distribution in the subsequent lemma.

<u>Lemma 1:</u> Let $S \in \mathcal{S}'(\mathbb{R}^4)$ be a temperate distribution with $\operatorname{supp} S \subseteq \{x^2 = 0\}$ and $\operatorname{supp} \tilde{S} \subseteq \{p^2 \geq 0\}$. Then

$$S(x) = \sum_{finite} c_{(m),n}\, x^{(m)}\, \varepsilon(x_o)\, \delta^{(n)}(x^2)$$

where $x^{(m)} = x_o^{m_o} x_1^{m_1} x_2^{m_2} x_3^{m_3}$ with $m_i \in (0, \mathbb{N})$. $\delta^{(n)}(\cdot)$ denotes the n-th derivative of the δ-function.

<u>Proof:</u> Since S has its support on the surface $x^2 = 0$ and is temperate there exists a minimal number $N \in \mathbb{N}$ such that $(x^2)^N S(x) = 0$, hence $\square_p^N \tilde{S}(p) = 0$. We divide the set of distributions S into subsets corresponding to N and

prove the lemma by induction in N : for N = 1 $\tilde{S}(p)$ is a solution of the wave equation and can be expressed by its (temperate) Cauchy data on the hyperplane $p_0 = 0$. Because of the fact that supp $\tilde{S} \subseteq \{p^2 \geq 0\}$, these data are localized at $\underline{p} = 0$ and therefore finite sums of δ-functions and their derivatives. Taking into account that $\varepsilon(p_0)\delta(p^2)$ is a solution of the wave equation with Cauchy data 0 and $2\pi.\delta(\underline{p})$ it is then easy to verify that

$$\tilde{S}(p) = P_1(\underline{\partial})\,\varepsilon(p_0)\,\delta(p^2) + P_2(\underline{\partial})\,\partial_0\,\varepsilon(p_0)\,\delta(p^2)$$

where P_1, P_2 are certain polynomials and $(\partial_0, \underline{\partial})$ are the derivatives with respect to $(p_0, \underline{p})$. This proves the statement for N = 1 after Fourier transformation. To complete the induction we must show that if $x^2 S(x)$ is of the form given in the lemma, i.e.

$$x^2.\,S(x) = \sum_{finite} c_{(m),n}\, x^{(m)}\, \varepsilon(x_0)\, \delta^{(n)}(x^2)$$

then S(x) itself has this form. To this end we must divide the above equation by x^2. An obvious solution to this problem with the desired properties is

$$S_0(x) = -\sum_{finite} c_{(m),n}\, x^{(m)}\, (n+1)^{-1}\, \varepsilon(x_0)\, \delta^{(n+1)}(x^2)$$

But the difference between S and S_0 is a distribution ΔS with supp $\widetilde{\Delta S} \subseteq \{p^2 \geq 0\}$ and $x^2.\Delta S(x) = 0$. We may therefore apply to ΔS the argument given above for N = 1 and conclude that $S = S_0 + \Delta S$ has the form given in the lemma.

This lemma shows that the commutator function K_λ given in (5) has a finite Laurent expansion at $\lambda = 0$,

$$K_\lambda(x) = \sum_{finite} \lambda^i K_i(x) \quad \text{with } i \in \mathbb{Z}. \qquad (6)$$

In fact K_λ is even a polynomical in λ; the degree of this polynomial is bounded by a number n which depends on ϕ but not on B_d. To verify this we estimate

$$\left| \int d^4x\, d^4y\, g(x) h(y) K_\lambda(x-y) \right| \leq \|\phi_\lambda(g)\Omega\| \cdot \{\|B_d(h)\Omega\| + \|B_d(h)^*\Omega\| \qquad (7)$$

with g, h real. Now $(\Omega, \phi\phi(x)\Omega)$ is the Fourier transform of a temperate measure and therefore $\|\phi_\lambda(g)\Omega\| \leq c_g(1+\lambda^n)$ for some $n \in \mathbb{N}$. Thus $|K_\lambda(f)| \leq c_f'(1+\lambda^n)$ and this limits the exponents i in relation (6) to $0 \leq i \leq n$. Taking into account the spectrum condition we conclude that for arbitrary $B_d \in \mathcal{B}(A)$

$$\frac{d^{n+1}}{d\lambda^{n+1}} (\Omega, B_d\, \phi_\lambda(x)\Omega) = 0. \qquad (8)$$

