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Gustafson integrals for $SL(2,\mathbb{C})$ spin magnet.

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Abstract. It was observed recently that the multidimensional Mellin–Barnes integrals (Gustafson's integrals) arise naturally in studies of the $SL(2,R)$ spin chain models. We extend this analysis to the noncompact $SL(2,\mathbb{C})$ spin magnets and obtain integral which generalizes Gustafson's integral to the complex case.

1. Introduction

It was shown by E. K. Sklyanin [1] that eigenfunctions of a monodromy matrix provide convenient bases for studies of spin chain magnets. In many cases these eigenfunctions can be constructed in explicit form. Rather (un)expectedly the most simple and elegant expressions arise for models with infinite dimensional Hilbert spaces. Such models include the so-called noncompact spin magnets and famous Toda chain. The eigenfunctions constructed with the help of Quantum Inverse Scattering Method [2–6] (QISM) are given by multi-variable integrals which have a hierarchical structure and can be represented as Feynman diagrams of certain type [7–11]. In many cases the calculation of scalar products between eigenfunctions or matrix elements can be, quite effectively, carried out on diagram level. For the Toda chain or the $SL(2,\mathbb{R})$ spin chains the result is given, as a rule, by a product of Euler's gamma functions depending on spectral parameters (separated variables). It was shown recently [12] that certain relations between scalar products and matrix elements for the $SL(2,\mathbb{R})$ spin chains take the form of multidimensional Mellin–Barnes integrals which are equivalent to the integrals derived by R. A. Gustafson in [13, 14].

In this work we apply the same program to the $SL(2,\mathbb{C})$ spin magnets and derive the counterparts of Gustafson's integral in the complex case. The eigenfunctions of the monodromy matrix for the $SL(2,\mathbb{C})$ magnet and the corresponding Sklyanin measures were obtained in [7, 15]. Using these results we derive an analog of the first Gustafson integral (Eq. (5.2) in Ref. [13]) by calculating matrix elements of the shift operator. It has exactly the same functional form as the Gustafson integral. The only changes amount to a modification of the integration measure and replacement of all Euler gamma functions (the gamma-function associated with the real field $\mathbb{R}$ in the classification of Ref. [16]) entering this integral by the gamma functions associated with the complex field $\mathbb{C}$ [16]. It allows one to suggest that all Gustafson's integrals admit the corresponding generalization.

The paper is organized in the following way. In sect. 2 we recall the formulation of the $SL(2,\mathbb{C})$ spin chain model and necessary facts from the QISM and SoV approach. In sect. 3 we calculate the relevant matrix elements and derive an analog of the first Gustafson integral. Elements of the diagrammatic technique are given in Appendix A. Finally, we present the Mellin–Barnes form of the star–triangle relation in Appendix B.

2. $SL(2,\mathbb{C})$ magnet

The quantum $SL(2,\mathbb{C})$ spin magnet is a generalization of the ordinary XXX_s spin chain. Dynamical variables of the model are spin generators which belong, at each site, to a unitary continuous principal series representation of the $SL(2,\mathbb{C})$ group. Such a representation, $T^{(s,\bar{s})}$, is determined by two complex spins, s and $\bar{s}$, which are parameterized by a (half)integer number n_s and a real number ν_s [17]

$$s=\frac{1+n_s}{2}+i\nu_s, \qquad \bar{s}=\frac{1-n_s}{2}+i\nu_s. \tag{1}$$

The group transformation takes the form

$$[T^{(s,\bar{s})}(g)\phi](z,\bar{z})=(a-cz)^{-2s}(\bar{a}-\bar{c}\bar{z})^{-2\bar{s}}\,\phi\left(\frac{dz-b}{a-cz},\frac{\bar{d}\bar{z}-\bar{b}}{\bar{a}-\bar{c}\bar{z}}\right), \tag{2}$$

where $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is a complex unimodular matrix. The transformation (2) is a unitary transformation on the space $L_2(\mathbb{C})$

$$\langle\phi\,|\psi\rangle=\int d^2z\,\overline{\phi(z,\bar{z})}\,\psi(z,\bar{z})\,, \qquad \langle\,T^{(s,\bar{s})}(g)\phi\,|\,T^{(s,\bar{s})}(g)\psi\,\rangle=\langle\phi\,|\psi\rangle. \tag{3}$$

The generators of infinitesimal transformations (spin operators) take the form

$$\begin{aligned} S_- &= -\partial_z, & S_0 &= z\partial_z + s, & S_+ &= z^2\partial_z + 2s\,z\,, \\ \bar{S}_- &= -\partial_{\bar{z}}, & \bar{S}_0 &= \bar{z}\partial_{\bar{z}} + \bar{s}, & \bar{S}_+ &= \bar{z}^2\partial_{\bar{z}} + 2\bar{s}\,\bar{z}\,. \end{aligned} \tag{4}$$

They are adjoint to each other up to a sign, $S_\alpha^\dagger = -\bar{S}_\alpha$, and satisfy the $sl(2)$ commutation relations

$$[S_+, S_-] = 2S_0, \qquad [S_0, S_\pm] = \pm S_\pm\,. \tag{5}$$

The anti-holomorphic generators satisfy exactly the same relations. Henceforth, if holomorphic and anti-holomorphic equations are the same we will write down only the holomorphic version.

