Matrix Elements of the Unitary Representations of the Homogeneous Lorentz Group

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Abstract
In the representation space of the Main Series of \( SL(2,\mathbb{C}) \) there exists an improper basis, in which the representation of a Cartan Subgroup is diagonal. The matrix elements with respect to this basis can be obtained with the help of differential equations. In this form the connection between the unitary representations and the spinor representations becomes more transparent.
Introduction

The group $SL(2,\mathbb{C})$ consists of all complex 2x2-matrices of determinant 1. The physical importance of this group lies in the fact that it is the universal covering group of the restricted Lorentz group $L_+^\uparrow$ in 4-dimensional Minkowski space. For

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 \end{pmatrix} \in SL(2,\mathbb{C})$$

the homomorphism $\alpha \mapsto \Lambda(\alpha)$ is defined by $\Lambda(\alpha)x = x'$ with $(x_o' + \dot{x}'\sigma) = \alpha(x_o + \dot{x}\sigma)\alpha^+$.\(^1\)

On the other hand, $SL(2,\mathbb{C})$ is also the complex Lorentz group in 3-dimensional Minkowski space. The complex 3-dimensional Lorentz group is defined as the set of all linear transformations on $\mathbb{C}^3$ leaving $z_o^2 - z_1^2 - z_2^2$ invariant. Setting $w_o = z_o$, $w_1 = -iz_1$, $w_2 = -iz_2$ we get $z_o^2 - z_1^2 - z_2^2 = w_o^2 + w_1^2 + w_2^2$; so this group is isomorphic to $0(3,\mathbb{C})$, the complex orthogonal group in three dimensions. If we put

$$Z = \begin{pmatrix} z_o + z_2 & z_1 \\ z_1 & z_o - z_2 \end{pmatrix},$$

we have $Z^T = Z$, det $Z = z_o^2 - z_1^2 - z_2^2$. For the transformed $Z' = \alpha Z \alpha^T$, $\alpha \in SL(2,\mathbb{C})$ follows $Z'^T = Z'$ and $z_o^2 - z_1^2 - z_2^2 = \det Z' = Z = z_o^2 - z_1^2 - z_2^2$. Therefore each $\alpha \in SL(2,\mathbb{C})$ defines a complex Lorentz transformation in $\mathbb{C}^3$. Instead of the symmetric spinor $Z$, which has only covariant indices, we introduce the spinor $\hat{Z} = Z\varepsilon$, $\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has covariant and contravariant indices. The condition $Z = Z^T$ is equivalent to $\hat{Z}^T = \varepsilon \hat{Z} \varepsilon$ or to Trace $\hat{Z} = 0$. The $Z$ transform like $\hat{Z}' = \alpha \hat{Z} \alpha^{-1}$. The stability group of the point $\hat{Z}_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is $S_p = \{ (\lambda, 0) \mid \lambda \neq 0 \}$. The orbit through $\hat{Z}_p$ is diffeomorphic to $SL(2,\mathbb{C})/S_p$. Our aim is to discuss the spherical harmonics on this homogeneous space of $SL(2,\mathbb{C})$.*

*Usually one considers the spherical harmonics on this 4-dimensional real hyperboloid $x_o^2 - x_1^2 = c^2$.\(^2\)
The parametrization of \( \alpha \in \text{SL}(2, \mathbb{C}) \) by complex Euler angles \( \phi, \theta, \psi \)

\[
\begin{pmatrix}
    e^{i/2(\phi+\psi)} \cos \frac{\theta}{2} & e^{i/2(\phi-\psi)} \sin \frac{\theta}{2} \\
    -e^{-i/2(\phi-\psi)} \sin \frac{\theta}{2} & e^{-i/2(\phi+\psi)} \cos \frac{\theta}{2}
\end{pmatrix} = e^{i\phi/2} \sigma_3 \quad e^{i\theta/2} \sigma_2 \quad e^{i\psi/2} \sigma_3
\]

gives a simple parametrization of the points of this orbit \( \text{SL}(2, \mathbb{C})/S_F \), a complex sphere, by complex polar coordinates \( \phi, \theta \). In order to find the spherical harmonics on this homogeneous space, we must look for the matrix elements of the unitary representations of \( \text{SL}(2, \mathbb{C}) \) in a basis in which \( S_F \) is in diagonal form. For a complete set of spherical harmonics it will be sufficient to consider the Main Series of \( \text{SL}(2, \mathbb{C}) \). These representations are defined on the Hilbert space of all complex valued functions \( f \) defined on \( \mathbb{C} \) for which

