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Integrable Hamiltonian Systems and Interactions through Quadratic Constraints

by

K. Pohlmeyer

II. Institut für Theoretische Physik, Universität Hamburg

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Integrable Hamiltonian Systems and Interactions through Quadratic Constraints

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K. Pohlmeyer

II. Institut für Theoretische Physik der Universität Hamburg, Germany

Abstract: O_n - invariant classical relativistic field theories in one time and one space dimension with interactions that are entirely due to quadratic constraints are shown to be closely related to integrable Hamiltonian systems.

I. Introduction

Even in one space dimension, relativistically invariant classical field theories defining integrable Hamiltonian systems with a non-trivial, momentum dependent scattering matrix, are not in oversupply. Actually, up to equivalence and slight modifications there is only one such model available, the celebrated sine-Gordon equation [1,2,3].

In this paper we shall present a whole series of non-equivalent relativistically invariant field theories in one time and one space dimension, each having a one parameter family of Backlund transformations and an infinite number of known integrals of motion. These conserved quantities are associated with covariant local conserved currents for which the family of Backlund transformations serves as a generating functional. Further, each one of these models has non-trivial momentum dependent scattering, and possesses stable stationary finite energy solutions, so-called solitons.
By a procedure explained below ("reduction"), the series of new models is obtained from O_n - invariant Lagrangian field theories whose interaction arises solely from the condition that the values of the field functions be constrained to the surface of a sphere (describing a homogeneous space for O_n). The new examples should be viewed as generalizations (involving more and more fields) of the sine-Gordon theory , which corresponds to the chiral symmetry group O_3 (To O_2 there corresponds the theory of a free massless field). The connection with the O_n - invariant chiral theories allows for a simple geometrical interpretation of various computational manipulations in the new models. For $n \leq 6$ we set up the linear

eigenvalue equation (for the characteristic initial value problem), which is the key to the inverse scattering method [4,5]. We determine the evolution of the spectral data and thereby solve the characteristic initial value problem.

Conversely, the analysis of the new models provides a significant first step towards the complete description of all finite energy solutions of the original O_n - invariant chiral theories e.g. supplying for them an infinite number of integrals of motion associated with covariant local conserved currents. Of particular interest is the $O_4 \underset{\text{loc.}}{\cong} SU(2) \times SU(2)$ - invariant chiral theory, the one-space-dimensional version of the non-linear σ- model [6]. Apropos, the original O_n - invariant chiral theories do not possess soliton solutions. However, the solutions related to the soliton solutions of the corresponding reduced model are expected to play a special role.

To sum up, the aim of this paper is twofold:

i) furnishing new examples with the same powerful structure as the one which is at the bottom of the sine-Gordon theory and

ii) contributing to the solution of theories with an effective Lagrangian comprising the results of current algebra [6].

The present communication grew out of joint work with H. Lehmann and G. Roepstorff in 1968 when the connection between the chiral O_3 - invariant theory and the sine-Gordon theory came to light.

II. Heuristic Considerations and Normalization of Coordinates

We start from the classical theory of n real-valued scalar or pseudoscalar fields $q_1(x^o, x^1), \ldots, q_n(x^o, x^1)$ in one time and one space dimension which in dimensionless units, with the help of a Lagrangian multiplier $\lambda(x^o, x^1)$, is described by the Lagrangian density

$$\text{(II.1)}\quad L(x^o, x^1) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\mu=0}^{1}\left(\frac{\partial}{\partial x_\mu} q_i(x^o,x^1)\right)\left(\frac{\partial}{\partial x^\mu} q_i(x^o,x^1)\right) + \frac{\lambda(x^o,x^1)}{2}\left(\sum_{i=1}^{n} q_i^2(x^o,x^1) - 1\right).$$

This Lagrangian is invariant under the action of the internal symmetry group O_n:

$$\left\{\begin{matrix} q_i(x^o, x^1) \\ i = 1,\ldots,n \end{matrix}\right\} \rightarrow \left\{\begin{matrix} q'_j(x^o, x^1) \\ j = 1,\ldots, n \end{matrix}\right\}, \quad \lambda(x^o, x^1) \rightarrow \lambda'(x^o,x^1)$$

(II.2)

$$q'_j(x^o,x^1) = \sum_{i=1}^{n} R_{ji}\, q_i(x^o,x^1), \quad R = (R_{ji}) \in O_n$$
$$\lambda'(x^o, x^1) = \lambda(x^o, x^1).$$

With the short hand notation:

(II.3)

$$x = (x^o, x^1), \quad p = (p_1, \cdots, p_n)$$
$$(p \cdot q) = \sum_{i=1}^{n} p_i q_i, \quad p^2 = \| p \|^2 = \sum_{i=1}^{n} p_i^2$$
$$q_\mu(x) = \frac{\partial}{\partial x^\mu} q(x), \quad q^\mu(x) = \frac{\partial}{\partial x_\mu} q(x)$$

together with the summation convention , the Lagrangian density takes the simple form

$$\text{(II.4)}\quad L(x) = \frac{1}{2}\left(q_\mu(x)\cdot q^\mu(x)\right) + \frac{\lambda(x)}{2}\left(q^2(x)-1\right) \quad .$$

The corresponding Euler-Lagrange equations of motion are

$$\text{(II.5)}\quad \begin{aligned} &\Box\, q(x) + \left(q_\mu(x)\cdot q^\mu(x)\right) q(x) = 0\,,\quad q^2(x) \equiv 1 \\ &\lambda(x) = -\left(q_\mu(x)\cdot q^\mu(x)\right) . \end{aligned}$$

It is convenient to introduce the characteristic coordinates

$$\text{(II.6)}\quad \xi = \frac{x^0 + x^1}{2} \quad , \qquad \eta = \frac{x^0 - x^1}{2}$$

in which the d'Alembertian $\Box \doteq \frac{\partial^2}{\partial (x^0)^2} - \frac{\partial^2}{\partial (x^1)^2}$ factorizes

$$\text{(II.7)}\quad \Box = \frac{\partial^2}{\partial\xi\,\partial\eta} \quad .$$

