CYCLIC TRIDIAGONAL PAIRS, HIGHER ORDER ONSAGER ALGEBRAS AND ORTHOGONAL POLYNOMIALS

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ABSTRACT. The concept of cyclic tridiagonal pairs is introduced, and explicit examples are given. For a fairly general class of cyclic tridiagonal pairs with cyclicity N, we associate a pair of 'divided polynomials'. The properties of this pair generalize the ones of tridiagonal pairs of Racah type. The algebra generated by the pair of divided polynomials is identified as a higher-order generalization of the Onsager algebra. It can be viewed as a subalgebra of the $q-$Onsager algebra for a proper specialization at q the primitive $2N$th root of unity. Orthogonal polynomials beyond the Leonard duality are revisited in light of this framework. In particular, certain second-order Dunkl shift operators provide a realization of the divided polynomials at $N = 2$ or $q = \mathrm{i}$.

1. INTRODUCTION

Let $\mathbb{K}$ denote a field. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Recall that a *Leonard pair* is a pair of linear transformations A, A^* such that there exist two bases for V with respect to which the matrix representing A (resp. A^*) is irreducible tridiagonal (resp. diagonal) and the matrix representing A^* (resp. A) is diagonal (resp. irreducible tridiagonal) [T99]. For the well-known families of one-variable orthogonal polynomials in the Askey-scheme including the Bannai–Ito polynomials [BI84], the bispectral problem they solve finds a natural interpretation within the framework of Leonard's theorem [BI84] and Leonard pairs [T03]: given a Leonard pair, the overlap coefficients between the two bases coincide with polynomials of the Askey-scheme ($q-$Racah, $q-$Hahn, $q-$Krawtchouk, etc.) or their specializations evaluated on a discrete support. The three-term recurrence relation, three-term difference equation and orthogonality property of the orthogonal polynomials follow from the corresponding Leonard pair [T03]. In the literature, examples of such correspondence first appeared in [BI84] and in the context of mathematical physics, see for instance [Z91, GLZ92].

The concept of *tridiagonal pair* (see Definition 2.1.1) introduced by Ito–Tanabe–Terwilliger arised as a natural extension of the concept of Leonard pair [T93, ITT99]. For a tridiagonal pair A, A^*, in the basis in which A (resp. A^*) is diagonal with multiplicities in the eigenvalues, then A^* (resp. A) is a block tridiagonal matrix. So, the main difference between a tridiagonal pair and a Leonard pair is the dimension of the eigenspaces of A, A^* that are not necessarily one-dimensional. In view of the connection between Leonard pairs and orthogonal polynomials of the Askey-scheme, given a

tridiagonal pair a correspondence with certain multivariable generalizations of orthogonal polynomials of the Askey-scheme was expected. This has been recently exhibited [BM15]: the so-called Gasper-Rahman polynomials [GR04] and their classical analogues discovered by Tratnik [T89] evaluated on a multidimensional discrete support are interpreted as the overlap coefficients between two distinct eigenbases of A and A^*. In particular, the bispectrality of these polynomials was shown in [GI07, Il08]: they satisfy a system of coupled three-term recurrence relations and three-term difference relations.

From a more general point of view, Leonard and tridiagonal pairs arise in the context of representation theory, from irreducible finite dimensional modules of a certain class of classical and quantum algebras with two generators A, A^* and unit, called tridiagonal algebras [T99] and the q−Onsager algebra [B04]. These algebras are defined through a pair of relations, see Definition 2.2.1, which can be viewed as deformations of the q−Serre relations of $U_q(\widehat{sl_2})$ or deformations of the Dolan-Grady relations [DG82]. Note that infinite dimensional modules have also been constructed, see [BB12] or [BM15]. In [BM15], A, A^* are mapped to certain multivariable q−difference/difference operators introduced by Geronimo and Iliev [GI07, Il08].

Having in mind a specialization of the tridiagonal and q−Onsager algebras relations at q a root of unity, in the present paper we introduce a natural generalization of the tridiagonal pairs that we call a 'cyclic tridiagonal pair'. Let N be a positive integer and $\mathbb{Z}_N$ denotes the cyclic group $\mathbb{Z}/N\mathbb{Z}$ of order N.

1.1. **Definition.** By a *cyclic tridiagonal pair* on V, we mean an ordered pair of linear transformations $C : V \to V$ and $C^* : V \to V$ that satisfy the following four conditions for some positive integer N.

  (i) Each of C, C^* is diagonalizable.
  (ii) There exists an ordering $\{V_p\}_{p \in \mathbb{Z}_N}$ of the eigenspaces of C such that

$$(1.1) \qquad\qquad\qquad C^* V_p \subseteq V_{p-1} + V_p + V_{p+1}.$$

  (iii) There exists an ordering $\{V_p^*\}_{p \in \mathbb{Z}_N}$ of the eigenspaces of C^* such that

$$(1.2) \qquad\qquad\qquad C V_p^* \subseteq V_{p-1}^* + V_p^* + V_{p+1}^*.$$

  (iv) There does not exist a subspace W of V such that $CW \subseteq W$, $C^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We call the number N *the cyclicity* of the cyclic tridiagonal pair.

The purpose of this paper is to study a general class of cyclic tridiagonal pairs of cyclicity N. The properties of cyclic tridiagonal pairs C, C^* (they satisfy certain polynomial equations) lead us to the introduction of a new pair of operators that we call the 'divided polynomials' (associated to the pair C, C^*). As an application of these new concepts, we propose a systematic approach to the construction of higher-order generalizations of tridiagonal pairs and of the tridiagonal algebras. Namely, we derive explicit pairs of square matrices such that there exist two bases with respect to which the first matrix (resp. the second) is irreducible block $(2N+1)$−diagonal (resp. diagonal) and the second matrix (resp. the first) is diagonal (resp. irreducible block $(2N + 1)$−diagonal), and any such given pair generates an algebra that generalizes the tridiagonal algebra. The case $N = 1$ specializes to tridiagonal pairs and the corresponding algebra is a tridiagonal algebra.

There are numerous motivations for the higher-order generalizations: for instance, the problem of polynomial solutions to higher-order recurrence relations [DVA95] (see also [K49] and Chapter VII of [B68]) that occur in the context of matrix valued orthogonal polynomials, sieved polynomials [AAA84, CIM94], orthogonal polynomials beyond Leonard duality [GVZ13, TVZ11] that are discussed in Section 5, and the bispectral problem for multidiagonal infinite, semi-infinite or finite dimensional

matrices [GIK12]. Another motivation comes from representation theory of the $q-$Onsager algebra and the non-perturbative analysis of quantum integrable systems associated with quantum groups or related coideal subalgebras for a deformation parameter q specialized at a root of unity [DFM99, V15].

We now describe the main results of this paper. The first main result is the following: under some reasonable assumptions (see Assumption 3.2 and the assumption in Corollary 3.5), to a tridiagonal pair of $q-$Racah type (see Case I after Definition 2.1.1) we associate a cyclic tridiagonal pair (by specializing q to the root of unity $e^{i\pi/N}$) and a pair of divided polynomials. For a large class of tridiagonal pairs of $q-$Racah type, the relations satisfied by the two pairs are identified (in Proposition 4.2.2, Theorems 4.3 and 4.4). In particular, the divided polynomials satisfy the defining relations of a higher-order generalization of the Onsager algebra, called $N-$Onsager algebra and denoted $OA^{[N]}$ (see Definition 4.5), and act on the vector space as higher-order generalizations of tridiagonal pairs.

The second main result is about orthogonal polynomials outside of the Askey scheme (like the complementary Bannai–Ito polynomials): starting from a special class of tridiagonal pairs A, A^* of $q-$Racah type, under the specialization $q = e^{i\pi/2}$ the operator A^* and the divided polynomial associated to A generate another (non-symmetric) higher-order generalization of the tridiagonal/Onsager algebra. The relations are given by (5.20). Importantly, a homomorphism from this algebra to the complementary Bannai–Ito algebra (5.6), introduced in [GVZ13] is exhibited, see (5.21). According to the relation between irreducible finite dimensional representations of (5.6) and the complementary Bannai–Ito polynomials, we conjecture that (spectrum-degenerate representations of) our new algebra provides multivariable generalizations of the complementary Bannai–Ito polynomials and their descendants, besides the one-variable case [GVZ13, TVZ11]. By construction, such multivariable orthogonal polynomials should satisfy a system of coupled three-term recurrence relations and *five-term* difference equations.

The paper is organized as follows. In Section 2, we review the concept of tridiagonal pairs introduced in [T93]. The $q-$Dolan-Grady relations satisfied by a class of tridiagonal pairs as well as their higher-order generalizations are also recalled. The concept of a cyclic tridiagonal pair is introduced in Section 3, with an example of a cyclic tridiagonal pair in a $4-$dimensional vector space reported in Appendix A.1. To any tridiagonal pair of $q-$Racah type and having certain assumptions, we also construct a cyclic tridiagonal pair. An example is reported in Appendix A.2. In Section 3.7, using the pair of minimal polynomials associated with the cyclic tridiagonal pair, we construct a second pair of elements that we call the 'divided polynomials'. We describe then the action of the two pairs with respect to the distinct eigenbases. In Section 4, the relations satisfied by the cyclic tridiagonal pair and the divided polynomials are identified. In particular, the subalgebra generated by the divided polynomials is (a representation of) the algebra $OA^{[N]}$ with the defining relations (4.19). In Section 5, we consider two operators built from a particular tridiagonal pair of $q-$Racah type A, A^*. They are obtained as A^* and the divided polynomial associated with A under the specialization $q = $ i. The algebra (5.20) generated by these two operators can be viewed as a non-symmetric generalization of the tridiagonal algebra. We then describe a homomorphism of this algebra onto the complementary Bannai–Ito algebra (5.6) and onto its degenerate version, the dual -1 Hahn algebra (5.15). In particular, a representation of the algebra (5.20) in terms of second-order Dunkl shift operators is given in Proposition 5.5.

Let us point out that the results here presented are independently of interest in relation with the construction of analogues of the Lusztig's specialization [Lu93] of the quantum affine algebra $U_q(\widehat{sl_2})$ at

q a root of unity and an analogue of Lusztig's "small" quantum group for the $q-$Onsager algebra[1]. For the reader familiar with this subject, the pair of divided polynomials here introduced can be viewed as an analogue of Lusztig's divided powers for elements in a certain coideal subalgebra of $U_q(\widehat{sl_2})$. In this vein, we conjecture that the relations (2.10) together with the polynomial relations (4.20) and the relations (4.15), (4.16) and (4.17) discussed in Section 4 are the defining relations of a specialization of the $q-$Onsager algebra at $q = e^{\frac{i\pi}{N}}$. We intend to discuss this important subject elsewhere.

Finally, let us mention that higher-rank generalizations of the $N-$Onsager algebra $OA^{[N]}$ may be introduced as well, having in mind the higher-rank generalizations of the $q-$Onsager algebra [BB09] and the higher-rank generalization of the Onsager algebra [UI95]. However, to our knowledge neither the analog of the concept of tridiagonal pair for higher-rank cases nor the representation theory of the generalized $q-$Onsager algebras has been investigated yet.