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(z, z^*)|^2 \, du \, dv < \infty , \quad z = u + iv , \quad z^* = u - iv
\]

and they are given by

\[
[D(\alpha)f](z, z^*) = (a_1^2 + a_2^2)^{(-is + j_0 - 1)} [(a_1^2 z + a_2^2)^*]^{(-is - j_0 - 1)} f(a_1^2 z + a_2^2, a_1^2 z^* + a_2^2, a_1^2 z + a_2^2)^* \]

\( s \) is a real number, \( 2j_0 \) is an integer. \((s, j_0)\) and \((-s, -j_0)\) define unitary equivalent representations. For \( 2j_0 \) uneven we have two valued representations of \( L^+ \).

The "eigenfunctions" of \( S_F \) are given by"
\[ f_{r,m}(z,z^*) = \frac{1}{2\pi} | z |^{1/2(m+ir-1)} z^{1/2(-m+ir-1)} \]

with \( m \) integer, \( r \) real and with the scalar product

\[ \langle f_{r,m}, f_{r',m'} \rangle = \delta(r-r') \delta_{m,m'} \tag{3a} \]

This follows directly from the application of (2) to \( \alpha(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \):

\[ [D(\alpha(\lambda)) f_{r,m}](z,z^*) = \lambda((is+ir-m j_0) \lambda (is+ir+m j_0) f_{r,m}(z,z^*) \tag{3b} \]

For convenience we introduce the following notation:

\[ 2j_1 = (-is+j_0-1), \quad 2j_2 = (-is-j_0-1), \quad 2n = is+ir+m j_0, \]

\[ 2n^* = -is-ir+m j_0 \quad \text{and} \quad f_{r,m}(z,z^*) \equiv f_{n,n^*}(z,z^*) \tag{3c} \]

Eq. (3b) reads then \( D(\exp \frac{i\phi}{2} \sigma_3) f_{n,n^*} = \exp(i(\phi n^* n^* f_{n,n^*} \cdot \exp(in^* \phi + in \psi) \langle n,n^*|d|j_1 j_2(\theta,\theta^*)|n',n'^*\rangle \exp(in' \psi + in'^* \phi) \tag{4a} \]

where

\[ \langle n,n^*|d|j_1 j_2(\theta,\theta^*)|n',n'^*\rangle \equiv d_{nn'}^{X}(\theta,\theta^*) = \langle f_{n,n^*}, D(\alpha(\theta,\theta^*)|n',n'^*\rangle \tag{4b} \]

and therefore we have to determine \( d_{nn'}^{X}(\theta,\theta^*) \) which is according to (2),
\[ d^X_{nn'}(\theta, \theta^*) = \]
\[
\frac{1}{4\pi \mathcal{C}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} du d\nu \left[ z^{-j_1-n-1} z^{*j_2+n^*-1} \left[ \sin^2_{\theta} z + \cos^2_{\theta} \right] j_1^{-n'} \times \right. \\
\left. \left[ \cos^2_{\theta} z - \sin^2_{\theta} \right] \left[ (\sin^2_{\theta} z + \cos^2_{\theta})^* \right] \left[ (\cos^2_{\theta} z - \sin^2_{\theta})^* \right] \right] \]

The Differential Equations for the Matrix Elements

In order to set up differential equations for the matrix elements we consider the Lie algebra \( \mathfrak{sl}(2,\mathbb{C}) \). We choose the usual \( \hat{M} = \frac{1}{2} \hat{x} \), \( \hat{N} = \frac{i}{2} \hat{N} \) as a base, in \( \mathbb{C} \)-plex notation we set

\[ i\hat{V} = \frac{1}{2}(\hat{M} - i\hat{N}), \quad 2\hat{V} = \frac{1}{2}(\hat{M} + i\hat{N}), \quad r\hat{V}_\pm = r\hat{V}_1 \pm i r\hat{V}_2, \quad r = 1, 2 \]

A unitary representation of \( \mathfrak{sl}(2,\mathbb{C}) \) gives a representation of the Lie algebra by essentially adjoint operators. In the representation defined by Eq. (2) the basis elements \( \hat{1}_V \) and \( \hat{2}_V \) are represented by the following differential operators \( r\hat{V}_i \):

\[ \hat{1}_V^+ = -z^2 \frac{\partial}{\partial z} + 2j_1 z \], \[ \hat{1}_V^- = \frac{\partial}{\partial z} \], \[ \hat{1}_V^3 = z \frac{\partial}{\partial z} - j_1 \]

\[ \hat{2}_V^+ = -\frac{\partial}{\partial z^*} \], \[ \hat{2}_V^- = z^*2 \frac{\partial}{\partial z^*} - 2j_2 z^* \], \[ \hat{2}_V^3 = -z^* \frac{\partial}{\partial z^*} + j_2 \]

