Counting the number of master-integrals for sunrise-type Feynman diagrams via Mellin-Barnes representation

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Abstract

A number of irreducible master-integrals for L-loop sunrise-type and bubble Feynman diagrams with generic values of masses and external momenta are explicitly evaluated via Mellin-Barnes representation.

1 Introduction

The sunrise-type or watermelon Feynman diagram (see Fig. 1) is a one of the simplest Feynman diagrams which have been studied by physics community as well as by mathematicians over the last 50 years [2]- [14].

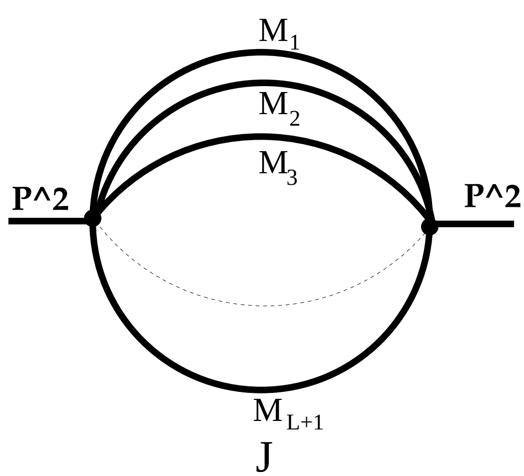

Figure 1: Sunrise Feynman Diagram

The diagrams of this type have a few different representations. Within dimensional regularization [1] in the momentum space it is defined as

$$J(\vec{M_j^2};\vec{\alpha_j};p^2) = \int \Pi_{j=1}^{L} \frac{d^n k_j}{[k_j^2 - M_j^2]^{\alpha_j}} \times \frac{1}{[(p-k_1-\cdots-k_L)^2 - M_{L+1}^2]^{\alpha_{L+1}}} \,, \tag{1}$$

where α_j are positive integers and M_j^2, p^2 are some (in general, complex) parameters and n is an (in general, non-integer) parameter of dimensional regularization. The parametric representation for this diagram has the following form [1]:

$$J \sim \Gamma\left(\alpha - \frac{n}{2}L\right) \int_0^\infty \Pi_{i=1}^{L+1} dx_i \frac{x_i^{\alpha_i - 1}}{\Gamma(\alpha_i)} \delta\left(1 - \sum_i x_i\right) \frac{F^{\frac{n}{2}L-\alpha}}{U(x)^{\frac{n}{2}(L+1)-\alpha}} , \tag{2}$$

where F and U are Symanzik polynomials [15] and $\alpha = \sum_{j=1}^{L+1} \alpha_j$. Using coordinate representation for Feynman propagator and performing an integration over the angle, it is easy to get [4] a one-fold integral representation for this type diagram:

$$J = \text{Const.} \times \frac{\Pi_{j=1}^{L+1} (M_j)^{\frac{d}{2}-\alpha_j}}{(\sqrt{-p^2})^{\frac{d}{2}-1}} \int_0^\infty dt t^{\sum_{k=1}^{L+1}\alpha_k - \frac{d}{2}L} \Pi_{j=1}^{L+1} K_{\frac{n}{2}-\alpha_j}(M_j t) J_{\frac{n}{2}-1}(t\sqrt{-p^2}) , \tag{3}$$

where J and K are the usual and modified Bessel functions, respectiverly, and Const. is some constant (see for details [10, 11]).

Applying the algorithm of [16], a Mellin-Barnes integral representation [17] for sunrise-type diagram can be deduced [6] (see Eq. (16)).

The aim of the present letter is to extend the approach described in [18, 19] to the multivariable case with reducible monodromy. As an illustration, the number of irreducible master-integrals of L-loop sunrise-type diagram is evaluated. Appendices A and B contain some additional details of the differential reduction technique.

2 Mellin-Barnes integral versus Horn hypergeometric function

Let us consider the function Φ defined through a K-fold Mellin-Barnes integral:

$$\Phi(A, B; \{C_k\}; z_1, \cdots, z_K) = \int \Pi_{j=1}^K dt_j \Gamma(-t_j) \Gamma(C_j - t_j) z_j^{t_j} \frac{\Gamma\left(A + \vec{t}\right)}{\Gamma\left(B - \vec{t}\right)} . \tag{4}$$

This function depends on $K+2$ discrete parameters, A, B and $\{C_j\}$, and K variables $z_1, \cdots, z_m$. Let us briefly remind some basic steps of the differential reduction procedure [20] in application to Mellin-Barnes integral. The differential contiguous relations for the function Φ follow directly from Mellin-Barnes representation (see Eq. (4) in [19]) and have the following form:

$$\Phi(A, B; \{\}, C_j + 1, \{\}; \vec{z}) = (C_j - \theta_j) \Phi(A, B; \{\}, C_j, \{\}; \vec{z}) , \tag{5a}$$

$$\Phi(A+1, B; \{C_k\}; \vec{z}) = (A + \sum_{j=1}^K \theta_j) \Phi(A, B; \{C_k\}; \vec{z}) , \tag{5b}$$

$$\Phi(A, B-1; \{C_k\}; \vec{z}) = (B - 1 - \sum_{j=1}^K \theta_j) \Phi(\vec{A}; B; \{C_k\}; \vec{z}) , \tag{5c}$$

[1] It could be also rewritten in the projective space [2, 3].

where

$$\theta_j = z_j \frac{d}{dz_j} \,, \quad j = 1, \cdots, m.$$

We denote the set of differential operators on the r.h.s. of Eq. (5) as $\mathbf{B}^+_{C_j,A,B}$, respectively. A linear system of partial differential equations (PDE) for the function Φ can be derived in two steps (see Eq. (3) in [19]): (I) at the first step we define the polynomials P and Q as:

$$\frac{P_j}{Q_j} = \frac{\phi(t_j+1)}{\phi(t_j)} \,.$$

These polynomials define the corresponding system of differential equations (step II):

$$L_j : \left(Q_j|_{t_j \to \theta_j} \frac{1}{z_j} \Phi = P_j|_{t_j \to \theta_j} \Phi \right) \,.$$

For the function under consideration we have:

$$\frac{P^\Phi_j}{Q^\Phi_j} = - \frac{\left(A + \sum_{j=1}^K t_j\right)\left(1 - B + \sum_{j=1}^K t_j\right)}{(1-C_j+t_j)(1+t_j)} \Rightarrow \tag{6a}$$

$$L^\Phi_j : (\theta_j - C_j)\,\theta_j \Phi = -z_j \left(\sum_{j=1}^K \theta_j + A \right) \left(\sum_{j=1}^K \theta_j + (1-B) \right) \Phi \,, \quad j = 1, \cdots, m. \tag{6b}$$

To get the full system of PDE for the function Φ, a prolongation procedure should be applied [21] (the prolongation procedure consists in applying new derivatives to the system of PDE) so that the differential system (6b) can be written in a Pfaffian form [2]:

$$d\phi = \Omega \phi \,, \tag{7}$$

where the matrix Ω depends only on the values of parameters and singular locus of differential system and vector-function ϕ is defined as $(\Phi, \theta_i \Phi, \theta_{ij}\Phi, \cdots, \theta_{j_1, j_2, \cdots, j_r}\Phi)$. The rank r of the matrix Ω at the point z_0 (in our case $z_0 = 0$) in Eq. (7) is equal to the number of independent solutions of the full system of differential equations (see Eq. (9) for function Φ).

