Relation between the fermion pole mass and $\overline{\text{MS}}$ Yukawa coupling in the standard model

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We present the full one-loop correction to the relation between the running $\overline{\text{MS}}$ Yukawa coupling $\bar{h}_f(\mu)$ at some renormalization scale μ and the pole mass m_f of a fermion f in the standard model. Our result complements previous analyses that included just the QCD correction to this relation in the case of quarks. This allows us to convert, without loss of information, the threshold value $\bar{h}_f(m_f)$ of $\bar{h}_f(\mu)$, which satisfies a renormalization-group equation and may be driven up to some high-energy scale of new physics, to physical observables at low energy. We investigate the limit of the Higgs boson and/or the top quark being much heavier than all the other particles. We also list simple and yet rather precise approximation formulas, which are convenient for applications.

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I. INTRODUCTION

Perhaps some of the most puzzling questions concerning the standard model (SM) are related to the origin of the Cabibbo-Kobayashi-Maskawa (CKM) matrix and the large ratio of fermion masses, e.g., the ratio of the top-quark mass to the electron mass is $m_t/m_e \approx 3 \times 10^5$. There have been many attempts to try and explain the seemingly random low-energy values of the Yukawa matrices as the result of some underlying structure imposed by a new symmetry at a higher scale, possibly in the framework of a grand-unified theory (GUT). Of course, the SM GUT is ruled out because (i) the gauge couplings do not meet within the experimental errors [1], (ii) the predicted proton lifetime is too low [2], and (iii) the large hierarchy between the GUT scale, M_{GUT} , and the electroweak scale is unstable under radiative corrections and requires a very severe fine-tuning [3]. However, all these shortcomings can be reconciled in the minimal supersymmetric extension of the SM (MSSM).

GUT models predict automatically relations among the fermion masses. The simplest and most successful example is the ratio m_b/m_τ , which can be predicted in SUSY GUT's assuming Yukawa unification at $M_{\text{GUT}} \approx 2 \times 10^{16}$ GeV, and there have been more ambitious attempts that tried to incorporate also the first two generations [4]. The dominant radiative corrections to these relations can be included via renormalization-

group (RG) evolution of the parameters from M_{GUT} to the electroweak scale using one-loop and two-loop β functions [5].

Other significant effects arise at the thresholds when one tries to derive a prediction for a physical observable in terms of the running fundamental parameters. The QCD portion of these effects is well under control [5,6]. The first step to also cover the electroweak part was taken by Sirlin and Zucchini [7], who derived the one-loop electroweak correction to the relation between the pole mass of the SM Higgs boson and its quartic self-coupling defined in the modified minimal-subtraction ($\overline{\text{MS}}$) scheme. In this paper, we shall extend that analysis to the fermion sector of the SM. Specifically, we shall compute the full one-loop radiative correction, $\delta_f(\mu)$, in the relation

$$\bar{h}_f(\mu) = 2^{3/4} G_F^{1/2} m_f [1 + \delta_f(\mu)] , \quad (1.1)$$

where $G_F = 1.16639 \times 10^{-5} \text{ GeV}^{-2}$ is Fermi's constant [2], m_f is the pole mass of fermion f , and $\bar{h}_f(\mu)$ is the running $\overline{\text{MS}}$ Yukawa coupling of f at renormalization scale μ . For the most part of our analysis we shall keep f generic, except for the numerical discussion, where we shall concentrate on the cases of special interest, $f = t, b$, and τ . We may view Eq. (1.1) also as the relation between $\bar{h}_f(\mu)$ and its counterpart in the on-shell scheme, $h_f = 2^{3/4} G_F^{1/2} m_f$. Notice that, although we focus attention on the effects due to the particles of the SM, our analysis can be immediately generalized to any extension thereof. We note in passing that the one-loop correction to the MSSM relation $h_f = \sqrt{g^2 + g'^2} m_f / (\sqrt{2} M_Z \sin \beta)$, where g and g' are the $\text{SU}(2)_L$ and $\text{U}(1)_Y$ gauge couplings, respectively, and $\tan \beta = v_2/v_1$, with v_1 and v_2 being the vacuum expectation values (VEV's) of the two Higgs doublets, has been found recently [8] in the framework of a minimal subtraction scheme in which the di-

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vergences are regularized by dimensional reduction ($\overline{\text{DR}}$) [9]. However, this result does not include the solution to our problem, since, in Ref. [8], g , g' , v_1 , and v_2 are defined as $\overline{\text{DR}}$ parameters, and their expressions in terms of G_F and the physical particle masses are not specified.

This paper is organized as follows. In Sec. II, we shall explain our treatment of the Higgs tadpoles and then derive the full SM one-loop correction, $\delta_f(\mu)$, in Eq. (1.1). We shall first assume that f is the heaviest particle of the SM. In a second step, we shall incorporate the effects of decoupling of weak physics at intermediate scales, which will eliminate the leading logarithmic (LL) corrections. Section III contains our numerical discussion. Our conclusions are summarized in Sec. IV. Lengthy expressions are relegated to the Appendix.