We have formally \( \hat{1}_V^* = \hat{2}_V \) and further

\[ \hat{1}_V^f n, n^* = nf_n, n^* \], \[ \hat{2}_V^3 n, n^* = n^f_n, n^* \]
The Casimir operators of SL(2, $\phi$) are $^1V^2$ and $^2V^2$. In an irreducible representation they are represented by multiples of the unit matrix:

\[ ^\frac{1}{2}V^2 = j_1(j_1 + 1) \cdot 1, \quad ^\frac{2}{2}V^2 = j_2(j_2 + 1) \cdot 1 \]  

(8b)

If we consider SL(2, $\phi$) as a transformation group on SL(2, $\phi$) by left multiplication, the elements $S_i$ of the Lie algebra correspond to differential operators

\[ S_i^k(\phi) \frac{\partial}{\partial \phi^k} \alpha(\phi) = S_i \alpha(\phi) \]  

(9a)

($\phi_1, \ldots, \phi_6$) being real local coordinates on SL(2, $\phi$). We choose the complex Euler angles as coordinates and obtain

\[ ^1V_3 = \frac{\partial}{\partial \phi^3}, \quad ^2V_3 = \frac{\partial}{\partial \phi^3}^* \]

\[ ^1V_+ = e^{-i\phi} (\text{ctg}^0 \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \theta} - \frac{1}{\sin^0} \frac{\partial}{\partial \psi}) \]

\[ ^1V_- = e^{+i\phi} (\text{ctg}^0 \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \theta} - \frac{1}{\sin^0} \frac{\partial}{\partial \psi}) \]

\[ ^2V_+ = e^{-i\phi^*} (\text{ctg}^0 \frac{\partial}{\partial \phi^*} + \frac{\partial}{\partial \theta^*} - \frac{1}{\sin^0} \frac{\partial}{\partial \psi^*}) \]

\[ ^2V_- = e^{+i\phi^*} (\text{ctg}^0 \frac{\partial}{\partial \phi^*} - \frac{\partial}{\partial \theta^*} - \frac{1}{\sin^0} \frac{\partial}{\partial \psi^*}) \]  

(10)
In a unitary representation in which $S_i$ is represented by $M_i$ we have according to Eq. (9a)

$$g_1^k(\omega) \frac{\partial}{\partial \omega} D(\alpha(\phi)) = M_i D(\alpha(\phi))$$

(9b)

where the derivative of an operator is defined as the derivatives of the matrix elements. We shall assume differentiability of the matrix elements (4a). Using (8), (9b) and (10), we obtain the following differential equations for the matrix elements:

$$\left[ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{n^2 + n'^2 - 2nn' \cos \theta}{\sin^2 \theta} + j_1(j_1 + 1) \right] d_{n,n'}^X = 0$$

(11a)

$$\left[ \frac{\partial^2}{\partial \theta^2} + \cot \theta' \frac{\partial}{\partial \theta'} - \frac{n^2 + n'^2 - 2nn' \cos \theta^*}{\sin^2 \theta^*} + j_2(j_2 + 1) \right] d_{n,n'}^Y = 0$$

Since the two differential operators are commuting, we make the product ansatz $d_{n,n'}^X(\theta, \theta^*) = G_{n,n'}^j(y) G_{n,n^*}^{j_2}(y^*)$ with $y = \frac{1}{2}(1 - \cos \theta)$ and $y^* = \frac{1}{2}(1 - \cos \theta^*)$. In order to show that the Eq. (11a) may than be solved by hypergeometric functions we put

$$G_{n,n'}^j(y) = y^{(n-n')/2} (1-y)^{(n+n')/2} F_{n,n'}^j(y)$$

$$G_{n,n^*}^{j_2}(y^*) = (y^*)^{(n-n^*)/2} (1-y^*)^{(n^*+n^*)/2} F_{n^*,n^*}^{j_2}(y^*)$$

From Eq. (11a) we get for the $F_{n,n'}^j(y)$ the hypergeometric differential equations.
\[ \left( z^{(1-z)} \right) \frac{d^2}{dz^2} + \left( n - n' + 1 - 2(1+n)z \right) \frac{d}{dz} + j_1(j_1+1)\gamma(n+1) \right) F_{n,n'}(z) = 0 \]

\[ \left( z^*(1-z^*) \right) \frac{d^2}{dz^*} + \left( n^*-n'^* + 1 - 2(1+n^*)z^* \right) \frac{d}{dz^*} + j_2(j_2+1)\gamma(n^*+1) \right) F_{n^*,n'^*}(z^*) = 0 \]

\[ (11c) \]