In accordance with the algorithm described in [20], the differential operators b^-_j inverse to the operators defined by Eqs. (5) can be constructed so that

$$b^-_j \mathbf{B}^+_j \Phi(A, B; \vec{C}_j; \vec{z}_j) = \Phi(A, B; \vec{C}_j; \vec{z}_j) \,, \quad j = 1, \cdots, m \,.$$

The operators b^-_a are defined [3] up to modulo of full system of PDE. The differential reduction has the form of a product of several operators b^-_i and B^+_j and maximal power of derivatives

[2] The condition of complete integrability is valid.

[3] Due to relation $\theta_p \Phi(A, B; \vec{C}_j; \vec{z}_j) = -z_p \Phi\left(1+A, B-1; C_p - 1, \{C_j\}\right) \equiv -z_p B^+_A B^+_B b^-_C \Phi(A, B; \vec{C}_j; \vec{z}_j)$, not all operators b^-_c are independent.

$\theta_{i_1} \cdots \theta_{i_r}$ is equal to the dimension of the solution space. In a symbolic form this can be written as

$$\Phi(\vec{I} + \{A, B, \vec{C_j}\}; \vec{z_j}) = \left[Q_0 + \sum_{j=1}^{m} Q_j \theta_j + \sum_{\substack{i,j=1 \\ i<j}}^{m} Q_{ij} \theta_{ij} + \cdots \right] \Phi(A, B, \vec{C_j}; \vec{z_j}) \;, \tag{8}$$

where $\vec{I}$ is a set of integers, $\{A, B, \vec{C}\}$ is a set of parameters, and Q_i are some rational functions with respect to $\{z_i\}$ and $A, B, \{C_j\}$. The number of symmetric derivatives $\theta_{i_1, i_2, \cdots, i_m}$, where $i_1 < i_2 < \cdots < i_m$, is equal to $\frac{K!}{m!(K-m)!}$.

The solution of Eq. (6b) for a generic, non-integer set of parameters can be written as linear combination of Lauricella [21] function $F_C^{(K)}$ of K variables. For example, closing all contours of integration to the right (see [16] for details) we get:

$$\begin{aligned}
&\Phi(A, B, \{C_j\}; \{z_j\}) = \frac{\Gamma(A)}{\Gamma(B)} \Pi_{j=1}^{K} \Gamma(C_j) \times F_C^{(K)}(A, 1-B; \{1-C_j\}; \{-z_j\}) \\
&+ \sum_{j=1}^{K} (z_j)^{C_j} \frac{\Gamma(A+C_j)\,\Gamma(-C_j)}{\Gamma(B-C_j)} \Pi_{\substack{a=1 \\ a\neq j}}^{K} \Gamma(C_a) \\
&\quad \times F_C(A+C_j, 1-B+C_j; 1-C_1, \cdots, 1-C_{j-1}, 1+C_j, 1-C_k; \{-z_j\}) \\
&+ \sum_{j_1, j_2 = 1}^{K} (z_{j_1})^{C_{j_1}} \Gamma(-C_{j_1}) \times (z_{j_2})^{C_{j_2}} \Gamma(-C_{j_2}) \times \frac{\Gamma(A+C_{j_1}+C_{j_2})}{\Gamma(B-C_{j_1}-C_{j_2})} \\
&\quad \times \Pi_{\substack{a=1 \\ a \neq j_1, j_2}}^{K} \Gamma(C_a) \times F_C^{(K)}(A+C_{j_1}+C_{j_2}, 1-B+C_{j_1}+C_{j_2}; \{1-C_k\}, 1+C_{j_1}, 1+C_{j_2}; \{-z_j\}) \\
&+ \cdots \\
&+ \sum_{b=1}^{K} \frac{\Gamma(C_{j_b})}{(z_{j_b})^{C_{j_b}} \Gamma(-C_{j_b})} \times \frac{\Gamma\left(A + \sum_{j=1}^{K} C_j - C_b\right)}{\Gamma\left(B - \sum_{j=1}^{K} C_j - C_b\right)} \times \Pi_{a=1}^{K} (z_{j_a})^{C_{j_a}} \Gamma(-C_{j_a}) \\
&\quad \times F_C^{(K)}(A + \sum_{j=1}^{K} C_j - C_b, 1 - B + \sum_{j=1}^{K} C_j - C_n; 1 - C_b, \{1 + C_j\}; \{-z_j\}) \\
&+ \Pi_{j_a=1}^{K} (z_{j_a})^{C_{j_a}} \Gamma(-C_{j_a}) \frac{\Gamma\left(A + \sum_{j=1}^{K} C_j\right)}{\Gamma\left(B - \sum_{j=1}^{K} C_j\right)} \\
&\quad \times F_C^{(K)}(A + \sum_{j=1}^{K} C_j, 1 - B + \sum_{j=1}^{K} C_j; \{1 + C_j\}; \{-z_j\}) \;,
\end{aligned} \tag{9}$$

where Lauricella function $F_C^{(K)}$ is defined as

$$F_C^{(K)}(a, b; \{c_j\}; \{z_j\}) = \sum_{j_1, \cdots, j_k = 0}^{\infty} (a)_{j_1 + \cdots + j_K} (b)_{j_1 + \cdots + j_K} \Pi_{p=1}^{K} \frac{z_p^{j_p}}{j_p! (c_p)_{j_p}} \;, \tag{10}$$

and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol.

The holonomic [4] rank [5] of system of differential equations, Eq. (6b), for a generic set of parameters $\{A, B, C_j\}$ is equal to 2^K [21] (see also [22–25]). However, when parameters $A, B, \{C_i\}$ satisfy some linear combinations, the additional differential operators are generated so that the Puiseux type solution appear. The main questions are how to find the corresponding linear combination of parameters and how to define the minimal set of additional differential equations.