But the linear span of $\{B_d(f)\Omega : B_d \in \mathcal{B}(A), f \in \mathcal{S}(\mathbb{R}^4), d \in (0, \mathbb{N})\}$ is dense in H [2] and therefore

$$\frac{d^{n+1}}{d\lambda^{n+1}} \phi_\lambda(x)\Omega = 0. \qquad (9)$$

This establishes relation (3) on the vacuum. Now it follows from Huyghens' principle that $[\phi_\lambda(x), A(y)] = 0$ for $(x-y)^2 > 0$ and all $\lambda > 0$. So relation (9) still holds if we replace Ω by a vector $\mathcal{P}(A)\Omega$, where $\mathcal{P}(A)$ is any smeared polynomical in A. This establishes relation (3) on the dense domain $\mathcal{D}_0 = \{\mathcal{P}(A)\Omega\}$.

In the second step of our argument we analyse the components ϕ_d in the decomposition $\phi_\lambda = \sum_{d=0}^{n} \lambda^d \phi_d$. Clearly the ϕ_d are scalar fields which are defined on $\mathcal{D}_0$; moreover they satisfy $[\phi_d(x), A(y)] = 0$ for $(x-y)^2 > 0$. Keeping this in mind and using assumption b) it is easy to see that $\phi_0 = (\Omega, \phi\Omega)\cdot 1$ and $\phi_1 = A$ [7]. In order to show that $\phi_d \in \mathcal{B}(A)$ for $d \geq 2$ we start from the relation $[\phi_\lambda(x), \phi_\lambda(y)] = 0$ which holds for $(x-y)^2 < 0$ because of locality. Then we replace ϕ_λ by the expression given above and arrive at the equations $(1 \leq m \leq n)$

$$\sum_{d=1}^{m} [\phi_d(x), \phi_{m+1-d}(y)] = 0 \qquad \text{for } (x-y)^2 < 0, \qquad (10)$$

which hold in the sense of bilinear forms on $\overline{\mathcal{D}_0} \times \mathcal{D}_0$. But $\phi_1 = A$ with $\Box A(y) = 0$ and therefore $(3 \leq m \leq n)$

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[2] This may be verified by direct computation or extracted from [6].

$$[A(x), \Box\phi_m(y)] = -\sum_{d=2}^{m-1} [\phi_d(x), \Box\phi_{m+1-d}(y)] \quad \text{for } (x-y)^2 < 0 \qquad (11)$$

Hence if $\phi_2, \dots \phi_{m-1} \in \mathcal{B}(A)$, the right hand side of this equation vanishes and $\Box\phi_m$ must be an element of $\mathcal{B}(A)$. To complete the argument it suffices therefore to show that $\Box\phi_d \in \mathcal{B}(A)$ implies $\phi_d \in \mathcal{B}(A)$ for $d \geq 2$.

Lemma 2: Let ϕ_d, $d \geq 2$ be a temperate field with properties specified above. If $\Box\phi_d \in \mathcal{B}(A)$ then also $\phi_d \in \mathcal{B}(A)$.

Proof: Since $\Box\phi_d \in \mathcal{B}(A)$ it can be represented in the form

$$\Box\phi_d(x) = \sum_{k=2}^{d+2} P_k(\partial_1, \dots, \partial_k) : A(x_1)\dots A(x_k): \Big|_{x_1=\dots=x_k=x} \quad (*)$$

where P_k are symmetric polynomials in the derivatives $\partial_i = (\partial_{io}, \underline{\partial}_i)$ with respect to $(x_{io}, \underline{x}_i)$. We shall see that each $P_k(\partial_1, \dots \partial_k)$ contains a factor $\Box_k = (\partial_1 + \dots + \partial_k)^2$. It is then obvious that ϕ_d itself has the form (*) up to a term which is a solution of the "homogenous equation" $\Box\phi_d = 0$. However, this equation has only trivial solutions if $d \geq 2$ since then $\phi_d\Omega$ has no one-particle contribution and the vacuum is separating for the operators ϕ_d .