We use the two linearly independent solutions of the hypergeometric equation:  \[ u_1 = F(-j_1+n, j_1+n+1, n-n'+1; y) \] and
\[ u_5 = y(n'-n) F(-j_1+n', j_1+n'+1, n'-n+1; y) \] etc. in order to determine
\[ d_n n'(\theta, \theta^*) \] up to free constants according to ansatz (11b):

\[ d_n n'(\theta, \theta^*) = A_{11} H^X(n,n^*,n',n'^*; \theta, \theta^*) + A_{12} H^X(n^*,n',n^*,n'^*; \theta, \theta^*) \]

\[ + A_{21} H^X(n',n^*,n',n'^*; \theta, \theta^*) + A_{22} H^X(n',n^*,n^*,n'^*; \theta, \theta^*) \]

\[ (12) \]

with

\[ H^X(n,n^*,n',n'^*; \theta, \theta^*) = \left( \sin \frac{\theta}{2} \right)^{n-n'} \left( \cos \frac{\theta}{2} \right)^{n+n'} \left( \sin \frac{\theta^*}{2} \right)^{n'^*-n'^*} \left( \cos \frac{\theta^*}{2} \right)^{n+n'^*} G^X(n,n^*,n'; \theta, \theta^*) \]

\[ G^X(n,n^*,n'; \theta, \theta^*) = F(-j_1+n, j_1+n+1, n-n'+1; \sin \frac{\theta}{2}) \times \]

\[ \times F(-j_2+n^*, j_2+n^*+1, n-\bar{n}^*+1; \sin \frac{\theta^*}{2}) \]

\[ (13) \]

In the following chapter we shall determine the coefficients \( A_{11}, A_{12}, A_{21}, \)
\[ A_{22}. \]
The Determination of the Integration Constants

In order to discuss the solution (12) we define powers \( x^{\alpha} \) which appear in the form \( (\sin^{\alpha} \theta) \frac{n-n'}{n+n'} \) etc. as unique analytic functions in the cut plane with a cut \(-\infty < x \leq 0\). The hypergeometric series \( F(a,b,c;z) \) is absolutely convergent in \( |z| < 1 \) and may be uniquely analytically continued into the cut plane \( 1 < z < \infty \). Therefore the expression (12) has cuts in the periodicity region \(-2\pi \leq \text{Re} \, \theta \leq 2\pi \) along the real line \( \pi \leq \text{Re} \, \theta \leq 4\pi \) and along the lines parallel to the imaginary axis \( \text{Re} \, \theta = \pm \pi \). (Fig.1) We have to require the solution (12) to be a one valued function of \( \theta \). Therefore the discontinuities across the cuts must be zero.

\[
\begin{array}{cccc}
\text{III} & \text{IV} & \text{I} & \text{II} \\
-2\pi & -\pi & 0 & \pi & 2\pi \\
\end{array}
\]

Fig.1.: \( \theta \) plane with cuts \(--.----.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.--.\n
In the following discussion of these conditions we use the notation \( G_{11} = G(n,n',n',n'',\theta,\theta^*) \), \( G_{12} = G_{11}(n \rightarrow n') \), \( G_{21} = G_{11}(n \leftarrow n') \), \( G_{22} = G_{11}(n \leftarrow n', n^* \rightarrow n'^*) \). We calculate first the discontinuity across the real line in the interval IV (Fig.1): \(-\pi < \theta < 0\).

The singular parts of expression (12) in IV are given by the powers of \( \sin^{\theta} \) and \( \sin^{\theta^*} \). Putting \( \Delta f(\theta) = f(\theta+i0) - i(\theta-i0) \) and
\[ \Delta f(\theta) = f(\theta + i0) + f(\theta - i0) \] we obtain

\[ \Delta (\sin \frac{\theta}{2})^\alpha = -\Delta (\sin \frac{\theta^*}{2})^\alpha = 2i \sin \alpha \pi (\sin(-\frac{\theta}{2}))^\alpha \]

and with help of \( \Delta (fg) = (\Delta f)g + \bar{f}(\Delta g) \) we get

\[ \Delta d^X_{n,n'}(\theta, \theta^*) = 2i(A_{11} G_{11} \sin \pi (n-n^*+n^*) \sin(-\frac{\theta}{2})^n+n^*-n'-n^*) \]

\[ + A_{21} G_{21} \sin \pi (n^*-n-n^*) \sin(-\frac{\theta}{2})^n+n^*-n'-n^*) \]

\[ + A_{12} G_{12} \sin \pi (n-n^*+n^*) \sin(-\frac{\theta}{2})^n+n^*-n'-n^*) \]

\[ + A_{22} G_{22} \sin \pi (n^*-n-n^*) \sin(-\frac{\theta}{2})^n+n^*-n'-n^*) \]