Our approach to these problems is based on studying the inverse differential operators: the exceptional case of parameters, when dimension of the solution space is reduced, corresponds to the condition that the denominators of functions Q_i entering in Eq. (8) are equal to zero for an arbitrary values of z_i [26]. The same recipe is valid (with some restrictions) in application to Mellin-Barnes integral [27] (which can be treated as a particular case of GKZ-hypergeometric system [28]). However, when Mellin-Barnes integral is expressible in terms of linear combination of Horn-type hypergeometric functions there is a more simple way to find conditions of reducibility and define the dimension of irreducible subspace of solution [18, 19]. In our case, the system of differential equations is irreducible when (see Appendix A):

$$F_C \quad : \quad \left\{ a \notin \mathbb{Z}, \quad b \notin \mathbb{Z}, \quad a - \sum_{s_1}^{r} c_p \notin \mathbb{Z}, \quad b - \sum_{s_2}^{r} c_p \notin \mathbb{Z} \right\} , \tag{11a}$$

$$\Phi \quad : \quad \left\{ A \notin \mathbb{Z}, \quad B \notin \mathbb{Z}, \quad A + \sum_{S_1}^{r} C_p \notin \mathbb{Z}, \quad B - \sum_{S_2}^{r} C_p \notin \mathbb{Z} \right\} , \tag{11b}$$

where s_i, S_j are any subsets of $\{c_i\}$ and $\{C_j\}$, respectivelyly, while the first line in Eq. (11) (F_C) corresponds to the set of exceptional values of parameters for hypergeometric function F_C [6], and the second line (Φ) defines the exceptional set of parameters for the function Φ. The conditions defined by Eq. (11b) are invariant with respect to a linear change of variables $\{t_j\} \to \{A_1 \pm \sum_{A \in \{1,\cdots,p\}} t_A\}$, in the Mellin-Barnes integral, Eq. (4).

In a similar manner we can consider the function Ψ defined as a K-fold Mellin-Barnes integral:

$$\Psi(A, D; \{C_k\}; z_1, \cdots, z_K) = \int \Pi_{j=1}^{K} dt_j \Gamma(-t_j) \Gamma\left(C_j - t_j\right) z_j^{t_j} \Gamma\left(A + \vec{t}\right) \Gamma\left(D + \vec{t}\right) \ . \tag{12}$$

[4] We accept the following definition of holonomic function [20]: A function is called holonomic if it satisfies a system of linear differential equations with polynomial coefficients whose solutions form a finite-dimensional vector space.

[5] The dimension of the space of solutions of system of differential equations near some generic point is called holonomic rank.

[6] This set of parameters coincides with condition that monodromy group of Appell function F_C is reducible [22]. For a recent results concerning evaluation of monodromy for GKZ-system see [29].

In this case we have:

$$\frac{P_j^\Psi}{Q_j^\Psi} = \frac{\left(A+\sum_{j=1}^K t_j\right)\left(D+\sum_{j=1}^K t_j\right)}{(1-C_j+t_j)(1+t_j)} \Rightarrow \tag{13a}$$

$$L_j^\Psi : (\theta_j - C_j)\,\theta_j \Psi = z_j \left(\sum_{j=1}^K \theta_j + A\right)\left(\sum_{j=1}^K \theta_j + D\right)\Psi\,, \quad j = 1, \cdots, m. \tag{13b}$$

The system of differential equations for function Ψ is irreducible when

$$\Psi \quad : \quad \left\{ A \notin \mathbb{Z}, \quad D \notin \mathbb{Z}, \quad A + \sum_{S_1}^r C_p \notin \mathbb{Z}, \quad D + \sum_{S_2}^r C_p \notin \mathbb{Z}, \quad \right\}, \tag{14}$$

and S_1 and S_2 are any subsets of $\{C_j\}$. The solution of Eq. (13b) for generic set of parameters can be written as linear combination of Lauricella function $F_C^{(K)}$ of K variables. For completeness, we present it here:

$$\begin{aligned}
&\Psi(A,D,\{C_j\};\{z_j\}) = \Gamma(A)\,\Gamma(D)\,\Pi_{j=1}^K \Gamma(C_j) \times F_C^{(K)}(A,D;\{1-C_j\};\{z_j\}) \\
&+ \sum_{j=1}^K (z_j)^{C_j}\Gamma(A+C_j)\,\Gamma(D+C_j)\,\Gamma(-C_j)\,\Pi_{\substack{a=1\\a\neq j}}^K \Gamma(C_a) \\
&\quad \times F_C(A+C_j, D+C_j; 1-C_1, \cdots, 1-C_{j-1}, 1+C_j, 1-C_k; \{z_j\}) \\
&+ \sum_{j_1,j_2=1}^K (z_{j_1})^{C_{j_1}}\Gamma(-C_{j_1}) \times (z_{j_2})^{C_{j_2}}\Gamma(-C_{j_2}) \times \Gamma(A+C_{j_1}+C_{j_2}) \times \Gamma(D+C_{j_1}+C_{j_2}) \\
&\quad \times \Pi^K_{\substack{a=1\\a\neq j_1,j_2}} \Gamma(C_a) \times F_C^{(K)}(A+C_{j_1}+C_{j_2}, D+C_{j_1}+C_{j_2}; \{1-C_k\}, 1+C_{j_1}, 1+C_{j_2}; \{z_j\}) \\
&+ \cdots \\
&+ \sum_{b=1}^K \frac{\Gamma(C_{j_b})}{(z_{j_b})^{C_{j_b}}\Gamma(-C_{j_b})} \times \Gamma\left(A + \sum_{j=1}^K C_j - C_b\right) \times \Gamma\left(D + \sum_{j=1}^K C_j - C_b\right) \\
&\quad \times \Pi_{a=1}^K (z_{j_a})^{C_{j_a}}\Gamma(-C_{j_a}) \times F_C^{(K)}(A + \sum_{j=1}^K C_j - C_b, D + \sum_{j=1}^K C_j - C_n; 1-C_b, \{1+C_j\}; \{z_j\}) \\
&+ \Pi_{j_a=1}^K (z_{j_a})^{C_{j_a}}\Gamma(-C_{j_a})\,\Gamma\left(A + \sum_{j=1}^K C_j\right)\Gamma\left(D + \sum_{j=1}^K C_j\right) \\
&\quad \times F_C^{(K)}(A + \sum_{j=1}^K C_j, D + \sum_{j=1}^K C_j; \{1+C_j\}; \{z_j\})\,.
\end{aligned} \tag{15}$$

3 L-loop sunrise diagram

The Mellin-Barnes representation for the diagram defined by Eq. (1) follows from the algorithm presented in [16] (see also [6]) and has the following form:

$$J^{(L)}(M_1^2,\cdots,M_{L+1}^2,\alpha_1,\cdots,\alpha_{L+1},p^2) = (p^2)^{\frac{n}{2}L-\alpha}[i^{1-n}\pi^{n/2}]^L$$
$$\times \int \left\{ \Pi_{j=1}^{L+1} dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2}-\alpha_j-t_j\right)}{\Gamma(\alpha_j)} \left(-\frac{M_j^2}{p^2}\right)^{t_j} \right\} \frac{\Gamma\left(\alpha-\frac{n}{2}L+\vec{t}\right)}{\Gamma\left(\frac{n}{2}(L+1)-\alpha-\vec{t}\right)} \,, \tag{16}$$

where

$$\alpha = \sum_{j=1}^{L+1}\alpha_j \,, \quad \vec{t} = \sum_{j=1}^{L+1} t_j \,,$$

and α_j, L are positive integers, M_j^2, p^2 are some (in general, complex) parameters and d is a parameter (in general, non-integer) of dimensional regularization [1].