Employing the notation

$$\text{(II.8)}\quad q_\xi = \frac{\partial}{\partial\xi} q(\xi,\eta)\,,\quad q_\eta = \frac{\partial}{\partial\eta} q(\xi,\eta) \quad \text{etc.}$$

and denoting the unit sphere in R^n by the symbol S_{n-1}, the equations of motion read in characteristic coordinates:

(II.9) $$q_{\xi\eta} + (q_\xi \cdot q_\eta) q = 0 \quad , \quad q \in S_{n-1} .$$

These equations are forminvariant under general coordinate transformations which map the light cône onto itself, i.e. under the local scale transformations

(II.10) $$(\xi, \eta) \longrightarrow (\xi', \eta')$$
$$d\xi' = |H(\xi)| \, d\xi \, , \quad d\eta' = |K(\eta)| \, d\eta$$

with
$$H(\xi) \neq 0 \neq K(\eta) .$$

The sum and difference of the energy and momentum densities of the fields are given by $\frac{1}{2} q_\eta^2$ and $\frac{1}{2} q_\xi^2$ respectively and the energy-momentum conservation is expressed by the equations

(II.11) $$\left\{ \tfrac{1}{2} q_\eta^2 \right\}_\xi = 0 = \left\{ \tfrac{1}{2} q_\xi^2 \right\}_\eta .$$

Hence q_η^2 and q_ξ^2 are functions solely of η and ξ respectively:

(II.12) $$q_\xi^2 = h^2(\xi) \quad , \quad q_\eta^2 = k^2(\eta)$$

which are already determined by the Cauchy data at a fixed time.

For the derivation of the continuity equations (II.11) note that

q_ξ and q_η are orthogonal to q in virtue of $q^2 \equiv 1$ and that $q_{\xi\eta}$ is parallel to q in virtue of the equation

$$q_{\xi\eta} + (q_\xi \cdot q_\eta)\, q = 0 .$$

We may choose

$$\text{(II.13)} \qquad \begin{aligned} |H(\xi)| &= |h(\xi)| \\ |K(\eta)| &= |k(\eta)| \end{aligned} \quad .$$

This amounts to an identically vanishing monumentum density and a constant energy density $\equiv \frac{1}{2}$ in the new "normalized" coordinates. (A situation with $h(\xi) = 0$ for some $\xi = \xi_0$ and/or $k(\eta) = 0$ for some $\eta = \eta_0$ is to be approximated by Cauchy data for which $h(\xi)$ and $k(\eta)$ are different form zero everywhere.) Without loss of generality we may take normalized coordinates as a basis for our discussion and omit the identifying primes.
Now, only one of the O_n - invariants formed from q and its first derivatives q_ξ and q_η is undetermined, namely $(q_\xi \cdot q_\eta)$:

$$\text{(II.14)} \qquad \begin{array}{lll} q^2 \equiv 1 , & q_\xi^2 \equiv 1, & q_\eta^2 \equiv 1 , \\ (q_\xi \cdot q) \equiv 0, & (q_\eta \cdot q) \equiv 0, & -1 \leq (q_\xi \cdot q_\eta) \leq 1 . \end{array}$$

We set

$$\text{(II.15)} \qquad (q_\xi \cdot q_\eta) = \cos\alpha .$$

III. The chiral O_3 model and the sine-Gordon equation

For n = 3 the vectors q, q_ξ and q_η already span the entire space $\mathbb{R}^3$. Without loss of generality the solutions of the equations of motion (II.9) satisfy the constraints

$$q^2 \equiv 1 \equiv q_\xi^2 \equiv q_\eta^2 \quad .$$

Conversely, for n = 3 the constraints $q^2 \equiv 1 \equiv q_\xi^2 \equiv q_\eta^2$ imply the equations of motion (II.9).

Hence, solving (II.9) for n = 3 is equivalent to the construction of all three component unit vector fields $q = q(\xi,\eta)$ with

$$q_\xi^2 \equiv 1 \equiv q_\eta^2 \quad .$$

Next we want to derive the sine-Gordon equation for arc cos $(q_\xi \cdot q_\eta)$. To this end, we express the second derivatives $q_{\xi\xi}$ and $q_{\eta\eta}$ as linear combinations of q, q_ξ and q_η

$$\text{(III.1)} \quad \begin{aligned} q_{\xi\xi} &= -q + 2\alpha_\xi \operatorname{ctg}\alpha \, q_\xi - 2\alpha_\xi \frac{1}{\sin\alpha} q_\eta \\ q_{\eta\eta} &= -q - 2\alpha_\eta \frac{1}{\sin\alpha} q_\xi + 2\alpha_\eta \operatorname{ctg}\alpha \, q_\eta \end{aligned}$$

and compute the mixed second derivative of α =arc cos $(q_\xi \cdot q_\eta)$

$$\text{(III.2)} \quad \begin{aligned} \alpha_{\xi\eta} &= -\left\{ \frac{(q_\xi \cdot q_\eta)_\xi}{\sin\alpha} \right\}_\eta = -\left\{ \frac{(q_{\xi\xi} \cdot q_\eta)}{\sin\alpha} \right\}_\eta \\ &= -\frac{\alpha_\xi \alpha_\eta \cos\alpha + (\{-\cos\alpha \, q\}_\xi \cdot q_\eta) + (q_{\xi\xi} \cdot q_{\eta\eta})}{\sin\alpha} \end{aligned}$$

Making use of the above expressions for $q_{\xi\xi}$ and $q_{\eta\eta}$ we obtain

$$\text{(III.3)} \quad \alpha_{\xi\eta} = -\frac{\alpha_\xi \alpha_\eta \cos\alpha - \cos^2\alpha + (1 - \alpha_\xi \alpha_\eta \cos\alpha)}{\sin\alpha}$$

$$\text{(III.4)} \quad \alpha_{\xi\eta} = -\sin\alpha$$

i.e. the sine-Gordon equation

which can be derived from the Lagrangian density $\hat{L} = \hat{L}(x)$

$$\hat{L} = \frac{1}{2}\left(\frac{\partial}{\partial x_\mu}\alpha\right)\left(\frac{\partial}{\partial x^\mu}\alpha\right) + (\cos\alpha - 1)\,. \tag{III.5}$$

Conversely, to every solution α of the sine-Gordon equation there exists a solution of the equations of motion (II.9) with $q_\xi^2 \equiv 1 \equiv q_\eta^2$ and $(q_\xi \cdot q_\eta) = \cos\alpha$.