From Eq. (3c) we see

\[ n-n^*+n^* = i(x-x') \quad , \quad n-n^*+n^* = m-m' \]

Therefore the condition \( \Delta d = 0 \) in the interval IV gives

\[ A_{11} = A_{22} = 0 \quad (14) \]

Next we consider the cut along the line \( \text{Re} \, \theta = \pi \). In order to continue our solution across this cut we use Kummer's relations for the hypergeometric functions:

\[ u_1 = c_{12} u_2 + c_{16} u_6 \quad , \quad u_5 = c_{52} u_2 + c_{56} u_6 \quad (15) \]

with

\[ u_2 = F(-j_1+n, j_1+n+1, n+n'+1; \cos^2 \frac{\theta}{2}) \]

\[ u_6 = (\cos^2 \frac{\theta}{2})^{-n-n'} F(j_1+1-n', -j_1-n', -n-n'+1; \cos^2 \frac{\theta}{2}) \]
and
\[
\begin{align*}
    c_{12} &= \frac{(n-n')(n-n'-1)!}{(j_1'-n')!(n-j_1'-1)!} \\
    c_{16} &= \frac{(n-n')!(n+n'-1)!}{(-j_1+n-1)!(j_1+n')!} \\
    c_{52} &= \frac{(n'-n)!(n-n'-1)!}{(j_1-n)!(j_1-n-1)!} \\
    c_{56} &= \frac{(n'-n)!(n+n'-1)!}{(-j_1+n-1)!(j_1+n')!}
\end{align*}
\]

(16)

Analogous relations hold for the complex conjugate solutions.
We obtain as the continued solution for the region \( \pi < \text{Re} \, \theta \leq \pi \):
\[
\begin{align*}
    d_{n,n'}^x(\theta, \theta^*) &= (\sin \frac{\theta}{2})^{n-n'}(\cos \frac{\theta}{2})^{n+n'}(\sin \frac{\theta^*}{2})^{n^*-n'^*}(\cos \frac{\theta^*}{2})^{n^*+n'^*} \\
    &\times (A_{21}(c_{52}c_{12}u_2^*u_2^*+c_{52}c_{12}^*u_2u_2^*)+c_{56}c_{12}^*u_2u_2+6c_{12}^*u_2u_2+6c_{56}^*u_2u_2) \\
    &+ A_{12}(c_{52}c_{12}^*u_2^*u_2^*+c_{52}c_{12}u_2u_2^*)+c_{56}c_{12}u_2u_2^*+6c_{12}u_2u_2^*+6c_{56}u_2u_2^*)
\end{align*}
\]

(17)

Now we regard the cut in the interval II, \( \pi \leq \theta \leq 2\pi \), by collecting the singular terms in \( \cos \frac{\theta}{2}, \cos \frac{\theta^*}{2} \). With the same arguments as in the case of the interval IV, the condition \( \Delta d = 0 \) on II gives
\[
\begin{align*}
    A_{21}c_{52}c_{12}^* + A_{12}c_{12}c_{52}^* &= 0 \\
    A_{21}c_{56}c_{12}^* + A_{12}c_{12}c_{56}^* &= 0
\end{align*}
\]

(18)

The condition for Eq. (18) to have a non-trivial solution is
\[
\frac{c_{52}c_{12}c_{56}^*c_{16}}{c_{12}c_{52}c_{56}c_{16}} = 1
\]

(19)
Using the relation \( \Gamma(z)\Gamma(1-z) = \pi/\sin\pi z \) one checks that (19) is always satisfied for \( j_1, j_2, n, n' \) given by (3c).

For \( A_{21} \) and \( A_{12} \) we get the following expressions

\[
A_{21} = \frac{N(n-n')!(n'^*-n^*)!}{(j_1-n')!(-n'-j_1-1)!(j_2-n^*)!(-n^*-j_2-1)!}
\]

\[
A_{21} = \frac{-N(n-n')!(n'^*-n^*)!}{(j_1-n)!(-j_1-n-1)!(j_2-n^*)!(-n^*-j_2-1)!}
\]

where \( N \) is not yet determined.