The assertion concerning the polynomials P_k will be proved by induction. For k = 2 the statement follows simply from Lorentz-invariance and the fact that $\Box\phi_d$ has dimension $d+2$; so $P_2(\partial_1, \partial_2)$ must be a homogenous polynomial of degree $d \geq 2$ in the invariant $(\partial_1 + \partial_2)^2$. (The invariants ∂_1^2, ∂_2^2 do not appear because of $\Box A = 0$.)

Now if each $P_k(\partial_1, \dots \partial_k)$, $2 \leq k \leq \ell - 1$ contains a factor $\Box_k$ we may subtract in relation (*) the corresponding contributions from $\Box\phi_d$ and assume that the sum starts with $\ell \geq 3$. Then the expressions $\int d^4x \, \Box\phi_d(x) D^{\binom{ret}{adv}}(x)$ and $\int d^4x \, \Box\phi_d(x) D(x)$ are well defined as bilinear forms on $\mathfrak{D}_o \times \mathfrak{D}_o$. (Here D denotes the zero mass Pauli-Jordan commutator function and $D^{\binom{ret}{adv}}$ the retarded and advanced parts of D.) Moreover we have $\int d^4x \, \Box\phi_d(x) D(x) = 0$ on $\mathfrak{D}_o \times \mathfrak{D}_o$ as expected from the Yang-Feldman equations. To confirm this

statement we consider the temperate distribution

$$\int d^4x \, D(x) \left(\Omega, A(x_1)\dots A(x_k) \Box\phi_d(x) A(x_{k+1})\dots A(x_\ell)\Omega\right) \qquad (**)$$

Taking into account the support properties of $D^{\binom{ret}{adv}}$, $D = D^{(ret)} - D^{(adv)}$ and the timelike commutation relations between ϕ_d and A it follows[3] that (**) coincides with

$$\int d^4x \, D^{(ret)}(x) (\Omega, \Box\phi_d(x) A(x_1)\dots A(x_\ell)\Omega) - \int d^4x \, D^{(adv)}(x) (\Omega, A(x_1)\dots A(x_\ell) \Box\phi_d(x)\Omega)$$

provided $x_1, \dots x_k \in V_-$ and $x_{k+1}, \dots x_\ell \in V_+$. But

$$\int d^4x \, D^{(ret)}(x) \Box\phi_d(x)\Omega = \int d^4x \, D^{(adv)}(x) \Box\phi_d(x)\Omega = \phi_d\Omega$$

if $d \geq 2$ and using the timelike commutation relations once more it becomes clear that (**) vanishes for the special configurations $x_1, \dots x_\ell$ mentioned above. On the other hand (**) is the boundary value of a function which is analytic in a tube because of the spectrum condition. We may therefore apply the "edge of the wedge" theorem [2] and conclude that (**) vanishes identically. If we insert now into (**) the expression given for $\Box\phi_d$ in relation (*) we get after a simple calculation

$$\int \prod_{j=1}^{\ell} d^4p_j \, \delta_+(p_j^2) \, \delta\Big(\big(\sum_{j=1}^{\ell} s_j p_j\big)^2\Big) \, \varepsilon\Big(\sum_{j=1}^{\ell} s_j p_{jo}\Big) P_\ell(is_1p_1, \dots is_\ell p_\ell) \exp\Big(i\sum_{j=1}^{\ell} s_j(p_j x_j)\Big) = 0$$

where $s_j = -1$ for $j \leq k$ and $s_j = 1$ for $j \geq k+1$. Since the measure

$$\prod_{j=1}^{\ell} d^4p_j \, \delta_+(p_j^2) \, \delta\Big(\big(\sum_{j=1}^{\ell} s_j p_j\big)^2\Big)$$

has support on the manifold $p_1^2 = \dots = p_\ell^2 = (\sum_{j=1}^{\ell} s_j p_j)^2 = 0$, $p_{jo} \geq 0$ if $1 \leq k \leq \ell - 1$ and $\ell \geq 3$, it follows from the above equation that $P_\ell(is_1p_1, \dots is_\ell p_\ell)$ vanishes at these points. Thus $P_\ell(ip_1, \dots ip_\ell)$ vanishes in particular in an open neighbourhood of a regular point on the manifold

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[3] Our argument is not completely rigorous in view of the distribution character of $D^{\binom{ret}{adv}}$ and D. However, it can be straightened out if one approximates these distributions by suitable testfunctions $D_n^{\binom{ret}{adv}}$, D_n and takes limits afterwards.