The sine-Gordon equation is known to possess a one parameter family of Backlund transformations T_γ, $\gamma \in R^1$:

$$\alpha \longrightarrow \alpha(\cdot\,;\gamma)$$

where $\alpha(\cdot\,;\gamma)$ is again a solution of the sine-Gordon equation

$$T_\gamma \begin{cases} \left\{\dfrac{\alpha(\cdot\,;\gamma)+\alpha}{2}\right\}_\xi = \gamma^{-1}\sin\left(\dfrac{\alpha(\cdot\,;\gamma)-\alpha}{2}\right) & (\text{III.6}_1) \\[2ex] \left\{\dfrac{\alpha(\cdot\,;\gamma)-\alpha}{2}\right\}_\eta = -\gamma\sin\left(\dfrac{\alpha(\cdot\,;\gamma)+\alpha}{2}\right) & (\text{III.6}_2) \end{cases}$$

γ being a constant independent of ξ and η [7].

By elementary manipulations we derive the conservation law

$$\gamma\left\{\cos\left(\frac{\alpha(\cdot\,;\gamma)+\alpha}{2}\right)\right\}_\xi + \gamma^{-1}\left\{\cos\left(\frac{\alpha(\cdot\,;\gamma)-\alpha}{2}\right)\right\}_\eta = 0. \tag{III.7}$$

If we expand $\alpha(\cdot\,;\gamma)$ in a formal power series in γ around the point $\gamma = 0$, insert this expansion into the above conservation law, collect terms involving the same power of γ and set the coefficients of the resulting power series in γ separately equal to zero, we obtain an infinite number of conservation laws for covariant local currents involving higher and higher powers of higher and higher derivatives of the sine-Gordon field α. The corresponding integrals of motion are independent of each other and in involution.

The soliton and antisoliton solutions are most easily obtained by applying the Backlund transformations T_γ with positive and negative γ respectively to the vacuum solution $\alpha \equiv 0$.

Next, we would like to remind the reader of Newell's derivation of the isospectral linear eigenvalue problems $L(\eta)\psi = \zeta\psi$ with the η- coordinate as the deformation parameter [8]. These eigenvalue problems play a central role in the inverse scattering method for the characteristic initial value problem. The derivation follows the pattern: one-parameter family of Backlund transformations $\rightarrow$ one parameter family of Riccati equations $\rightarrow$ linearization of these Riccati equations $\equiv$ isospectral linear eigenvalue problems.

We start from the first of the two ordinary differential equations defining T_γ:

$$\text{(III.6}_1\text{)} \qquad \left\{\frac{\alpha(\cdot\,;\gamma)+\alpha}{2}\right\}_\xi = \gamma^{-1}\sin\left(\frac{\alpha(\cdot\,;\gamma)-\alpha}{2}\right)$$

and reduce the transcendental non-linearity to a quadratic one by rewriting the equation for

$$\text{(III.8)} \qquad \Gamma = Tg\left(\frac{\alpha(\cdot\,;\gamma)-\alpha}{4}\right):$$

$$\text{(III.9}_1\text{)} \qquad \Gamma_\xi + \frac{\alpha_\xi}{2}\left(1+\Gamma^2\right) = \gamma^{-1}\Gamma .$$

This is the above mentioned one parameter family of Riccati equations.

The linearization is achieved by substituting

(III.10) $$\Gamma = \frac{\psi_1}{\psi_2} .$$

The resulting differential equation

(III.11) $$\psi_2 \psi_{1\xi} - \psi_1 \psi_{2\xi} + \frac{\alpha_\xi}{2} (\psi_1^2 + \psi_2^2) = \gamma^{-1} \psi_1 \psi_2$$

is satisfied if ψ_1 and ψ_2 solve the following linear system of first order ordinary differential equations

(III.12$_1$) $$\begin{aligned} \psi_{1\xi} + \frac{\alpha_\xi}{2} \psi_2 &= \frac{1}{2\gamma} \psi_1 \\ -\psi_{2\xi} + \frac{\alpha_\xi}{2} \psi_1 &= \frac{1}{2\gamma} \psi_2 \end{aligned}$$

i.e. the linear eigenvalue problems for each value of η

(III.13) $$L(\eta)\psi = \left(\frac{i}{2\gamma}\right)\psi$$

where

(III.14) $$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$
$$L(\eta) = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{d\xi} + \frac{i}{2} \begin{pmatrix} 0 & \alpha_\xi \\ \alpha_\xi & 0 \end{pmatrix}.$$

Remember that γ is independent of η (and ξ).

The η-evolution of ψ can be determined from the second of the two ordinary differential equations defining T_γ:

(III.6$_2$) $$\left\{\frac{\alpha(\cdot\,;\gamma) - \alpha}{2}\right\}_\eta = -\gamma \sin\left(\frac{\alpha(\cdot\,;\gamma) + \alpha}{2}\right):$$

(III.9$_2$) $$\Gamma_\eta + \frac{\gamma}{2} \sin\alpha (1 - \Gamma^2) = -\gamma \cos\alpha\, \Gamma$$

leading to

(III.12$_2$) $$\psi_\eta = B\psi$$

with

$$B = -\frac{\gamma}{2}\begin{pmatrix} \cos\alpha, & \sin\alpha \\ \sin\alpha, & -\cos\alpha \end{pmatrix}.$$

We confirm the relation

(III.15) $$\frac{\partial L}{\partial \eta} = [B, L]$$

the necessary and sufficient condition for the η - independence of the spectrum of $L(\eta)$.