Conventions: In this paper, $\{x\}$ denotes the integer part of x. Let m, n be integers. We define $\overline{m,n}$ for the set $\{m, m+1, ..., n-1, n\}$. We also use the q-numbers $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, the q-factorials $[n]_q! = \prod_{r=1}^n [r]_q$, with $[0]_q! = 1$, and the q-commutators $[X, Y]_q = qXY - q^{-1}YX$, and $[X, Y] = [X, Y]_1$ is the ordinary commutator.

2. Tridiagonal pairs and relations

In this Section, we first recall the concept of tridiagonal pairs developed in [T93, T99, ITT99, NT09]. Given a tridiagonal pair A, A^*, we recall the basic and higher-order relations satisfied by A, A^*. As an example, we consider a tridiagonal pair associated with a reduced parameter sequence and indeterminate q. In this case, the pair satisfies the so-called $q-$Dolan-Grady relations and their higher-order generalizations.

2.1. **Definitions and the classification.** Let $\mathbb{K}$ be an arbitrary field unless otherwise noted. Throughout the paper, we always denote by V a finite-dimensional vector space over $\mathbb{K}$.

2.1.1. **Definition** (See [ITT99], Definition 2.1). *A tridiagonal pair* (or TD pair) on V is an ordered pair of linear transformations[2] $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable on V.
(ii) There exists an ordering $\{V_p\}_{p=0}^d$ of the eigenspaces of A such that

$$(2.1) \qquad A^* V_p \subseteq V_{p-1} + V_p + V_{p+1} \qquad\qquad 0 \le p \le d,$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.
(iii) There exists an ordering $\{V_p^*\}_{p=0}^\delta$ of the eigenspaces of A^* such that

$$(2.2) \qquad A V_p^* \subseteq V_{p-1}^* + V_p^* + V_{p+1}^* \qquad\qquad 0 \le p \le \delta,$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.
(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^* W \subseteq W$, $W \ne 0$, $W \ne V$.

2.1.2. **Remark.** In Definition 2.1.1, the scalars d and δ are necessarily equal [ITT99, Lemma 4.5].

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[1]The $q-$Onsager algebra is isomorphic to a certain coideal subalgebra of $U_q(\widehat{sl_2})$ [B06, BB12]. For the proof of injectivity of the map, see [Ko12].

[2]According to a common notational convention, for a linear transformation A the conjugate-transpose of A is denoted A^*. This convention is not used here.

Let A, A^* denote a TD pair on V, as in Definition 2.1.1. By [ITT99, Lemma 4.5] the integer $d = \delta$ from (ii), (iii) is called *the diameter* of the pair. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (2.1) (resp. (2.2)). Let $\{V_p\}_{p=0}^d$ (resp. $\{V_p^*\}_{p=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). For $0 \leq p \leq d$, let θ_p (resp. θ_p^*) denote the eigenvalue of A (resp. A^*) associated with V_p (resp. V_p^*). According to the classification [T99, Theorem 4.4], the eigenvalues θ_p and θ_p^*, $p = 0, 1, ..., d$, can take three different forms:

- Case I: the $q-$Racah type (with $b, c, b^*, c^* \neq 0$)

(2.3)
$$\theta_p = a + bq^{2p} + cq^{-2p},$$
$$\theta_p^* = a^* + b^* q^{2p} + c^* q^{-2p};$$

- Case II: the Racah type[3] (with $c, c^* \neq 0$)

$$\theta_p = a + bp + cp(p-1)/2,$$
$$\theta_p^* = a^* + b^* p + c^* p(p-1)/2;$$

- Case III (the characteristic of $\mathbb{K}$ is not 2)

$$\theta_p = a + b(-1)^p + cp(-1)^p,$$
$$\theta_p^* = a^* + b^*(-1)^p + c^* p(-1)^p.$$

Here a, a^*, b, b^*, c, c^* are scalars and $q \neq 0$, $q^2 \neq 1$, $q^2 \neq -1$. In this case, we say that A, A^* is a tridiagonal pair of $q-$Racah (case I) or Racah (case II) type [NT09, Theorem 5.3]. Note that the cases listed above are the 'generic' cases where the parameters are nonzero. Other examples with some of the parameters set to zero can be found in [T05, IT10].

For the analysis presented below, the notion of TD system [ITT99] is needed. Let $\operatorname{End}(V)$ denote the $\mathbb{K}$-algebra of all linear transformations from V to V. Let A denote a diagonalizable element of $\operatorname{End}(V)$. Let $\{V_p\}_{p=0}^d$ (resp. $\{\theta_p\}_{p=0}^d$) denote an ordering of the eigenspaces (resp. eigenvalues) of A. For $0 \leq p \leq d$, define $E_p \in \operatorname{End}(V)$ such that

$$(E_p - \mathbf{1})V_p = 0 \qquad \text{and} \qquad E_p V_m = 0 \quad \text{for} \quad m \neq p \ (0 \leq m \leq d) ,$$

where $\mathbf{1}$ denotes the identity of $\operatorname{End}(V)$. We call E_p the *primitive idempotent* of A corresponding to V_p (or θ_p). Observe that (i) $\mathbf{1} = \sum_{p=0}^d E_p$; (ii) $E_p E_m = \delta_{p,m} E_p$ ($0 \leq p, m \leq d$); (iii) $V_p = E_p V$ ($0 \leq p \leq d$); (iv) $A = \sum_{p=0}^d \theta_p E_p$. Let A, A^* denote a TD pair on V. An ordering of the primitive idempotents of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

2.1.3. **Definition.** [ITT99, Definition 2.1] A *tridiagonal system* (or *TD system*) on V is a sequence

$$\Phi = (A; \{E_p\}_{p=0}^d; A^*; \{E_p^*\}_{p=0}^d)$$

that satisfies:

(i) A, A^* is a TD pair on V,
(ii) $\{E_p\}_{p=0}^d$ is a standard ordering of the primitive idempotents of A,
(iii) $\{E_p^*\}_{p=0}^d$ is a standard ordering of the primitive idempotents of A^*.

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[3]If the characteristic of $\mathbb{K}$ is 2, one identifies $p(p-1)/2$ as 0 if $p = 0$ or 1 (mod 4), and as 1 if $p = 2$ or $p = 3$ (mod 4).

2.1.4. **Lemma.** [INT10, Lemma 2.2] *Let* $(A; \{E_p\}_{p=0}^d; A^*; \{E_p^*\}_{p=0}^d)$ *denote a TD system. Then the following holds for* $0 \le p, m, r \le d$:

(i) $E_p^* A^r E_m^* = 0$

(ii) $E_p A^{*r} E_m = 0$ *if* $|p - m| > r$.

2.2. **Tridiagonal algebras.** The theory of tridiagonal pairs is closely related with the representation theory of tridiagonal algebras. Tridiagonal algebras are defined as follows:

2.2.1. **Definition** (See [T99], Definition 3.9)**.** Let $\beta, \gamma, \gamma^*, \rho, \rho^*$ denote scalars in $\mathbb{K}$. *The tridiagonal algebra* (or TD algebra) $T = T(\beta, \gamma, \gamma^*, \rho, \rho^*)$ is the associative $\mathbb{K}$-algebra with unit generated by two elements A, A^* subject to the relations

(2.4) $[\mathsf{A}, \mathsf{A}^2\mathsf{A}^* - \beta\mathsf{A}\mathsf{A}^*\mathsf{A} + \mathsf{A}^*\mathsf{A}^2 - \gamma(\mathsf{A}\mathsf{A}^* + \mathsf{A}^*\mathsf{A}) - \rho\mathsf{A}^*] = 0,$

(2.5) $[\mathsf{A}^*, \mathsf{A}^{*2}\mathsf{A} - \beta\mathsf{A}^*\mathsf{A}\mathsf{A}^* + \mathsf{A}\mathsf{A}^{*2} - \gamma^*(\mathsf{A}^*\mathsf{A} + \mathsf{A}\mathsf{A}^*) - \rho^*\mathsf{A}] = 0.$

A and A^* are called the standard generators of T. The relations (2.4) and (2.5) are called the tridiagonal (TD) relations.

Every TD pair comes from an irreducible finite dimensional module of the tridiagonal algebra. The explicit connection between the representation theory of tridiagonal algebras and the theory of tridiagonal pairs, as defined in Definition 2.1.1, is established by the following theorem:

2.2.2. **Theorem** (See [ITT99], Theorem 10.1)**.** *Let* A, A^* *denote a TD pair over* $\mathbb{K}$. *Then there exists a sequence of scalars* $\beta, \gamma, \gamma^*, \rho, \rho^*$ *taken from* $\mathbb{K}$ *such that* A, A^* *satisfy the tridiagonal relations (2.4) and (2.5). The parameter sequence* $\beta, \gamma, \gamma^*, \rho, \rho^*$ *is uniquely determined by the pair if the diameter is at least* 3.

2.3. **Higher-order relations.** Given a TD pair A, A^* it is possible to show that, in addition to the two basic relations (2.4), (2.5), A and A^* satisfy an infinite family of pairs of polynomial relations of higher degree. Indeed, recall that $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$ denote the eigenvalues of A and A^*, respectively.

2.3.1. **Lemma.** [BV13] *For each positive integer s, there exist scalars* $\beta_s, \gamma_s, \gamma^*{}_s, \delta_s, \delta_s^*$ *in* $\mathbb{K}$ *such that*

$$\theta_i^2 - \beta_s\theta_i\theta_j + \theta_j^2 - \gamma_s(\theta_i + \theta_j) - \delta_s = 0 \;,$$
$$\theta_i^{*2} - \beta_s\theta_i^*\theta_j^* + \theta_j^{*2} - \gamma_s^*(\theta_i^* + \theta_j^*) - \delta_s^* = 0 \;, \qquad for \quad 0 \le i, j \le d \quad with \quad |i - j| = s.$$

2.3.2. **Definition.** [BV13] Let x, y denote commuting indeterminates. For each positive integer r, we define the polynomials $p_r(x, y)$ and $p_r^*(x, y)$ as follows:

(2.6) $$p_r(x, y) = (x - y) \prod_{s=1}^r (x^2 - \beta_s xy + y^2 - \gamma_s(x + y) - \delta_s) \;,$$

(2.7) $$p_r^*(x, y) = (x - y) \prod_{s=1}^r (x^2 - \beta_s xy + y^2 - \gamma_s^*(x + y) - \delta_s^*) \;.$$

Observe $p_r(x, y)$ and $p_r^*(x, y)$ have a total degree $2r + 1$ in x, y.

As a consequence of Lemma 2.3.1, observe that $p_r(\theta_i, \theta_j) = 0$ and $p_r^*(\theta_i^*, \theta_j^*) = 0$ if $|i - j| \le r$, for $0 \le i, j \le d$. Combining these facts together with Lemma 2.1.4, higher-order polynomial relations generalizing (2.4), (2.5) that are satisfied by a TD pair A, A^* can be derived. Namely:

2.3.3. **Theorem.** [BV13] *Let A, A^* denote a TD pair over $\mathbb{K}$. For each positive integer r,*

$$(2.8) \qquad \sum_{i,j=0}^{i+j\leq 2r+1} a_{ij} A^i A^{*r} A^j = 0 \ , \qquad \sum_{i,j=0}^{i+j\leq 2r+1} a_{ij}^* A^{*i} A^r A^{*j} = 0$$

where the scalars a_{ij}, a_{ij}^* are defined by:

$$(2.9) \qquad p_r(x,y) = \sum_{i,j=0}^{i+j\leq 2r+1} a_{ij} x^i y^j \quad and \quad p_r^*(x,y) = \sum_{i,j=0}^{i+j\leq 2r+1} a_{ij}^* x^i y^j \ .$$

In the next subsection, we describe an explicit example of a TD pair and the relations it satisfies.