The normalization of the matrix elements \( d^X_{n,n'}(\theta,\theta^*) \) is given by the Plancherel formula, which was derived by Gelfand and Neumark\(^6\) for \( \text{SL}(2,\mathbb{F}) \):

\[
\delta(\alpha a^{-1}) = \int^{+\infty}_{-\infty} \int^{+\infty}_{-\infty} \frac{s^{2+j^2}}{2j_0} \frac{S^2+j^2}{8\pi^2} \text{Trace} \left( \frac{D^{j_1j_2}_{\alpha a^{-1}}}{8\pi^2} \right)
\]

\[
= \int^{+\infty}_{-\infty} \int^{+\infty}_{-\infty} \frac{s^{2+j^2}}{8\pi^2} \text{Trace} \left( \frac{D^{j_1j_2}_{\alpha a^{-1}}}{8\pi^2} \right)
\]

with \( \alpha_1, \alpha \) \( \text{SL}(2,\mathbb{F}) \), \( \delta(\alpha a^{-1}) \) the Dirac-function on \( \text{SL}(2,\mathbb{F}) \) with respect to the invariant measure \( du(\alpha) = |\alpha_2^2|^{-2} d\alpha_1^2 d\alpha_1^* d\alpha_2^* d\alpha_2^2 d\alpha_2^{2*} \cdot\cdot\cdot \cdot = \frac{1}{16} |\sin^2 \theta| d\phi \psi d\theta d\theta^* d\psi d\psi^* \), and with the abbreviation

\[
\int^{+\infty}_{-\infty} \int^{+\infty}_{-\infty} d(2\text{Im} n) \text{ where } n+n^* \text{ runs through all integers if}
\]
\( 2j_o \) is even and through all half integers if \( 2j_o \) is odd. Since the matrix elements \( \langle f_{nn'}|D_{1j_2}(\alpha)|f_{n',n''}\rangle \) are solutions of the differential equations (8) and (11), they are orthogonal functions on \( \text{SL}(2,\mathbb{C}) \). With help of the orthogonality conditions we derive from (21) the normalization condition

\[
\int d\mu(\alpha) \langle f_{pp'}|D^{k_1k_2}(\alpha)|f_{p',p'',*}\rangle \delta(n-n') \delta(s-s') \delta(j_o-j'_o) + e^{i\phi} \delta(s+s') \delta(j_o,-j'_o) = \frac{8\pi^4}{s^2+j_o^2} \delta(n-p) \delta(n'-p') \delta(s-s') \delta(j_o,j'_o) + e^{i\phi} \delta(s+s') \delta(j_o,-j'_o) \delta(n-n*) \delta(p-p*) \delta(i(n-n*)-i(p-p*)),
\]

(22)

with \( k_1 = \frac{1}{2}(-is+j'_o-1) \), \( j_1 = \frac{1}{2}(-is+j_o-1) \) etc.,

\( s^2 + j_o^2 = -(2j_1+1)(2j_2+1) \) and \( \delta(n-p) = \delta_{n+n*,p+p*} \delta(i(n-n*)-i(p-p*)) \).

The term \( \delta(s+s') \delta(j_o,-j'_o) \) arises because \( D_{1j_2} \) and \( D_{-j_1-1,-j_2-1} \) are unitary equivalent. Their matrix elements may differ by a phase factor

\[
e^{i\phi} = \exp(i\phi(j_1,j_2,n,n*) - i\phi(j_1,j_2,n',n'*))
\]

(23)

We now normalize our solution \( d^X_{n,n'}(\theta,\theta^*) \) according to (22). After a straightforward but lengthy calculation we may so determine the constant \( N \) in (20) up to a phase and get

\[
A_{21} = \frac{ig(j_1,j_2,n,n*,n',n'*)}{4\sin^2(n-n'-n*n'*) \cdot (n-n)! (n*n*)!} \times
\]

\[
\times \left[ \frac{(j_1-n)!(-j_1-n-1)! (j_2-n'*)!(-j_2-n'*-1)!}{(j_1-n')!(-j_1-n'-1)! (j_2-n)!(-j_2-n*-1)!} \right]^{1/2}
\]
\[ A_{12} = \frac{-e^{i \gamma}}{4 \sin \frac{\pi}{2} (n-n'-n^*+n'^*)(n-n')!(n^*-n^*)!} \times \]
\[ \times \left[ \frac{(j_1-n')!(-j_1-n'-1)! (j_2-n^*)!(-j_2-n^*-1)!}{(j_1-n)!(-j_1-n-1)! (j_2-n^!)(-j_2-n^*-1)!} \right]^{1/2} \]