II. RELATION BETWEEN THE FERMION POLE MASS AND $\overline{\text{MS}}$ YUKAWA COUPLING

To start with, we consider the Higgs sector and describe how we handle the tadpoles. The most general gauge-invariant renormalizable potential of one complex $\text{SU}(2)_L$ -doublet scalar field, Φ , with $Y = -1$ can be written as

$$V = \frac{\lambda_0}{2} (\Phi^\dagger \Phi)^2 - m_0^2 \Phi^\dagger \Phi. \quad (2.1)$$

Here and in the following we denote all bare quantities by a subscript 0. With the replacement $\Phi \rightarrow (v_0 + H)/\sqrt{2}$, where v_0 is the VEV of the CP -even component of Φ , we obtain

$$V = tH + \frac{M_H^2}{2} H^2 + O(H^3). \quad (2.2)$$

Here, we have defined $t = v_0(\lambda_0 v_0^2/2 - m_0^2)$ and $M_H^2 = 3\lambda_0 v_0^2/2 - m_0^2$. Since v_0 is not a physical quantity, its definition at the one-loop level is arbitrary and a matter of convenience. There are two obvious choices one could make. One possibility is to define v_0 such that the bare one-point vertex and the one-loop tadpole contribution T defined in Fig. 1 cancel: i.e., $t + T = 0$. This has the advantage that the tadpoles never enter a calculation that does not involve other observables of the Higgs sector, such as the mass or the self-coupling of the Higgs boson, at the tree level. Examples for calculations with an explicit dependence on the tadpoles are presented in Refs. [7,10]. Another possibility is to define $v_0 = m_0 \lambda_0^{-1/2}$, which has the advantage that v_0 is expressed only in terms of bare parameters and is thus manifestly gauge independent. Since we wish to use gauge independence and the explicit cancellation of the tadpoles as a welcome check of our calculation, we shall choose the second definition.

The tree-level relation between the mass and the Yukawa coupling of an up-type fermion U is fixed by the Yukawa Lagrangian

$$\mathcal{L} = h_{U0} \overline{Q}_L \Phi U_R + \text{H.c.} \supset m_{U0} \overline{U}_L U_R + \text{H.c.}, \quad (2.3)$$

and similarly for a down-type fermion D . Here, the sub-

$$\begin{aligned} \text{Diagram 1: } \text{Feynman diagram with a fermion loop and a gauge boson (W or Z) insertion.} &= -i[g_{\mu\nu}\Pi_{\text{WW}}(p^2) + p_\mu p_\nu \lambda_{\text{WW}}(p^2)] \\ \text{Diagram 2: } \text{Feynman diagram with a fermion loop and a fermion (f) insertion.} &= i\not{p} [\Sigma_V^f(p^2) + \gamma_5 \Sigma_A^f(p^2)] + i m_f \Sigma_S^f(p^2) \\ \text{Diagram 3: } \text{Feynman diagram with a Higgs boson (h) tadpole contribution.} &= iT \end{aligned}$$

FIG. 1. Definitions of the gauge-boson and fermion self-energies and the Higgs tadpole contribution.

scripts L and R label left- and right-handed fields, respectively, and $Q_L = (U_L, D_L)$. Thus, one has

$$m_{f0} = 2^{-1/2} v_0 h_{f0}, \quad (2.4)$$

with $f = U, D$. In the following, we shall incorporate the full one-loop correction to this relation in the SM. For applications, it is most useful to express this relation in terms of G_F , $\bar{h}_f(\mu)$, and m_f [see Eq. (1.1)].

The pole mass of a particle is defined as the real part of the complex pole position of its propagator and is a constant of nature [11]. In applications, it is often more convenient to deal with the on-shell mass, which corresponds to the zero of the real part of the inverse propagator. In the approximation of neglecting the momentum dependence of the imaginary part of the self-energy, the two mass definitions coincide. For the purposes of the one-loop analysis carried out here, we need not distinguish between them. At one loop, m_f and m_{f0} are related by

$$\begin{aligned} m_f = m_{f0} \left\{ 1 - \text{Re}[\Sigma_V^f(m_f^2) + \Sigma_S^f(m_f^2)] \right. \\ \left. + 2^{1/4} G_F^{1/2} \frac{T}{M_H^2} \right\}, \end{aligned} \quad (2.5)$$

where $\Sigma_V^f(q^2)$ and $\Sigma_S^f(q^2)$ are the tadpole-free vector and scalar components of the self-energy of f at four-momentum q , respectively. The precise definitions of these amplitudes may be seen from Fig. 1; analytic expressions in the 't Hooft-Feynman gauge may be found in Ref. [12]. For the reader's convenience, we list $\Sigma_V^f(m_f^2) + \Sigma_S^f(m_f^2)$ and T in the Appendix. Throughout this work, we use dimensional regularization with $n = 4 - 2\epsilon$ being the dimensionality of space-time. Adopting the R_ξ gauge, we have verified that the right-hand side of Eq. (2.5) is independent of the four SM gauge parameters, ξ_W , ξ_Z , ξ_γ , and ξ_g , as it should be.