2.1. L operators and monodromy matrices

The Hilbert space of the model is given by a direct product of N copies of $L_2(\mathbb{C})$,

$$\mathbb{H}_N = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_N, \qquad \mathbb{V}_k = L_2(\mathbb{C})\,, \qquad k = 1,\ldots,N. \tag{6}$$

We will consider only the homogeneous chains, i.e. the spins $s_k, \bar{s}_k$ are the same for all k, $s_k = s$, $\bar{s}_k = \bar{s}$. We also assume that $s - \bar{s} = n_s$ is an integer number †.

In the QISM approach one defines (at each site) the so - called L - operator

$$L_k(u) = \begin{pmatrix} u + iS_0^{(k)} & iS_-^{(k)} \\ iS_+^{(k)} & u - iS_0^{(k)} \end{pmatrix}, \qquad \bar{L}_k(\bar{u}) = \begin{pmatrix} \bar{u} + i\bar{S}_0^{(k)} & i\bar{S}_-^{(k)} \\ i\bar{S}_+^{(k)} & \bar{u} - i\bar{S}_0^{(k)} \end{pmatrix} \tag{7}$$

and constructs a monodromy matrix as a product of these operators,

$$T(u) = L_1(u)L_2(u)\ldots L_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}. \tag{8}$$

The anti-holomorphic monodromy matrix is given by the same expression with $L_k(u) \to \bar{L}_k(\bar{u})$. The spectral parameters $u, \bar{u}$ are two arbitrary complex numbers.

It is shown in the QISM [1, 4, 6] that the entries of the monodromy matrix form commuting operator families, i.e.

$$[A_N(u), A_N(v)] = 0, \quad [B_N(u), B_N(v)] = 0, \quad [C_N(u), C_N(v)] = 0, \quad [D_N(u), D_N(v)] = 0\,. \tag{9}$$

Moreover, in the case under consideration the operators A_N and D_N (B_N and C_N) are related by the inversion transformation [15]. In what follows we consider the operators A_N and B_N only. These operators commute with the total generators $S_\alpha = S_\alpha^{(1)} + \ldots + S_\alpha^{(N)}$ as follows

$$[S_0\,, A_N(u)] = 0\,, \qquad [S_-\,, B_N(u)] = 0. \tag{10}$$

Due to commutativity the operators $A_N(u)$, $\bar{A}_N(\bar{u})$ ($B_N(u)$, $\bar{B}_N(\bar{u})$) can be diagonalized simultaneously and their eigenfunctions do not depend on the spectral parameters $u, \bar{u}$. These eigenfunctions play a distinguished role in the QISM formalism and form the basis of the Sklyanin representation of Separated Variables [5]. For the $SL(2,\mathbb{C})$ magnet the eigenfunctions of B_N and A_N operators were constructed in Refs. [7, 15]. We present the explicit expressions for them in the next subsection.

† The final expression for the $SL(2,\mathbb{C})$ analog of the Gustafson integral does not depend at all on the spins $s, \bar{s}$.

2.2. SoV representation

Since by construction the operators $A_N(u)$ and $B_N(u)$ are polynomials of degree N and $N-1$ in u, respectively, their eigenvalues are polynomials in u as well. It turns out very convenient to label the eigenfunction by roots of the corresponding eigenvalue. We accept the following notations for the eigenfunctions – $\Psi_A(\boldsymbol{x}|\boldsymbol{z})$ and $\Psi_B(p,\boldsymbol{x}|\boldsymbol{z})$.

- The eigenfunction $\Psi_A(\boldsymbol{x}|\boldsymbol{z})$ satisfies the equations

$$\begin{aligned} A_N(u)\,\Psi_A(\boldsymbol{x}|\boldsymbol{z}) &= (u-x_1)\cdots(u-x_N)\,\Psi_A(\boldsymbol{x}|\boldsymbol{z})\,, \\ \bar{A}_N(\bar{u})\,\Psi_A(\boldsymbol{x}|\boldsymbol{z}) &= (\bar{u}-\bar{x}_1)\cdots(\bar{u}-\bar{x}_N)\,\Psi_A(\boldsymbol{x}|\boldsymbol{z})\,. \end{aligned} \tag{11}$$

  Here we introduced the shorthand notations $\boldsymbol{z} = \{z_1,\ldots,z_N\}$ and $\boldsymbol{x} = \{x_1,\ldots,x_N\}$. The separated variables take the form

$$x_k = -\frac{in_k}{2} + \nu_k\,, \qquad\qquad \bar{x}_k = \frac{in_k}{2} + \nu_k\,, \tag{12}$$

  where $\nu_k \in \mathbb{R}$ and $n_k \in \mathbb{Z}$, so that $\bar{x}_k = x_k^*$. .

- The eigenfunctions of the operator B_N are determined by the equations

$$\begin{aligned} B_N(u)\,\Psi_B(p,\boldsymbol{x}|\boldsymbol{z}) &= p(u-x_1)\cdots(u-x_{N-1})\,\Psi_B(p,\boldsymbol{x}|\boldsymbol{z})\,, \\ \bar{B}_N(\bar{u})\,\Psi_B(p,\boldsymbol{x}|\boldsymbol{z}) &= \bar{p}(\bar{u}-\bar{x}_1)\cdots(\bar{u}-\bar{x}_{N-1})\,\Psi_B(p,\boldsymbol{x}|\boldsymbol{z})\,, \end{aligned} \tag{13}$$

  where $\bar{p} = p^*$ and the separated variables $\boldsymbol{x} = \{x_1,\ldots,x_{N-1}\}$ have the form (12).