$$p_1^2 = \dots = p_\ell^2 = 0 \; , \; \Big(\sum_{j=1}^{\ell} p_j\Big)^2 = 0 \; .$$

Each polynomial with this property is of the form

$$\Big(\sum_{j=1}^{\ell} p_j\Big)^2 Q(p_1, \dots p_\ell) + \sum_{j=1}^{\ell} (p_j)^2 R_j(p_1, \dots p_\ell)$$

where Q, R_j are also polynomials (see the Appendix). Again the terms containing $p_1^2, \dots p_\ell^2$ do not contribute and therefore $P_\ell(ip_1, \dots ip_\ell)$ contains a factor $\big(\sum_{j\geq 1} p_j\big)^2$. This finishes the proof of the lemma.

Collecting the results in this chapter we arrive at the

Theorem: Let ϕ be a temperate field with properties a) to d) given above. Then $\phi \in \mathcal{B}(A)$.

An immediate consequence of this theorem is the

Corollary: Let ϕ be a temperate field with properties a) to c) given above. Then ϕ leads to a trivial S-matrix if and only if

$$[\phi(x), \phi(y)] = 0 \qquad \text{for} \quad (x-y)^2 > 0$$

* * *

APPENDIX

a) The Borchers Class of A

For the convenience of the reader we recall here some facts about the Borchers class $\mathcal{B}(A)$ of the free, massless, scalar field A: a temperate field B is said to be an element of $\mathcal{B}(A)$ if it has the following properties:

a) $B(f), f \in \mathcal{S}(\mathbb{R}^4)$ is defined on the dense set of vectors $\mathcal{D}_o = \{\mathcal{P}(A)\Omega\}$ where $\mathcal{P}(A)$ are the smeared polynomials in A.

b) B transforms under the same unitary representation $x \longrightarrow U(x)$ of the translations as A.

c) $[A(x), B(y)] = 0$ for $(x-y)^2 < 0$ in the sense of bilinear forms on $\overline{\mathcal{D}}_o \times \mathcal{D}_o$.

Following an argument of Epstein [4] we sketch the proof that $\mathcal{B}(A)$ consists of all Wick polynomials of A and its derivatives. For this purpose we consider the n-fold commutator function

$$(\Omega, [A(x_1), [A(x_2), \dots [A(x_n), B] \dots] \Omega) \; ,$$

which is a solution of the wave equation in each of its arguments. This expression is symmetric in $x_1, \dots x_n$ (because A has c-number commutation relations) and therefore vanishes if any one of its arguments x_i becomes spacelike. Hence if we express the commutator function by its Cauchy data at time $x_{1_0} = \quad = x_{n_0} = 0$ we realize that it is of the form

$$P_n(\partial_1, \dots \partial_n) \prod_{i=1}^{n} D(x_i) \quad ,$$

where P_n is some symmetric polynomial in the derivatives $\partial_i = (\partial_{i_0}, \underline{\partial}_i)$ and D is the zero mass Pauli-Jordan commutator function. We get therefore in the sense of bilinear forms on $\overline{\mathcal{D}}_o \times \mathcal{D}_o$:

$$B = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(-\partial_1, \dots -\partial_n) : A(x_1) \dots A(x_n): \Big|_{x_1 = \dots = x_n = 0}$$

But this series must terminate at some finite n because B is a temperate field.

b) Structure of the Polynomials P .

In this appendix we want to show that a polynomial $P(p_1, \ldots p_\ell)$, $\ell \geq 3$ which vanishes in a real neighbourhood of a regular point on the manifold $p_1^2 = \cdots = p_\ell^2 = (\sum_{j=1}^{\ell} p_j)^2 = 0$ is of the form

$$P(p_1, \cdots p_n) = \left(\sum_{j=1}^{\ell} p_j\right)^2 Q(p_1, \cdots p_n) + \sum_{j=1}^{\ell} p_j^2 R_j(p_1, \cdots p_\ell)$$

with polynomials Q and R_j. Unfortunately, we have not been able to establish this result only by elementary means. We shall heavily rely on results of the theory of polynomial rings, as expounded e.g. in [8] and [9].