As far as QCD corrections are concerned, the term $\delta_f(\mu)$ in Eq. (1.1) is well known [5,6]. In fact, it is the very correction that appears in the relation between m_f and the $\overline{\text{MS}}$ mass, $\bar{m}_f(\mu)$, evaluated at renormalization scale μ , so that $\bar{h}_f(\mu) = 2^{3/4} G_F^{1/2} \bar{m}_f(\mu)$. Here it is understood that only QCD corrections are taken into account. The purely electromagnetic correction may be inferred

from this relation in the usual way, by substituting $Q_f^2 \alpha$ for $C_F \alpha_s(\mu)$, where Q_f is the electric charge of f in units of the positron charge, α is the fine-structure constant, $C_F = (N_c^2 - 1)/(2N_c)$, with $N_c = 3$, is the quadratic Casimir operator of $SU(3)_c$, and $\alpha_s(\mu)$ is the strong-coupling constant at scale μ . In the case of the weak correction, things are more involved, since the renormalization of the VEV must be included, too. As a consequence, in the full SM, $\bar{h}_f(\mu)$ will not be proportional to some $\overline{\text{MS}}$ fermion mass to be defined.

Nevertheless, it is interesting in its own right to think about the definition of $\bar{m}_f(\mu)$ in the presence of weak interactions—especially, since we were not able to locate a discussion of this matter in the literature [13]. By analogy to the QCD case, we may convert Eq. (2.5) from the on-shell scheme to the $\overline{\text{MS}}$ scheme by keeping just the terms proportional to $\Delta = 1/\epsilon - \gamma_E + \ln(4\pi)$, where γ_E is the Euler-Mascheroni constant, within the curly brackets. We may then eliminate m_{f0} to obtain a relation between $\bar{m}_f(\mu)$ and m_f ,

$$\bar{m}_f(\mu) = m_f \left\{ 1 + \text{Re}[\Sigma_V^f(m_f^2) + \Sigma_S^f(m_f^2)] - 2^{1/4} G_F^{1/2} \frac{T}{M_H^2} \right\}_{\Delta=0}. \quad (2.6)$$

It is interesting to observe that we need to keep the tadpole contribution in order to obtain a gauge-independent definition of $\bar{m}_f(\mu)$. This introduces corrections that grow like $1/M_H^2$ for M_H decreasing, which seems peculiar, since there is no stringent theoretical principle that prohibits the Higgs boson from being very light. On the other hand, if we chose to discard the tadpole contribution in Eq. (2.6), $\bar{m}_f(\mu)$ would depend on ξ_W and ξ_Z . One might argue that this does not create a problem because, in contrast to m_f , $\bar{m}_f(\mu)$ does not represent a physical observable and is thus entitled to be gauge dependent [14]. This avenue was taken recently in Ref. [15] for the definition of $\bar{m}_t(\mu)$ in the high- m_t approximation. Since the light-Higgs-boson scenario is now excluded experimentally, we express a preference for the definition according to Eq. (2.6).

We now return to our original problem of determining the full one-loop expression for $\delta_f(\mu)$ in Eq. (1.1). The Fermi constant is not a parameter of the SM Lagrangian. In order to express it in terms of SM parameters, we need to compute a physical process both in the Fermi model and the SM and equate the two results. This was done first in a pioneering paper by Sirlin [16], who considered the muon lifetime. In our notation, the result of Ref. [16] may be written as

$$G_F = \frac{1}{\sqrt{2}v_0^2} \left[1 + \frac{\Pi_{WW}(0)}{M_W^2} - 2^{5/4} G_F^{1/2} \frac{T}{M_H^2} + E \right]. \quad (2.7)$$

Here, $\Pi_{WW}(q^2)$ is the tadpole-free W -boson self-energy at four-momentum q , which may be found in Ref. [17]; for completeness, we list $\Pi_{WW}(0)$ in the Appendix. E comprises the wave-function renormalizations and the vertex and box corrections that the SM introduces on top of the calculation of the muon decay width in the QED-improved Fermi model. In Ref. [16], E is presented in the 't Hooft-Feynman gauge, with $\xi_W = \xi_Z = \xi_\gamma = 1$:

$$E = \frac{\alpha}{4\pi s_W^2} \left[4 \left(\Delta - \ln \frac{M_Z^2}{\mu^2} \right) + \left(\frac{7}{2s_W^2} - 6 \right) \ln c_W^2 + 6 \right]. \quad (2.8)$$