The eigenfunctions of both operators can be constructed recursively [7,15]. Namely, let us define two (layer) operators, $\Lambda_k(x)$ and $\widetilde{\Lambda}_k(x)$ which map functions of $k-1$ variables to functions of k variables. The operator $\Lambda_k(x)$ is an integral operator defined as follows [15]

$$\begin{aligned} \big[\Lambda_k(x)\Phi\big](z_1,\ldots,z_k) = r_k(x)\prod_{i=1}^{k-1}[z_i-z_{i+1}]^{1-2s} \\ \times\prod_{i=1}^{k-1}\int d^2w_i\,[w_i-z_i]^{s+ix-1}[w_i-z_{i+1}]^{s-ix-1}\;\Phi(w_1\,,\ldots\,,w_{k-1})\,. \end{aligned} \tag{14}$$

Here $[z]^\alpha \equiv z^\alpha\bar{z}^{\bar{\alpha}}$, the parameter x has the form (12) and the normalization factor $r_k(x)$ is given by

$$r_k(x) = \big(a(s+ix)a(\bar{s}-i\bar{x})\big)^{k-1}\,, \qquad\qquad a(\alpha) = \Gamma(1-\bar{\alpha})/\Gamma(\alpha). \tag{15}$$

The second layer operator reads

$$\widetilde{\Lambda}_k(x) \equiv [z_k]^{-s+ix}\Lambda_k(x) = \Lambda_k(x)[z_k]^{-s+ix}\,. \tag{16}$$

The eigenfunctions have the following form [15]

$$\Psi_B(p\,,\boldsymbol{x}|\boldsymbol{z}) = |p|^{N-1}\Lambda_N\,(x_1)\cdots\Lambda_2(x_{N-1})\,\mathrm{e}^{ipz_1+i\bar{p}\bar{z}_1}\,, \tag{17}$$

$$\Psi_A(\boldsymbol{x}|\boldsymbol{z}) = \widetilde{\Lambda}_N\,(x_1)\cdots\widetilde{\Lambda}_2(x_{N-1})\,\widetilde{\Lambda}_1(x_N)\,. \tag{18}$$

The normalization of the layer operators is chosen in such a way that they satisfy the exchange relation,

$$\Lambda_k(x_1)\Lambda_{k-1}(x_2) = \Lambda_k(x_2)\Lambda_{k-1}(x_1) \tag{19}$$

and similar for $\widetilde{\Lambda}$. This ensures that the eigenfunctions (17), (18) are symmetric function of the separated variables $\boldsymbol{x}$, see Ref. [7,15] for details.

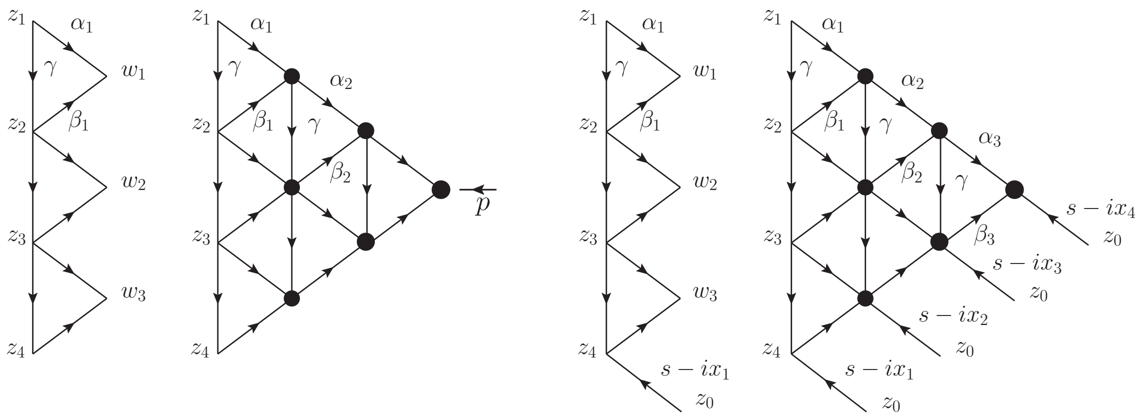

Figure 1. The diagrammatic representation for the layer operator $\Lambda_N(x_1)$ and the eigenfunction $\Psi_B(p,\boldsymbol{x}|\boldsymbol{z})$ for $N=4$ (two left figures) and the layer operator $\widetilde{\Lambda}_N(x_1)$ and the eigenfunction $\Psi_A(\boldsymbol{x}|\boldsymbol{z})$ for $N=4$ (two right figures). The line with arrow from w to z with the index α stands for the propagator, $[z-w]^{-\alpha}$, and the black dots – for the integration vertices. The indices have the following values: $\alpha_k = 1-s-ix_k$, $\beta_k = 1-s+ix_k$ and $\gamma = 2s-1$. For the operator $\widetilde{\Lambda}_N$ and the eigenfunction $\Psi_A(\boldsymbol{x}|\boldsymbol{z})$ the variable $z_0=0$.

