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Representation Theory of the Universal Covering of the Euclidean Conformal Group and Conformal Invariant Green's Functions

by

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Representation Theory of the Universal Covering of the Euclidean Conformal Group and Conformal Invariant Green's Functions.

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G. Grensing

Abstract:

We present a special realization for the universal covering of the euclidean conformal group. This group can be defined as a transformation group on the euclidean version of compactified Minkowski space such that the action on this space coincides with the usual one of the euclidean conformal group. We construct the representations of the principal and complementary series and derive the intertwining kernels for the equivalent representations. The connection between representation theory and conformal invariant quantum field theory is studied. To this end we also give the reduction of the tensor product of two representations of the supplementary series.

The group SU*(4) [1] is the universal covering of the euclidean conformal group $SO_o(5,1)$. The unitary irreducible representations of this group are completely known on the pure Lie algebra level of representation theory [2,3,4,5]. It is the main goal of this paper to develop the global representation theory.

The investigation is motivated by the recent interest in conformal invariant euclidean quantum field theory [6,7,8], the underlying symmetry group being the twofold covering of $SO_o(5,1)$. It is generally accepted that such a theory becomes relevant if interpreted as a Gell-Mann Low limit [9] of a renormalizable field theory [10].

As is well known, the two- and three-point functions of a conformal invariant quantum field theory are, up to a multiplicative constant, uniquely determined [11,12]. For the four-point function a conformal invariant partial wave decomposition has been given, which contains only products of two- and three-point functions [13]. With this knowledge the system of coupled integral equations the n-point functions are known to obey [14] becomes tractable. This has been shown by Mack [6]. Furthermore, he has drawn attention to the fact that the two- and three-point functions have a pure group theoretical meaning. A thorough investigation of this point will be given in this paper.

We present a special realization G of SU*(4) (Sect.1), which will be useful in exhibiting the relation between representation theory and conformal invariant quantum field theory. A factorization of this group can be derived (Sec.2) such that the euclidean version of compactified Minkowski space is obtained as the factor space G/G', where G' is the inducing subgroup for the unitary irreducible representations of G. Our choice of the universal covering group proves to be significant because the action of G on G/G' coincides with the usual one of the conformal group (Sec.3). Thus, the representations of the principal (Sec.4) and supplementary series (Sec.5) of G act on fields over Euclidean space and they yield a transformation law, which is adapted to the conformal group. The two-point functions appear in this context as kernels for the scalar product of the supplementary series or as intertwining kernels for equivalent representations (Sec.6), and the three-point functions as Clebsch-Gordan kernels for the tensor product of two representations of the supplementary series (Sec.7).

After completion of this work we have received a preprint by Koller [15], in which a similar program is carried out for the groups $SO_o(n,1)$. He gives an independent derivation of the intertwining kernels by means of the nontrivial element of the Weyl group.

1. The universal covering group of $SO_o(5,1)$

We use the following realization of the universal covering group of $SO_o(5,1)$: G consists of the elements g of $SL(4,\mathbb{C})$ which obey the condition

$$g\,E = E\,g^* \tag{1.1}$$

where

$$E = \begin{pmatrix} -\varepsilon & 0 \\ 0 & +\varepsilon \end{pmatrix} \quad \text{with} \quad \varepsilon = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \tag{1.2}$$

and $*$ means complex conjugation.

If g is split into 2x2-matrices such that

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{1.3}$$

the condition (1.1) reads

$$\begin{aligned} g_{11}\varepsilon &= \varepsilon\, g_{11}^* \qquad & g_{12}\,\varepsilon &= -\varepsilon\, g_{12}^* \\ g_{21}\varepsilon &= -\varepsilon\, g_{21}^* \qquad & g_{22}\,\varepsilon &= \varepsilon\, g_{22}^*. \end{aligned} \tag{1.4}$$

It is readily verified that G is isomorphic to $SU^*(4)$ [1] by virtue of a suitable real, orthogonal transformation. Furthermore, it can be shown that there exists a homomorphism of G onto $SO_o(5,1)$ which has kernel $Z_2 = \{+e, -e\}$, this yielding the well known isomorphism

$$SU^*(4)\;/\;Z_2 \mathrel{\hat{=}} SO_o(5,1). \tag{1.5}$$

2. Factorization of G

The set of conditions (1.4) on g is now used to obtain a parametrization for G, which will be valid for all elements of G with the exception of a lower dimensional manifold.

The elements g of G with subdeterminant $|g_{11}| \neq 0$ can uniquely be factorized into

$$g = g_1 \, g_2 \, g_3 \, g_4, \tag{2.1}$$

each factor constituing a subgroup G_a ($a = 1,2,3,4$) of G. They are explicitely given by:

(1)
$$g_1 = \begin{pmatrix} \mathbb{1} & 0 \\ A & \mathbb{1} \end{pmatrix} \tag{2.2}$$

The 2×2-matrix A is built up by a four-vector $a = (a^1, a^2, a^3, a^4)$ according to

$$A = \begin{pmatrix} -a^1 - ia^2 & ia^4 + a^3 \\ -ia^4 + a^3 & a^1 - ia^2 \end{pmatrix} \quad . \tag{2.3}$$

G_1 is an abelian, four-dimensional subgroup of G.