Here and in the following, we use the abbreviation

$c_W^2 = 1 - s_W^2 = M_Z^2/M_W^2$ [16] and identify $\alpha/s_W^2 = \sqrt{2}G_F M_W^2/\pi$ at the one-loop level. Note that there are no counterterms on the right-hand side of Eq. (2.7), since the tree-level amplitude is still parametrized in terms of the bare (and divergent) quantity v_0 . An explicit check of the gauge independence of the right-hand side of Eq. (2.7) would require the calculation of Δr [16] in an arbitrary R_ξ gauge, which is beyond the scope of this work. However, if we assume that Δr is gauge independent then the gauge independence of Eq. (2.7) follows immediately from the results of Refs. [16,18]. To prove this assumption at one loop, we recall that Δr is defined as the difference of the radiative corrections to the muon decay width calculated in the QED-improved Fermi model and the SM. The latter result is gauge independent because it represents the complete radiative correction to a physical observable at a given order in a renormalizable theory. On the other hand, the ξ_γ independence of the photonic correction to the muon decay width at one loop in the Fermi theory has been established more than 30 years ago [19].

Next, we have to define the normalized Yukawa coupling. At one loop in the $\overline{\text{MS}}$ scheme, we have

$$\bar{h}_f(\mu) = h_{f0} - \delta h_f, \quad (2.9)$$

where the counterterm

$$\delta h_f = \frac{\beta_f}{2} \Delta, \quad (2.10)$$

only subtracts the divergent piece proportional to Δ . Here, β_f is the familiar β function of the SM Yukawa sector [5], which may be written in the compact form

$$\beta_f = \frac{h_f}{16\pi^2} \left[\frac{3}{2} (h_f^2 - h_{f'}^2) + \sum_i N_i h_i^2 - \frac{9}{4} g^2 - \frac{3}{4} g'^2 (Y_{fL}^2 + Y_{fR}^2) - 6 C_F g_s^2 \right], \quad (2.11)$$

where $g^2 s_W^2 = g'^2 c_W^2 = 4\pi\alpha$, $g_s^2 = 4\pi\alpha_s$, f' stands for the weak isopartner of f , i runs over all massive fermions, $N_i = 1$ (3) for leptons (quarks), and it is understood

that the QCD term is to be omitted if f represents a lepton. Furthermore, Y_{fL} and Y_{fR} denote the weak hypercharges of the left- and right-handed components of f , respectively. Specifically,

$$(Y_{eL}, Y_{eR}, Y_{uL}, Y_{uR}, Y_{dL}, Y_{dR}) = (-1, -2, \frac{1}{3}, \frac{4}{3}, \frac{1}{3}, -\frac{2}{3})$$

and similarly for the other generations. The right-hand side of Eq. (2.9) is gauge independent, since h_{f0} is a bare quantity and Eq. (2.10) does not carry any gauge parameters.

Inserting Eqs. (2.5), (2.7), and (2.9) into Eq. (2.4) and comparing the outcome with Eq. (1.1), we obtain the desired result

$$\delta_f(\mu) = \text{Re}[\Sigma_V^f(m_f^2) + \Sigma_S^f(m_f^2)] - \frac{\Pi_{WW}(0)}{2M_W^2} - \frac{E}{2} - \frac{\beta_f}{2h_f} \Delta. \quad (2.12)$$

This is manifestly finite and gauge independent. Furthermore, we observe that the tadpole contributions have canceled, so that there are no inverse powers of M_H left. Equation (2.12) may be decomposed into a weak, an electromagnetic, and a QCD part, which are separately finite and gauge independent:

$$\delta_f(\mu) = \delta_f^W(\mu) + \delta_f^{\text{QED}}(\mu) + \delta_f^{\text{QCD}}(\mu). \quad (2.13)$$

The latter, which is present only if f represents a quark, is quite familiar [5,6]:

$$\delta_f^{\text{QCD}}(\mu) = C_F \frac{\alpha_s(\mu)}{4\pi} \left(3 \ln \frac{m_f^2}{\mu^2} - 4 \right), \quad (2.14)$$

and even the $O(\alpha_s^2)$ term is known¹ [20]. As mentioned before, $\delta_f^{\text{QED}}(\mu)$ emerges from Eq. (2.14) by substituting $Q_f^2 \alpha$ for $C_F \alpha_s(\mu)$. $\delta_f^{\text{QCD}}(\mu)$ is gauge independent because it is the complete $O(\alpha_s)$ contribution to the gauge-independent quantity $\delta_f(\mu)$. This is true also for $\delta_f^{\text{QED}}(\mu)$, since it is obtained from the very set of Feynman diagrams with the gluon replaced by a photon. Thus, also, $\delta_f^W(\mu)$ is gauge independent. In the remainder of this paper, we shall concentrate on $\delta_f^W(\mu)$.