It appears quite useful to represent the kernels of operators and eigenfunctions as Feynman diagrams. Several such diagrams are shown in Fig. 1. For more examples of the diagrammatic technique used in this paper see Ref. [7]. Taking into account that $\widetilde{\Lambda}_1(x_N) = [z_N]^{-s+ix_N}$ one can write the eigenfunction of the operator A_N in the equivalent form

$$\Psi_A(\boldsymbol{x}|\boldsymbol{z}) = \Lambda_{N-1}(x_1)\cdots\Lambda_2(x_{N-1})\,[z_N]^{-s+ix_1}\cdots[z_1]^{-s+ix_N}\,. \tag{20}$$

Using this representation and taking into account that the shift operator $T_{z_0} = e^{z_0 S_- + \bar z_0 \bar S_0}$ ($T_{z_0}\Phi(\boldsymbol{z}) = \Phi(z_1 - z_0,\ldots,z_N - z_0)$) commutes with the layer operators one derives

$$\begin{aligned} T_{z_0}\Psi_A(\boldsymbol{x}|\boldsymbol{z}) &= \Lambda_{N-1}(x_1)\cdots\Lambda_2(x_{N-1})\,[z_N - z_0]^{-s+ix_1}\cdots[z_1 - z_0]^{-s+ix_N} \\ &= \widetilde{\Lambda}_N^{(z_0)}(x_1)\cdots\widetilde{\Lambda}_2^{(z_0)}(x_{N-1})\,\widetilde{\Lambda}_1^{(z_0)}(x_N)\,, \end{aligned} \tag{21}$$

where $\Lambda_k^{(z_0)} = \Lambda_k(x)[z_k - z_0]^{-s+ix}$. The diagram for the function (21) is shown in Fig. 1.

The functions $\Psi_A(\boldsymbol{x}|\boldsymbol{z})$ and $\Psi_B(p\,,\boldsymbol{x}|\boldsymbol{z})$ being eigenfunctions of the self-adjoint operators form a complete orthogonal basis in the Hilbert space of the model

$$\int d^{2N}\boldsymbol{z}\,\overline{\Psi_A(\boldsymbol{x}'|\boldsymbol{z})}\,\Psi_A(\boldsymbol{x}|\boldsymbol{z}) = \left(\boldsymbol{\mu}_N^{(A)}(\boldsymbol{x})\right)^{-1}\delta_N(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{22}$$

$$\int d^{2N}\boldsymbol{z}\,\overline{\Psi_B(p'\,,\boldsymbol{x}'|\boldsymbol{z})}\,\Psi_B(p\,,\boldsymbol{x}|\boldsymbol{z}) = \left(\boldsymbol{\mu}_N^{(B)}(\boldsymbol{x})\right)^{-1}\delta^2(\vec p - \vec p\,')\,\delta_{N-1}(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{23}$$

Here $d^{2N}\boldsymbol{z} = d^2z_1\ldots d^2z_N$ and the delta function $\delta_N(\boldsymbol{x}-\boldsymbol{x}')$ is defined as follows:

$$\delta_N(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{N!}\sum_{s\in S_N}\delta^{(2)}(x_1 - x'_{s(1)})\cdots\delta^{(2)}(x_N - x'_{s(N)})\,, \tag{24}$$

where the sum goes over all permutations of N elements and

$$\delta^{(2)}(x-x') \equiv \delta_{nn'}\delta(\nu-\nu'). \tag{25}$$

The weight functions $\boldsymbol{\mu}_N(\boldsymbol{x})$ and $\boldsymbol{\mu}_N(p,\boldsymbol{x})$ are the so-called Sklyanin measures. They were calculated in Refs. [7, 15] and take the following form

$$\boldsymbol{\mu}_N^{(A)}(\boldsymbol{x}) = \frac{1}{N!}\frac{\pi^{-N^2}}{(2\pi)^N}\prod_{k<j}[x_k - x_j]\,, \tag{26}$$

$$\boldsymbol{\mu}_N^{(B)}(\boldsymbol{x}) = \frac{1}{(N-1)!}\frac{2\pi^{-N^2}}{(2\pi)^N}\prod_{k<j}[x_k - x_j]\,. \tag{27}$$

The completeness condition for the functions $\Psi_A(\boldsymbol{x}|\boldsymbol{z})$ and $\Psi_B(\boldsymbol{x}\,,\boldsymbol{p}|\boldsymbol{z})$ reads

$$\prod_{k=1}^{N}\delta^{(2)}(\vec{z}_k - \vec{z}_k^{\,\prime}) = \int \mathcal{D}_N\boldsymbol{x}\,\boldsymbol{\mu}_N^{(A)}(\boldsymbol{x})\Psi_A(\boldsymbol{x}|\boldsymbol{z})\,\overline{\Psi_A(\boldsymbol{x}|\boldsymbol{z'})}\,, \tag{28}$$

$$\prod_{k=1}^{N}\delta^{(2)}(\vec{z}_k - \vec{z}_k^{\,\prime}) = \int d^2p\int \mathcal{D}_{N-1}\boldsymbol{x}\,\boldsymbol{\mu}_N^{(B)}(\boldsymbol{x})\Psi_B(\boldsymbol{x}\,,\boldsymbol{p}|\boldsymbol{z})\,\overline{\Psi_B(\boldsymbol{x}\,,\boldsymbol{p}|\boldsymbol{z'})}\,, \tag{29}$$

where the symbol $\mathcal{D}_N\boldsymbol{x}$ stands for

$$\int \mathcal{D}_N\boldsymbol{x} = \prod_{k=1}^{N}\left(\sum_{n_k=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu_k\right) \tag{30}$$

and the sum goes over all integers. The relations (28), (29) can be easily checked for $N=1,2$. The proof for general N will be given elsewhere.