2.4. **The $q-$Dolan-Grady relations and generalizations.** We now consider the *reduced* parameter sequence determining the tridiagonal relations (2.4) and (2.5): setting $\gamma = \gamma^* = 0$. In this case, the tridiagonal pair A and A^* will be denoted by $\mathcal{W}_0$ and $\mathcal{W}_1$, respectively. Parametrizing β as $\beta = q^2 + q^{-2}$, the two operators then satisfy the $q-$*Dolan-Grady* relations, also known as the defining relations of the $q-$*Onsager algebra* [T99, B04]:

$$(2.10) \qquad \begin{aligned} \big[\mathcal{W}_0, \big[\mathcal{W}_0, [\mathcal{W}_0, \mathcal{W}_1]_q\big]_{q^{-1}}\big] &= \rho\big[\mathcal{W}_0, \mathcal{W}_1\big] \ , \\ \big[\mathcal{W}_1, \big[\mathcal{W}_1, [\mathcal{W}_1, \mathcal{W}_0]_q\big]_{q^{-1}}\big] &= \rho^*\big[\mathcal{W}_1, \mathcal{W}_0\big] \ . \end{aligned}$$

In terms of the parameters in the spectra (2.3) of the TD pair $\mathcal{W}_0, \mathcal{W}_1$ corresponding to the choice $a = a^* = 0$, one has the identification [T99]:

$$(2.11) \qquad \rho = -bc(q^2 - q^{-2})^2 \quad and \quad \rho^* = -b^* c^* (q^2 - q^{-2})^2.$$

Besides the two basic relations above, the tridiagonal pair $\mathcal{W}_0, \mathcal{W}_1$ also satisfies higher-order relations generalizing (2.10), see Theorem 2.3.3. Let r be a positive integer. We refer to these relations as the *r-th higher-order $q-$Dolan-Grady relations.* Explicitely, they read [BV13, V15][4]:

$$(2.12) \qquad \begin{aligned} \sum_{p=0}^{r} \sum_{j=0}^{2r+1-2p} (-1)^{j+p} \rho^p \, c_j^{[r,p]} \, \mathcal{W}_0^{2r+1-2p-j} \mathcal{W}_1^r \mathcal{W}_0^j &= 0 \ , \\ \sum_{p=0}^{r} \sum_{j=0}^{2r+1-2p} (-1)^{j+p} (\rho^*)^p \, c_j^{[r,p]} \, \mathcal{W}_1^{2r+1-2p-j} \mathcal{W}_0^r \mathcal{W}_1^j &= 0 \ , \end{aligned}$$

where $c_{2(r-p)+1-j}^{[r,p]} = c_j^{[r,p]}$ and

$$(2.13) \qquad c_j^{[r,p]} = \sum_{k=0}^{j} \frac{(r-p)!}{(\{\frac{j-k}{2}\})!(r-p-\{\frac{j-k}{2}\})!} \sum_{\mathcal{P}} [s_1]_{q^2}^2 \cdots [s_p]_{q^2}^2 \frac{[2s_{p+1}]_{q^2} \cdots [2s_{p+k}]_{q^2}}{[s_{p+1}]_{q^2} \cdots [s_{p+k}]_{q^2}}$$

$$(2.14) \qquad \text{with} \quad \begin{cases} j \in \overline{0, r-p} \ , \quad s_i \in \{1, 2, ..., r\} \ , \\ \mathcal{P}: \quad \begin{array}{l} s_1 < \cdots < s_p \ ; \quad s_{p+1} < \cdots < s_{p+k} \ , \\ \{s_1, \ldots, s_p\} \cap \{s_{p+1}, \ldots, s_{p+k}\} = \emptyset \end{array} \end{cases} \quad .$$

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[4]The coefficients a_{ij} and a_{ij}^* introduced in (2.9) are such that: $a_{2r+1-2p-j\ j} = (-1)^{j+p} \rho^p \, c_j^{[r,p]}$, $a_{2r+1-2p-j\ j}^* = (-1)^{j+p} (\rho^*)^p \, c_j^{[r,p]}$ for $p = 0, ..., r$ and a_{ij}, a_{ij}^* are zero otherwise.

Explicit examples of TD pairs of $q-$Racah type that are not Leonard pairs[5] have been constructed in [B06]. In particular, a simple example of a TD pair of $q-$Racah type is considered in Appendix A.2.

3. Cyclic tridiagonal pairs and the divided polynomials

In this Section, we first recall the new concept of 'cyclic tridiagonal pairs' and define the notion of cyclic tridiagonal systems. Explicit examples of cyclic TD pairs are given in Appendix A. Then, we show that under Assumption 3.2 and the assumption in Corollary 3.5, a TD pair of q-Racah type gives a cyclic TD pair. We also show that each operator of the cyclic TD pair in any finite-dimensional representation satisfies a polynomial equation of degree N. Based on that, we introduce a pair of operators that we call the divided polynomials and study their action.

3.1. **Conventions.** We introduce here several assumptions that are used in the paper.

We start with the definition of a cyclic tridiagonal pair, recall Definition 1.1 in the Introduction. We do not repeat it here for brevity and only note that we will use the notation $V_p^{(N)}$ and $V_p^{*(N)}$ instead of V_p and V_p^* in Definition 1.1 to avoid a confusion with the notation used for the eigenspaces of TD pairs from Definition 2.1.1.

We also note that our definition of cyclic TD pairs could be naturally generalized by introducing a different cyclicity N^* instead of N in the point (iii) of Definition 1.1, in accordance with the definition of usual TD pairs that involves two different diameters d and δ. But because the two diameters are necessarily equal, recall Remark 2.1.2, and most of our examples of cyclic TD pairs come from specializations of the TD pairs of $q-$Racah type we fixed $N^* = N$. We also say that an ordering of the eigenspaces of C (resp. C^*) is *standard* whenever it satisfies (1.1) (resp. (1.2)).

In Appendix A.1, we give a simple and explicit example of a non-trivial cyclic TD pair acting on a $4-$dimensional vector space, see (A.1). This example naturally comes from the theory of cyclic representations of the quantum algebra $U_q(sl_2)$ at $2N$-th roots of unity.

3.1.1. **Remark.** We note that each TD pair of diameter d is a cyclic TD pair of cyclicity $d+1$ (with trivial contribution of V_0 while acting on V_d by A^*, etc.). But the reverse is of course not true.

By analogy with the Definition 2.1.3 of tridiagonal systems, the notion of *cyclic TD system* can be introduced. Let C denote a diagonalizable element of $\mathrm{End}(V)$. Let $\{V_p^{(N)}\}_{p\in\mathbb{Z}_N}$ denote an ordering of the eigenspaces of C and let $\{\theta_p\}_{p\in\mathbb{Z}_N}$ denote the corresponding ordering of the eigenvalues of C. For $p \in \mathbb{Z}_N$, define $E_p^{(N)} \in \mathrm{End}(V)$ such that

(3.1) $(E_p^{(N)} - \mathbf{1})V_p^{(N)} = 0$ and $E_p^{(N)}V_m^{(N)} = 0$ for $m \neq p \ (m \in \mathbb{Z}_N)$.

We call $E_p^{(N)}$ the *primitive idempotent* of C corresponding to $V_p^{(N)}$ (or θ_p), *i.e.,* we have (i) $\mathbf{1} = \sum_{p\in\mathbb{Z}_N} E_p^{(N)}$ and (ii) $E_p^{(N)} E_m^{(N)} = \delta_{p,m} E_p^{(N)}$, for $m, p \in \mathbb{Z}_N$, and (iii) $C = \sum_{p\in\mathbb{Z}_N} \theta_p E_p^{(N)}$. Let C, C^* denote a cyclic TD pair on V, as defined in Definition 1.1. An ordering of the primitive idempotents of C (resp. C^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of C (resp. C^*) is standard.

3.1.2. **Definition.** By a *cyclic tridiagonal system* (or *cyclic TD system*) on V we mean a sequence

$$\Phi^{(N)} = (C; \{E_p^{(N)}\}_{p\in\mathbb{Z}_N}; C^*; \{E_p^{*(N)}\}_{p\in\mathbb{Z}_N})$$

that satisfies (i)–(iii) below.

---

[5]For a Leonard pair, by definition eigenspaces of A, A^* are one-dimensional [ITT99].

(i) C, C^* is a cyclic TD pair on V.

(ii) $\{E_p^{(N)}\}_{p\in\mathbb{Z}_N}$ is a standard ordering of the primitive idempotents of C.

(iii) $\{E_p^{*(N)}\}_{p\in\mathbb{Z}_N}$ is a standard ordering of the primitive idempotents of C^*.

According to the Definitions 1.1 and 3.1.2, the proof of the following lemma is straightforward:

3.1.3. Lemma. *Let* $(C; \{E_p^{(N)}\}_{p\in\mathbb{Z}_N}; C^*; \{E_p^{*(N)}\}_{p\in\mathbb{Z}_N})$ *denote a cyclic TD system. Then the following holds for* $p, m, r \in \mathbb{Z}_N$ *and* $r < N/2$:

(i) $E_p^{*(N)} C^r E_m^{*(N)} = 0 \quad$ *if* $\quad r < |p - m| < N - r$,

(ii) $E_p^{(N)} C^{*r} E_m^{(N)} = 0 \quad$ *if* $\quad r < |p - m| < N - r$.

We assume $\mathbb{K} = \mathbb{C}(q)$, for q an indeterminate. Given a TD pair (A, A^*) of q−Racah type, recall that we have the decompositions

$$(3.2) \qquad V = \bigoplus_{n=0}^{d} V_n(q) \qquad \text{or} \qquad V = \bigoplus_{n=0}^{d} V_n^*(q),$$

where $V_n(q)$ (resp. $V_n^*(q)$) denotes the A-eigenspace (resp. A^*-eigenspace) associated with the eigenvalue $\theta_n(q)$ (resp. $\theta_n^*(q)$) for $0 \le n \le d$ given by (2.3). According to (2.1), with respect to an A-eigenspace the action of A^* is:

$$(3.3) \qquad A^* V_n(q) \subseteq V_{n-1}(q) + V_n(q) + V_{n+1}(q) \qquad\qquad 0 \le n \le d,$$

where $V_{-1}(q) = 0$ and $V_{d+1}(q) = 0$. For the rest of our discussion we will use the following assumption.

3.2. Assumption. *Let* A, A^* *be a TD pair of q-Racah type of diameter* d. *We assume there exist two bases in* V *where the matrix representing* A *is diagonal (resp. block tridiagonal) and the one representing* A^* *is block tridiagonal (resp. diagonal) and such that, specializing* q *to* $e^{i\pi/N}$, *we have*[6]

(i) *all entries of* A *and* A^* *are finite with respect to the bases,*

(ii) *all entries of the transition matrix between the two bases are finite,*

(iii) b, c, b^*, c^* *from (2.3) are such that* $c/b \ne q^{2m}$ *and* $c^*/b^* \ne q^{2m'}$ *for any* $m, m' \in \overline{0, N-1}$.