It is interesting to study the limit where M_H and/or m_t are large as compared to the masses of all the other particles of the SM. By putting the latter equal to zero, we obtain

$$\delta_t^W(\mu) = \frac{G_F m_t^2}{8\pi^2 \sqrt{2}} \left[- \left(N_c + \frac{3}{2} \right) \ln \frac{m_t^2}{\mu^2} + \frac{N_c}{2} + 4 - r + 2r(2r-3) \ln(4r) - 8r^2 \left(1 - \frac{1}{r} \right)^{3/2} \text{arccosh} \sqrt{r} \right], \quad (2.15)$$

$$\delta_b^W(\mu) = \frac{G_F}{8\pi^2 \sqrt{2}} \left\{ m_t^2 \left[\left(-N_c + \frac{3}{2} \right) \ln \frac{m_t^2}{\mu^2} + \frac{N_c}{2} - \frac{5}{4} \right] + \frac{M_H^2}{4} \right\}, \quad (2.16)$$

$$\delta_f^W(\mu) = \frac{G_F}{8\pi^2 \sqrt{2}} \left[N_c m_t^2 \left(-\ln \frac{m_t^2}{\mu^2} + \frac{1}{2} \right) + \frac{M_H^2}{4} \right], \quad (2.17)$$

where $f \neq t, b$. Equation (2.15) is valid for $r = (M_H^2/4m_t^2) \geq 1$. For $r < 1$, one has to replace $(1-1/r)^{3/2} \text{arccosh} \sqrt{r}$ by $(1/r-1)^{3/2} \text{arccos} \sqrt{r}$. Equation (2.15) agrees with Eq. (15) of Ref. [15]. Furthermore, we can expand Eq. (2.15) for $M_H \gg 2m_t$ and $M_H \ll 2m_t$, which yields

$$\delta_t^W(\mu) = \frac{G_F}{8\pi^2 \sqrt{2}} \left\{ \frac{M_H^2}{4} + m_t^2 \left[N_c \left(-\ln \frac{m_t^2}{\mu^2} + \frac{1}{2} \right) - \frac{3}{2} \ln \frac{M_H^2}{\mu^2} + \frac{7}{4} + O \left(\frac{m_t^2}{M_H^2} \ln \frac{M_H^2}{m_t^2} \right) \right] \right\}, \quad (2.18)$$

$$\delta_t^W(\mu) = \frac{G_F m_t^2}{8\pi^2 \sqrt{2}} \left[- \left(N_c + \frac{3}{2} \right) \ln \frac{m_t^2}{\mu^2} + \frac{N_c}{2} + 4 - 2\pi \frac{M_H}{m_t} + O \left(\frac{M_H^2}{m_t^2} \ln \frac{m_t^2}{M_H^2} \right) \right], \quad (2.19)$$

respectively. Equation (2.17) emerges from the W -boson self-energy and is thus flavor independent, while Eqs. (2.15) and (2.16) receive additional contributions from the top- and bottom-quark self energies, respectively. The last term in Eq. (2.12) just subtracts the ultraviolet divergence, and E does not survive in the limit of large M_H and/or m_t anyway. From Eqs. (2.16)–(2.18) we see that a heavy Higgs boson induces through quantum effects power corrections quadratic in M_H . This is not surprising when we recall that the screening theorem [22] applies only to the gauge sector, whereas $h_f(\mu)$ is of Higgs origin. For instance, $\Pi_{WW}(0)/M_W^2$ appears also in the expression for the ρ parameter, but there is also a similar contribution due to the Z boson to cancel the term proportional to M_H^2 . At the other extreme, a Higgs boson with $M_H \ll 2m_t$ does not produce large corrections as may be seen from Eqs. (2.16), (2.17), and (2.19).

Equations (2.15)–(2.17) have been obtained from the weak part of Eq. (2.12) by omitting subleading corrections. Detailed inspection of the latter reveals that their M_H dependence is approximately logarithmic, while they do not depend on m_t , except for the subleading correction to $\delta_t^W(\mu)$, which, to good approximation, depends logarithmically on m_t . We may exploit these observations by constructing very simple approximations for the electroweak part of Eq. (2.12). We find that $\delta_f^W(m_f) + \delta_f^{\text{QED}}(m_f)$ are very well approximated by Eqs.

¹A numerically insignificant error in the relevant formula of Ref. [20] is corrected in Ref. [21].

(2.15)–(2.17) if the terms

$$d_f = a_f + b_f \ln \frac{M_H}{300 \text{ GeV}} + c_f \ln \frac{m_t}{175 \text{ GeV}} \quad (2.20)$$

are included on their right-hand sides. The coefficients, a_f , b_f , and c_f for the most important cases $f = t, b$, and τ are listed in Table I. We shall illustrate the goodness of these approximations in the next section.