(2)
$$g_2 = \begin{pmatrix} \mathfrak{a}' & 0 \\ 0 & \mathfrak{a}''^{*} \end{pmatrix} \tag{2.4}$$

with $\mathfrak{a}', \mathfrak{a}'' \in SU(2)$.

Because G_2 is simply $SU(2) \otimes SU(2)$ we use the notation $\mathfrak{a} = (\mathfrak{a}', \mathfrak{a}'')$.

(3)
$$g_3 = \begin{pmatrix} e^{-\lambda/2}\mathbb{1} & 0 \\ 0 & e^{+\lambda/2}\mathbb{1} \end{pmatrix} \tag{2.5}$$

with $\lambda \in \mathbb{R}$.

This subgroup is isomorphic to the multiplicative group of the positive real numbers.

(4)
$$g_4 = \begin{pmatrix} \mathbb{1} & -C^{\dagger} \\ 0 & \mathbb{1} \end{pmatrix} \tag{2.6}$$

The matrix C has the same form as has been given in (2.3), the four-vector a replaced by c.

3. The connection with the euclidean conformal group

Obviously, the elements of $G_1 \cdot G_2$ form a subgroup of G isomorphic to the twofold covering group of the euclidean Poincaré group. That suggests to examine the action of G on the translational part G_1 of G, the elements of which we now write

$$g_x = \begin{pmatrix} \underline{1} & 0 \\ X & \underline{1} \end{pmatrix} . \tag{3.1}$$

According to the results of Sect. 2 we can uniquely factorize $g\, g_x$ for $|(g\, g_x)_{11}| \neq 0$ into

$$g\, g_x = g_{x'}\, g' \tag{3.2}$$

where $g' \varepsilon G' = G_2 G_3 G_4$. Hence, we obtain

$$X' = g_{21} + g_{22}\, X/g_{11} + g_{12}\, X. \tag{3.3}$$

The computation of (3.3) for the various subgroups G_a (a=1,2,3,4) is facilitated by means of the identity

$$\frac{1}{2}\,(X_1 X_2^{\dagger} + X_2 X_1^{\dagger}) = x \cdot x_2\, \underline{1}\,, \tag{3.4}$$

the right hand side being the usual scalar product over Euclidean space M; the result is:

$$\begin{aligned} &(1) \quad x' = x + a \\ &(2) \quad x' = \Lambda x \\ &(3) \quad x' = e^{\lambda} x \\ &(4) \quad x' = x - x^2 c / 1 - 2x \cdot c + x^2 c^2 . \end{aligned} \tag{3.5}$$

In deriving Equation (2) of (3.5) we have used the covering map π of $SU(2) \otimes SU(2)$ onto the euclidean Lorentz group $L_o = SO(4)$ given by

$$\Lambda^{\mu}{}_{\nu} = \frac{1}{2}\, \mathrm{tr}\; \{\sigma^{\mu} \mathfrak{a}\, \tilde{\sigma}_{\nu}\, \mathfrak{a}'^{\dagger}\} \tag{3.6}$$

where $\sigma = (\sigma^k, -i\mathbb{1})_{k=1,2,3}$ and $\tilde{\sigma} = (\sigma^k, +i\mathbb{1})_{k=1,2,3}$.

From (3.5) we see that the elements of G with $|g_{11}| \neq 0$ act on $g_x G'$ in exactly the same way as the euclidean conformal group on elements x of M.

Thus we can identify the factors occuring in (2.1) as translations, Lorentz transformations, dilatations, and special conformal transformations. For that reason we shall use the notation

$$g = (a, \mathfrak{a} \,|\, d, c) \tag{3.7}$$

for the elements with $|g_{11}| \neq 0$.

4. The principal series of representations of G

In this section we will construct the representations of the principal series. To do this we must know the Iwasawa decomposition G = K A N of G with K being the maximal compact, A an abelian and N a nilpotent subgroup [1]. In our specific case, K is the unitary subgroup of G

$$K = \{g \in G \mid g^{\dagger} g = e\}. \tag{4.1}$$

It is easy to prove that K is isomorphic to Sp(2). The remaining factors occured already, $A = G_3$ and $N = G_4$.

The representations of the principal series are obtained as induced representations on the homogeneous space G/G', where G' = K' A N and K' being the centralizer of A in K [16]. We have $K' = G_2$, so that

$$G' = G_2\, G_3\, G_4. \tag{4.2}$$

The factor space G/G' is diffeomorphic to K/K'. However, we want to use the compactified M as homogeneous space, this leading to the decomposition (2.1).