3.3. Remark. Two different choices of the pairwise bases from Assumption 3.2 may not give the same pair of linear transformations under the specialization q to $e^{i\pi/N}$, we do not study this important question here. But our statements about properties of cyclic TD pairs obtained from TD pairs of q−Racah type do not depend on such a choice.

From now on, we abuse the convention that q stands for an indeterminate by writing $q = e^{\frac{i\pi}{N}}$ for the specialization of the indeterminate q to the root of unity $e^{\frac{i\pi}{N}}$. A large family of examples satisfying Assumption 3.2 can be explicitly constructed. A simple example is given in Appendix A.2.

3.4. Proposition. *For* $q = e^{\frac{i\pi}{N}}$ *and integer* $2 \le N \le d$, *the operators* A *and* A^* *from Assumption 3.2 have* N *distinct eigenvalues. They read:*

$$(3.4) \qquad \begin{aligned} \theta_t &= a + bq^{2t} + cq^{-2t}, \\ \theta_t^* &= a^* + b^* q^{2t} + c^* q^{-2t}, \end{aligned} \qquad \text{for} \quad t = 0, 1, ..., N-1.$$

---

[6]We thank P. Terwilliger for stressing us the importance of the point (ii).

Let $V_t^{(N)}$ (resp. $V_t^{*(N)}$) denotes the eigenspace associated with θ_t (resp. θ_t^*). One has the decomposition:

$$(3.5) \qquad V_t^{(N)} = \bigoplus_{k=0}^{\lfloor \frac{d}{N} \rfloor} V_{t+kN}(q)\Big|_{q=e^{\frac{i\pi}{N}}} \qquad and \qquad V_t^{*(N)} = \bigoplus_{k=0}^{\lfloor \frac{d}{N} \rfloor} V_{t+kN}^*(q)\Big|_{q=e^{\frac{i\pi}{N}}}$$

with the action

$$(3.6) \qquad\qquad A^* V_t^{(N)} \subseteq V_{t+1}^{(N)} + V_t^{(N)} + V_{t-1}^{(N)} \qquad\qquad (0 \leq t \leq N-1),$$

where $V_{-1}^{(N)} = V_{N-1}^{(N)}$ and $V_N^{(N)} = V_0^{(N)}$ and

$$(3.7) \qquad\qquad A V_t^{*(N)} \subseteq V_{t+1}^{*(N)} + V_t^{*(N)} + V_{t-1}^{*(N)} \qquad\qquad (0 \leq t \leq N-1),$$

where $V_{-1}^{*(N)} = V_{N-1}^{*(N)}$ and $V_N^{*(N)} = V_0^{*(N)}$.

For $N > d$, the operator A (resp. A^*) from Assumption 3.2 has $d+1$ distinct eigenvalues θ_t (resp. θ_t^*) and the corresponding eigenspaces $V_t^{(N)} = V_t(e^{\frac{i\pi}{N}})$ (resp. $V_t^{*(N)} = V_t^*(e^{\frac{i\pi}{N}})$), and acts as in (2.2) (resp. (2.1)).

Proof. First, consider $2 \leq N \leq d$ and recall the expressions (2.3) with $b, c, b^*, c^* \neq 0$. The conditions $\theta_t - \theta_{t'} = 0$ and $\theta_t^* - \theta_{t'}^* = 0$ for some $t \neq t'$ reduce to the equalities $c/b = q^{2m}$ and $c^*/b^* = q^{2m'}$, for some $m, m' \in \overline{0, N-1}$, which are excluded by Assumption 3.2 (iii). So, the eigenvalues are distinct. Under Assumption 3.2, the eigenvectors of A are well-defined at $q = e^{\frac{i\pi}{N}}$ for any positive integer $N > 1$, and a vector in $V_n(e^{i\pi/N})$ corresponds to the eigenvalue $\theta_n(e^{i\pi/N})$. For $0 \leq n \leq d$, there exist unique non-negative integers t, k such that $n = t + kN$, with $0 \leq t \leq N-1$. Using (2.3), we immediately find (3.4). Then for each fixed t, we note that by Assumption 3.2 the eigenspace $V_t^{(N)}$ of the operator A is spanned by the eigenvectors in $V_t(e^{\frac{i\pi}{N}}), V_{t+N}(e^{\frac{i\pi}{N}}), \ldots$ and thus we have the decomposition in (3.5). We have obviously the similar statement for the A^* eigenspaces $V_t^{*(N)}$ of the eigenvalue θ_t^*. The action (3.6)-(3.7) of A and A^* on these spaces follows from our Assumption 3.2 where the specialization of A^* with finite entries means that the action is block tridiagonal on each $V_n(e^{i\pi/N})$ and thus by construction on $V_t^{(N)}$.

For $N > d$, we have obviously $d+1$ distinct eigenvalues θ_t (resp. θ_t^*) by Assumption (3.2) (iii) and each $V_t^{(N)}$ is $V_t(e^{\frac{i\pi}{N}})$ (resp. $V_t^{*(N)}$ is $V_t^*(e^{\frac{i\pi}{N}})$). The statement then follows. $\square$

We emphasize that given a TD pair of q−Racah type A, A^* its specialization to q a root of unity does not necessary give an irreducible action of A and A^*. Studying the irreducibility is a difficult problem. In Appendix A.2, we give an explicit example of a cyclic TD pair obtained from a TD pair of q−Racah type with $a = a^* = 0$, where the irreducibility is only conjectured. Combining the results in Proposition 3.4 and Assumption 3.2 (ii)[7], we have the following corollary.

3.5. Corollary. *For $q = e^{\frac{i\pi}{N}}$ and integer $2 \leq N \leq d$, assume the vector space V is irreducible under the action of A and A^* in Assumption 3.2. Then, the pair of operators A, A^* form a cyclic tridiagonal pair of cyclicity N.*

---

[7]The point (ii) about existence of the specialization of the transition matrix between the two eigenbases allows us to show that the two pairs of matrices (A, A^*) specialized in the first eigenbasis and in the second eigenbasis, correspondingly, give the same pair of linear maps $A : V \to V$ and $A^* : V \to V$.

Note that our first example of cyclic TD pairs given in Appendix A.1 does not appear as a root-of-unity specialization of a TD pair of $q-$Racah type.

3.6. Cyclicity and the minimal polynomials.

From now on, we will assume that $2 \leq N \leq d$. Then according to Proposition 3.4, any cyclic TD pair A, A^* obtained from a TD pair of $q-$Racah type of diameter d has exactly N distinct eigenvalues denoted by θ_t and θ_t^*, respectively, for $t = 0, 1, ..., N-1$. As a consequence, the operator A satisfies an algebraic equation determined by the minimal polynomial

$$(3.8) \qquad P_N\big(x; \{\theta_t\}\big) = \prod_{t=0}^{N-1}(x - \theta_t),$$

and similarly for A^*, and $\{\theta_t\}$ stands for the set $\{\theta_t\}_{t=0}^{N-1}$ for brevity (and we follow this convention below).

3.6.1. Proposition. *Assume $q = e^{\frac{i\pi}{N}}$. Let A, A^* denote a cyclic TD pair with spectra of the form (3.4). Then, the operators A and A^* satisfy the relations*

$$(3.9) \qquad P_N\big(A; \{\theta_t\}\big) = 0 \qquad and \qquad P_N\big(A^*; \{\theta_t^*\}\big) = 0.$$

Note that the minimal polynomials can be expanded as follows. Let $e_k(\theta_0, \cdots, \theta_{N-1})$, with $k = 0, 1, ..., N-1$, denote the elementary symmetric polynomials in the variables θ_n, i.e. given by

$$(3.10) \qquad e_k(\theta_0, \cdots, \theta_{N-1}) = \sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq N-1} \theta_{j_1} \theta_{j_2} \cdots \theta_{j_k} \quad \text{with} \quad e_0(\theta_0, \cdots, \theta_{N-1}) = 1.$$

For $N \geq 2$, we then have

$$(3.11) \qquad P_N\big(x; \{\theta_t\}\big) = \sum_{k=0}^{N} e_k(\theta_0, \cdots, \theta_{N-1}) x^{N-k}$$

and similarly for $P_N\big(x; \{\theta_t^*\}\big)$ with the replacement $(a, b, c) \to (a^*, b^*, c^*)$. For instance[8]:

- For $N = 2$, $q^4 = 1$,

$$P_2(x; \theta_0, \theta_1) = x^2 - 2ax + (a+b+c)(a-b-c);$$

- For $N = 3$, $q^6 = 1$,

$$P_3(x; \theta_0, \theta_1, \theta_2) = x^3 - 3ax^2 + 3(a^2 - bc)x - (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac);$$

- For $N = 4$, $q^8 = 1$,

$$P_4(x; \theta_0, \theta_1, \theta_2, \theta_3) = x^4 - 4ax^3 + 2(3a^2 - 2bc)x^2 - 4a(a^2 - 2bc)x$$
$$+ (a+b+c)(a-b-c)(a^2 + b^2 + c^2 - 2bc).$$

---

[8]For $N = 1$, $q^2 = 1$ there is no polynomial. In this special case, the operators $\mathcal{W}_0, \mathcal{W}_1$ however reduce to elements that generate the undeformed Onsager algebra.

3.7. **The divided polynomials.** For two operators A and A^* that form a cyclic tridiagonal pair with spectra (3.4), for q specialized at a root of unity, we now introduce and study two additional operators that are called below the divided polynomials.

In view of the fact that the polynomials (3.11) for A and A^* vanish for $q^{2N} = 1$, see Proposition 3.6.1, we renormalise them by the q-factorial $[N]_q!$ and take then the root of unity limit called the *divided polynomials* for the cyclic TD pair A, A^*:

$$(3.12) \qquad A^{[N]} = \lim_{q \to e^{\frac{i\pi}{N}}} \frac{P_N(A; \{\theta_t\})}{[N]_q!} \qquad \text{and} \qquad A^{*[N]} = \lim_{q \to e^{\frac{i\pi}{N}}} \frac{P_N(A^*; \{\theta_t^*\})}{[N]_q!}.$$

Here, the limit is taken in the sense of matrix elements and the entries of A and A^* are (rational) functions in q, we also assume that θ_t in (3.8) is fixed for given N and does not depend on q, i.e. taken as the eigenvalue at the root of unity. We now give a general proof that (3.12) is well defined in the eigenbases of A and A^*, respectively.

According to the spectral properties (3.4) of the cyclic tridiagonal pair A, A^*, the eigenvalues of the divided polynomials (3.12) can be derived in a closed form. Let us first consider the special case $N = 2$ which is the most degenerate situation. The spectra are particularly simple.