Since $\bar{h}_f(\mu)$ is a running parameter that satisfies the RG equation,

$$\frac{d\bar{h}_f(\mu)}{d \ln \mu} = \beta_f, \quad (2.21)$$

we can, in general, expect to absorb all the LL corrections by a suitable choice of μ . In particular, we expect the LL terms in $\delta_f(\mu)$ to vanish if we choose $\mu = m_f$. However, this is only true in a theory without any intermediate scale; i.e., all other particle masses have to be small as compared to μ . Obviously, this is only true for m_t when we assume that $M_H = O(M_Z)$, which we choose to do for the time being. In general, there are other particles X_i ($i = 1, 2, \dots$), with masses $M_i > m_f$ that are decoupled from the theory for $\mu < M_i$. This induces in $\delta_f(\mu)$ LL corrections proportional to $\ln(M_i/\mu)$. Clearly, this only affects $\delta_f^W(\mu)$.

For a quantitative discussion of this issue, we rewrite Eq. (2.21) as

$$\frac{d\delta_f^W(\mu)}{d \ln \mu} = \frac{\beta_f^W}{h_f}, \quad (2.22)$$

where $\beta_f^W = \beta_f + 3h_f(Q_f^2\alpha + C_F\alpha_s)/(2\pi)$ is the weak part of Eq. (2.11). Next, we split β_f^W/h_f into a part proportional to h_t and the remainder, which, for simplic-

TABLE I. Coefficients a_f , b_f , and c_f to be inserted in Eq. (2.20) for $f = t, b$, and τ .

f	a_f	b_f	c_f
t	6.90×10^{-3}	1.73×10^{-3}	-5.82×10^{-3}
b	1.52×10^{-2}	1.73×10^{-3}	0
τ	1.59×10^{-2}	1.73×10^{-3}	0

ity, we associate with the scale M_Z , i.e., $\beta_f^W = \beta_f^t + \beta_f^Z$. The effective running Yukawa coupling in the presence of decoupling is thus determined by

$$\frac{d\delta_f^{\text{eff}}(\mu)}{d \ln \mu} = \frac{1}{h_f} [\beta_f^t \theta(\mu - m_t) + \beta_f^Z \theta(\mu - M_Z)], \quad (2.23)$$

where θ is the step function. At sufficiently high scales, M , with $M \geq m_t$, this effective Yukawa coupling coincides with the original one, so that $\delta_f^W(M) = \delta_f^{\text{eff}}(M)$. Solving Eqs. (2.22) and (2.23) and using this boundary condition, we obtain, for arbitrary μ ,

$$\delta_f^W(\mu) = \delta_f^{\text{eff}}(\mu) + \delta_f^{\text{LL}}(\mu), \quad (2.24)$$

where the LL terms are contained in

$$\delta_f^{\text{LL}}(\mu) = -\frac{1}{h_f} \left[\theta(m_t - \mu) \beta_f^t \ln \frac{m_t}{\mu} + \theta(M_Z - \mu) \beta_f^Z \ln \frac{M_Z}{\mu} \right]. \quad (2.25)$$

Of course, $\delta_t^{\text{LL}}(m_t) = 0$. For the reader's convenience, we list the corresponding threshold expressions for $f = b$ and τ :

$$\begin{aligned} \delta_b^{\text{LL}}(m_b) &= -\frac{1}{16\pi^2} \left\{ h_t^2 \left(N_c - \frac{3}{2} \right) \ln \frac{m_t}{m_b} + \left[h_b^2 \left(N_c + \frac{3}{2} \right) + \sum_{i \neq t, b} N_i h_i^2 - \frac{9}{4} g^2 - \frac{5}{12} g'^2 + \frac{2}{3} e^2 \right] \ln \frac{M_Z}{m_b} \right\}, \\ \delta_\tau^{\text{LL}}(m_\tau) &= -\frac{1}{16\pi^2} \left[N_c h_t^2 \ln \frac{m_t}{m_\tau} + \left(\frac{5}{2} h_\tau^2 + \sum_{i \neq \tau, t} N_i h_i^2 - \frac{9}{4} g^2 - \frac{15}{4} g'^2 + 6e^2 \right) \ln \frac{M_Z}{m_\tau} \right]. \end{aligned} \quad (2.26)$$

We have seen above that a heavy Higgs boson does not decouple but produces corrections proportional to M_H^2 . We have, therefore, refrained from separating out the logarithms involving M_H .

In any RG analysis of the Yukawa couplings in GUT models, one decouples the top quark by setting $h_t = 0$ for $\mu < m_t$. The W and Z bosons can be decoupled by replacing the β functions of the unbroken $SU(2)_L \otimes U(1)_Y$ theory with the ones of the $U(1)_{\text{em}}$ theory for $\mu < M_Z$. This automatically takes care of the LL terms, which are summed over by solving the RG equation numerically. We have also separated out the QCD contribution,

which is universal, since the electroweak gauge and Higgs sectors do not interact strongly. We have proceeded similarly also for the purely electromagnetic contribution.