We notice that the parameter γ of the family of Backlund transformations plays the role of the continuous eigenvalue in the family of isospectral linear eigenvalue problems associated with the sine-Gordon equation.

Returning to the Backlund transformations T_γ, let q be a solution of the equations of motion (II.9) for the chiral O_3 model with $q_\xi^2 \equiv 1 \equiv q_\eta^2$ and $(q_\xi \cdot q_\eta) = \cos\alpha$.

Then

(III.16) $$q(\cdot\,;1) = \cos\frac{\alpha(\cdot\,;1)}{2}\left(\frac{q_\xi + q_\eta}{2}\right) + \sin\frac{\alpha(\cdot\,;1)}{2}\left(\frac{q_\xi - q_\eta}{2}\right)$$

is a solution of the equations of motion (II.9) with

(III.17) $$\left(q(\cdot\,;1)_\xi\right)^2 \equiv 1 \equiv \left(q(\cdot\,;1)_\eta\right)^2$$

$$\left(q(\cdot\,;1)_\xi \cdot q(\cdot\,;1)_\eta\right) = \cos\alpha(\cdot\,;1).$$

The vectors q and $q(\cdot\,;1)$ are orthogonal to each other. The requirement that the component of $q(\cdot;1)_{\xi\eta}$ in the direction of q vanishes, is equivalent to the conservation law

$$\left\{\cos\left(\frac{\alpha(\cdot;1)+\alpha}{2}\right)\right\}_{\xi}+\left\{\cos\left(\frac{\alpha(\cdot;1)-\alpha}{2}\right)\right\}_{\eta}=0. \tag{III.18}$$

However, for general values of the parameter γ, the geometrical relation between the solution vector q and a solution $q(\cdot;\gamma)$ of the equations of motion with

$$\begin{aligned}&\left(q(\cdot;\gamma)_{\xi}\right)^2 \equiv 1 \equiv \left(q(\cdot;\gamma)_{\eta}\right)^2\\ &\left(q(\cdot;\gamma)_{\xi}\cdot q(\cdot;\gamma)_{\eta}\right)=\cos\alpha(\cdot;\gamma)\end{aligned} \tag{III.19}$$

is not so simple. The resolution of this point is presented in section VI for general values of n, after the geometrical meaning of the parameter γ has been clarified in section V.

IV Goals and Strategy

We aim at associating a one-parameter family of isospectral linear eigenvalue problems (with the η-coordinate as the deformation parameter) to the evolution equations for the invariants of the general chiral group O_n in normalized coordinates. So far, this task is accomplished for $n \leq 6$ only. A less ambitious goal is the construction of a generating functional for the infinite number of covariant local conserved currents.

Lacking a better idea for the realization of this program we maintain the spirit of Newells' pattern [8]:

1) introduction of γ as an expansion or contraction parameter for the sum and difference of the energy and momentum densities respectively

2) adjunction of one discrete Backlund transformation for the chiral fields q_i $i = 1, \ldots, n$

Remark: 1) and 2) yield a one-parameter continuum of Backlund transformations T_γ which serves as a generating functional for the infinite number of conservation laws.

3) transition from the chiral fields to O_n - invariants

4) derivation of a one-parameter family of systems of (n - 2) coupled non-linear first order ordinary differential equations involving at most bilinear terms

5) construction of a one-parameter family of genuine Riccati equations

6) linearization of these Riccati equations to obtain the desired isospectral linear eigenvalue equations.

V The geometrical meaning of the parameter γ

In this section we shall show that for every solution q of the equations of motion (II.9) there exists a one-parameter family of solutions $q^{(\gamma)}$, $\gamma \in \mathbb{R}^1$ with

(V.I) $$q_\xi^{(\gamma)\,2} = \gamma^{-2} q_\xi^2 \,, \quad q_\eta^{(\gamma)\,2} = \gamma^2 q_\eta^2 \,,$$
$$\left(q_\xi^{(\gamma)} \cdot q_\eta^{(\gamma)}\right) = \left(q_\xi \cdot q_\eta\right)$$

(in a general system of coordinates). Actually we shall prove even more, namely the existence of coordinate dependent rotation matrices $\mathcal{R}^{(\gamma)} = \mathcal{R}^{(\gamma)}(\xi,\eta;q)$:

$$\mathcal{R}^{(\gamma)}\mathcal{R}^{(\gamma)tr} = \mathcal{R}^{(\gamma)tr}\mathcal{R}^{(\gamma)} = \mathbb{1} \tag{V.2}$$

the superscript tr denoting transposition

such that the equations of motion are satisfied for

$$q^{(\gamma)} \doteq \mathcal{R}^{(\gamma)} q \tag{V.3}$$

and

$$q^{(\gamma)}_{\xi} = \gamma^{-1}\mathcal{R}^{(\gamma)}q_{\xi}\,, \quad q^{(\gamma)}_{\eta} = \gamma\,\mathcal{R}^{(\gamma)}q_{\eta}\,. \tag{V.4}$$

Obviously, for $q^{(\gamma)}$ so defined the relations (V.1) hold.

The existence of such rotation matrices $\mathcal{R}^{(\gamma)}$ follows from the compatibility of the equations

$$\mathcal{R}^{(\gamma)}_{\xi} = (1-\gamma^{-1})\,\mathcal{R}^{(\gamma)} M_{(+)} \tag{V.5$_1$}$$

$$\mathcal{R}^{(\gamma)}_{\eta} = (1-\gamma)\,\mathcal{R}^{(\gamma)} M_{(-)} \tag{V.5$_2$}$$

and

$$\mathcal{R}^{(\gamma)}\mathcal{R}^{(\gamma)tr} = \mathcal{R}^{(\gamma)tr}\mathcal{R}^{(\gamma)} = \mathbb{1} \tag{V.5$_3$}$$

where

$$(V.6_1) \qquad M_{(\pm)} = M_{(\pm)}(\xi,\eta;q) = \left(M_{(\pm)kl}\right)$$

and

$$(V.6_2) \qquad M_{(+)kl} = q_k q_{l\xi} - q_{k\xi} q_l \,, \; M_{(-)kl} = q_k q_{l\eta} - q_{k\eta} q_l \,.$$

$M_{(\pm)kl}$ are the sum and difference respectively of the zero and one components of the current densities for the chiral field vector q corresponding to a rotation in the (k,l)-plane.