3.7.1. **Lemma.** *Let d be the diameter of a TD pair of $q-$Racah type. Assume that the derivatives $\frac{\partial \theta_n(q)}{\partial q}$ (resp. $\frac{\partial \theta_n^*(q)}{\partial q}$) are finite at $q = i$, for $n = 0, 1, ..., d$. For $N = 2$, the eigenvalues $\tilde{\theta}_n$, with $n = 0, \ldots, d$ (resp. $\tilde{\theta}_n^*$, $n = 0, \ldots, d$) of the divided polynomials $A^{[2]}$ (resp. $A^{*[2]}$) take the form*

$$(3.13) \qquad \tilde{\theta}_n = i(c^2 - b^2)2n \quad and \quad \tilde{\theta}_n^* = i(c^{*2} - b^{*2})2n.$$

Proof. For $N = 2$, the minimal polynomial $P_2(x; \theta_0, \theta_1)$ is given below (3.11). Then, evaluating the right-hand side of the first expression (3.12) in the A-eigenbasis in $V_n(i)$ we can replace A simply by its eigenvalue and define the numbers

$$\tilde{\theta}_n = \lim_{q \to e^{i\pi/2}} \frac{\theta_n{}^2 - 2a\theta_n + (a + b + c)(a - b - c)}{q + q^{-1}}.$$

Using the l'Hôpital's rule, we obtain

$$\tilde{\theta}_n = \lim_{q \to e^{i\pi/2}} \frac{4nq^{-1}(bq^{2n} + cq^{-2n})(bq^{2n} - cq^{-2n})}{1 - q^{-2}}$$

which reduces to (3.13). A similar expression is obtained for $\tilde{\theta}_n^*$, replacing $(b, c) \to (b^*, c^*)$. $\qquad \square$

The analysis above can be generalized in a straightforward manner.

3.7.2. **Lemma.** *Let d be the diameter of a TD pair of $q-$Racah type. Assume that the derivatives $\frac{\partial \theta_n(q)}{\partial q}$ (resp. $\frac{\partial \theta_n^*(q)}{\partial q}$) are finite at $q = e^{\frac{i\pi}{N}}$, for $n = 0, 1, ..., d$ and integer $N > 2$. Then, the divided polynomials $A^{[N]}$ (resp. $A^{*[N]}$) have $d + 1$ eigenvalues denoted by $\tilde{\theta}_n$ (resp. $\tilde{\theta}_n^*$), $n = 0, 1, ..., d$. If k, t are non-negative integers such that $n = t + kN$, $0 \le t \le N - 1$, the eigenvalues read as*

$$(3.14) \qquad \tilde{\theta}_n = C_0^{(n)} \left(\frac{\partial a}{\partial q}q + \frac{\partial b}{\partial q}q^{2n+1} + \frac{\partial c}{\partial q}q^{-2n+1} + 2n(bq^{2n} - cq^{-2n}) \right)\Big|_{q=e^{\frac{i\pi}{N}}}$$

and

$$C_0^{(n)} = -\frac{(q - q^{-1})}{2N[N-1]_q!} \prod_{j=0, j\neq t}^{N-1} (\theta_t - \theta_j),$$

where $q = e^{\frac{i\pi}{N}}$. A similar expression is obtained for $\tilde{\theta}_n^*$, replacing $(a, b, c) \to (a^*, b^*, c^*)$. The eigenspace of $A^{[N]}$ corresponding to $\tilde{\theta}_n$ is $V_n \equiv V_n(q = e^{\frac{i\pi}{N}})$, and similarly for $A^{*[N]}$.

Proof. In terms of the indeterminate q, recall that A has $d + 1$ distinct eigenvalues

$$\theta_n(q) = a + bq^{2n} + cq^{-2n}, \qquad n = 0, \ldots, d,$$

and corresponding eigenspaces $V_0(q)$, $V_1(q)$, $\ldots$, $V_d(q)$. By Proposition 3.4 we have that A has N distinct eigenvalues $\theta_0, \theta_1, \ldots, \theta_{N-1}$ at the specialization $q = e^{\frac{i\pi}{N}}$. Let $\tilde{\theta}_n$ (resp. $\tilde{\theta}_n^*$) denote the eigenvalues of $A^{[N]}$ (resp. $A^{*[N]}$). We calculate the spectrum of the right-hand side of the first expression (3.12) in the A-eigenbasis. Restricting to the subspace $V_n(e^{\frac{i\pi}{N}})$, we can replace A by its eigenvalue θ_n. Therefore, $\tilde{\theta}_n$ can be then written as

$$\tilde{\theta}_n = \lim_{q \to e^{\frac{i\pi}{N}}} \frac{(\theta_n(q) - \theta_0)(\theta_n(q) - \theta_1) \ldots (\theta_n(q) - \theta_{N-1})}{[N]_q!}, \qquad n = 0, \ldots, d.$$

For all $n = 0, \ldots, d$, there exist unique k, t non-negative integers, such that $n = kN + t$, $0 \le t \le N - 1$. It yields to

$$(3.15) \qquad \tilde{\theta}_n = \frac{(q - q^{-1}) \prod\limits_{j=0, j \neq t}^{N-1} (\theta_t - \theta_j)}{[N-1]_q!} \lim_{q \to e^{\frac{i\pi}{N}}} \frac{\theta_n(q) - \theta_t}{q^N - q^{-N}}.$$

Using l'Hôpital's rule, one has

$$\lim_{q \to e^{\frac{i\pi}{N}}} \frac{\theta_n(q) - \theta_t}{q^N - q^{-N}} = \lim_{q \to e^{\frac{i\pi}{N}}} \frac{\frac{\partial \theta_n(q)}{\partial q}}{-2Nq^{-1}}.$$

Plugging this expression into (3.15), the equation (3.14) follows. A similar expression is obtained for $\tilde{\theta}_n^*$, replacing $(a, b, c) \to (a^*, b^*, c^*)$. Finally by the construction, the eigenspaces of $A^{[N]}$ and $A^{*[N]}$ corresponding to $\tilde{\theta}_n$ and $\tilde{\theta}_n^*$ are $V_n(e^{\frac{i\pi}{N}})$ and $V_n^*(e^{\frac{i\pi}{N}})$, respectively. $\qquad \square$

3.8. **Remark.** We note that the assumption on the first derivatives in the Lemmas 3.7.1 and 3.7.2 is only a technical assumption, to make the formulas less complicated. One can easily generalize the statement by assuming that the higher derivatives $\frac{\partial^k \theta_n(q)}{\partial q^k}$ (resp. $\frac{\partial^k \theta_n^*(q)}{\partial q^k}$) for some finite k are finite at $q = e^{\frac{i\pi}{N}}$, for $n = 0, 1, ..., d$. The expression for $\tilde{\theta}_n$ and $\tilde{\theta}_n^*$ is then obtained by applying l'Hôpital's rule k times. In Section 5, we will have a situation where the first derivatives are diverging but the second ones are finite.

Using the expressions (3.12) of the divided polynomials in terms of A and A^*, we now consider the action of the two operators on the eigenspaces $V_n \equiv V_n(e^{\frac{i\pi}{N}})$ and $V_n^* \equiv V_n^*(e^{\frac{i\pi}{N}})$ of $A^{[N]}$ and $A^{*[N]}$, respectively.

3.9. **Proposition.** *For any eigenbasis of A (resp. A^*) where $A^{*[N]}$ (resp. $A^{[N]}$) is finite, the divided polynomials $A^{*[N]}$ and $A^{[N]}$ act as*

$$(3.16) \qquad A^{[N]} V_n \subseteq V_n,$$
$$A^{*[N]} V_n \subseteq V_{n-N} + \cdots + V_n + \cdots + V_{n+N} \qquad (0 \le n \le d),$$
$$A^{*[N]} V_s^* \subseteq V_s^*,$$
$$A^{[N]} V_s^* \subseteq V_{s-N}^* + \cdots + V_s^* + \cdots + V_{s+N}^* \qquad (0 \le s \le d).$$

Proof. The first and third equations follow from Lemmas 3.7.1 and 3.7.2. In any eigenbasis of A for indeterminate q the polynomial $P_N(A^*; \{\theta_t^*\})/[N]_q!$ in A^* is of order N (recall (3.11)) and acts as a block $(2N+1)$-diagonal matrix. We consider then any eigenbasis of A from Assumption 3.2 and where $A^{*[N]}$ is finite and therefore it acts as a block $(2N+1)$-diagonal matrix as well. Therefore, it acts as in the second line of (3.16). Similar argument gives the proof for the action of the divided polynomial for A. □

Note that in the basis which diagonalizes A (resp. A^*), the matrix representing $A^{*[N]}$ (resp. $A^{[N]}$) is block $(2N+1)$-diagonal[9]. From this point of view, the two operators (3.12) can be understood as objects generalizing the properties of tridiagonal pairs. According to the relationship between TD pairs and the representation theory of TD algebras (see Section 2), an important problem is to identify the relations satisfied by the four operators. In the next Section, we investigate such relations for a rather large class of cyclic TD pairs and corresponding divided polynomials.

4. THE ALGEBRA GENERATED FOR A CLASS OF CYCLIC PAIRS

The purpose of this section is to identify the set of relations satisfied by a class of cyclic tridiagonal pairs and the corresponding divided polynomials

$$(4.1) \qquad \mathcal{W}_i^{[N]} = \lim_{q \to e^{\mathrm{i}\pi/N}} \frac{P_N\big(\mathcal{W}_i; \{\theta_t^{(i)}\}\big)}{[N]_q!} , \qquad \text{with} \qquad i \in \mathbb{Z}_2 ,$$

where the operators $\mathcal{W}_0$ and $\mathcal{W}_1$ form a TD pair of q−Racah type, as discussed in Section 2. The class of cyclic TD pair considered here is associated with operators having the spectra of the form (3.4), with $a = a^* = 0$. First, we identify the relations for $N = 2$ and then for $N > 2$. All together, the four operators generate a new algebra. We conjecture that the relations we present here (Theorems 4.3 and 4.4) are the defining relations for a specialization of the q−Onsager algebra at the root of unity. We also show that the subalgebra generated by $\mathcal{W}_0^{[N]}$, $\mathcal{W}_1^{[N]}$ is a higher-order generalization of the (classical) Onsager algebra with defining relations given by (4.19).

4.1. **The relations for $N = 2$.** We study here the algebra generated by the cyclic TD pair $\mathcal{W}_0, \mathcal{W}_1$ and the divided polynomials (4.1) at $N = 2$ for $a = a^* = 0$. The generators $\mathcal{W}_i$ satisfy the q−Dolan-Grady relations (2.10) evaluated at $q = e^{\mathrm{i}\pi/2}$ or, because $\rho = \rho^* = 0$ (see (2.11)), the q-Serre relations at $q^2 = -1$:

$$(4.2) \qquad \mathcal{W}_i^3 \mathcal{W}_{i+1} + \mathcal{W}_i^2 \mathcal{W}_{i+1} \mathcal{W}_i - \mathcal{W}_i \mathcal{W}_{i+1} \mathcal{W}_i^2 - \mathcal{W}_{i+1} \mathcal{W}_i^3 = 0$$

together with the polynomial relations (3.9) that take the form

$$(4.3) \qquad \mathcal{W}_0^2 = (b + c)^2 \mathbf{1} \quad \text{and} \quad \mathcal{W}_1^2 = (b^* + c^*)^2 \mathbf{1}$$

Note that the relations (4.2) are actually consequences of (4.3), i.e., they are identities if we take into account (4.3) only. It simply means that, for the special case $N = 2$ the basis elements in the subalgebra generated by $\mathcal{W}_i$ are monomials of the form

$$\mathcal{W}_i \mathcal{W}_{i+1} \mathcal{W}_i \mathcal{W}_{i+1} \cdots$$

---

[9]An example of such operators is given in Appendix A.2.2.