III. NUMERICAL ANALYSIS

We are now in a position to explore the phenomenological consequences of our results. To start with, we specify the input values for our numerical analysis. We use $M_W = 80.24 \text{ GeV}$ [23], $M_Z = 91.1895 \text{ GeV}$ [24],

FIG. 2. Electroweak threshold corrections to Eq. (1.1), $\delta_f^W(m_f) + \delta_f^{\text{QED}}(m_f)$, for (a) $f = t$, (b) $f = b$, and (c) $f = \tau$ at one loop in the SM as functions of M_H assuming $m_t = (175 \pm 25)$ GeV. In (b) and (c), decoupling effects are not corrected for. The dotted lines correspond to the approximations by Eqs. (2.15)–(2.17) plus Eq. (2.20).

FIG. 3. Electroweak threshold corrections to Eq. (1.1), $\delta_f^{\text{eff}}(m_f) + \delta_f^{\text{QED}}(m_f)$, for (a) $f = b$, (b) $f = \tau$, and (c) their difference at one loop in the SM as functions of M_H assuming $m_t = (175 \pm 25)$ GeV. In contrast with Figs. 2(b) and 2(c), decoupling effects are corrected for.

$m_\tau = 1.777$ GeV [25], $m_b = 4.72$ GeV [26],² and $m_t = (175 \pm 25)$ GeV [28]. We adopt the other quark masses from Ref. [29]. All other input parameters are taken from Ref. [2].

In Fig. 2, we show the electroweak threshold corrections to Eq. (1.1), $\delta_f^W(m_f) + \delta_f^{\text{QED}}(m_f)$, for Fig. 2(a) $f = t$, Fig. 2(b) $f = b$, and Fig. 2(c) $f = \tau$ at one loop in the SM as functions of M_H assuming $m_t = (175 \pm 25)$ GeV. Obviously, Eqs. (2.15)–(2.17) plus Eq. (2.20) lead to excellent approximations in all cases throughout the full parameter space (dotted lines). For $f \neq t$, decoupling effects are not corrected for, i.e., the results still include the large, negative LL terms given in Eq. (2.26). Subtraction of these terms leads to Fig. 3(a) and 3(b) for $f = b$ and τ , respectively. Looking at Figs. 2(a), 3(a), and 3(b), we see that, for $M_H = O(M_Z)$, the threshold corrections are modest, well below 1% in size. Due to their quadratic dependence on M_H , they can become as large as 3% for $M_H = 1$ TeV. In addition, we have presented in Fig. 3(c) the difference of the results contained in Figs. 3(a) and 3(b), which is relevant in SU(5) GUT's for the τ - b Yukawa unification. We find a negative effect, which is insensitive to M_H and whose magnitude increases with m_t . However, even for $m_t = 200$ GeV, it only amounts to approximately -0.25% .

IV. CONCLUSIONS

We have calculated the full one-loop correction to the relation between the running $\overline{\text{MS}}$ Yukawa coupling, $\bar{h}_f(\mu)$, and the pole mass m_f of a fermion in the SM and shown that this correction is gauge independent. As a by-product, we have introduced a gauge-independent definition of the running $\overline{\text{MS}}$ fermion mass $\bar{m}_f(\mu)$. In a way, our work complements the paper by Sirlin and Zucchini [7] where the relation between the running $\overline{\text{MS}}$ quartic coupling and the pole mass of the Higgs boson has been evaluated at one loop in the SM. Our result is particularly useful in the context of GUT's, where one wishes to determine the implications of Yukawa-coupling unification at

the GUT scale for low-energy observables, which depend on the physical particle masses. To our knowledge, previous analyses have only included the familiar QCD term in the quark case. We found that the additional electroweak threshold corrections are relatively modest, below 1% in the low- M_H region, after the LL terms originating from the decoupling of weak physics at scales M_Z and m_t have been subtracted. At the other extreme, these threshold corrections have a quadratic high- M_H behavior and may reach values up to 3% at $M_H = 1$ TeV.

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APPENDIX: ONE-LOOP AMPLITUDES

Here, we list the residual amplitudes that enter the evaluations of $\bar{m}_f(\mu)$ and $\bar{h}_f(\mu)$ at one loop in the SM. Having verified that these two quantities are gauge independent, we may choose the 't Hooft–Feynman gauge. It is convenient to introduce the following shorthand notations: $\Delta(m^2) = \Delta - \ln(m^2/\mu^2)$, $w = M_W^2$, $z = M_Z^2$, $h = M_H^2$, $x = m_f^2$, $y = m_{f'}^2$, $v_f = (I_f - 2s_W^2 Q_f)/(2c_W s_W)$, and $a_f = I_f/(2c_W s_W)$, where I_f is the third component of weak isospin of f_L .