The construction of the matrices $\mathcal{R}^{(\gamma)}$ follows the standard iterative procedure for the generation of the resolvents for systems of homogeneous linear first order ordinary differential equations [9] .

Moreover, the following transitivity equation holds

$$(V.7) \qquad \mathcal{R}^{(\gamma_2\cdot\gamma_1)}(\cdot\,;q) = \mathcal{R}^{(\gamma_2)}(\cdot\,;\mathcal{R}^{(\gamma_1)}(\cdot\,;q)q)\cdot \cdot\,\mathcal{R}^{(\gamma_1)}(\cdot\,;q) \,.$$

To sum up, the parameter γ describes expansions or contractions of the respective sum and difference of the energy and momentum densities of the chiral field vector q without at the same time changing the angle between q_ξ and q_η .

VI Backlund Transformations and Conservation Laws

We adjoin to the transformations R_γ , $\gamma \in \mathbb{R}^1$:

$$(VI.1) \qquad q \xrightarrow{R_\gamma} q^{(\gamma)} = \mathcal{R}^{(\gamma)}(\cdot\,;q)\,q$$

of the solutions of (II.9) onto themselves essentially one more such discrete non-linear transformation B_+:

(VI.2) $$q \xrightarrow{B_+} q'$$

which changes neither the energy nor the momentum density, but the angle between the ξ- and η-derivatives:

(VI.3) $$q'^2_\xi = q^2_\xi \quad , \quad q'^2_\eta = q^2_\eta$$
$$(q'_\xi \cdot q'_\eta) \neq (q_\xi \cdot q_\eta) .$$

B_+ is defined up to some coordinate independent rotations by the four compatible equations:

(VI.4^+_1) $$(q' + q)_\xi = \frac{(q' \cdot q_\xi) - (q'_\xi \cdot q)}{2} (q' - q)$$

(VI.4^+_2) $$(q' - q)_\eta = \frac{(q_\eta' \cdot q) - (q' \cdot q_\eta)}{2} (q' + q)$$

(VI.4^+_3) $$q'^2 \equiv 1$$

(VI.4^+_4) $$(q' \cdot q) = 0 .$$

This transformation corresponds to the Backlund transformation T_I in the sine-Gordon theory. Along with B_+ goes the conservation law:

(VI.5) $$(q' \cdot q_\xi)_\eta + (q' \cdot q_\eta)_\xi = 0 \, .$$

We obtain a one-parameter family of Backlund transformations T_γ - to be compared with the maps $\alpha \longrightarrow \alpha(\cdot\,;\gamma)$ in the sine-Gordon theory, cf. equation (III.6) - if we combine B_+ and the R_γ's to form

(VI.6) $$T_{\gamma+} = R_\gamma^{-1} B_+ R_\gamma$$
$$q \xrightarrow{T_{\gamma+}} q(\cdot\,;\gamma+) = \left((q^{(\gamma)})' \right)^{(1/\gamma)}$$

taking the non-commutativity of the diagram

(VI.7) $$\begin{tikzcd} q \arrow[r, "R_\gamma"] \arrow[d, "B_+"'] & q^{(\gamma)} \arrow[dd, "B_+"] \\ q' & \\ q(\cdot\,;\gamma+) & (q^{(\gamma)})' \arrow[l, "R_\gamma^{-1}"] \end{tikzcd}$$

into account : the angle between $q(\cdot\,;\gamma+)_\xi$ and $q(\cdot\,;\gamma+)_\eta$ depends on γ .

Along with $T_{\gamma+}$ goes the conservation law:

$$(q^{(\gamma)\prime} \cdot q^{(\gamma)}_\xi)_\eta + (q^{(\gamma)\prime} \cdot q^{(\gamma)}_\eta)_\xi$$

(VI.8) $$= \gamma^{-1} (q(\cdot\,;\gamma+) \cdot \mathcal{V}\, q_\xi)_\eta + \gamma (q(\cdot\,;\gamma+) \cdot \mathcal{V}\, q_\eta)_\xi = 0$$

with
$$\mathcal{V} \doteq \mathcal{R}^{(\gamma)}(\cdot\,; q(\cdot\,;\gamma))^{tr} \, \mathcal{R}^{(\gamma)}(\cdot\,; q) \, .$$

By expanding $q^{(\gamma)'}$ near the asymptote of $\dfrac{\mathcal{R}^{(\gamma)}(\cdot\,;q)\,q_\xi}{\|q_\xi\|}$ for $\gamma \sim 0$ into an asymptotic series in γ, inserting this expansion into (VI.8), collecting all terms of the same order in γ and setting the resulting coefficients separately equal to zero, we obtain an infinite number of covariant local non-polynomial conservation laws leading to independent integrals of motion for the chiral fields which are in involution. The first three continuity equations are given explicitly by

$$\text{(VI.9}_1\text{)} \qquad \left\{\frac{1}{2}\, q_\xi^2\right\}_\eta = 0$$

$$\text{(VI.9}_2\text{)} \qquad \left\{\frac{1}{2\|q_\xi\|}\left(\frac{\partial}{\partial\xi}\left(\frac{q_\xi}{\|q_\xi\|}\right)\right)^2\right\}_\eta = \left\{\frac{(q_\xi\cdot q_\eta)}{\|q_\xi\|}\right\}_\xi$$

$$\text{(VI.9}_3\text{)} \qquad \left\{\frac{1}{2\|q_\xi\|}\left(\frac{\partial}{\partial\xi}\left[\frac{1}{\|q_\xi\|}\frac{\partial}{\partial\xi}\left(\frac{q_\xi}{\|q_\xi\|}\right)\right]\right)^2 - \frac{5}{8\|q_\xi\|^3}\left[\left(\frac{\partial}{\partial\xi}\left(\frac{q_\xi}{\|q_\xi\|}\right)\right)^2\right]^2\right\}_\eta = \left\{-\frac{(q_\xi\cdot q_\eta)}{2\|q_\xi\|^3}\left(\frac{\partial}{\partial\xi}\left(\frac{q_\xi}{\|q_\xi\|}\right)\right)^2\right\}_\xi \,.$$