For the mixed relations (between $\mathcal{W}_i$ and $\mathcal{W}_{i+1}^{[2]}$) we obtain:

$$(4.4) \qquad \mathcal{W}_i^3 \mathcal{W}_{i+1}^{[2]} + \mathcal{W}_i^2 \mathcal{W}_{i+1}^{[2]} \mathcal{W}_i - \mathcal{W}_i \mathcal{W}_{i+1}^{[2]} \mathcal{W}_i^2 - \mathcal{W}_{i+1}^{[2]} \mathcal{W}_i^3 \;\; = \;\; 0,$$

$$(4.5)\; \left(\mathcal{W}_i^{[2]}\right)^3 \mathcal{W}_{i+1} - 3\left(\mathcal{W}_i^{[2]}\right)^2 \mathcal{W}_{i+1} \mathcal{W}_i^{[2]} + 3 \mathcal{W}_i^{[2]} \mathcal{W}_{i+1} \left(\mathcal{W}_i^{[2]}\right)^2 - \mathcal{W}_{i+1}\left(\mathcal{W}_i^{[2]}\right)^3 \;\; = \;\; \rho_i^{[2]}\left[\mathcal{W}_i^{[2]}, \mathcal{W}_{i+1}\right],$$

where

$$(4.6) \qquad \rho_0^{[2]} = -4(b^2 - c^2)^2, \qquad \rho_1^{[2]} = -4(b^{*2} - c^{*2})^2.$$

Finally, we have also relations between the divided polynomials $\mathcal{W}_0^{[2]}$ and $\mathcal{W}_1^{[2]}$:

$$(4.7) \quad \left[\mathcal{W}_i^{[2]}, \left(\mathcal{W}_i^{[2]}\right)^4 \mathcal{W}_{i+1}^{[2]} + \mathcal{W}_{i+1}^{[2]}\left(\mathcal{W}_i^{[2]}\right)^4 \right.$$

$$- 4\left(\mathcal{W}_i^{[2]}\right)^3 \mathcal{W}_{i+1}^{[2]} \mathcal{W}_i^{[2]} + 6\left(\mathcal{W}_i^{[2]}\right)^2 \mathcal{W}_{i+1}^{[2]}\left(\mathcal{W}_i^{[2]}\right)^2 - 4\mathcal{W}_i^{[2]} \mathcal{W}_{i+1}^{[2]}\left(\mathcal{W}_i^{[2]}\right)^3$$

$$\left. - 5\rho_i^{[2]}\left\{ \left(\mathcal{W}_i^{[2]}\right)^2 \mathcal{W}_{i+1}^{[2]} - 2\mathcal{W}_i^{[2]} \mathcal{W}_{i+1}^{[2]} \mathcal{W}_i^{[2]} + \mathcal{W}_{i+1}^{[2]}\left(\mathcal{W}_i^{[2]}\right)^2 \right\} + 4(\rho_i^{[2]})^2 \mathcal{W}_{i+1}^{[2]} \right] = 0.$$

The proof of the relations (4.4), (4.5) and (4.7) will be given below for $N > 2$.

4.2. The relations for $N > 2$.
For the indeterminate q, a TD pair of $q-$Racah type with spectra (2.3) for $a = a^* = 0$ satisfies the $q-$Dolan-Grady relations (2.10) [T99], as well the higher-order $q-$Dolan-Grady relations (2.12) derived in [BV13] (see Section 2). In particular, for the specialization $q = e^{\frac{i\pi}{N}}$ it implies that any cyclic TD pair C, C^* that is obtained from such a TD pair satisfies (2.10), (2.12) with $\mathcal{W}_0 \to C$, $\mathcal{W}_1 \to C^*$. Below, we present an independent proof of these relations.

4.2.1. Proposition.
Assume $q = e^{\frac{i\pi}{N}}$. Let $\mathcal{W}_0, \mathcal{W}_1$ denote a cyclic TD pair with spectra given by (3.4) with $a = a^* = 0$. Then, the operators $\mathcal{W}_0, \mathcal{W}_1$ satisfy the relations (2.10).

Proof. Let Δ_1 denote the expression of the left-hand side minus the right-hand side of the first equation in (2.10). We show now that $\Delta_1 = 0$. Recall that $q = e^{\frac{i\pi}{N}}$ and $\mathcal{W}_0, \mathcal{W}_1$ is a cyclic TD pair by the assumption of the proposition, we thus use the primitive idempotents defined in (3.1). For $i, j \in \mathbb{Z}_N$, one has $E_i^{(N)} \Delta_1 E_j^{(N)} = p_1(\theta_i, \theta_j)\, E_i^{(N)} \mathcal{W}_1 E_j^{(N)}$ with

$$(4.8) \qquad p_1(x, y) = (x - y)(x^2 - (q^2 + q^{-2})xy + y^2 - \rho)\,.$$

For each pair i, j it is straightforward to check that $p_1(\theta_i, \theta_j) = 0$ if $|i - j| \leq 1$ or $|i - j| = N - 1$, while according to Lemma 3.1.3 we have $E_i^{(N)} \mathcal{W}_1 E_j^{(N)} = 0$ if $1 < |i - j| < N - 1$. It implies $\Delta_1 = 0$ and thus the first equation in (2.10) holds. Similar arguments are used to show the second equation in (2.10). $\qquad \square$

Similar arguments are used to derive the higher-order $q-$Dolan-Grady relations for the cyclic TD pair.

4.2.2. Proposition.
Assume $q = e^{\frac{i\pi}{N}}$. Let $\mathcal{W}_0, \mathcal{W}_1$ denote a cyclic TD pair with spectra given by (3.4) with $a = a^* = 0$. Let r be a positive integer such that $1 \leq r \leq N - 1$. Then, the operators $\mathcal{W}_0, \mathcal{W}_1$ satisfy the r-th higher-order $q-$Dolan-Grady relations (2.12).

Proof. Let Δ_r denote the expression of the left-hand side of the first equation in (2.12). We show $\Delta_r = 0$. For $i, j \in \mathbb{Z}_N$, one has $E_i^{(N)} \Delta_r E_j^{(N)} = p_r(\theta_i, \theta_j) \, E_i^{(N)} \mathcal{W}_1^r E_j^{(N)}$ with

$$(4.9) \qquad p_r(x, y) = (x - y) \prod_{s=1}^{r} \left(x^2 - \frac{[2s]_{q^2}}{[s]_{q^2}} xy + y^2 - [s]_{q^2}^2 \rho \right) \; .$$

For each i, j it is straightforward to check that $p_r(\theta_i, \theta_j) = 0$ if $N - r \le |i - j| \le r$, while according to Lemma 3.1.3 we have $E_i^{(N)} \mathcal{W}_1^r E_j^{(N)} = 0$ if $N - r > |i - j| > r$. It implies the first equation in (2.12). Similar arguments are used to show the second equation in (2.12).   □

Note that any higher-order relation for $r \ge N$ can be reduced through the relations (3.9) with $A \to \mathcal{W}_0$, $A^* \to \mathcal{W}_1$. Finally, recall that the two operators $\mathcal{W}_0$ and $\mathcal{W}_1$ act on a finite dimensional vector space. It implies that $\mathcal{W}_0, \mathcal{W}_1$ should satisfy additionnal polynomial relations with coefficients depending on the representation considered. These relations can be undertstood as generalizations of the Askey–Wilson relations. Examples of such relations can be found in [BB10] for the indeterminate q.

4.2.3. *Mixed relations.* Let us now turn to the derivation of the mixed relations, combining the cyclic TD pair and the divided polynomials. According to the action of the divided polynomials on the eigenspaces V_n, V_s^* in (3.16), polynomial relations are indeed expected. Below, for simplicity we focus on the class of cyclic TD pairs with spectra (3.4) and $a = a^* = 0$ and the rational functions b, c, b^*, and c^* of the form

$$(4.10) \qquad b = b_0 + b_1 q^\beta, \qquad c = c_0 + c_1 q^{-\beta}, \qquad b^* = b_0^* + b_1^* q^{\beta^*}, \qquad c^* = c_0^* + c_1^* q^{-\beta^*} \;,$$

for $b_i, b_i^*, c_i, c_i^* \in \mathbb{C}$, for $i = 0, 1$, and integer β and β^*. For this choice, the spectra (3.14) of the divided polynomials (4.1) simplifies. For instance, using (3.14) the spectrum of $\mathcal{W}_0^{[N]}$ simplifies to

$$(4.11) \qquad \tilde{\theta}_n = (2n + \beta) C_N, \qquad n = 0, 1, ..., d$$

with

$$(4.12) \qquad C_N = \frac{(q - q^{-1})^N}{2N} q^{-N(N+1)/2} \left(b^N - c^N \right)$$

and similarly for $\mathcal{W}_1^{[N]}$, replacing $C_N \to C_N^*$ and $(b, c) \to (b^*, c^*)$ in the above expressions. Importantly, the spectra have an arithmetic structure with respect to the integer $n = 0, 1, 2, ..., d$. Our basic example in Appendix A.2 does indeed satisfy the assumption (4.10) on the coefficient functions a, b, c, etc.

Now, to derive the set of relations satisfied by the cyclic TD pair and the divided polynomials, we proceed by analogy with [T99, BV13].

4.2.4. **Definition.** Let x and y denote commuting indeterminates. For each positive integer N and $q = e^{\frac{i\pi}{N}}$, we define the two-variable polynomials $p_k^{(i)}(x, y)$, $i = 0, 1$, as follows:

$$(4.13) \qquad p_k^{(i)}(x, y) = (x - y) \prod_{s=1}^{k} (x^2 - 2xy + y^2 - \rho_i^{[N]} s^2)$$

with

$$(4.14) \qquad \rho_0^{[N]} = 4 \left(C_N \right)^2, \qquad \rho_1^{[N]} = 4 \left(C_N^* \right)^2 \;.$$

Observe $p_k^{(i)}(x, y)$ have a total degree $2k + 1$ in x, y.

4.2.5. **Lemma.** *Let k be a positive integer. We have*

$$p_k^{(0)}(\tilde{\theta}_r, \tilde{\theta}_s) = 0, \qquad p_k^{(1)}(\tilde{\theta}_r^*, \tilde{\theta}_s^*) = 0 \qquad for \qquad |r - s| \leq k .$$

Proof. For $r = s$, the relation obviousy holds. Due to the arithmetic progression of $\tilde{\theta}_s$ and $\tilde{\theta}_s^*$, one observes $(2s)^2 - 2(2s)(2s \pm 2k) + (2s \pm 2k)^2 = 4k^2$. The two equalities then follow. $\square$

4.3. **Theorem.** *Let r be a positive integer. Then, the operators $\mathcal{W}_i$ and the divided polynomials $\mathcal{W}_i^{[N]}$, for $i \in \mathbb{Z}_2$, satisfy the mixed relations*

$$(4.15) \qquad \sum_{p=0}^{N-1} \sum_{k=0}^{2N-1-2p} (-1)^{p+k} \rho_i^p c_k^{[N-1,p]} \mathcal{W}_i^{2N-1-2p-k} \mathcal{W}_{i+1}^{[N]} \mathcal{W}_i^k = 0,$$

$$(4.16)$$
$$\left(\mathcal{W}_i^{[N]}\right)^3 \mathcal{W}_{i+1} - 3\left(\mathcal{W}_i^{[N]}\right)^2 \mathcal{W}_{i+1}\mathcal{W}_i^{[N]} + 3\mathcal{W}_i^{[N]}\mathcal{W}_{i+1}\left(\mathcal{W}_i^{[N]}\right) - \mathcal{W}_{i+1}\left(\mathcal{W}_i^{[N]}\right)^3 = \rho_i^{[N]}\left[\mathcal{W}_i^{[N]}, \mathcal{W}_{i+1}\right],$$

where $\rho_0 = \rho$ and $\rho_1 = \rho^*$ are given by (2.11) and $c_k^{[N-1,p]}$ by (2.13) with $q = e^{i\pi/N}$, and $\rho_i^{[N]}$ by (4.14).