According to the general theory, the representations are induced by the unitary and irreducible representations of G' which are trivial on N. The complete system of representations of G_2G_3, which is just $SU(2) \otimes SU(2) \otimes \mathbb{R}_+$, can immediately be written down

$$D(g') = (e^{\lambda})^{-i\rho}\, D^{(\ell_1,\ell_2)}(\mathfrak{a}), \tag{4.3}$$

where ρ is real and $D^{(\ell_1,\ell_2)}(\mathfrak{a}) = D^{(\ell_1)}(\mathfrak{a}') \otimes D^{(\ell_2)}(\mathfrak{a}'')$ is a unitary and irreducible representation of $SU(2) \otimes SU(2)$ on the tensor product $\mathbb{C}^{2\ell_1+1} \otimes \mathbb{C}^{2\ell_2+1}$ with $\ell_1, \ell_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

The representation space H consists of the functions

$$\phi : M \to \mathbb{C}^{2\ell_1+1} \otimes \mathbb{C}^{2\ell_2+1} \tag{4.4}$$

infinitely differentiable on M including infinity. Finally, we may define the representations of the principal series

$$U(g)\ \phi(x) = \left|\frac{\partial x'}{\partial x}\right|^{1/2} D(g')\ \phi(x') \tag{4.5}$$

with

$$g^{-1}\ g_x = g_{x'} g'^{-1} \quad x' = g^{-1} \cdot x. \tag{4.6}$$

and where we have to exclude the point at infinity. The representation is unitary relative to the scalar product

$$(\phi_1, \phi_2) = \int \phi_1^\dagger(x)\ \phi_2(x)\ d^4x \tag{4.7}$$

and known to be irreducible, too [16,17].

To get a more explicit form of the transformation law (4.5), we use the decomposition (2.1); the factors $g_a (a = 1,2,3,4)$ yield:

$$\begin{aligned}
&(1)\ U^\chi(a)\ \phi(x) = \phi(x-a) \qquad (4.8)\\
&(2)\ U^\chi(\mathfrak{a})\phi(x) = D^{(\ell_1,\ell_2)}(\mathfrak{a})\ \phi(\Lambda^{-1}x)\\
&(3)\ U^\chi(d)\ \phi(x) = (e^\lambda)^{-\Delta}\phi(e^{-\lambda}x)\\
&(4)\ U^\chi(c)\ \phi(x) = (1+2x\ c+x^2c^2)^{-\Delta}\ D^{(\ell_1,\ell_2)}(\mathfrak{a}(x,c))\ \phi(x+x^2c/1+2x\cdot c+x^2c^2)
\end{aligned}$$

In the last equation the SU(2) ⊗ SU(2) element

$$\mathfrak{a}(x,c) = (\mathfrak{a}'(x,c), \mathfrak{a}''(x,c)) \tag{4.9}$$

is given by

$$\begin{aligned}
\mathfrak{a}'(x,c) &= (\mathbb{1} + X^\dagger C)/1+2x\cdot c+x^2c^2\\
\mathfrak{a}''(x,c) &= (\mathbb{1} + X\ C^\dagger)^* / 1+2x\cdot c+x^2c^2.
\end{aligned} \tag{4.10}$$

Furthermore, we have defined $\Delta = 2 + i\rho$ and more accurately written U^χ with χ being an abbreviation for the labels of the representation of G'

$$\chi = (\ell_1, \ell_2;\ \Delta = 2 + i\rho). \tag{4.11}$$

After all, there remains to be shown that the integral (4.7) is actually convergent for $\phi \varepsilon H^{\chi}$. This will be proved by the aid of the element

$$g_{\infty} = \begin{pmatrix} 0 & -i\varepsilon \\ +i\varepsilon & 0 \end{pmatrix} \tag{4.12}$$

of the maximal compact subgroup of G, which maps x into

$$g_{\infty} \cdot x = \Lambda_T \, x/x^2 \tag{4.13}$$

where Λ_T is the time inversion. Obviously, this element is appropriate for studying the behaviour of ϕ for large x. To compute the action of $U^{\chi}(g_{\infty})$ on ϕ, we observe that $\mathfrak{a}(x,c)$ can be cast into the form

$$\mathfrak{a}(x,c) = \mathfrak{a}_x \; \mathfrak{a}^{\dagger}_{x'} \tag{4.14}$$

where $x' = x + x^2 c \,/\, 1+2x\cdot c+x^2c^2$ and $\mathfrak{a}_x = (\mathfrak{a}'_x, \mathfrak{a}''_x)$ with

$$\mathfrak{a}'^{\dagger}_x = \frac{i}{(x^2)^{\frac{1}{2}}} X = \mathfrak{a}''^{*}_x . \tag{4.15}$$

Then the final result is

$$U^{\chi}(g_{\infty}) \, \phi(x) = (x^2)^{-\Delta} \, D^{(\ell_1,\ell_2)} \, (\mathfrak{a}_x) \, D^{(\ell_1,\ell_2)}(\mathfrak{a}_{\varepsilon})\phi(\Lambda_T x/x^2) \tag{4.16}$$

with $\mathfrak{a}_{\varepsilon}=(-\varepsilon,+\varepsilon)$. From (4.16) we may derive the asymptotic behaviour

$$|\phi(x)| \sim c \, (x^2)^{-\frac{1}{2}(\Delta+\bar{\Delta})} \quad \text{for } x^2 \to \infty \tag{4.17}$$

which yields the convergence of the integral (4.7) for arbitrary $\bar{\rho}$ in $\mathbb{R}$.

With (4.8) we come into contact with conformal quantum field theory: If we continue x^4 to imaginary values, that is, define $ix^4 = x^o$ such that SO(4) gets $SO_o(3,1)$, we obtain the integrated form of the representation of the conformal Lie algebra acting on fields over Minkowski space [18].