Obviously, instead of the discrete non-linear transformation B_+ we could have taken the transformation B_-, up to some coordinate independent rotations defined by

$$\text{(VI.4}_1^-\text{)} \qquad (q' - q)_\xi = \frac{(q'_\xi\cdot q) - (q'\cdot q_\xi)}{2}\,(q' + q)$$

(VI.4$_2^-$) $$(q' + q)_\eta = \frac{(q' \cdot q_\eta) - (q'_\eta \cdot q)}{2} (q' - q)$$

(VI.4$_3^-$) $$q'^2 \equiv 1$$

(VI.4$_4^-$) $$(q' \cdot q) = 0 \quad .$$

To B_- there corresponds the same conservation law as to B_+: eq. (VI.5) and a one-parameter family of Backlund transformations

(VI.10) $$T_{\gamma -} = R_\gamma^{-1} B_- R_\gamma \quad .$$

Going through the same routine as before, we obtain an infinite number of new covariant local non-polynomial conservation laws, which arise from the above ones by replacing ξ-derivatives by η - derivatives and vice versa.

VII Differential Equations for the O_n - Invariants

From now on, we shall work with normalized coordinates. In these coordinates the conservation laws look much simpler than before e.g.

(VII.1$_1$) $$\left\{ \tfrac{1}{2} q_\xi^2 \right\}_\eta = 0$$

(VII. 1_2) $$\left\{ \tfrac{1}{2}\, q_{\xi\xi}^{2} \right\}_{\eta} = \left\{ (q_{\xi}\cdot q_{\eta}) \right\}_{\xi}$$

(VII.1_3) $$\left\{ \tfrac{1}{2}\, q_{\xi\xi\xi}^{2} - \tfrac{5}{8}\,(q_{\xi\xi}^{2})^{2} \right\}_{\eta} = \left\{ - \frac{(q_{\xi}\cdot q_{\eta})}{2}\, q_{\xi\xi}^{2} \right\}_{\xi} \quad \text{etc.}$$

The coordinates normalized for the solution q serve at the same time as normalized coordinates for the Backlund transformed solutions

$$q(\cdot\,;\gamma\pm) = T_{\gamma\pm}\, q \quad .$$

From the solution vector q we construct a basis in $\mathbb{R}^n$ with the first three basisvectors specified

(VII.2) $$\underline{b_{-1}} = q\,,\ \underline{b_0} = q_{\xi}\,,\ \underline{b_1} = \frac{q_{\eta} - \cos\alpha\, q_{\xi}}{\sin\alpha}\,,\ \underline{b_i}$$
$$i = 2, \cdots, n-2\,.$$

such as to give $M_{(+)}$ a simple form with (ξ,η)-independent entries:

(VII.3) $$M_{(+)}\,\underline{b_{-1}} = -\underline{b_0}\,,\ M_{(+)}\,\underline{b_0} = \underline{b_{-1}}\,,\ M_{(+)}\,\underline{b_j} = 0 \quad j = 1,\cdots,n-2.$$

In this basis also $M_{(-)}$ takes a fairly simple form:

(VII.4) $$\begin{aligned} &M_{(-)}\,\underline{b_{-1}} = \cos\alpha\,\underline{b_0} + \sin\alpha\,\underline{b_1}\,,\ M_{(-)}\,\underline{b_0} = \cos\alpha\,\underline{b_{-1}} \\ &M_{(-)}\,\underline{b_1} = \sin\alpha\,\underline{b_{-1}}\,,\ M_{(-)}\,\underline{b_i} = 0 \quad i = 2,\cdots,n-2. \end{aligned}$$

$\mathcal{R}^{(\gamma)}(\cdot;q)$ maps this basis into a new one

$$\text{(VII.5)} \qquad \underline{b}_k^{(\gamma)} = \mathcal{R}^{(\gamma)}(\cdot\,;q)\,\underline{b}_k \quad , \quad k=-1,0,1,\dots,n-2.$$

We may construct a similar basis $\{\underline{b}_k^{(\gamma)\prime}\}$ starting from the solution vector $q(\cdot\,;\gamma)$ and envisage a derivation of a differential equation for the rotation $\mathcal{O}^{(\gamma)}$. This describes the change of basis $\{\underline{b}_k^{(\gamma)}\}\to\{\underline{b}_k^{(\gamma)\prime}\}$ and is symmetric with respect to primed and unprimed quantities.
The entries of the orthogonal matrix $\mathcal{O}^{(\gamma)}$:

$$\text{(VII.6)} \qquad \mathcal{O}_{kk'}^{(\gamma)} = \left(\underline{b}_{k'}^{(\gamma)\prime}\cdot\underline{b}_k^{(\gamma)}\right)$$

are O_n - invariant.