To begin with we mention that a polynomial P with properties given above, automatically vanishes on the whole complex manifold $p_1^2 = \cdots = p_\ell^2 = (\sum_{j=1}^{\ell} p_j)^2 = 0$. This may be seen either by function theoretical techniques or from the fact that this manifold is algebraically irreducible. (It is of course crucial here that P vanishes in a whole neighbourhood of a regular point on this manifold).

In a second step we show that P has the desired form up to a factor. To this end we introduce an auxiliary variable λ and study the polynomial $P'(\lambda, p_1, \cdots p_\ell) = P(\lambda p_1, p_2, \cdots p_\ell)$. P' vanishes if $p_1^2 = \cdots = p_\ell^2 = 0$ and

$$\lambda = \lambda_0 = -\left(\sum_{j>2} (p_1 p_j)\right)^{-1} \left(\sum_{j>k>2} (p_j p_k)\right).$$

Hence if m is the degree of P in the variable λ it follows that

$$R'(p_1, \cdots p_\ell) = \left(\sum_{j=2}^{\ell} (p_1 p_j)\right)^m P'(\lambda_0, p_1, \cdots p_\ell)$$

is a polynomial which vanishes if $p_1^2 = \ldots = p_\ell^2 = 0$. A straightforward application of the Euclidean division algorithm then gives

$$R'(p_1, \cdots p_\ell) = \sum_{j=1}^{\ell} p_j^2 R_j'(p_1, \cdots p_\ell)$$

with polynomials R_j'. Since $P'(\lambda = 1) = P$ it is also obvious that $[(\sum_{j=2}^{\ell} (p_1 p_j))^m P - R']$ can be divided by $\sum_{j>k>2} (p_j p_k)$. Therefore we get

$$NP = \left(\sum_{j=1}^{\ell} p_j\right)^2 Q + \sum_{j=1}^{\ell} p_j^2 R_j \qquad (*)$$

with polynomials Q, R_j and $N = (\sum_{j=2}^{\ell} (p_1 p_j))^m$.

The last (and non-trivial) step in our analysis consists of showing that we may choose the polynomials Q and R_j in the above expression in such a way that they can be divided by N. For this purpose we introduce some algebraic notions:

Let $f_1, \ldots f_k$ be fixed polynomials. We call the set of all polynomials of the form $(g_i f_i + \ldots + g_k f_k)$, g_i being arbitrary polynomials, the ideal $I = (f_i, \ldots f_n)$ generated by $f_1, \ldots f_k$. I is called primary if $f \cdot g \in I$ and $f \notin I$ implies $g^m \in I$ for some $m \in \mathbb{N}$. It is one of the fundamental results in the theory of polynomial rings that each ideal I is a finite intersection of primary ideals, $I = \bigcap_{i=1}^{m} J_i$. Such a representation is of course not unique. However, there exist the distinguished representations in which one cannot omit any J_i. Let us fix such a representation and consider the varieties of the primary ideals J_i, i.e. the set of all common zeros of the polynomials in J_i. It turns out that these varieties are uniquely determined by I. So let us call them associated varities. Now it is crucial for our argument that if f is a polynomial, which does not identically vanish on any of these varieties and if $f \cdot g \in I$ then g must be an element of I [7, Chapter 16].

So in our case we have only to show that N does not identically vanish on any of the associated varieties of the ideal $I_0 = (p_1^2, \cdots, p_\ell^2, (\sum_{j=1}^{\ell} p_j)^2)$. To this end we must determine them. First of all we realize that the associated varieties of I_0 are subsets of the variety of I_0, i.e. the manifold $V_0 = \{p_1^2 = \cdots = p_\ell^2 = (\sum_{j=1}^{\ell} p_j)^2 = 0\}$: therefore their dimensions must be less or equal to $(3\ell - 1)$. But then Macaulay's theorem [8, Theorem 26] tells us that each of these varieties has dimension $(3\ell - 1)$, because I_0 is generated by $4\ell - (3\ell - 1)$ functions. Bearing in mind that the manifold V_0 is algebraically irreducible we conclude that the associated varieties of I_0 coincide with V_0. Now we come back to relation (*): since $N \cdot P \in I_0$ and N does not identically vanish on V_0 if $\ell \geq 3$, P must be an element of I_0. This, finally, proves the statement.

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