The sum of the vector and scalar components of the self-energy of an on-mass-shell fermion, f , may be written as [12]

$$\begin{aligned} \Sigma_V^f(x) + \Sigma_S^f(x) = & \frac{\alpha}{4\pi} \left\{ \left[\frac{3}{8s_W^2} \left(1 + \frac{x-y}{w} \right) + \frac{1}{8c_W^2} - 3(v_f^2 - a_f^2 + Q_f^2) \right] \Delta(x) \right. \\ & + \frac{1}{4s_W^2 w} \left[\frac{(x-y)^2 + w(x+y) - 2w^2}{2x} F(x, y, w) + \left(2x - \frac{h}{2} \right) F(x, x, h) \right. \\ & + \frac{(x-y)^3 + 4wx^2 + 3w^2(x+y) - 2w^3}{4x^2} \ln \frac{y}{w} + \left(\frac{x-3y}{2} + w \right) \ln \frac{x}{y} \\ & + \frac{h}{2} \left(3 - \frac{h}{2x} \right) \ln \frac{x}{h} + 4x - \frac{5}{2}y + \frac{w-h}{2} + \frac{y^2 + yw - 2w^2}{2x} \left. \right] \\ & + \left[-v_f^2 \left(2 + \frac{z}{x} \right) + a_f^2 \left(4 - \frac{z}{x} \right) \right] F(x, x, z) \\ & + \left[-v_f^2 \frac{z^2}{2x^2} + a_f^2 \left(2 + 3\frac{z}{x} - \frac{z^2}{2x^2} \right) \right] \ln \frac{x}{z} - v_f^2 \left(4 + \frac{z}{x} \right) + a_f^2 \left(6 - \frac{z}{x} \right) - 4Q_f^2 \left. \right\}, \end{aligned} \quad (\text{A1})$$

²The value $m_b = (4.72 \pm 0.05)$ GeV [26] has been determined by means of QCD sum rules and is consistent with a recent lattice computation [27], which has yielded $m_b = (4.94 \pm 0.15)$ GeV.

where

$$F(x, y, z) = \begin{cases} \frac{\sqrt{\lambda}}{x} \operatorname{arccosh} \frac{y+z-x}{2\sqrt{yz}} & \text{if } x \leq (\sqrt{y} - \sqrt{z})^2, \\ -\frac{\sqrt{-\lambda}}{x} \arccos \frac{y+z-x}{2\sqrt{yz}} & \text{if } (\sqrt{y} - \sqrt{z})^2 < x \leq (\sqrt{y} + \sqrt{z})^2, \\ -\frac{\sqrt{\lambda}}{x} \left(\operatorname{arccosh} \frac{x-y-z}{2\sqrt{yz}} - i\pi \right) & \text{if } x > (\sqrt{y} + \sqrt{z})^2, \end{cases} \quad (\text{A2})$$

with $\lambda = x^2 + y^2 + z^2 - 2(xy + yz + zx)$. For $x = y$, Eq. (A2) simplifies to

$$F(x, x, z) = \begin{cases} \frac{z}{x} \sqrt{1 - (4x/z)} \operatorname{arccosh} \left(\frac{1}{2} \sqrt{z/x} \right) & \text{if } x \leq \frac{z}{4}, \\ -\frac{z}{x} \sqrt{(4x/z) - 1} \arccos \left(\frac{1}{2} \sqrt{z/x} \right) & \text{if } x > \frac{z}{4}. \end{cases} \quad (\text{A3})$$

The bosonic and fermionic contributions to the self-energy of a W boson with zero invariant mass, $\Pi_{WW}(0) = \Pi_{WW}^{\text{bos}}(0) + \Pi_{WW}^{\text{fer}}(0)$, are given by [17]

$$\Pi_{WW}^{\text{bos}}(0) = \frac{\alpha w}{4\pi s_W^2} \left[\left(-2 + \frac{1}{c_W^2} \right) \Delta(w) + \left(2 + \frac{1}{c_W^2} - \frac{17}{4s_W^2} \right) \ln c_W^2 - \frac{3}{4} \frac{h}{w-h} \ln \frac{w}{h} - \frac{17}{4} + \frac{7}{8c_W^2} - \frac{h}{8w} \right], \quad (\text{A4})$$

$$\Pi_{WW}^{\text{fer}}(0) = \frac{\alpha}{8\pi s_W^2} \sum_{(U,D)} N_{(U,D)} \left\{ -m_U^2 \left[\Delta(m_U^2) + \frac{1}{2} \right] - m_D^2 \left[\Delta(m_D^2) + \frac{1}{2} \right] + \frac{m_U^2 m_D^2}{m_U^2 - m_D^2} \ln \frac{m_U^2}{m_D^2} \right\},$$

where (U, D) runs over all fermion doublets. Finally, the tadpole contribution reads [17]

$$T = \frac{e}{16\pi^2 s_W M_W} \left\{ w \left(3w + \frac{h}{2} \right) \Delta(w) + \frac{z}{2} \left(3z + \frac{h}{2} \right) \Delta(z) + \frac{3}{4} h^2 \Delta(h) + w^2 + \frac{z^2}{2} + \frac{h}{2} \left(w + \frac{z}{2} + \frac{3}{2} h \right) - 2 \sum_i N_i m_i^4 [\Delta(m_i^2) + 1] \right\}, \quad (\text{A5})$$

where i runs over all massive fermions.

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