Proof. We give first a proof of (4.15) for $i = 0$. Let $\Delta^{(1)}$ denote the corresponding left-hand side of (4.15). To show that $\Delta^{(1)} = 0$, we recall that $q = e^{\frac{i\pi}{N}}$ and $\mathcal{W}_0, \mathcal{W}_1$ is a cyclic TD pair, we thus use the primitive idempotents $E_n^{(N)}$ defined in (3.1) which are projectors onto the eigenspaces $V_n^{(N)}$ in (3.5). Observe then

$$E_n^{(N)}\Delta^{(1)}E_m^{(N)} = p_{N-1}(\theta_n, \theta_m)E_n^{(N)}\mathcal{W}_1^{[N]}E_m^{(N)}, \qquad 0 \leq n, m \leq N - 1,$$

where the two-variable polynomial (4.9) has to be expanded as a power series in the variable x, y. Recall then (4.9) with $r = N - 1$, we thus have $p_{N-1}(\theta_n, \theta_m) = 0$ for $|n - m| \leq N - 1$. This implies $\Delta^{(1)} = 0$.

We then prove (4.16) for $i = 0$ in a similar way. Let $\Delta^{(2)}$ denote the corresponding left-hand side of (4.16). To show $\Delta^{(2)} = 0$, we use now the primitive idempotents E_n which are the projectors onto the eigenspaces V_n associated with the eigenvalues $\tilde{\theta}_n$. We have

$$E_n\Delta^{(2)}E_m = p_1^{(0)}(\tilde{\theta}_n, \tilde{\theta}_m)E_n\mathcal{W}_1 E_m$$

where the two-variable polynomial (4.13) has to be expanded as a power series in the variables x, y. According to (3.3) one has $E_n\mathcal{W}_1 E_m = 0$ for $|n - m| > 1$. Using Lemma 4.2.5 for $k = 1$, it implies $\Delta^{(2)} = 0$. Similar arguments are used to show (4.15), (4.16) for $i = 1$. $\square$

Finally, we identify relations satisfied by the divided polynomials.

4.4. **Theorem.** *The divided polynomials $\mathcal{W}_i^{[N]}$, for $i \in \mathbb{Z}_2$, satisfy the following relations:*

$$(4.17) \qquad \sum_{p=0}^{N} \sum_{k=0}^{2N+1-2p} (-1)^{k+p}\left(\rho_i^{[N]}\right)^p d_k^{[N,p]}\left(\mathcal{W}_i^{[N]}\right)^{2N+1-2p-k} \mathcal{W}_{i+1}^{[N]}\left(\mathcal{W}_i^{[N]}\right)^k = 0 ,$$

with

$$(4.18) \qquad d_k^{[N,p]} = \binom{2N+1-2p}{k} \sum_{1 \leq s_1 < \cdots < s_p \leq N} s_1^2 s_2^2 \ldots s_p^2 \qquad with \qquad k = 0, 1, ..., N - p.$$

Proof. Let $\Delta^{[N]}$ denote the expression on the left-hand side of (4.17) for $i = 0$. We show $\Delta^{[N]} = 0$. Recall that E_n denotes the projector onto the eigenspace V_n associated with the eigenvalue $\tilde{\theta}_n$. We have then

$$E_n \Delta^{[N]} E_m = p_N^{(0)}(\tilde{\theta}_n, \tilde{\theta}_m) E_n \mathcal{W}_1^{[N]} E_m.$$

According to Proposition 3.9, one has $E_n \mathcal{W}_1^{[N]} E_m = 0$ for $|n-m| > N$. Using Lemma 4.2.5 for $k = N$, it implies $\Delta^{[N]} = 0$. Similar arguments are used to show (4.17) for $i = 1$. □

4.4.1. **Remark.** Note that the defining relations (4.17) do not coincide with the higher-order q−Dolan−Grady relations (2.13) for $r = N$ and $q = 1$. The main difference is the power of the operator $\mathcal{W}_{i+1}^{[N]}$ entering in (4.17) compared with (2.12) specialized at $q = 1$.

Based on the previous results, we now introduce a new algebra that can be viewed as a higher-order generalization of the Onsager algebra.

4.5. **Definition.** Let N be a positive integer and $\rho_i^{[N]}, i \in \mathbb{Z}_2$, denote scalars in $\mathbb{K}$. The N−Onsager algebra $OA^{[N]}$ is the associative $\mathbb{K}$-algebra with unit generated by two elements $\mathsf{W}_0^{[N]}$, $\mathsf{W}_1^{[N]}$ subject to the relations

$$(4.19) \qquad \sum_{p=0}^{N} \sum_{k=0}^{2N+1-2p} (-1)^{k+p} \left(\rho_i^{[N]}\right)^p d_k^{[N,p]} \left(\mathsf{W}_i^{[N]}\right)^{2N+1-2p-k} \mathsf{W}_{i+1}^{[N]} \left(\mathsf{W}_i^{[N]}\right)^k = 0,$$

with $d_k^{[N,p]}$ defined by (4.18).

Note that the classical Onsager algebra [O44, DG82] corresponds to $N = 1$ and $\rho_0^{[N]} = \rho_1^{[N]} = 16$.

4.6. **Corollary.** *The divided polynomials $\mathcal{W}_0^{[N]}$ and $\mathcal{W}_1^{[N]}$ generate a (finite-dimensional) algebra which is a quotient of the N−Onsager algebra $OA^{[N]}$ with $\rho_0^{[N]}$ and $\rho_1^{[N]}$ given in (4.14).*

We finally conclude that the four operators $\mathcal{W}_i$ and $\mathcal{W}_i^{[N]}$, $i \in \mathbb{Z}_2$, satisfy

(1) the q−Dolan–Grady relations (2.10) at $q = e^{i\pi/N}$,
(2) the polynomial relations

$$(4.20) \qquad P_N\big(\mathcal{W}_0; \{\theta_t\}\big) = 0 \qquad \text{and} \qquad P_N\big(\mathcal{W}_1; \{\theta_t^*\}\big) = 0 \,,$$

where the polynomials P_N are defined in (3.8) together with (3.4),
(3) the mixed relations (4.15) and (4.16),
(4) and the N−Onsager algebra $OA^{[N]}$ relations (4.17).

We also note that during the derivation of these relations we did not use the condition on irreducibility of the action of the pair $\mathcal{W}_0$, $\mathcal{W}_1$. It is thus natural to define an (infinite-dimensional) algebra generated by the four generators subject to the (1)-(4) defining relations. Note that the algebra depends on 5 parameters: positive integer $N \geq 2$, and 4 complex parameters b, c, and b^*, c^*. The first two relations give an analogue of the "small quantum group" for $U_q \widehat{s\ell}(2)$ at $q = e^{\frac{i\pi}{N}}$ while the relations for the divided polynomials are analogues for the relations of the Lusztig's divided powers [Lu93] that form the classical algebra $U\widehat{s\ell}(2)$. We thus conjecture that the algebra described in (1)-(4) is a specialization of the q−Onsager algebra at $q = e^{\frac{i\pi}{N}}$.

We finally note that our analysis in the last two sections can be easily extended to $q = e^{i\pi m'/m}$ for rational m'/m and coprime m, m': the only conditions we actually use in Proposition 3.4 is $q^{2m} = 1$, so one has just to replace N by m. The rest follows without changes.

5. Orthogonal polynomials beyond Leonard duality revisited

All known one-variable orthogonal polynomials of the Askey scheme and their specializations satisfy a *bispectral problem* associated with a three-term recurrence relation and a second-order differential or $(q-)$difference equation [KS96]. Generalizing Leonard's theorem [Le82], Bannai and Ito [BI84] showed furthermore that all orthogonal polynomials satisfying this type of bispectral problem should coincide with polynomials of the Askey scheme including the so-called Bannai–Ito polynomials [BI84]. From the point of view of representation theory, all these orthogonal polynomials in one variable find a natural interpretation within the framework of Leonard pairs [T03]. Namely, for a pair of elements A, A^* that act on an irreducible finite dimensional module and are diagonalizable, and satisfy the so-called Askey–Wilson relations

$$
(5.1) \qquad
\begin{aligned}
A^2 A^* - (q^2 + q^{-2})AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \rho A^* &= \gamma^* A^2 + \omega A + \eta \mathbf{1}, \\
A^{*2} A - (q^2 + q^{-2})A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \rho^* A &= \gamma A^{*2} + \omega A^* + \eta^* \mathbf{1},
\end{aligned}
$$

with scalars $\beta, \gamma, \gamma^*, \rho, \rho^*, \omega, \eta, \eta^*$, it is known that there exist two bases with respect to which the two matrices representing A, A^* are diagonal (resp. tridiagonal) and tridiagonal (resp. diagonal). Importantly, the overlap coefficients between the two bases coincide with the Askey–Wilson polynomials evaluated on a discrete support [Z91, T03]. They satisfy a three-term recurrence relation and a three-term difference equation.

5.1. Polynomials and five-term difference equations.
In the literature, there are however isolated examples of one-variable orthogonal polynomials defined on a discrete support that *do not* satisfy the Leonard duality property. Instead, the corresponding orthogonal polynomials defined on a discrete support satisfy a bispectral problem associated with a three-term recurrence relation and a *five-term* difference equation. To our knowledge, the first example of such type is provided by the one-variable complementary Bannai–Ito polynomials $I_n(x) \equiv I_n(x; \rho_1, \rho_2, r_1, r_2)$ with scalars ρ_1, ρ_2, r_1, r_2 and variable x. They have been introduced in [GVZ13] through a Christoffel transformation of the Bannai–Ito polynomials [BI84]. By construction, the complementary Bannai–Ito polynomials solve the following bispectral problem. On one hand, they satisfy the three-term recurrence relation

$$
I_{n+1}(x) + (-1)^n \rho_2 I_n(x) + \tau_n I_{n-1}(x) = x I_n(x),
$$

where

$$
\begin{aligned}
\tau_{2p} &= -\frac{p(p + \rho_1 - r_1 + 1/2)(p + \rho_1 - r_2 + 1/2)(p - r_1 - r_2)}{(2p + g)(2p + g + 1)}, \\
\tau_{2p+1} &= -\frac{(p + g + 1)(p + \rho_1 + \rho_2 + 1)(p + \rho_2 - r_1 + 1/2)(p - \rho_2 - r_2 + 1/2)}{(2p + g + 1)(2p + g + 2)}
\end{aligned}
$$

with $g = \rho_1 + \rho_2 - r_1 - r_2$. One the other hand, they are eigenfunctions of a second-order Dunkl shift operator. Namely, define $T^{\pm 1} f(x) = f(x \pm 1)$ and the reflection operator $Rf(x) = f(-x)$. The complementary Bannai–Ito polynomials satisfy a *five-term* difference equation of the form

$$
(5.2) \qquad\qquad \mathcal{D}' I_n(x) = \tilde{\theta}'_n I_n(x)
$$

where

$$\mathcal{D}' = d_1'(x)T + d_2'(x)T^{-1} + d_3'(x)R + d_4'(x)TR - (d_1'(x) + d_2'(x) + d_3'(x) + d_4'(x))\mathbf{1}$$

and the spectrum reads

(5.3) $$\tilde{\theta}_{2p}' = p^2 + (g+1)p, \quad \tilde{\theta}_{2p+1}' = p^2 + (g+2) + \tilde{c}_0'.$$

Here, $\tilde{c}_0'$ is an arbitrary scalar. The corresponding explicit expressions for the rational functions $d_i'(x), i = 1, ..., 4$, can be obtained from [GVZ13, eq. (4.6)].