5. The supplementary series of representations of G

We proceed in analogy to the analysis for the universal covering of $SO_o(3,1)$ [19,20] and try to generalize Δ in (4.8) to arbitrary complex values δ and the scalar product (4.7) to

$$(\phi_1,\phi_2) = \int \phi_1^{\dagger}(x_1)\, K(x_1,x_2)\, \phi_2(x_2) \quad d^4x_1 \quad d^4x_2 \qquad (5.1)$$

such that the resulting representation is unitary and irreducible. Thus, U^{χ} must leave invariant the bilinear form (5.1), which will serve to fix the dependence of the kernel $K(x_1,x_2)$ on (x_1,x_2). Furthermore, the integral (5.1) must exist and be positive definite, which will yield the admissible range of δ.

At first, we analyze the invariance condition on the kernel:

$$\begin{aligned}
&(1)\quad K(x_1-a,\ x_2-a) = K(x_1,x_2)\\
&(2)\quad D^{(\ell_1,\ell_2)}(\mathfrak{a})\, K(\Lambda^{-1}x_1,\ \Lambda^{-1}x_2)\, D^{(\ell_1,\ell_2)}(\mathfrak{a})^{\dagger} = K(x_1,x_2)\\
&(3)\quad (e^{\lambda})^{-(4-\delta^*)}\, K(e^{-\lambda}x_1,\ e^{-\lambda}x_2)(e^{\lambda})^{-(4-\delta)} = K(x_1,x_2) \qquad (5.2)\\
&(4)\quad (x_1'^2|x_1^2)^{-(4-\delta^*)}\, D^{(\ell_1,\ell_2)}(\mathfrak{a}_{x_1}\mathfrak{a}_{x_1'}^{\dagger})\, K(x_1',x_2')\, D^{(\ell_1,\ell_2)}(\mathfrak{a}_{x_2}\mathfrak{a}_{x_2'}^{\dagger})^{\dagger}(x_2'^2|x_2^2)^{-(4-\delta)}\\
&\qquad = K(x_1,x_2)
\end{aligned}$$

Choosing $a = -x_2$ in Equation (1) of (5.2), we see that $K(x_1,x_2)$ is a function of x_1-x_2 only

$$K(x_1,x_2) = K(x_1-x_2,0) =: K(x_1-x_2). \qquad (5.3)$$

We take $x_2 = 0$, $x_1 = x$ and define

$$K(x) = \frac{1}{(x^2)^{4-\delta^*}} \quad \hat{K}(x) \qquad (5.4)$$

to simplify Equation (4) of (5.2), this yielding

$$\begin{aligned}
&(2')\quad D^{(\ell_1,\ell_2)}(\mathfrak{a})\, \hat{K}(\Lambda^{-1}x)\, D^{(\ell_1,\ell_2)}(\mathfrak{a})^{\dagger} = \hat{K}(x)\\
&(3')\quad (e^{\lambda})^{\delta-\delta^*}\, \hat{K}(e^{-\lambda}x) = \hat{K}(x) \qquad (5.5)\\
&(4')\quad D^{(\ell_1,\ell_2)}(\mathfrak{a}_{x'})^{\dagger}\, \hat{K}(x') = D^{(\ell_1,\ell_2)}(\mathfrak{a}_x)^{\dagger}\, \hat{K}(x)
\end{aligned}$$

where $x' = x + x^2 c \mid 1 + 2\, x\cdot c + x^2c^2$.

The Equation(4') of (5.5) requires

$$\hat{K}(x) = D^{(\ell_1,\ell_2)}(\mathfrak{a}_x)\,\hat{K}_o \tag{5.6}$$

with $\hat{K}_o$ being a constant, invertible matrix. To see whether Equation(2') of (5.5) is satisfied by (5.6), we use

$$(\mathfrak{a}'_{\Lambda x}, \mathfrak{a}''_{\Lambda x}) = (\mathfrak{a}', \mathfrak{a}'')(\mathfrak{a}'_x, \mathfrak{a}''_x)(\mathfrak{a}''^{\dagger *}, \mathfrak{a}'^{\dagger *}) \tag{5.7}$$

to transform (2') into

$$K_o D^{(\ell_1)}(\mathfrak{a}') \otimes D^{(\ell_2)}(\mathfrak{a}'') = D^{(\ell_1)}(\mathfrak{a}'') \otimes D^{(\ell_2)}(\mathfrak{a}')K_o \tag{5.8}$$

where we have set

$$\hat{K}_o = D^{(\ell_1,\ell_2)}(\mathfrak{a}_\varepsilon)\,K_o. \tag{5.9}$$

The condition (5.8) requires $\ell_1 = \ell_2$ and K_o is determined to be

$$K_o{}^{i_1 i_2}{}_{k_1 k_2} = k\,\delta^{i_1}{}_{k_2}\,\delta^{i_2}{}_{k_1} \tag{5.10}$$

Finally, Equation(3') of (5.5) shows that δ must be real.