Let us introduce variables, $z_j = \frac{M_j^2}{p^2}$, $j = 1,2,\cdots,L+1$, and define functions Φ_J as

$$\Phi_J = \Pi_{k=1}^{L+1} \frac{\Gamma(\alpha_j)}{[i^{1-n}\pi^{n/2}]^{\frac{n}{2}L}(p^2)^{\frac{n}{2}L-\alpha}} \times J^{(L)}(M_1^2,\cdots,M_{L+1}^2,\alpha_1,\cdots,\alpha_{L+1},p^2) \,. \tag{17}$$

After this redefinition, the results of Section 2 and of Appendix A are directly applicable to the analysis of the sunrise diagram. In particular, in application to L-loop sunrise diagram we have:

$$A = \alpha - \frac{n}{2}L \,, \quad B = \frac{n}{2}(L+1) - \alpha \,, \quad C_j = \frac{n}{2} - \alpha_j \,, \tag{18}$$

and the system of differential equations Eq. (6b) is irreducible when

$$\frac{n}{2}L - \sum_{S_1}\frac{n}{2} \notin \mathbb{Z} \,, \tag{19a}$$

$$\frac{n}{2}(L+1) - \sum_{S_2}\frac{n}{2} \notin \mathbb{Z} \,, \tag{19b}$$

where S_1, S_2 are any subsets of $1,\cdots,L+1$, $\mathbb{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$ and n is non-integer.

The following lemma represents the first step in our analysis:

Lemma

The holonomic rank H_J of the sunrise diagram with generic values of masses and momenta in the neighborhood of $\vec{z}=0$ is equal to

$$H_J = 2^{L+1} - 1 \,. \tag{20}$$

Proof: this Lemma automatically follows from Eq. (9): there are 2^{L+1} solutions, and one of them, the term proportional to $1/\Gamma(0)$, is explicitly equal to zero $\square$.

Among H_J solutions of the system of PDE related to the sunrise diagram, there are $L+1$ Puiseux-type solutions (see Appendix A). Let us explain this result by using an explicit hypergeometric representation defined by Eq. (9). This equation includes $L+1$ terms of the following type: $1/z_a^{c_a} \times \left[\Pi_{j=1}^{L+1} z_j^{c_j}\right] \times F_C^{(L+1)}(I, b; \{c_a\}; \{z_b\})$, where I is integer and $b, c_a \notin \mathbb{Z}$. When one of the upper parameters of a Horn-type hypergeometric function is a positive integer, the step-by-step application to this function of the inverse differential operator b_a^- gives rise to unit. As consequence, each of the above mentioned terms can be rewritten in a more simple form, $1/z_a^{c_a} \times \left[\Pi_{j=1}^{L+1} z_j^{c_j}\right]$. In this way, Eq. (9) includes $L+1$ linearly independent Puiseux-type solutions.

It was pointed out in [19] that Puiseux-type solution corresponds to a product of one-loop bubble diagrams. Then the following Theorem is valid:

Theorem

The number of irreducible master-integrals of L-loop sunrise diagram with generic values of masses and momenta is equal to

$$\boxed{N_J = 2^{L+1} - L - 2} \, . \tag{21}$$

For example, for $L = 1, 2, 3, 4, 5, 6$ we have $N_J = 1, 4, 11, 26, 57, 120$, correspondingly.

The result of differential reduction can be written in a more familiar form via propagators with dots [7]: the term without derivative in the r.h.s of Eqs. (8) corresponds to diagram itself, whereas terms with derivative(s) $\theta_i \Phi$ correspond to diagrams with dot(s). In application to sunrise diagram we have

$$J^{(L)}(\vec{M}, \vec{I} + \{\alpha\}; p^2) = \sum_{j=0}^{L-1} Q_j \vec{\partial}^j J^{(L)}(\vec{M}, \{1\}; p^2) + \sum_{j=1}^{L+1} P_j(\vec{M}) \, , \tag{22}$$

where Q_j and P_j are some rational functions and $\vec{\partial}^j$ means symmetric derivatives with respect to masses M_j^2: $\vec{\partial}^j \equiv \frac{\partial^j}{\partial M_{i_1}^2 \partial M_{i_2}^2, \cdots, \partial M_{i_j}^2}$ and $i_1 < i_2 < \cdots i_j$. Indeed, the number of symmetric derivatives $\vec{\partial}^j$ in the considered case is equal to $\frac{(L+1)!}{j!(L+1-j)!}$, so that

$$2^{L+1} - (L+2) = \sum_{j=0}^{L-1} \frac{(L+1)!}{j!(L+1-j)!} \, . \tag{23}$$

In other words, for the L-loop sunrise diagram, any propagator with two or more derivatives on the line is reducible to diagrams with propagator without derivatives and propagators with no more than one derivative on the lines and the number of the lines with one dot are less or equal to $L-1$. As illustration of this relation: at the two-loop level only first derivatives with respect to masses enter in the reduction procedure (in agreement with [30]), at three-loop level, the second symmetric derivatives with respect to masses $\frac{\partial^2}{\partial M_i^2 \partial M_j^2} J^{(3)}(M_i^2; \vec{1})$, where

[7] The derivative with respect to z_i can be rewritten as derivative with respect to mass M_i^2 and external momenta p^2: $z_i \frac{\partial}{\partial z_i} \equiv M_i^2 \frac{\partial}{\partial M_i^2} - p^2 \frac{\partial}{\partial p^2}$.

$i < j$ and $i, j = 1, 2, 3, 4$, are generated $(1, 4, 6$ terms), at four-loop level, the third symmetric derivatives with respect to masses $\frac{\partial^3}{\partial M_i^2 \partial M_j^2 \partial M_k^2} J^{(4)}(M_i^2; \vec{1})$, where $i < j < k$ and $i, j, k = 1, 2, 3, 4, 5$ are generated $(1, 5, 10, 10$ terms). In general, at the L-loop level, the symmetric derivatives up to order $L-1$ with respect to masses $\frac{\partial^J}{\partial M_{i_1} \partial M_{i_2} \cdots \partial M_{i_J}} J^{(L)}(\vec{M}_i^2; \vec{1})$ are generated.

The case of even n demands an extra analysis [32].