Let us note that for the case $N=1$ the eigenfunctions of the operators $B_1 = -i\partial_z$ and $A_1(u) = u + i\big(s + z\partial_z\big)$ are the exponential and power functions, respectively. Namely,

$$\Psi_B(p|z) = e^{ipz+i\bar{p}\bar{z}}, \qquad \Psi_A(x|z) = [z]^{ix-s}\,. \tag{31}$$

The orthogonality and completeness relations for the power functions read

$$\begin{aligned}\int d^2z\,[z]^{ix_1-s}([z]^{ix_2-s})^* &= \int d^2z\,[z]^{-1+i(x_1-x_2)} = 2\pi^2\delta^{(2)}(x_1-x_2)\,,\\ \int Dx[z_1]^{ix-s}([z_2]^{ix-s})^* &= [z_1]^{-s}([z_2]^{-s})^*\int Dx[z_1/z_2]^{ix} = 2\pi^2\delta^2(\vec{z}_1-\vec{z}_2)\end{aligned} \tag{32}$$

and agree with (22), (28).

3. Matrix elements and integrals identities

Let us calculate the matrix element of the shift operator between the eigenstates of the operator A_N. We define

$$T_{z_0}(\boldsymbol{x},\boldsymbol{x}') = \langle\Psi_A(\boldsymbol{x}')|T_{z_0}|\Psi_A(\boldsymbol{x})\rangle\,. \tag{33}$$

The calculation of (33) goes along the following lines: first, using the representations (18) and (21) we write the matrix element as follows

$$\langle\Psi_A(\boldsymbol{x}')|T_{z_0}|\Psi_A(\boldsymbol{x})\rangle = \widetilde{\Lambda}_1^{\dagger}(x_N')\widetilde{\Lambda}_2^{\dagger}(x_{N-1}')\cdots\widetilde{\Lambda}_N^{\dagger}(x_1')\,\widetilde{\Lambda}_N^{(z_0)}(x_1)\cdots\widetilde{\Lambda}_2^{(z_0)}(x_{N-1})\,\widetilde{\Lambda}_1^{(z_0)}(x_N)\,. \tag{34}$$

Second, representing the product $\widetilde{\Lambda}_N^{\dagger}(x_1')\,\widetilde{\Lambda}_N^{(z_0)}(x_1)$ as a Feynman diagram and simplifying it with the help of the identities (A.4), (A.5), (A.6) in Appendix A one gets

$$\widetilde{\Lambda}_N^{\dagger}(x')\,\widetilde{\Lambda}_N^{(z_0)}(x) = [z_0]^{i(x-x')}(-1)^{[s-ix]}\,q(x,x')\,(Q_{N-1}^{(z_0=0)}(x))^{\dagger}Q_{N-1}^{(z_0)}(x')\,, \tag{35}$$

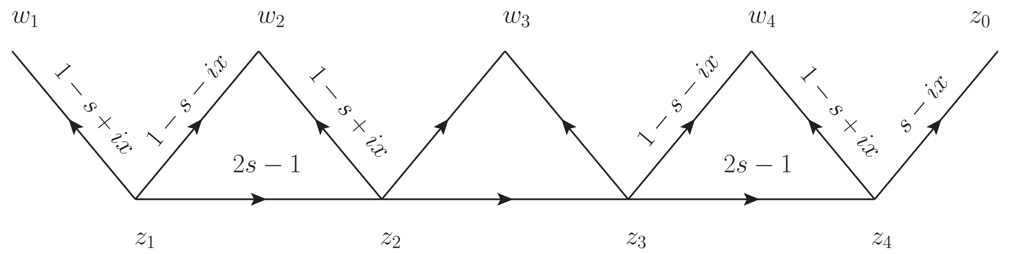

Figure 2. The diagrammatic representation for the operator $Q_N^{(z_0)}(x)$ for $N=4$. Note that the index of the rightmost arrow differs from others.

where the sign factor $(-1)^{[a]} \equiv (-1)^{a-\bar a}$ and the diagram for the operator $Q_N^{(z_0)}(x)$ is shown in Fig. 2. The factor $q(x,x')$ reads

$$q(x,x') = \pi a\big(1+i(x-x')\big)\frac{a(\bar s - i\bar x)}{a(s-ix')}\,. \tag{36}$$

Third, using the diagrammatic technique it is straightforward to check that

$$Q_k^{(z_0)}(x')\widetilde\Lambda_k^{(z_0)}(x) = q(x,x')\widetilde\Lambda_k^{(z_0)}(x)Q_{k-1}^{(z_0)}(x')\,. \tag{37}$$

This allows one to reduce the N-point scalar product to the $N-1$-point scalar product and obtain for the matrix element (33)

$$T_{z_0}(\boldsymbol{x},\boldsymbol{x}') = (-1)^{[A_X]}[z_0]^{i(X-X')}\prod_{k,j=1}^{N} q(x_k,x'_j)\,. \tag{38}$$

Here we introduced the notations

$$X=\sum_k x_k,\qquad \bar X=\sum_k \bar x_k,\qquad A_X=\sum_k (s-ix_k),\qquad \bar A_{\bar X}=\sum_k(\bar s - i\bar x_k).$$

The calculation of the scalar product between the eigenfunctions $\Psi_A(\boldsymbol{x}|\boldsymbol{z})$ and $\Psi_B(p\,,\boldsymbol{x}|\boldsymbol{z})$ follows exactly the same lines so we give the final answer only

$$\langle\Psi_B(p,\boldsymbol{u})|\Psi_A(\boldsymbol{x})\rangle = i^{[A_X]}\pi^N|p|^{-N-1}[p]^{A_X}\prod_{k=1}^{N}a(\bar s-i\bar x_k)\prod_{j=1}^{N-1}q(x_k,u_j)\,. \tag{39}$$

Note that the expressions (38) and (39) are symmetric functions of the separated variables as it should be. We also remark here that these equations have a striking resemblance to the analogous expressions in the spin chain models with the $SL(2,\mathbb{R})$ symmetry group, see Refs. [11,12].