Collecting the results, we get:

$$\begin{aligned} K(x) &= \frac{1}{(x^2)^{4-\delta}}\, D^{(\ell_1,\ell_2)}(\mathfrak{a}_x)\, D^{(\ell_1,\ell_2)}(\mathfrak{a}_\varepsilon)\, K_o \\ K_o{}^{i_1 i_2}{}_{k_1 k_2} &= k\,\delta^{i_1}{}_{k_2}\,\delta^{i_2}{}_{k_1} \qquad \begin{array}{l} i_1,k_1 = -\ell_1,\dots,+\ell_1 \\ i_2,k_2 = -\ell_2,\dots,+\ell_2 \end{array} \end{aligned} \tag{5.11}$$

with $\delta = \delta^*$ and $\ell_1 = \ell_2$.

At this stage, it is convenient to use another realization of $D^{(\ell_1,\ell_2)}(\mathfrak{a})$ with $\ell_1 = \ell_2$. This representation is equivalent to the ℓ-fold tensor product $\overset{\ell}{\otimes}\Lambda$ with $\ell_1 = \ell_2 = \ell/2$ acting on the completely symmetric, traceless tensors of rank ℓ over M [21]. We denote these representations by $D^{(\ell)}(\Lambda)$.

To determine the analogue of (5.11) in this realization, we must know

$$\pi(\mathfrak{a}(x,c)) = \Lambda(x,c). \tag{5.12}$$

This is done with the help of (3.6), the result is

$$\Lambda(x,c) = g(x)\ g(x+x^2c/1+2x\cdot c+x^2c^2) \tag{5.13}$$

where

$$g_{\mu\nu}(x) = g_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}. \tag{5.14}$$

We remark that $g(x)$ is not contained in L_o, $\pi(\mathcal{a}_x)$ is the product of $g(x)$ and the reflection with respect to the second axis. However, the representation $D^{(\ell)}(\Lambda)$ of L_o can be extended to a representation of the complete Lorentz group. This property of $D^{(\ell)}(\Lambda)$ will be used in the sequel.

Repeating the arguments used to derive (5.11), we get

$$K^{\bar{\chi}}(x)^{\mu_1\ldots\mu_\ell\nu_1\ldots\nu_\ell} = k(\bar{\chi})\frac{1}{(x^2)^{4-\delta}}\frac{1}{\ell!}\sum_{\pi\epsilon S_\ell}\{g(x)^{\mu_1\nu_{\pi(1)}}\ldots g(x)^{\mu_2\nu_{\pi(1)}} \text{-traces}\} \tag{5.15}$$

where S_ℓ with elements π is the permutation group of ℓ objects. We slightly changed the notation in (5.15) in writing more specifically K^χ for K because of the invariance property

$$U^{\bar{\chi}}(g) \otimes U^{\bar{\chi}}(g)\ K^{\bar{\chi}}(x_1,x_2) = K^{\bar{\chi}}(x_1,x_2) \tag{5.16}$$

with $\bar{\chi} = (\bar{\ell};\bar{\delta})$ being defined by

$$\bar{\ell} = \ell \qquad \bar{\delta} = 4 - \delta. \tag{5.17}$$

The last requirement we have to fulfill is that the invariant bilinear form

$$(\phi_1,\phi_2)_\chi = \int \phi_1^\dagger(x_1)\ K^{\bar{\chi}}(x_1,x_2)\ \phi_2(x_2)\ d^4x_1\ d^4x_2 \tag{5.18}$$

be finite and positive. For this integral to be convergent, it is obviously necessary that $\delta > 2$. To give a sufficient answer, we use Fourier transformation because $K^\chi(x_1,x_2)$ is a function of x_1-x_2 only. The Fourier transform

$$\hat{\phi}(p) = \int d^4x\ e^{-ipx}\ \phi(x) \tag{5.19}$$

of $\phi(x)$ exists owing to $\delta > 2$ and the asymptotic behaviour (4.17). The computation of

$$\hat{K}^{\bar{\chi}}(p) = \int d^4x\ e^{-ipx}\ K^{\bar{\chi}}(x) \tag{5.20}$$

will be reduced to [22]

$$(x^2)^{-\delta} = \frac{(4\pi)^2}{2^{2\delta}} \frac{\Gamma(2-\delta)}{\Gamma(\delta)} \int \frac{d^4p}{(2\pi)^4} e^{ipx}(p^2)^{-2+\delta} , \tag{5.21}$$

which in turn may be used to show that

$$\hat{\bar{K}}^{\chi}(p) \sim C(p^2)^{2-\delta} \text{ for } p^2 \to 0 \tag{5.22}$$

and leads to the restriction $\delta < 4$. Hence, δ must lie in the interval $2 < \delta < 4$.