3.1 L-loop sunrise diagram with R massive lines

Let us consider the L-loop sunrise-diagram when only R lines have different masses:

$$J_R(\{M_R^2\}; \{\alpha_j\}, \{\beta_k\}; p^2) = \int \Pi_{j=1}^R \frac{d^n(k_1 \cdots k_L)}{[k_j^2 - M_k^2]^{\alpha_j} [k_{R+1}^2]^{\beta_1} \cdots [(p - k_1 - \cdots - k_L)^2]^{\beta_{L+1-R}}} . \tag{24}$$

The Mellin-Barnes representation for this diagram is

$$\begin{aligned} & J_R(\{M_R^2\}; \{\alpha_j\}, \{\beta_k\}; p^2) = (p^2)^{\frac{n}{2}L - \alpha - \beta} [i^{1-n} \pi^{n/2}]^L \\ & \times \left\{ \Pi_{k=1}^{L+1-R} \frac{\Gamma\left(\frac{n}{2} - \beta_k\right)}{\Gamma(\beta_k)} \right\} \frac{\Gamma\left(\beta - \frac{n}{2}(L-R)\right)}{\Gamma\left(\frac{n}{2}(L-R+1) - \beta\right)} \\ & \times \int \left\{ \Pi_{j=1}^R dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2} - \alpha_j - t_j\right)}{\Gamma(\alpha_j)} \left(-\frac{M_j^2}{p^2} \right)^{t_j} \right\} \frac{\Gamma\left(\alpha + \beta - \frac{n}{2}L + \vec{t}\right)}{\Gamma\left(\frac{n}{2}(L+1) - \alpha - \beta - \vec{t}\right)} , \end{aligned} \tag{25}$$

where

$$\alpha = \sum_{j=1}^R \alpha_j , \quad \beta = \sum_{j=1}^{L-R+1} \beta_j , \quad \vec{t} = \sum_{j=1}^R t_j .$$

In this case we have:

$$A = \alpha + \beta - \frac{n}{2}L , \quad B = \frac{n}{2}(L+1) - \alpha - \beta , \quad C_j = \frac{n}{2} - \alpha_j , \quad j = 1, \cdots, R.$$

Applying the procedure described in Appendix A and results of the previous section, we got that the number $N_{J,R}$ of irreducible master-integrals of L-loop sunrise diagram with R massive lines ($R \leq L$) is equal to

$$\boxed{N_{L,R} = 2^R - \delta_{0,L-R}} . \tag{26}$$

As follows from this relation, the sunrise diagram with R massive and two or more massless lines are irreducible and its holonomic rank near $\vec{z} = 0$ coincides with holonomic rank of hypergeometric functions $F_C^{(R)}$ with irreducible monodromy.

There is one dimensional invariant subspace (Puiseux-type solution) when sunrise diagram has one massless line [8] .

Example:

$$N_{1,1} = 1 , \quad N_{2,2} = 3 , \quad N_{2,1} = 2 , \quad N_{3,3} = 7 , \quad N_{3,2} = 4 , \quad N_{3,1} = 2 .$$

[8] At the two-loop level it was shown explicitly by O.Tarasov in [30] (see also discussion in [31]).

4 L-loop bubble

In a similar manner let us consider the L-loop bubble diagram (see Fig. 2) defined as

$$B(\{M_{L+1}^2\};\{\alpha_{L+1}\}) = \int \frac{d^n(k_1\cdots k_L)}{[k_1^2-M_1^2]^{\alpha_1}\cdots[k_L^2-M_L^2]^{\alpha_L}[(k_1-\cdots-k_L)^2-M_{L+1}^2]^{\alpha_{L+1}}} \ . \tag{27}$$

where α_j are integer.

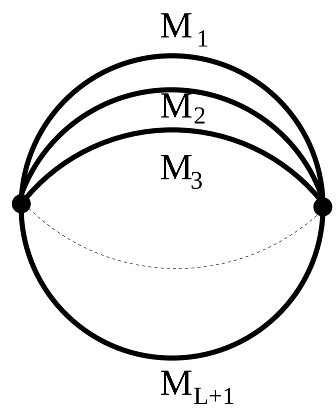

Figure 2: Bubble Feynman Diagram

The Mellin-Barnes representation is

$$B(M_1^2,\cdots,M_{L+1}^2;\alpha_1,\cdots,\alpha_{L+1}) = (-M_{L+1}^2)^{\frac{n}{2}L-\alpha} \times \frac{[i^{1-n}\pi^{n/2}]^L}{\Gamma\left(\frac{n}{2}\right)\Gamma(\alpha_{L+1})}$$

$$\times \int \left\{ \Pi_{j=1}^L dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2}-\alpha_j-t_j\right)}{\Gamma(\alpha_j)} \left(\frac{M_j^2}{M_{L+1}^2}\right)^{t_j} \right\} \Gamma\left(\alpha+\alpha_{L+1}-\frac{n}{2}L+\vec{t}\right) \Gamma\left(\alpha-\frac{n}{2}(L-1)+\vec{t}\right) \ , \tag{28}$$

where

$$\alpha = \sum_{j=1}^L \alpha_j \ , \quad \vec{t} = \sum_{j=1}^L t_j \ .$$

Let us introduce variables $z_j = -\frac{M_j^2}{M_{L+1}}$, $j=1,\cdots,K$, and define functions Ψ_B as

$$\Psi_B = \Pi_{k=1}^{L+1}\Gamma(\alpha_j) \times \frac{\Gamma\left(\frac{n}{2}\right)}{[i^{1-n}\pi^{n/2}]^{\frac{n}{2}L}(-M_{L+1}^2)^{\frac{n}{2}L-\alpha}} \times B(M_1^2,\cdots,M_{L+1}^2,\alpha_1,\cdots,\alpha_{L+1}) \ .$$

After this redefinition, the results of Section 2 are applicable to the analysis of bubble diagrams. In this case we have:

$$A = \alpha+\alpha_{L+1}-\frac{n}{2}L \ , \quad D = \alpha-\frac{n}{2}(L-1) \ , \quad C_j = \frac{n}{2}-\alpha_j \ ,$$

In this case (we treat n as non-integer parameter) there are $L+1$ Puiseux-type solutions (see Appendix A and section 3), and the number N_B of irreducible master-integrals for L-loop bubble with an generic values of masses is equal to:

$$\boxed{N_B = 2^L - L - 1} \,. \tag{29}$$

For example, for $L = 1,2,3,4,5,6$ there are $N_B = 0,1,4,11,26,57$ nontrivial master-integrals. Let us explain the zero result for $L = 1$. The one-loop tadpole A_0 is treated as constant in framework of differential algebra (see discussion in [18, 19]).

Remark 1: The result, Eq. (29), can be derived from Eq. (21) by consideration of bubble-diagram as sunrise diagram with external momenta equal to zero (which effectively reduces the number of independent variables by one).

Remark 2: In contrast to the sunrise diagram, the reduction for the bubble diagrams with $L \geq 3$ does not have completely symmetric structure with respect to mass derivatives. At the three-loop level this statement was confirmed in [33]. In the framework of IBP relations [34] there is so called {`dim`}-relation (see discussion in [35]) connecting the diagram (27) without derivative with a linear combination of diagrams with the first derivative:

$$\left\{ \sum_{j=1}^{L+1} M_j^2 \frac{\partial}{\partial M_j^2} - \left(\frac{n}{2} L - \sum_{j=1}^{L+1} \alpha_j \right) \right\} B(M_1^2, \cdots, M_{L+1}^2; \alpha_1, \cdots, \alpha_{L+1}) = 0 \,, \tag{30}$$

so that

$$B(\{M_{L+1}^2\}; \vec{1}) = \frac{1}{\left[\frac{n}{2}L - (L+1)\right]} \left(\sum_{j=1}^{L+1} B(\{M_{L+1}^2\}; \vec{1} + \vec{e_j}) \right) \,, \tag{31}$$

where e_j is unit vector with unity in the j-th place.