(24)

\[ e^{i \gamma} \text{ is the undetermined phase factor. Since } (e^{-i \gamma A_{12}})^* = e^{-i \gamma A_{21}}, \]
\[ \exp(i \gamma) \text{ is, up to a sign, the phase of } d^X_{n,n'}(\theta, \theta^*). \]

Now we consider the limit \( \theta \to 0 \), i.e. \( z = \sin^2 \frac{\theta}{2} \to 0 \), for which we expect that \( d^X_{n,n'} \)
becomes a Dirac function in the lower indices. We consider the limit in the form

\[ \lim_{z,z^* \to 0} F(z,z^*) = \lim_{\phi \to 0} \frac{1}{4\pi} \int_0^{4\pi} F(\phi e^{i \phi}, \phi e^{-i \phi}) d\phi \]

(25)

and indeed we get

\[ \lim_{\theta, \theta^* \to 0} d^X_{n,n'}(\theta, \theta^*) = \delta_{m,m'} \delta(r-r') e^{i \gamma(j_1,j_2,n,n^*,n,n^*)} \]

(26)

which is in agreement with (3a) and (4a) provided
\[ e^{i \gamma(j_1,j_2,n,n^*,n,n^*)} = 1. \]

The phase factors in the case \( n \neq n' \), \( n^* \neq n'^* \) remains to be determined. We know that

\[ 2i e^{i \gamma(j_1,j_2,n,n^*,n,n'^*)} = d^X_{n,n'}(\theta, \theta^*) [d^X_{n,n'}(\theta, \theta^*)]^* \]

(27)
The right hand side of (27) is independent of \( \theta \) and we may try to perform the limit \( \theta \to 0 \). For this we use expression (5) \( d_{n,n',n}^k(\theta, \theta^*) \), in which we extend the integration only over \( R^{-1} < |z| < R \). For sufficiently small \( \theta \) we may approximate the single factors in the integrand; careful use of the relations:

\[
(1-x)^{-a} = (1-x)^{-b} F(b-a,b,c; x/1-x)|_{c=b}
\]

and of \( F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \) for \( \text{Re} \ c > \text{Re} \ b > 0 \)

\( \text{Re} \ (c-b-b) > 0 \)

gives in the cases

(a) \( \text{Re} \ (n'-n) \geq 0 \), \( \text{Re} \ (n-j_1) > 0 \), \( \text{Re} \ (j_1+n+1) > 0 \)
(b) \( \text{Re} \ (n-n') \geq 0 \) \( \text{Re} \ (n'-j_1) > 0 \) \( \text{Re} \ (j_1+n'+1) > 0 \)
(c) \( \text{Re} \ (n-n') \geq 0 \) \( \text{Re} \ (-j_1-n) > 0 \) \( \text{Re} \ (j_1+1-n) > 0 \)
(d) \( \text{Re} \ (n'-n) \geq 0 \) \( \text{Re} \ (-j_1-n') > 0 \) \( \text{Re} \ (j_1-1-n') > 0 \)

the result

\[
2\text{ig}(j_1, j_2, n, n', n', n'^*) = e^{i \pi (n+n'^*+n'+n'^*)} \frac{(j_1-n!)(j_2-n'^*)!(-j_1-n-1)!(-j_2-1-n'^*)!}{(j_1-n)! (j_2-n'^*)! (-j_1-n'-1)! (-j_2-1-n'^*)!}
\]

This and the condition \( e^{i \pi} = 1 \) can be used as the starting point for multiple applications of the operators \( V_{\pm} \) in the well-known fashion in order to determine the phase generally as

\[
\text{ig}(j_1, j_2, n, n'^*, n'^*, n'^*) = e^{i \pi (n+n'^*-n'-1)! (n+n'^*-n'^*-1)!} \left[ \frac{(j_1-n!) (j_2-n'^*)! (-j_1-n-1)! (-j_2-1-n'^*)!}{(j_1-n)! (j_2-n'^*)! (-j_1-n'-1)! (-j_2-1-n'^*-1)!} \right]^{1/2}
\]

(28)
We have the relation \( e^{ig(j_1,j_2,n,n^*,n',n'^*)} = e^{-ig(j_1,j_2,n^*,n,n^*)} \).