The function $a(1+i(x-x'))$ entering (36) becomes singular for $x=x'$. Indeed,

$$\begin{aligned} a(1+i(x-x')) &= \frac{\Gamma(i(\bar x'-\bar x))}{\Gamma(1+i(x-x'))} \\ &= \frac{\Gamma(i(\nu'-\nu)+(n-n')/2)}{\Gamma(1-i(\nu'-\nu)+(n-n')/2)} = (-1)^{n-n'}\frac{\Gamma(i(\nu'-\nu)-(n-n')/2)}{\Gamma(1-i(\nu'-\nu)-(n-n')/2)}\,. \end{aligned} \tag{40}$$

Thus the function $a(1+i(x-x'))$ is singular only when $n=n'$ and $\nu=\nu'$. The divergency comes from the integration of propagator chains. It can be regularized by giving the variable ν a small imaginary part, i.e. $x=-in/2+i\nu$, $\bar x = in/2+i\nu$ where $\operatorname{Im}\nu>0$ (Note, that the variable x is related to the function on the right side of the scalar product). So from now on we assume, whenever it is necessary, that the parameters ν_k in Eq. (12) have a positive imaginary part.

3.1. Gustafson integrals for $SL(2,\mathbb{C})$

In this section we present a generalization of Gustafson's integrals (Eq. 5.2 in Ref. [13]) to the complex case. Using the completeness condition (29) for the B–system one can represent the matrix element (33) in the form

$$T_{z_0}(\boldsymbol{x},\boldsymbol{x}') = \int d^2p\, e^{-ipz_0-i\bar{p}\bar{z}_0} \int \mathcal{D}_{N-1}\boldsymbol{u}\, \boldsymbol{\mu}_N^{(B)}(\boldsymbol{u}) \overline{\langle \Psi_B(p,\boldsymbol{u})|\Psi_A(\boldsymbol{x}')\rangle} \langle \Psi_B(p,\boldsymbol{u})|\Psi_A(\boldsymbol{x})\rangle\,, \tag{41}$$

where we take into account that $T_{z_0}\Psi_B(p,\boldsymbol{u}) = e^{-ipz_0-i\bar{p}\bar{z}_0}\Psi_B(p,\boldsymbol{u})$. Substituting the expressions (38) and (39) into (41) one gets after some algebra

$$\frac{1}{(N-1)!}\left(\prod_{k=1}^{N-1}\sum_{n_k=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\nu_k}{2\pi}\right)\frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1}a(1+i(x_k-u_j))\,a(1+i(u_j-x'_k))}{\prod_{m<j}a(1+i(u_j-u_m))a(1+i(u_m-u_j))} =$$
$$= \frac{\prod_{k,j=1}^{N}a(1+i(x_k-x'_j))}{a(1+i(X-X'))}\,. \tag{42}$$

We recall here that the integration variables u_k take the values: $u_k = -in_k/2+\nu_k$, $\bar{u}_k = in_k/2+\nu_k$, where n_k is an integer and ν_k is a real number. The external parameters x_k, x'_k are

$$x_k = -\frac{im_k}{2}+\mu_k, \qquad \bar{x}_k = \frac{im_k}{2}+\mu_k, \qquad x'_k = -\frac{im'_k}{2}+\mu'_k, \qquad \bar{x}_k = \frac{im'_k}{2}+\mu'_k\,,$$

where m_k, m'_k are integers and μ_k and μ'_k are complex numbers such that $\operatorname{Im}\mu_k > 0$ and $\operatorname{Im}\mu'_k < 0$. It can be checked that for such a prescription the ν-poles of the functions $a(1+i(x_k-u_j))$ and $a(1+i(u_j-x'_k))$ are separated by the integration contour.

We also recall that the function $a(\alpha) \equiv a(\alpha,\bar{\alpha}) = \Gamma(1-\bar{\alpha})/\Gamma(\alpha)$, see Eq. (A.3), is a function of two complex variables, $\alpha,\bar{\alpha}$, such that $\alpha-\bar{\alpha}=n$. This function is related to the gamma function for the complex field $\mathbb{C}$ defined in [16]

$$\Gamma(\alpha,\bar{\alpha}) = i^{\alpha-\bar{\alpha}}\frac{\Gamma(\alpha)}{\Gamma(1-\bar{\alpha})} = i^{\alpha-\bar{\alpha}}a(1-\bar{\alpha})\,. \tag{43}$$

Thus, Eq. (42) is a direct analog of the first Gustafson integral (Eq. (5.2) in Ref. [13]) — the only difference consists in replacing the Euler gamma functions by the function (43) and the appropriate modification of the integration measure. In was shown in [12] that many of Gustafson's integrals arise in studies of matrix elements in the $SL(2,\mathbb{R})$ spin chain models. There is little doubt that such an analysis can be extended to the $SL(2,\mathbb{C})$ magnet and, therefore, it seem very plausible that many of Gustafson's integrals admit an extension to the complex case.