4.1 L-loop bubble when only R lines are massive

Let us consider the L-loop bubble diagram for the case when only R lines have different masses:

$$B^R(\{M_R^2\}; \{\alpha_j\}, \{\beta_k\}) = \int \frac{d^n(k_1 \cdots k_L)}{[k_1^2 - M_1^2]^{\alpha_1} \cdots [k_R^2 - M_R^2]^{\alpha_R} [k_{R+1}^2]^{\beta_1} \cdots [(k_1 - \cdots - k_L)^2]^{\beta_{L+1-R}}} \,. \tag{32}$$

The Mellin-Barnes representation for this diagram is

$$\begin{aligned} B^R(M_1^2, \cdots, M_R^2, \{\alpha_j\}, \{\beta_k\}) = {} & (-M_R^2)^{\frac{n}{2}L - \alpha - \beta} \frac{[i^{1-n}\pi^{n/2}]^L}{\Gamma\left(\frac{n}{2}\right)\Gamma(\alpha_{L+1})} \times \left\{ \Pi_{k=1}^{L+1-R} \frac{\Gamma\left(\frac{n}{2} - \beta_k\right)}{\Gamma(\beta_k)} \right\} \\ & \times \frac{\Gamma\left(\beta - \frac{n}{2}(L-R)\right)}{\Gamma\left(\frac{n}{2}(L-R+1) - \beta\right)} \times \int \left\{ \Pi_{j=1}^{R-1} dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2} - \alpha_j - t_j\right)}{\Gamma(\alpha_j)} \left(\frac{M_j^2}{M_R} \right)^{t_j} \right\} \\ & \times \Gamma\left(\alpha + \beta - \frac{n}{2}(L-1) + \vec{t}\right) \Gamma\left(\alpha + \beta + \alpha_R - \frac{n}{2}L + \vec{t}\right) \,, \end{aligned} \tag{33}$$

where

$$\alpha = \sum_{j=1}^{R-1} \alpha_j \;, \quad \beta = \sum_{j=1}^{L-R+1} \beta_j \;, \quad \vec{t} = \sum_{j=1}^{R-1} t_j \;.$$

In terms of notations of Section 2 we have:

$$A = \alpha + \beta - \frac{n}{2}(L-1) \;, \quad D = \alpha + \beta + \alpha_R - \frac{n}{2} L \;, \quad C_j = \frac{n}{2} - \alpha_j \;.$$

In this case (we treat n as non-integer parameter) there are $L+1$ Puiseux-type solutions (see Appendix A and section 3) and the number $NB_{L,R}$ of irreducible master-integrals of L-loop bubble with R massive lines ($R < L$) is equal to

$$\boxed{NB_{L,R} = 2^{R-1} - \delta_{0,L-R-1}} \;. \tag{34}$$

Example:

$$NB_{2,2} = 1 \;, \quad NB_{3,3} = 3 \;, \quad NB_{3,2} = 2 \;, \quad NB_{4,4} = 7 \;, \quad NB_{4,3} = 4 \;, \quad NB_{4,2} = 2 \;.$$

5 Independent verification

Fortunately, there are a few others ways to cross check our results, Eqs. (21),(26),(29),(34). One is based on the reduction of Feynman diagrams (sunrise/bubble) to some basis by using the IBP relations [34] (however, there is no guarantee, that free available programs would do the complete reduction, see discussion in [14, 36, 37]).

Another way for cross-checking of our result is application of Lee-Pomenransky approach [38] (see also discussion in [39]). The algorithm is based on counting the critical points of the sum of Symanzik polynomials $F + U$ defined by Eq. (2). It was pointed out by the authors of [38] that there are situations when program does not reproduce the correct number of master-integrals. Here, we present a pedagogical example, when differential reduction allows to predict and get an algebraic relation between two master-integrals [36] which is not predictable by the algorithm described in [38] or by results of **LiteRed** [40]. Let us consider the two-loop sunrise Feynman diagram in an arbitrary kinematics. The corresponding Symanzik polynomial has the following form:

$$G \equiv F + U = z_1 z_2 + z_1 z_3 + z_2 z_3 + z_1 z_2 z_3 p^2 - (z_1 z_2 + z_1 z_3 + z_2 z_3)\left(z_1 M_1^2 + z_2 M_2^2 + z_3 M_3^2\right) \;.$$

In this case, there are eight critical points, defined by condition $\partial_{z_i} G = 0, i = 1,2,3$. Four of them are trivial, $G(\{q_1, q_2, q_3, q_4\}) = 0$, where $q_1 = (1/M_1^2, 0, 0), q_2 = (0, 1/M_2^2, 0), q_3 = (0, 0, 1/M_3^2), q_4 = (0,0,0)$. The remaing four points are algebraically independent (for a generic set of mass and momenta) so that there are four independent master-integrals (in agreement with the result of [30]). The number and the values of critical points do not depend on the value of α_i (power of propagators) and the dimension of space-time n, and the product of one-loop bubble diagrams does not enter in counting of master-integrals.

Let us consider another particular case, the two-loop on-shell diagram (it is denoted as **J011** in [36]): $M_3 = 0$ and all other masses on-mass shell: $M_1^2 = M_2^2 = p^2 = 1$. In this case there are six non-degenerate [9] critical points,

$$
\begin{aligned}
&q_1 = (0,0,0), \quad q_2 = (1,0,0), \quad q_3 = (0,1,0), \\
&q_4 = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right), q_5 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right), \quad q_6 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right),
\end{aligned}
$$

and

$$
G(q_1) = G(q_2) = G(q_3) = 0; \quad G(q_4) = G(q_5) = G(q_6) = -\frac{4}{27}.
$$

The index [10] of points $q_1 = q_2 = q_3$ is equal to 2 and the index of points $q_4 = q_5 = q_6$ is equal to 1. In accordance with the criteria suggested in [38], there are two independent critical points q_4, q_6 and there are two master-integrals which are not reducible to the product of one-loop bubble diagrams. However, it was shown in [36,42] that there is only one non-trivial master-integral in this case (the second one is a product of Γ-functions only [11]).

The difference between differential reduction technique and counting the critical points is related to the treatment of values of propagators and space-time dimension. In framework of differential reduction technique one can consider the parameters α_i (powers of propagators) as well as the dimension of space-time n as non-integer and, as consequence, the number of additional differential equations for Mellin-Barnes integral would be different in this case. In Lee-Pomeransky approach, the dimension of space-time and power of propagators do not enter the analysis. Nevertheless, when α_i are integer, the results of our evaluations and results of application of **Mint** should be equal. Roman Lee kindly agreed to cross check our results by help of his packages [38,40] and got the full agreement with our expressions up to $L = 5$ (for the sunrise) and up to $L = 6$ (for the bubble).