If (5.18) is expressed in terms of the Fourier transforms we get

$$(\phi,\phi)_\chi = \int \frac{d^4p}{(2\pi)^4} \hat{\phi}(p) \hat{\bar{K}}^{\chi}(p) \hat{\phi}(p). \tag{5.23}$$

Thus, the positivity condition amounts to the requirement that

$$z_{\mu_1\ldots\mu_\ell} \hat{\bar{K}}^{\chi}(p)^{\mu_1\ldots\mu_\ell \, \nu_1\ldots\nu_\ell} z_{\nu_1\ldots\nu_\ell} > 0 \tag{5.24}$$

for a nonzero, completely symmetric and traceless tensor z of rank ℓ. This condition is trivially fulfilled for $\ell = 0$, if the constant $k(o,4-\delta)$ appearing in (5.15) is chosen to be real and positive. For $\ell = 1$ it yields because of

$$\hat{K}^{(1,4-\delta)}(p)=k(1,4-\delta) \frac{(4\pi)^2}{2^{2(4-\delta)}} \frac{\Gamma(\delta-2)}{\Gamma(5-\delta)} (p^2)^{2-\delta}\{(3-\delta)g_{\mu\nu}+(\delta-2)2\frac{p_\mu p_\nu}{p^2}\} \tag{5.25}$$

that $\delta \leq 3$ and $k(1,4-\delta) > 0$. This condition is equally valid for $\ell = 2$, as can be shown by a somewhat lengthy computation, and is indeed known to hold true for all $\ell \neq 0$ [4].

We state the final results: If δ is restricted to the interval

$$\begin{aligned} 2 < \delta < 4 \quad &\text{for} \quad \ell = 0 \\ 2 < \delta < 3 \quad &\text{for} \quad \ell \neq 0 \end{aligned} \tag{5.26}$$

the representation (4.8) with $\chi = (\ell;\delta)$ and $\ell_1 = \ell_2 = \ell/2$ is unitary with respect to the scalar product (5.18), where the kernel is given by (5.15). These representations are said to belong the supplementary series. We assert, but do not prove that they are irreducible. At the integer points $\chi = (\ell;\delta=3)$ the exceptional series occurs.

The kernel $\bar{K}^{\chi}(x_1,x_2)$ is the inverse dressed propagator of conformal invariant euclidean quantum field theory. The two-point function for a field with half-integer spin can be obtained by taking an appropriate direct sum of representations (4.3). But these representations are not irreducible.

6. Equivalence of representations of G

The series of representations derived in Sections 4. and 5. are known to exhaust the irreducible and unitary representations of G [4]. These representations are, however, not all inequivalent.

To study this question, we look for a bounded operator K of H^{χ} into $H^{\chi'}$ such that K intertwines the action of $U^{\chi}(g)$ and $U^{\chi'}(g)$ for arbitrary g in G

$$U^{\chi'}(g)\ K = K\ U^{\chi}(g). \tag{6.1}$$

The kernel K(x,x') of K with

$$\phi'(x') = (K\phi)(x) = \int K(x',x)\ \phi(x)\ d^4x \tag{6.2}$$

thus has to satisfy

$$U^{\chi'}(g)\ \phi'(x') = \int K(x',x)\ U^{\chi}(g)\ \phi(x)\ d^4x. \tag{6.3}$$

The computation proceeds along the lines of Sect. 5. We only state the results:
An intertwining operator for the representations U^{χ} and $U^{\chi'}$ exists if $\chi' = \bar{\chi}$ with

$$\overline{(\ell_1,\ell_2)} = (\ell_2,\ell_1) \qquad \bar{\Delta} = 4 - \Delta. \tag{6.4}$$

The kernel $K(x',x) = K(x'-x)$ has the form

$$K(x) = \frac{1}{(x^2)^{4-\Delta}}\ D^{(\ell_1,\ell_2)}(\mathfrak{a}_x)\ D^{(\ell_1,\ell_2)}(\mathfrak{a}_\varepsilon)\ K_o \tag{6.5}$$

where $K_o{}^{i_1 i_2}{}_{k_1 k_2} = k\,\delta^{i_1}{}_{k_2}\,\delta^{i_2}{}_{k_1} \quad \begin{array}{l} i_1, k_2 = -\ell_1, \dots, +\ell_1 \\ i_2, k_1 = -\ell_2, \dots, +\ell_2 \end{array}$.

We note that, if necessary, the kernel has to be appropriately regularized [22].

The operator K is invertible, this statement yielding that the following representations are equivalent

$$U^{\chi}(g) \,\hat{=}\, U^{\bar{\chi}}(g) \qquad (6.6)$$

$$\chi = (\ell_1, \ell_2; \Delta) \qquad \bar{\chi} = (\ell_2, \ell_1; 4-\Delta).$$

However, we do not give the proof that K is invertible and will treat with some detail only the case of the principal and supplementary series with $\ell_1 = \ell_2 = \ell/2$. As has been explained in Section 5, we may change to an equivalent representation acting on completely symmetric, traceless tensors over M such that the kernel takes the form

$$K^{\bar{\chi}}(x)^{\mu_1 \dots \mu_\ell\, \nu_1 \dots \nu_\ell} = k(\bar{\chi}) \frac{1}{(x^2)^{4-\Delta}} \frac{1}{\ell!} \sum_{\pi \in S_\ell} \{ g(x)^{\mu_1 \nu_{\pi(1)}} \dots g(x)^{\mu_\ell \nu_{\pi(1)}} - \text{traces} \} \qquad (6.7)$$

Making the substitution $\delta \to \Delta$ in (5.15), we observe that the resulting expression is identical with (6.7). Thus the kernel (6.7) plays a twofold role for the representations of the supplementary series as intertwining kernel and as kernel for the scalar product (5.18). The kernel for the principal series of representations may be obtained from the kernel of the supplementary series by analytic continuation in δ.