Acknowledgments

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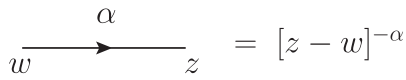
Appendix A. Diagram technique

This Appendix contains elements of a diagram technique which were used throughout the paper. The eigenfunctions and kernels of operators can be represented in the form of two - dimensional Feynman diagrams. The propagator is shown by the arrow directed from w to z with the index α attached to it



The propagator is given by the following expression

$$\frac{1}{[z-w]^{\alpha}} \equiv \frac{1}{(z-w)^{\alpha}(\bar z-\bar w)^{\bar\alpha}} = \frac{(\bar z-\bar w)^{\alpha-\bar\alpha}}{|z-w|^{2\alpha}} = \frac{(-1)^{\alpha-\bar\alpha}}{[w-z]^{\alpha}}\,, \tag{A.1}$$

where $\alpha-\bar\alpha = n_\alpha$ is integer. Performing the Fourier transform one defines the propagator in the momentum representation

$$\int d^2z e^{i(pz+\bar p\bar z)}[z]^{-\alpha} = \pi\, i^{\alpha-\bar\alpha}\, a(\alpha)\, [p]^{\alpha-1}\,, \tag{A.2}$$

where the function $a(\alpha)$ is defined by

$$a(\alpha) \equiv a(\alpha,\bar\alpha) = \frac{\Gamma(1-\bar\alpha)}{\Gamma(\alpha)}\,,\quad a(\bar\alpha) = \frac{\Gamma(1-\alpha)}{\Gamma(\bar\alpha)}\,,\quad a(\alpha,\beta,\gamma,\ldots) = a(\alpha)a(\beta)a(\gamma)\ldots \tag{A.3}$$

This function has the following properties

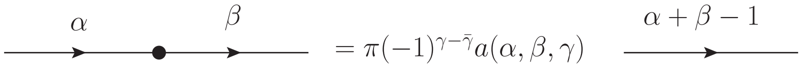
$$a(\alpha)a(1-\bar\alpha)=1\,,\quad a(1+\alpha) = -\frac{a(\alpha)}{\alpha\bar\alpha}\,,\quad a(\alpha)a(1-\alpha) = (-1)^{\alpha-\bar\alpha}\,,\quad a(\alpha) = (-1)^{\alpha-\bar\alpha}a(\bar\alpha)\,.$$

The evaluation of Feynman diagrams is based on their transformation with the help of certain rules

- Chain relation:

$$\int d^2w \frac{1}{[z_1-w]^{\alpha}[w-z_2]^{\beta}} = \pi(-1)^{\gamma-\bar\gamma}a(\alpha,\beta,\gamma)\frac{1}{[z_1-z_2]^{\alpha+\beta-1}}\,, \tag{A.4}$$

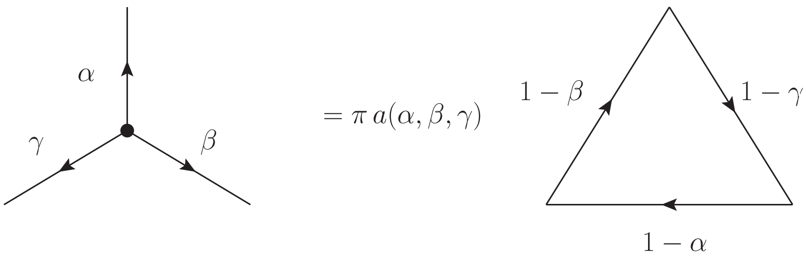
  where $\gamma = 2-\alpha-\beta,\ \bar\gamma = 2-\bar\alpha-\bar\beta$.



- Star - triangle relation:

$$\int d^2w \frac{1}{[z_1-w]^{\alpha}[z_2-w]^{\beta}[z_3-w]^{\gamma}} = \frac{\pi a(\alpha,\beta,\gamma)}{[z_2-z_1]^{1-\gamma}[z_1-z_3]^{1-\beta}[z_3-z_2]^{1-\alpha}}\,, \tag{A.5}$$

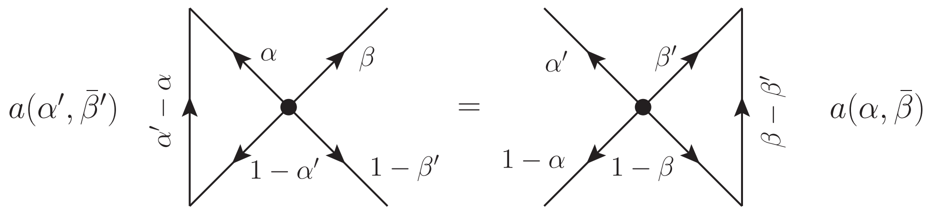
  where $\alpha+\beta+\gamma=2$ and $\bar\alpha+\bar\beta+\bar\gamma=2$.



- Cross relation:

$$\frac{1}{[z_1-z_2]^{\alpha'-\alpha}}\int d^2w\frac{a(\alpha',\bar\beta')}{[w-z_1]^{\alpha}[w-z_2]^{1-\alpha'}[w-z_3]^{\beta}[w-z_4]^{1-\beta'}}=$$
$$=\frac{1}{[z_3-z_4]^{\beta'-\beta}}\int d^2\zeta\frac{a(\alpha,\bar\beta)}{[w-z_1]^{\alpha'}[w-z_2]^{1-\alpha}[w-z_3]^{\beta'}[w-z_4]^{1-\beta}}\,, \tag{A.6}$$

where $\alpha+\beta=\alpha'+\beta'$.