6 Conclusion

The method of counting the master-integrals described in [18,19] was applied to the multi-variable case with a reducible monodromy. In contrast to our previous considerations [36,41], the case when a few additional differential equations are generated simultaneously has been analysed in the present paper. Our technique is based on getting a system of linear (partial) differential equations for a Feynman diagram via its Mellin-Barnes representation; it does not explore IBP relations [34]. The crucial point in our analysis is the evaluation of dimension of the space of nontrivial solutions of the linear system of PDE related to Mellin-Barnes integral. This procedure is performed by using the differential reduction technique. As an

[9] The corresponding Hessian matrix is non-singular at these points.

[10] The index of critical point is the dimension of the negative eigenspace of the corresponding Hessian at this point.

[11] Roman Lee make the following comment on this subject: since algebraic relation between master-integrals established in [14,36] does not follow from IBP relations related to the sunrise diagram [42] our results do not contradict the algorithm described in [38].

application of this technique, the number of irreducible master-integrals for L-loop sunrise-type and bubble Feynman diagrams with generic set of masses and momenta was evaluated, see Eqs. (21),(26),(29),(34).

As a by-product, we have got the following interesting consequences of our analysis: (i) as follows from Eq. (22), the basis for L-loop sunrise-type diagrams includes $L+1$ product of one-loop tadpole diagrams and watermelon diagrams with symmetric derivatives with respect to masses $\frac{\partial^J}{\partial M_{i_1}^2 \partial M_{i_2}^2 \cdots \partial M_{i_J}^2} J^{(L)}(\vec{M}_j^2; \vec{1})$, where $J = 0, 1, \cdots, L-1$ and $i_1 < i_2 < \cdots < i_J$. In other words, only watermelon diagrams with powers of propagators equals to one or two ($\alpha_j \leq 2$) and with number of lines with one dot not exceeding $L-1$ enter in the basis; (ii) as follows from Eq. (26) the sunrise diagram with R massive and two or more massless lines are irreducible and its holonomic rank near $\vec{z} = 0$ coincides with holonomic rank of hypergeometric functions $F_C^{(R)}$ with irreducible monodromy. The last property is relevant for analysis of special functions generated by ε-expansion of multiloop watermelon diagrams [11, 13, 43].

Acknowledgments
We are grateful to V. Bytev for discussion of corresponding problems related with differential reduction and R. Lee for crosschecking some of our results. We thank Andrei Davydychev for useful discussion and for careful reading of the manuscript. This work was supported in part by the German Research Foundation DFG through the Collaborative Research Centre No. 676 *Particles, Strings and the Early Universe—The Structure of Matter and Space-Time.*

A What is happen when monodromy is reducible

Here we present a short algebraic deviation of condition, Eq. (11), when the system of differential equations related to Mellin-Barnes integral, Eq. (4), are irreducible. We follow an idea presented in [44] (see also [45]).

Let Φ be the set of solutions for the system of linear differential operators L_j^Φ defined by Eq. (6b):

$$L_j^\Phi(A, B, \vec{C}) : \left[(\theta_j - C_j)\, \theta_j + z_j \left(\sum_{j=1}^K \theta_j + A \right) \left(\sum_{j=1}^K \theta_j + (1-B) \right) \right] , \quad j = 1, \cdots, K.$$

Let $S(A, B, \vec{C})$ denote the local solution space of operators $L_j^\Phi(A, B, \vec{C})$ around some point z_0. The contiguous differential operators B^+_{A,B,C_j} defined by Eq. (5), map the solution space $S(A, B, \vec{C})$ to the solution space $S(A \pm I_1, B \pm I_2, \vec{C} \pm \vec{I}; \vec{z})$, where $\{I_a\}$ are set of integers. When monodromy is reducible there is a monodromy invariant subspace (invariant under action of monodromy) in the space of solutions. In this case the contiguous differential operators B^+_{A,B,C_j} have a non-trivial kernel and it is necessary to evaluate their dimension.

Let us consider the equation $B_A^+ \Phi = 0$ where B_A^+ is defined by Eq. (5b). Then the system of equations defined by Eq. (6b) reduces to

$$L_j^\Phi(A, B, \vec{C})\Phi \equiv (\theta_j - C_j)\, \theta_j \Phi \equiv 0 , \quad j = 1, \cdots, K. \tag{35}$$

When $C_j \neq 0$ the solution Φ_0 of Eqs. (35) has the following form:

$$\Phi_0 = c_0 + \sum_{i=1}^{K} c_i z_i^{C_i} + \sum_{\substack{i,j=1 \\ i<j}}^{K} c_{i,j} z_i^{C_i} z_j^{C_j} + \cdots + \text{Const.} \times \Pi_{i=1}^{K} z_i^{C_i} \ . \tag{36}$$

Applying operator B_A^+ to function Φ_0 we get:

$$\begin{aligned} B_A^+ \Phi_0 \equiv 0 \ &= \ A c_0 + \sum_{i=1}^{K} c_i (A + C_i) z_i^{C_i} + \sum_{\substack{i,j=1 \\ i<j}}^{K} c_{i,j} (A + C_i + C_j) z_i^{C_i} z_j^{C_j} + \cdots \\ &+ \left(\Pi_{i=1}^{K} z_i^{C_i}\right) \times \sum_{a=1}^{K} \frac{\tilde{c}_a}{z_a^{C_a}} \left(A + \sum_{j=1}^{K} C_j - C_a \right) + \text{Const.} \times \Pi_{i=1}^{K} z_i^{C_i} \left(A + \sum_{j=1}^{K} C_j \right) , \end{aligned} \tag{37}$$

where $c_i, c_{i,j}, \cdots, \tilde{c}_i$ are some constants. Under conditions that

$$A, C_j, \notin \mathbb{Z} \ , \quad A + \sum_{j=1}^{K} C_j - C_a = 0 \ , \quad a = 1, \cdots, K,$$

there is an invariant subspace of dimension K for operator B_A^+.

Similar consideration can be done also for operator B_B^+ defined by Eq. (5c). In particular, it is easy to show that under conditions that

$$B, C_j \notin \mathbb{Z} \ , \quad B + \sum_{j=1}^{K} C_j = 0 \ ,$$

there is one-dimensional invariant subspace.