As can immediately be shown, K^{χ} and $K^{\bar{\chi}}$ must obey the relation

$$\int K^{\chi}(x-x'')\, K^{\bar{\chi}}(x''-x')\, d^4x'' = 1\, \delta^4(x-x'), \qquad (6.8)$$

which imposes a restriction on the constant $k(\bar{\chi})$ in (6.7). The value of this constant may be guessed by treating the low dimensional cases, e.g. for $\ell=2$ one must use

$$K^{(2,\Delta)}(x)^{\mu_1\mu_2\nu_1\nu_2} = k(2,\Delta) \frac{1}{(x^2)^{\Delta}} \frac{1}{2} \{ g(x)^{\mu_1\nu_1} g(x)^{\mu_2\nu_2} + g(x)^{\mu_1\nu_2} g(x)^{\mu_2\nu_1} - \frac{1}{2} g^{\mu_1\mu_2} g^{\nu_1\nu_2} \}. \qquad (6.9)$$

A possible choice, which is compatible with the results of Section 5, is

$$k(\bar{\chi}) = \frac{1}{\pi^2} \frac{\Gamma(4-\delta+\ell)}{\Gamma(-2+\delta)} \{ (\delta-1)\, \delta(\delta+1)\dots(\delta+\ell-2)\}^{-1} . \tag{6.10}$$

Thus, for χ in the principal series the representations with $\chi=(\ell;2+i\rho)$ and $\bar{\chi}=(\ell;2-i\rho)$ are equivalent. For χ in the supplementary series we get new representations, because $\bar{\delta}$ takes the values $0 < \bar{\delta} < 2$ for $\ell = 0$ and $1 < \bar{\delta} < 2$ for $\ell \neq 0$. Hence, the admissible range of δ for the supplementary series is:

$$(\delta \neq 2) \qquad \begin{array}{ll} 0 < \delta < 4 & \text{for} \quad \ell = 0 \\ & \\ 1 < \delta < 3 & \text{for} \quad \ell \neq 0. \end{array} \tag{6.11}$$

If $\delta < 2$ the kernel (6.7) must be regularized such that the integral (5.18) is convergent.

7. Clebsch-Gordan coefficients for the tensor product of representations of the supplementary series

In this concluding section we make some qualitative remarks about Clebsch-Gordan kernels of two representations of the supplementary series, which play the role of three-point functions in conformal invariant euclidean quantum field theory.

The problem we want to solve is the decomposition of the tensor product $U^1(g) \otimes U^2(g)$ acting on $\phi_{12}(x_1,x_2)$ into irreducible components. We will restrict ourselves to the spinless case, $\chi_1=(0;\delta_1)$ and $\chi_2 = (0;\delta_2)$, the general case can be dealt with in an analogous way. The decomposition of the Kronecker product is implemented by a kernel $C^{\chi}(x_1,x_2;x_3)$ such that

$$\phi_{12}(x_1,x_2) = \int d\chi \quad C^{\bar{\chi}}(x_1,x_2;x_3)\ \phi^{\chi}(x_3)\ d^4x_3 \tag{7.1}$$

where $d\chi$ means summation over (ℓ_1,ℓ_2) and integration over Δ.

We have to specify which series of representations of G contribute to the integral (7.1): From the analogous problem for the universal covering group of $SO_o(3,1)$ we know that only representations of the principal series occur [23]. We shall assume this to hold true in our case.

The decomposition (7.1) must accomplish

$$U_1(g) \otimes U_2(g)\ \phi_{12}(x_1,x_2) = \int d\chi \int \bar{C}^{\chi}(x_1,x_2;x_3)U^{\chi}(g)\phi^{\chi}(x_3)\ d^4x_3 \quad , \qquad (7.2)$$

which serves to determine the explicit form of the kernel:

$$\begin{aligned}
&(1)\quad \bar{C}^{\chi}(x_1-a,x_2-a;x_3-a) = \bar{C}^{\chi}(x_1,x_2;x_3)\\
&(2)\quad \bar{C}^{\chi}(\Lambda^{-1}x_1,\Lambda^{-1}x_2;\Lambda^{-1}x_3) = \bar{C}^{\chi}(x_1,x_2;x_3)\ D^{(\ell_1,\ell_2)}(\mathfrak{a})\\
&(3)\quad (e^{\lambda})^{-\delta_1}(e^{\lambda})^{-\delta_2}\ \bar{C}^{\chi}(e^{-\lambda}x_1,e^{-\lambda}x_2;e^{-\lambda}x_3) = \bar{C}^{\chi}(x_1,x_2;x_3)(e^{\lambda})^{4-\Delta}\\
&(4)\quad (x_1^2/x_1'^2)^{-\delta_1}(x_2^2/x_2'^2)^{-\delta_2}\ \bar{C}^{\chi}(x_1',x_2';x_3') = \bar{C}^{\chi}(x_1,x_2;x_3)D^{(\ell_1,\ell_2)}(\mathfrak{a}(x_3,c))(x_3^2/x_3'^2)^{4-\Delta}
\end{aligned} \qquad (7.3)$$