This phase determines also the phase appearing in the normalization condition (22) and (23)

\[
\begin{align*}
&\text{if}(j_1, j_2, n, n^*) - \text{if}(j_1, j_2, n', n'^*) = \\
&= e^{ig(j_1, j_2, n, n^*) - ig(j_1, j_2, n', n'^*)}
\end{align*}
\]

(29)

Now we are able to write down the complete matrix element:

\[
d^X_{n,n'}(\theta, \theta^*) = \frac{\frac{i}{2}(n-n'+n^*-n'^*)}{4\sin\frac{\pi}{2}(n-n'-n^*+n'^*)} \frac{(-j_1-n-1)!(-j_2-n'^*)!}{(-j_1-n'-1)!(-j_2-n'^*)!(n-n')!(n^*-n'^*)!} \times \\
\times (\sin\frac{\theta}{2})^{n-n'} (\cos\frac{\theta}{2})^{n+n'} (\sin\frac{\theta^*}{2})^{n'-n'^*} (\cos\frac{\theta^*}{2})^{n^*-n'^*} \times \\
\times F(-j_1+n', j_1+n'+1, n'-n+1; \sin^2\frac{\theta}{2})F(-j_2+n^*, j_2+n^*+1, n^*-n'^*+1; \sin^2\frac{\theta^*}{2}) \\
\times \frac{\frac{i}{2}(n-n'+n^*-n'^*)}{4\sin\frac{\pi}{2}(n-n'-n^*+n'^*)} \frac{(j_1-n')!(j_2-n'^*-1)!}{(j_1-n-1)!(-j_2-n'^*-1)!(n-n')!(n^*-n'^*)!} \times \\
\times (\sin\frac{\theta}{2})^{n-n'} (\cos\frac{\theta}{2})^{n+n'} (\sin\frac{\theta^*}{2})^{n'-n'^*} (\cos\frac{\theta^*}{2})^{n^*-n'^*} \times \\
\times F(-j_1+n, j_1+n+1, n-n'+1, \sin^2\frac{\theta}{2})F(-j_2+n^*, j_2+n^*+1, n'^*-n^*+1; \sin^2\frac{\theta^*}{2})
\]

(30)

This representation holds in the strip \(-\pi < \text{Re} \theta < +\pi\). The representation in the strips \(-2\pi < \text{Re} \theta \leq -\pi\) and \(\pi < \text{Re} \theta < 2\pi\) are then obtained.
by using Kummer's relations according to Eq. (16) and (17).

Concluding Remarks

The above calculations, especially the result Eq. (30), show a strong resemblance of the finite dimensional spinor representations of $SL(2,\phi)$ and the unitary irreducible representations of the Main Series of $SL(2,\phi)$ in our special base. Analytic continuation makes this point even more evident. For this we combine (4a) and (30) in order to obtain

\[
<f_{n,*}^{j_1} | D_{j_1,j_2}^{j_2}(\phi,\theta,\psi) | f_{n',*}^{n'*}>.
\]

Letting $(j_1, n, n')$ vary independently of $(j_2, n, n')$, we may ask for what values of $(j_1, j_2, n, n', n', n'* )$ the matrix elements of $D_{j_1,j_2}^{j_2}(\phi,\theta,\psi)$ are analytic in $(\phi,\theta,\psi)$. Inspection of (4a) immediately gives $n' = n' = 0$, whereas (30) gives $j_2(j_2 + 1) = 0$. Writing $F(a,b,c;z)$ as a series in $z$, we obtain

\[
<f_{n,*}^{j_1} | D_{j_1,j_2}^{j_2}(\phi,\theta,\psi) | f_{n',*}^{n'*}>= \frac{i}{\pi} \left[ \frac{(j_1 + n')!(j_1 - n')!}{(j_1 + n)!(j_1 - n)!} \right]^{1/2} D_{j_1,j_2}^{j_2}(\phi,\theta,\psi)
\]

where $D_{j_1,j_2}^{j_2}(\phi,\theta,\psi)$ are the matrix elements of the undotted irreducible spinor representations in case $2j_1$ is an integer, and where

$-j_1 < n, n' < j_1$; $n, n'$ are integers or half-integers according to whether $2j_1$ is even or odd.

We believe that this connection between the set of unitary representations and the set of "analytic" spinor representations of the $SL(2,\mathbb{C})$ will turn out to be useful in the applications of spherical harmonics to relativistic physics. Besides this, our calculations have illustrated some typical problems arising in the calculation of matrix elements of unitary representations of non-compact groups in a base where a 2-dimensional Cartan subgroup is diagonal.
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