In application to the Lauricella function $F_C^{(K)}(A, B; \{C_j\}; \vec{z})$ of K-variables our analysis is equivalent to the following Lemma A:

Lemma A

Under conditions that

$$\{A, B, C_j\} \notin \mathbb{Z} \ , \quad B - \sum_{j=1}^{K} C_j \in \mathbb{Z} \ , \quad A + \sum_{j=1}^{K} C_j - C_a \in \mathbb{Z} \ , \quad a = 1, \cdots, K \ , \tag{38}$$

there is $K+1$ dimension invariant subspace (Puiseux type solutions) so that the dimension of the space of non-trivial solutions is equal to

$$N_F = 2^K - (K+1) \ , \tag{39}$$

and the differential reduction in application to the hypergeometric function $F_C^{(K)}(A, B; \{C_j\}; \vec{z})$ of K-variables has the following form:

$$F_C^{(K)}(\vec{I} + \{A, B, C_j\}; \vec{z}) = \sum_{j=0}^{K-2} \vec{Q}_j \vec{\theta}^j F_C^{(K)}(A, B; \{C_j\}; \vec{z}) + \sum_{j=1}^{K+1} P_j(\vec{z}) \ , \tag{40}$$

where $\vec{I}$ are set of integers, $\vec{Q}_j, P_j$ are some rational functions (some of the P_j may be equal to zero) and $\vec{\theta}^j$ means $\theta_{\substack{i_1,i_2,\cdots,i_j \\ i_1<i_2<\cdots i_p}}$.

The number of symmetric derivatives $\theta_{i_1,i_2,\cdots,i_m}$, where $i_1 < i_2 < \cdots < i_m$, is equal to $\frac{K!}{m!(K-m)!}$ and the number of terms entering in the reduction procedure, Eq. (40), is equal to $\sum_{j=0}^{K-2} \frac{K!}{j!(K-j)!} = 2^K - (K+1)$ that coincides with Eq. (39). Consequently, the highest differential operators – one of order K,$\theta_{i_1,i_2,\cdots,i_K}$, and K operators of order $K-1$, $\theta_{i_1,i_2,\cdots,i_{K-1}}$ – are expressible in terms of low dimensional operators.

B L-loop V-type diagram

In the recent analysis of getting of the full set of IBP relations for complete reduction of Feynman diagram to a minimal set of master-integrals [14, 19, 42, 46] the Feynman diagram of V-type (see Fig. 3) has played an important role. In this section we prove the following statement:

For generic values of masses and external momenta, the diagram V (see Fig. 3) is reducible to sunrise diagram with the same masses and momenta.

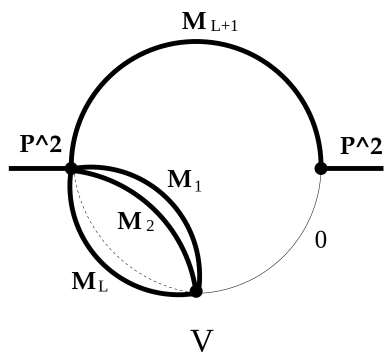

Figure 3: V-type Feynman Diagram

Let us consider the L-loop V-type Feynman diagram defined in momentum space as

$$
\begin{aligned}
& V^{(L)}(M_1^2, \cdots, M_{L+1}^2, \alpha_1, \cdots, \alpha_{L+1}; \sigma; p^2) \\
& \quad = \int \frac{d^n(k_1 \cdots k_L)}{[k_1^2 - M_1^2]^{\alpha_1} \cdots [(k_1 - \cdots - k_L)^2 - M_L^2]^{\alpha_L} [(p-k_L)^2 - M_{L+1}^2]^{\alpha_{L+1}} [k_L^2]^{\sigma}} \,.
\end{aligned} \tag{41}
$$

The Mellin-Barnes representation for this diagram is

$$
\begin{aligned}
& V^{(L)}(M_1^2, \cdots, M_{L+1}^2, \alpha_1, \cdots, \alpha_{L+1}; \sigma; p^2) = (p^2)^{\frac{n}{2}L - \alpha - \alpha_{L+1} - \beta - \sigma} [i^{1-n} \pi^{n/2}]^L \\
& \times \int \left\{ \Pi_{j=1}^{L+1} dt_j \frac{\Gamma(-t_j) \Gamma\left(\frac{n}{2} - \alpha_j - t_j\right)}{\Gamma(\alpha_j)} \left(-\frac{M_j^2}{p^2} \right)^{t_j} \right\} \frac{\Gamma\left(\frac{n}{2}L - \alpha - \sigma - \vec{t}\right) \Gamma\left(\alpha - \frac{n}{2}(L-1) + \vec{t}\right)}{\Gamma\left(\frac{n}{2}L - \alpha - \vec{t}\right) \Gamma\left(\alpha + \sigma - \frac{n}{2}(L-1) + \vec{t}\right)} \\
& \times \frac{\Gamma\left(\alpha + \alpha_{L+1} + \sigma - \frac{n}{2}L + \vec{t} + t_{L+1}\right)}{\Gamma\left(\frac{n}{2}(L+1) - \alpha - \alpha_{L+1} - \sigma - \vec{t} - t_{L+1}\right)} \,,
\end{aligned} \tag{42}
$$

where

$$L \geq 2 \; , \quad \alpha = \sum_{j=1}^{L} \alpha_j \; , \quad \vec{t} = \sum_{j=1}^{L} t_j \; .$$

For $\sigma = 0$ the diagram V coincides with the L-loop sunrise diagram. The integral, Eq. (42), includes the following ratio of Γ-functions with arguments different by integers:

$$\frac{\Gamma\left(\frac{n}{2}L-\alpha-\sigma-\vec{t}\right)}{\Gamma\left(\frac{n}{2}L-\alpha-\vec{t}\right)} \times \frac{\Gamma\left(\alpha-\frac{n}{2}(L-1)+\vec{t}\right)}{\Gamma\left(\alpha+\sigma-\frac{n}{2}(L-1)+\vec{t}\right)} \; .$$

Let us introduce two differential operators K_1 and K_2 defined as

$$K_1 : \Pi_{j=0}^{\sigma-1}[(\sum_{k=1}^{L}\theta_k + b + j)] \; , \quad K_2 : \Pi_{j=1}^{\sigma}[(-\sum_{k=1}^{L}\theta_k + a - j)] \; ,$$

where $a = \frac{n}{2}L - \alpha$ and $b = \alpha - \frac{n}{2}(L-1)$. Applying these operators to the function V we got:

$$(K_1 \circ K_2) \, V^{(L)} = J^{(L)},$$

where J is sunrise diagram defined by Eq. (16). Indeed

$$K_1 \circ K_2 \left(\Pi_{j=1}^{L} z_j^{t_j} \left(\frac{\Gamma\left(b+\vec{t}\right)}{\Gamma\left(b+\sigma+\vec{t}\right)} \right) \left(\frac{\Gamma\left(a-\sigma-\vec{t}\right)}{\Gamma\left(a-\vec{t}\right)} \right) \right) = \Pi_{j=1}^{L} z_j^{t_j} \; .$$

The operators K_1 and K_2 are products of differential operators of the first order, so that their inverse operators correspond to one-fold integral over linear form. Integral of this type convert Puiseux-type solutions of diagram J into Puiseux-type solutions of diagram V. As consequence, the dimension of the space of nontrivial solutions of differential operator related to diagram of V-type coincides with dimension of the space of nontrivial solutions of differential operator of sunrise diagram J. $\Box$.

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