where $x_i' = x_i + x_i^2c/1+2x_i\cdot c+x_i^2c^2 \qquad i = 1,2,3.$
We use Equation (1) to define

$$\bar{C}^{\chi}(x_1-x_2,x_2-x_3) := \bar{C}^{\chi}(x_1-x_3,x_2-x_3;0) \qquad (7.4)$$

and, furthermore, set

$$\bar{C}^{\chi}(x_1,x_2) = (x_1^2)^{-\delta_1}\ (x_2^2)^{-\delta_2}\ \hat{\bar{C}}^{\chi}(x_1,x_2) \qquad (7.5)$$

so that the remaining Equations of (7.3) take the simple form

$$\begin{aligned}
&(2')\quad \hat{\bar{C}}^{\chi}(\Lambda^{-1}x_1,\Lambda^{-1}x_2) = \hat{\bar{C}}^{\chi}(x_1,x_2)\ D^{(\ell_1,\ell_2)}(\mathfrak{a})\\
&(3')\quad (e^{\lambda})^{-4+\Delta+\delta_1+\delta_2}\ \hat{\bar{C}}^{\chi}(e^{-\lambda}x_1,e^{-\lambda}x_2) = \hat{\bar{C}}^{\chi}(x_1,x_2)\\
&(4')\quad \hat{\bar{C}}^{\chi}(x_1',x_2') = \hat{\bar{C}}^{\chi}(x_1,x_2)
\end{aligned} \qquad (7.6)$$

Equation (4') requires $\hat{C}^{\bar{\chi}}(x_1,x_2)$ to be invariant with respect to special conformal transformations. Because

$$\frac{x_1}{x_1^2} - \frac{x_2}{x_2^2} \tag{7.7}$$

and

$$\left(\frac{x_1}{x_1^2} - \frac{x_2}{x_2^2} \right)^2 = (x_1-x_2)^2 \;/\; x_1^2\, x_2^2 \tag{7.8}$$

are the only invariant expressions which can be formed out of x_1 and x_2, we get that

$$\hat{C}^{\bar{\chi}}(x_1,x_2) = \hat{C}^{\bar{\chi}}((x_1-x_2))^2 \;/\; x_1^2\, x_2^2 \; ; \frac{x_1}{x_1^2} - \frac{x_2}{x_2^2}) \; . \tag{7.9}$$

The dependence of $\hat{C}^{\bar{\chi}}$ on $x_1/x_1^2 - x_2/x_2^2$ must be used to build up the tensor character of the kernel.

Now we investigate Equation (2') of (7.6): Obviously, the sum $\ell_1+\ell_2$ must be an integer. For these representations it is known how to construct the representation space with the help of tensor products of M [21]. Thus, only those representations of the principal series contribute to (7.1) which satisfy $\ell_1 = \ell_2$, because on ℓ-fold tensor product of $x_1/x_1^2 - x_2/x_2^2$ is already symmetric so that the antisymmetrization with respect to two indices yields zero.

Hence, we obtain that $\hat{C}^{\bar{\chi}}(x_1,x_2)$ looks as follows:

$$\hat{C}^{\bar{\chi}}(x_1,x_2)^{\mu_1\ldots\mu_\ell} = c^{\bar{\chi}}((x_1-x_2)^2/x_1^2x_2^2) \;\{\; (\frac{x_1}{x_1^2} - \frac{x_2}{x_2^2})^{\mu_1} \ldots (\frac{x_1}{x_1^2} - \frac{x_2}{x_2^2})^{\mu_\ell} - \text{traces} \;\} \tag{7.10}$$

The determination of the scalar factor in (7.10) is easily done with the remaining Equation (3') of (7.6),

$$c^{\bar{\chi}}((x_1-x_2)^2/x_1^2x_2^2) = c(\bar{\chi})\; [x_1^2x_2^2/(x_1-x_2)^2]^{\frac{1}{2}(-2+\ell+i\rho+\delta_1+\delta_2)} \tag{7.11}$$

To write down the final result it is convenient to introduce the abbreviation

$$x_{ik} = x_i - x_k \qquad (i,k = 1,2,3) \quad \hat{x} = \frac{x_{13}}{x_{13}^2} - \frac{x_{23}}{x_{23}^2} \tag{7.12}$$

so that

$$C^{\bar{\chi}}(x_1,x_2;x_3)^{\mu_1\cdots\mu_\ell} = \tag{7.13}$$

$$c(\bar{\chi})\; (x_{12}^2)^{\frac{1}{2}(-2+\ell+i\rho+\delta_1+\delta_2)}\; (x_{13}^2)^{\frac{1}{2}(-2+\ell+i\rho-\delta_1+\delta_2)}\; (x_{23}^2)^{\frac{1}{2}(-2+\ell+i\rho+\delta_1-\delta_2)}$$

$$\{\; \hat{x}^{\mu_1} \cdots \hat{x}^{\mu_\ell} - \text{traces} \;\}.$$

This kernel has the following invariance property

$$U_1(g) \otimes U_2(g) \otimes \bar{U}^{\chi}(g)\ \bar{C}^{\chi}(x_1,x_2;x_3) = \bar{C}^{\chi}(x_1,x_2;x_3). \tag{7.14}$$

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