The d-dimensional versions of these relations are an essential part of the modern technique of multi-loop calculations in QFT, see e.g. [19]. The d-dimensional star - triangle relation (for scalar propagators) was used to construct an integrable ("fishnet") model in arbitrary dimension [20] .

Appendix B. Mellin transform and star-triangle relation

In the simplest case of the spin chain with one site the transformation to the SoV representation related to the eigenfunctions of the operator A coincides with the Mellin - Barnes transformation. In this Appendix we show that the star - triangle identity obtained in [18] is equivalent to the star - triangle relation (A.5) in the Mellin - Barnes representation.

Let $f(z,\bar z)=f(r,\varphi)$ be a function on the complex plane. Combining the Fourier transform with respect to the angle variable φ and the Mellin transform with respect to the radial variable r one gets

$$f(z,\bar z)=\sum_{n=-\infty}^{\infty}e^{i\varphi n}\int_{-\infty}^{\infty}d\nu\,r^{-2i\nu-1}f_n(\nu)=\int D\alpha\,[z]^{-1/2-\alpha}\hat f(\alpha), \tag{B.1}$$

where $\alpha=i\nu-n/2$, $\bar\alpha=i\nu+n/2$ and

$$\hat f(\alpha)\equiv f_n(\nu)=\frac{1}{2\pi^2}\int_0^{2\pi}d\varphi\,e^{-i\varphi n}\int_0^{\infty}dr\,r^{2i\nu}f(r,\varphi)=\frac{1}{2\pi^2}\int d^2z[z]^{-1/2+\alpha}\,f(z,\bar z)\,. \tag{B.2}$$

These formulae are equivalent to the relations

$$\int d^2z\,[z]^{-1+\alpha}=2\pi^2\delta^{(2)}(\alpha)\,,\qquad\qquad \int D\alpha\,[z_1]^{-\alpha}[z_2]^{\alpha}=2\pi^2[z_1]\,\delta^2(\vec z_1-\vec z_2)\,, \tag{B.3}$$

which are nothing else as the orthogonality and completeness relations (32). In order to avoid misunderstanding we recall that

$$\int D\alpha=\sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty}\mathrm{d}\nu,\qquad\qquad \delta^{(2)}(\alpha-\alpha')\equiv\delta_{nn'}\delta(\nu-\nu').$$

Let us transform the star - triangle relation ($\beta_1+\beta_2+\beta_3=1$)

$$\int d^2w\frac{1}{[z_1-w]^{1-\beta_1}[z_2-w]^{1-\beta_2}[z_3-w]^{1-\beta_3}}=\frac{\pi a(1-\beta_1\,,1-\beta_2\,,1-\beta_3)}{[z_2-z_1]^{\beta_3}[z_1-z_3]^{\beta_2}[z_3-z_2]^{\beta_1}} \tag{B.4}$$

to the Mellin - Barnes form. First of all, making use of the chain relation (A.4) we derive the following representation for the propagator

$$[w-z]^{\beta-1} = \frac{1}{2\pi}a(1-\beta)\int D\alpha\,\frac{a(1/2+\beta/2-\alpha)}{a(1/2-\beta/2-\alpha)}\,[z]^{-1/2+\beta/2-\alpha}\,[w]^{-1/2+\beta/2+\alpha}\,, \tag{B.5}$$

Next, multiplying both sides of Eq. (B.4) by $\prod_{i=1}^{3}[z_i]^{-1/2-\beta_i/2+\alpha_i}$ and integrating over all variables z_i we obtain with the help of Eq. (B.3)

$$(2\pi^2)\,\pi^3\,\delta^{(2)}\left(\sum\alpha_i\right)\prod_{i=1}^{3}a(1-\beta_i)\frac{a\,(1/2+\beta_i/2-\alpha_i)}{a\,(1/2-\beta_i/2-\alpha_i)} =$$

$$= \pi^4\prod_{i=1}^{3}\int D\gamma_i\frac{a(1-\beta_i/2-\gamma_i)}{a(\beta_i/2-\gamma_i)}\delta^{(2)}(\gamma_3-\gamma_2-\alpha_1)\delta^{(2)}(\gamma_1-\gamma_3-\alpha_2)\delta^{(2)}(\gamma_2-\gamma_1-\alpha_3)\,. \tag{B.6}$$

Comparing the coefficient at the delta function $\delta^{(2)}\left(\sum\alpha_i\right)$ on both sides we get

$$2\pi\prod_{i=1}^{3}a(1-\beta_i)\frac{a\,(1/2+\beta_i/2-\lambda_{i,i+1})}{a\,(1/2-\beta_i/2-\lambda_{i,i+1})} = \int D\gamma\prod_{i=1}^{3}\frac{a(1-\beta_i/2-\gamma+\lambda_{i-1})}{a(\beta_i/2-\gamma+\lambda_{i-1})}\,, \tag{B.7}$$

where we put $\alpha_i=\lambda_i-\lambda_{i+1}$, $\gamma_i=-\lambda_{i-1}+\gamma$, $\lambda_{i+3}\equiv\lambda_i$. For a special choice of the parameters this relation is reduced to the star-triangle identity, Eq. (22) in Ref. [18].

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