However, it turns out that the ξ- derivatives of the entries

$$\text{(VII.7)} \qquad X_k = \left(q^{(\gamma)\prime}\cdot\underline{b}_k^{(\gamma)}\right)$$

of the first column of $\mathcal{O}^{(\gamma)}$ can be expressed by at most quadratic terms in the entries of this very first column, and the same goes for the η- derivatives of X_k $k = (-1), 0,1,\dots, n-2$.
Hence we confine our attention to the O_n-invariants X_k, and study two systems of (n-1) coupled non-linear first order ordinary differential equations involving at most bilinear terms:

$$\text{(VII.8}_1\text{)} \qquad \begin{aligned} X_{0\xi} &= -\gamma^{-1}+\gamma^{-1}X_0^2-\sum_{j=1}^{n-2}\left(\underline{b}_0\cdot\underline{b}_{j\,\xi}\right)X_j \\ X_{j\xi} &= \gamma^{-1}X_0X_j+\sum_{\ell=0}^{n-2}\left(\underline{b}_{j\,\xi}\cdot\underline{b}_\ell\right)X_\ell \qquad j=1,\dots,n-2 \end{aligned}$$

and

(VII.8$_2$)
$$\begin{aligned} X_{0\eta} &= \gamma\cos\alpha\,(1-X_0^2) - \gamma\sin\alpha\, X_0 X_1 \\ X_{1\eta} &= -\gamma\cos\alpha\, X_0 X_1 + \gamma\sin\alpha\,(1-X_1^2) + \sum_{i=2}^{n-2} (\underline{b}_{1\eta}\cdot\underline{b}_i) X_i \\ X_{i\eta} &= -\gamma\cos\alpha\, X_0 X_i - \gamma\sin\alpha\, X_1 X_i + \sum_{j=1}^{n-2} (\underline{b}_{i\eta}\cdot\underline{b}_j) X_j \\ i &= 2,\cdots,n-2 \end{aligned}$$

subject to the constraint

(VII.9)
$$\begin{aligned} \underline{X} &= (X_0,\cdots,X_{n-2}) \in S_{n-2} \\ &\sum_{\ell=0}^{n-2} X_\ell^2 = 1\,. \end{aligned}$$

X_{-1} is identically equal to zero since $(q^{(r)'}\cdot q^{(r)})=0$.
The constraint (VII.9) reflects the condition $(q^{(r)'})^2=1$.
As we see, explicit knowledge of $\mathcal{R}^{(r)}$ is not required.

We eliminate the constraint (VII.9) with the help of the stereographic projection

(VII.10)
$$Y_j = \frac{X_j}{1+X_0} \qquad j=1,\cdots,n-2$$

and arrive at two systems of (n-2) coupled non-linear first order ordinary differential equations also involving at most bilinear terms.

(VII.11$_1$)
$$\begin{aligned} Y_{j\xi} &= \gamma^{-1} Y_j + \sum_{m=1}^{n-2} S^{(+)}_{jm} Y_m \\ &\quad + \Big(\sum_{m=1}^{n-2} t_m Y_m\Big) Y_j + \frac{t_j}{2}\Big(1-\sum_{m=1}^{n-2} Y_m^2\Big) \\ j &= 1,\cdots,n-2 \end{aligned}$$

with

$$S^{(+)}_{jm} = - S^{(+)}_{mj} = \left(\underline{b}_{j_\xi} \cdot \underline{b}_m \right)$$

$$t_m = \left(\underline{b}_0 \cdot \underline{b}_{m_\xi} \right)$$

and

$$Y_{j\eta} = - \gamma \cos\alpha \, Y_j + \sum_{m=1}^{n-2} S^{(-)}_{jm} Y_m \tag{VII.11$_2$}$$
$$- \gamma \sin\alpha \, Y_1 Y_j + \delta_{j1} \frac{\gamma}{2} \sin\alpha \left(1 + \sum_{m=1}^{n-2} Y_m^2 \right)$$
$$j = 1, \cdots, n-2$$

with

$$S^{(-)}_{jm} = - S^{(-)}_{mj} = \left(\underline{b}_{j_\eta} \cdot \underline{b}_m \right) .$$

These systems of differential equations are to be compared with the Riccati equations (III.9$_1$)and (III.9$_2$) of the sine-Gordon theory.

VIII Linearization and the chiral SU(2) x SU(2)

All points of our program (cf.section IV) have been realized but the final one which consists of the linearization of the system of equations (VII.11$_1$) (and (VII.11$_2$).

For n $\leq$ 6 the system (VII.11$_1$) can be cast into a single equation for

$$\chi = Y_1 + i Y_2 + j Y_3 + k Y_4 \tag{VIII.1}$$

where 1,i,j,k is the conventional basis for the quaternions:

$$\chi_\xi = \gamma^{-1} \chi - \frac{1}{2} \sigma^{(+)}_{(1)} \chi - \frac{1}{2} \chi \sigma^{(+)}_{(2)} + \frac{1}{2} \chi \bar{\tau} \chi + \frac{1}{2} \tau \tag{VIII.2$_1$}$$

(VIII.2$_2$) $$\left(\chi_\eta = -\gamma\cos\alpha\,\chi - \tfrac{1}{2}\sigma^{(-)}_{(1)}\chi - \tfrac{1}{2}\chi\sigma^{(-)}_{(2)} + \tfrac{\gamma}{2}\sin\alpha\,(1-\chi^2)\right)$$

with

(VIII.3) $$\sigma^{(\pm)}_{(r)} = i\left(S^{(\pm)}_{12} - (-1)^r S^{(\pm)}_{34}\right) + j\left(S^{(\pm)}_{13} - (-1)^r S^{(\pm)}_{42}\right) + k\left(S^{(\pm)}_{14} - (-1)^r S^{(\pm)}_{23}\right), \qquad r = 1,2$$

$$\tau = t_1 + i t_2 + j t_3 + k t_4$$

and where the bar denotes quaternion conjugation.

Linearization is achieved by the ansatz

(VIII.4) $$\chi = (a\varphi)(b\psi)^{-1}$$

with a and b satisfying the compatible equations:

(VIII.5) $$a_\xi = -\tfrac{1}{2}\sigma^{(+)}_{(1)} a, \qquad b_\xi = \tfrac{1}{2}\sigma^{(+)}_{(2)} b$$
$$a_\eta = -\tfrac{1}{2}\sigma^{(-)}_{(1)} a, \qquad b_\eta = \tfrac{1}{2}\sigma^{(-)}_{(2)} b$$

(VIII.6) $$a^{-1} = \bar{a}, \qquad b^{-1} = \bar{b}$$

giving

(VIII.7$_1$) $$\varphi_\xi = \frac{\gamma^{-1}}{2}\varphi + \tfrac{1}{2}(\bar{a}\tau b)\psi$$
$$-\psi_\xi = \frac{\gamma^{-1}}{2}\psi + \tfrac{1}{2}\overline{(\bar{a}\tau b)}\varphi,$$

(VIII.7$_2$) $$\varphi_\eta = -\frac{\gamma}{2}\cos\alpha\,\varphi + \frac{\gamma}{2}\sin\alpha\,(\bar{a}b)\psi$$
$$\psi_\eta = \frac{\gamma}{2}\cos\alpha\,\psi + \frac{\gamma}{2}\sin\alpha\,\overline{(\bar{a}b)}\varphi.$$

More conventionally this can be written as

(VIII.8$_1$) $$L(\eta)\,\Psi = \left(\frac{i}{2\gamma}\right)\Psi$$

(VIII.8$_2$) $$\frac{\partial}{\partial\eta}\Psi = B\,\Psi$$

where

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_4 \end{pmatrix}$$

(VIII.9) $$L(\eta) = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\frac{d}{d\xi} - \frac{i}{2}\begin{pmatrix} 0 & ((\bar{a}\tau b)_{ij}) \\ ((\overline{\bar{a}\tau b})_{ij}) & 0 \end{pmatrix}$$

(VIII.10) $$B = \frac{\gamma}{2}\begin{pmatrix} -\cos\alpha\, I\ , & \sin\alpha\,((\bar{a}b)_{ij}) \\ \sin\alpha\,((\overline{\bar{a}b})_{ij})\ , & \cos\alpha\, I \end{pmatrix}$$

(VIII.11) $$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \qquad (c_{ij}) = \begin{pmatrix} c_1, & -c_2, & -c_3, & -c_4 \\ c_2, & c_1, & -c_4, & c_3 \\ c_3, & c_4, & c_1, & -c_2 \\ c_4, & -c_3, & c_2, & c_1 \end{pmatrix}.$$

By setting up equation (VIII.10) we have determined the η-evolution of the spectral data for the linear eigenvalue problem (VIII.8$_1$) and hence solved (in the sense of the inverse scattering method) the characteristic initial value problem for the models obtained by reduction from the O_n - invariant chiral theories for $n \leq 6$.

To complete the proof that these models define integrable Hamiltonian systems we would have to solve (in the sense of the inverse scattering method) the Cauchy initial value problem and carry out an analysis similar to the one of ref. [2].

Among the models under discussion there is a particularly interesting one, n = 4, corresponding to the one-space -dimensional version of the non-linear σ-model [6].

The SU(2) x SU(2) - invariants in normalized coordinates are:

(VIII.12)
$$\alpha = \arccos(q_\xi \cdot q_\eta)$$
$$u = q_{\xi\xi} \cdot \frac{[q, q_\xi, q_\eta]}{\sin\alpha} \quad , \quad v = q_{\eta\eta} \cdot \frac{[q, q_\xi, q_\eta]}{\sin\alpha}$$

$[\ ,\ ,\ ]$ denoting the vector product.

The equations of motion for these invariants are:

(VIII.13)
$$\alpha_{\xi\eta} + \sin\alpha + \frac{uv}{\sin\alpha} = 0$$
$$u_\eta = \frac{\alpha_\xi}{\sin\alpha} v \quad , \quad v_\xi = \frac{\alpha_\eta}{\sin\alpha} u .$$

It was noted by H. Lehmann that the last two equations possess $\operatorname{tg}\frac{\alpha}{2}$ as an integrating factor. Thus we set

(VIII.14)
$$u = \beta_\xi \operatorname{tg}\frac{\alpha}{2} \quad , \quad v = -\beta_\eta \operatorname{tg}\frac{\alpha}{2}$$

and obtain two hyperbolic equations for the scalar fields α and β :

(VIII.15)
$$\alpha_{\xi\eta} + \sin\alpha - \frac{tg^2\frac{\alpha}{2}}{\sin\alpha}\beta_\xi\beta_\eta = 0$$
$$\beta_{\xi\eta} + \frac{\alpha_\xi\beta_\eta + \alpha_\eta\beta_\xi}{\sin\alpha} = 0 \quad .$$

These equations can be derived from the Lagrangian

(VIII.16) $$\mathcal{L}(\xi,\eta) = \frac{1}{2}\alpha_\xi\alpha_\eta + \frac{1}{2}tg^2\frac{\alpha}{2}\beta_\xi\beta_\eta + \cos\alpha - 1 .$$

For the formulation of the family of linear isospectral eigenvalue problems associated with this model, we can dispense with the quaternions:

(VIII.17$_1$)
$$L(\eta)\psi = \left(\frac{i}{2\gamma}\right)\psi$$
$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$L(\eta) = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{d\xi} + \frac{i}{2}\begin{pmatrix}0 & , (\alpha_\xi + i\beta_\xi tg\frac{\alpha}{2})e^{i\omega}\\ (\alpha_\xi - i\beta_\xi tg\frac{\alpha}{2})e^{-i\omega}, & 0\end{pmatrix}$$

where ω is defined up to some constant by the equations

(VIII.18) $$\omega_\xi = \frac{\beta_\xi\cdot\cos\alpha}{2\cos^2\frac{\alpha}{2}} \quad , \quad \omega_\eta = \frac{\beta_\eta}{2\cos^2\frac{\alpha}{2}} \quad .$$

The η-evolution of α and β, i.e. the solution of the characteristic initial value problem, is obtained from the η-evolution of the spectral data for (VIII.17$_1$) which in turn can be read off from

$$\text{(VIII.17}_2\text{)} \qquad \frac{\partial \psi}{\partial \eta} = B\psi$$

$$B = \frac{\gamma}{2}\begin{pmatrix} -\cos\alpha , & \sin\alpha\, e^{i\omega} \\ \sin\alpha\, e^{-i\omega}, & \cos\alpha \end{pmatrix} .$$

This completes our analysis of the classical theory related to the one-space-dimensional version of the non-linear σ-model.

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