The second explicit example is given by the dual -1 Hahn polynomials $Q_n(x) \equiv Q_n(x; \rho_2, r_1, r_2)$ with scalars ρ_2, r_1, r_2 and variable x, considered either as a limiting case $q \to -1$ of the dual $q-$Hahn polynomials or as a limiting case $\rho_1 \to \infty$ of the complementary Bannai–Ito polynomials [TVZ11, GVZ13]. By construction, they solve a bispectral problem which is defined as follows. They satisfy a three-term recurrence relation:

$$Q_{n+1}(x) + (-1)^n \rho_2 Q_n(x) + \sigma_n Q_{n-1}(x) = xQ_n(x)$$

where

$$\begin{aligned} \sigma_{2p} &= -p(p - r_1 - r_2), \\ \sigma_{2p+1} &= -(p + \rho_2 - r_1 + 1/2)(p + \rho_2 - r_2 + 1/2). \end{aligned}$$

The dual -1 Hahn polynomials also satisfy a *five-term* difference equation. They are the eigenfunctions of a difference operator of the Dunkl type:

(5.4) $$\mathcal{D}Q_n(x) = \tilde{\theta}_n Q_n(x)$$

where

$$\mathcal{D} = d_1(x)T + d_2(x)T^{-1} + d_3(x)R + d_4(x)TR - (d_1(x) + d_2(x) + d_3(x) + d_4(x))\mathbf{1}$$

and the spectrum reads [GVZ13]:

(5.5) $$\tilde{\theta}_{2p} = p, \qquad \tilde{\theta}_{2p+1} = p + \tilde{c}_0.$$

Here, $\tilde{c}_0$ is an arbitrary scalar. The explicit expressions of the rational functions $d_i(x), i = 1, ..., 4$, can be obtained from [GVZ13, page 16] (see also [TVZ11]). Note that other known examples of one-variable polynomials beyond Leonard duality can be understood as descendants of the two above. For instance, the para-Krawtchouk and the symmetric Hahn polynomials can be seen as special cases of the complementary Bannai–Ito polynomials [GVZ13].

5.2. **The complementary Bannai–Ito algebra.** Whereas the algebraic framework behind the orthogonal polynomials of the Askey scheme – including the Bannai–Ito polynomials – is encoded by the Askey–Wilson algebra with the defining relations (5.1) and the concept of Leonard pairs, the above examples of orthogonal polynomials outside of Leonard duality arise within the representation theory of the so-called complementary Bannai–Ito algebra with two generators κ_1, κ_2 and involution r. A

presentation[10] of this algebra is given by the defining relations:

$$(5.6) \qquad [\kappa_1, r] = 0, \qquad \{\kappa_2, r\} = 2\delta_3, \qquad r^2 = \mathbf{1},$$

$$(5.7) \qquad [\kappa_1, [\kappa_1, \kappa_2]] = \frac{1}{2}\{\kappa_1, \kappa_2\} - \delta_2[\kappa_1, \kappa_2]r - \delta_3\kappa_1 r + \delta_1\kappa_2 - \delta_1\delta_3 r,$$

$$(5.8) \qquad [\kappa_2, [\kappa_2, \kappa_1]] = \frac{1}{2}\kappa_2^2 + \delta_2\kappa_2^2 r + 2\delta_3\kappa_1 r + 2\delta_3[\kappa_1, \kappa_2]r + \kappa_1 + \delta_4 r + \delta_5,$$

where $\{x, y\} = xy + yx$ and δ_i, for $i = 1, \dots, 5$, are arbitrary scalars. Irreducible polynomial representations for the complementary Bannai–Ito algebra (5.6) have been constructed in [GVZ13]. For the complementary Bannai–Ito polynomials, the homomorphism

$$(5.9) \qquad \kappa_1 \mapsto \mathcal{D}', \qquad \kappa_2 \mapsto x$$

with (5.2) is considered. In this case, the explicit expressions of the structure constants δ_i, for $i = 1, \dots, 5$, in terms of r_1, r_2, ρ_1, ρ_2 are given in [GVZ13, eq. (5.6)]. In particular:

$$(5.10) \qquad \delta_1 = \tilde{c}_0'(g + 1 - \tilde{c}_0'), \qquad \delta_2 = g + 3/2 - 2\tilde{c}_0' \qquad \text{and} \qquad \delta_3 = \rho_2.$$

In this picture, the complementary Bannai–Ito polynomials defined on a discrete support are interpreted as the overlap coefficients between the two eigenbases in an irreducible finite dimensional vector space. With respect to the first basis, the generator κ_1 acts as a diagonal matrix with spectrum (5.3) and the generator κ_2 acts as a tridiagonal matrix. With respect to the second basis, the generator κ_2 acts as a diagonal matrix with Bannai–Ito's discrete spectrum (see [GVZ13]):

$$(5.11) \qquad \theta_n^{*\prime} = (-1)^n(n/2 + h + 1/4) - 1/4, \quad n \in \mathbb{Z}$$

where h is a scalar, whereas the generator κ_1 acts as a *five-diagonal* matrix.

5.2.1. *Relation to the divided polynomials.* We are now in position to reconsider the example of [GVZ13] in light of the results presented in previous Sections. To this end, we choose b, b^*, c, and c^* such that they have a pole at $q = e^{i\pi/2}$ but the spectra of some of the operators among the corresponding cyclic TD pair and the divided polynomials remain finite and non-trivial for $q = e^{i\pi/2}$, and agree with the spectra for κ_1 and κ_2. We fix

$$(5.12) \qquad a = 0, \qquad b = \frac{i}{4(q + q^{-1})^{1/2}}q^{2g+2}, \qquad c = -\frac{i}{4(q + q^{-1})^{1/2}}q^{-2g-2},$$

$$(5.13) \qquad a^* = -\frac{1}{4}, \qquad b^* = -\frac{1}{4(q + q^{-1})}q^{4h+1}, \qquad c^* = \frac{1}{4(q + q^{-1})}q^{-4h-1}$$

where g, h are arbitrary scalars. In this case, by straightforward calculations one finds that the spectrum of A is trivial ($\theta_n = 0$) and the spectrum of $A^{*[2]}$ is singular ($\tilde{\theta}_n^* \to \infty$). On the other hand, one finds that the spectra of $A^{[2]}$ and A^* read, respectively:

$$(5.14) \qquad \tilde{\theta}_n = \frac{n^2}{4} + \frac{n}{2}(g + 1), \qquad \theta_n^* = (-1)^n(n/2 + h + 1/4) - 1/4.$$

We note that Lemma 3.7.1 can not be applied for our choice of b and c, however the second derivative of $\theta_n(q)$ at $q = i$ exists and we compute the spectrum following Remark 3.8. Observe that these expressions coincide with the spectra of the operators κ_1 in (5.3), for the identification $\tilde{c}_0' = g/2 + 3/4$, and κ_2 in (5.11), respectively, associated with the complementary Bannai–Ito polynomials defined on a discrete support [TVZ11].

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[10]In [GVZ13], an alternative presentation is given in which case a third generator $\kappa_3 = [\kappa_1, \kappa_2]$ is introduced.

5.3. The dual -1 Hahn algebra. As a limiting case of the complementary Bannai–Ito polynomials, for the dual -1 Hahn polynomials the algebraic structure that encodes all the properties of the polynomials is a degenerate version of the complementary Bannai–Ito algebra (5.6) [GVZ13]. Using a certain limiting procedure, the defining relations instead read:

$$(5.15) \qquad [\kappa_1, r] = 0, \qquad \{\kappa_2, r\} = 2\gamma_3, \qquad r^2 = \mathbf{1},$$

$$[\kappa_1, [\kappa_1, \kappa_2]] = -\gamma_2[\kappa_1, \kappa_2]r + \gamma_1\kappa_2 - \gamma_1\gamma_3,$$

$$[\kappa_2, [\kappa_2, \kappa_1]] = \gamma_2\kappa_2^2 r + 2\gamma_3\kappa_1 r + 2\gamma_3[\kappa_1, \kappa_2]r + \kappa_1 + \gamma_4 r + \gamma_5.$$

In this case, the homomorphism:

$$(5.16) \qquad\qquad\qquad\qquad \kappa_1 \to \mathcal{D}, \qquad \kappa_2 \to x$$

with (5.4) is considered. The explicit expressions of the structure constants $\gamma_i, 1, ..., 5$, in terms of r_1, r_2, ρ_2 is given in [GVZ13, page 17]. In particular:

$$(5.17) \qquad\qquad \gamma_1 = \tilde{c}_0(1 - \tilde{c}_0), \qquad \gamma_2 = 1 - 2\tilde{c}_0 \qquad \text{and} \qquad \gamma_3 = \rho_2.$$

By analogy with the complementary Bannai–Ito polynomials, the dual -1 Hahn polynomials defined on a discrete support are interpreted as the overlap coefficients between the two eigenbases (for κ_1 and κ_2). Compared with the previous case, the spectrum of the generator κ_1 is now given by (5.5).

5.3.1. *Relation to the divided polynomials.* For our second example, instead of (5.12) let us now choose:

$$(5.18) \qquad\qquad\qquad\qquad a = 0, \qquad b = \frac{e^{i\pi/4}}{2}, \qquad c = 0$$

and (5.13). In this case, the spectrum of A is now given by $\theta_n = e^{i\pi/4}(-1)^n/2$. On the other hand, the spectrum of $A^{[2]}$ simplifies to:

$$(5.19) \qquad\qquad\qquad\qquad\qquad \tilde{\theta}_n = n/2,$$

whereas the spectrum of A^* remains unchanged, given by θ_n^* in (5.14). Observe that these expressions coincide with the spectra of the operators κ_1 (5.3) provided $\tilde{c}_0 = 1/2$ and κ_2 (5.11), respectively, associated with the dual -1 Hahn polynomials defined on a discrete support.

5.4. Non-symmetric generalization of the tridiagonal algebra. Recall that the Askey–Wilson algebra (5.1) is a quotient of the tridiagonal algebra (2.4)-(2.5), i.e., the defining relations (5.1) imply the defining relations (2.4)-(2.5) but the reverse is not true in general. By analogy with this, we introduce a non-symmetric generalization of the tridiagonal algebra as the associative $\mathbb{K}$-algebra with unit generated by two elements $A^{[2]}$ and A^* subject to the relations

$$\left[A^{[2]}, (A^{[2]})^2 A^* - 2A^{[2]}A^*A^{[2]} + A^*(A^{[2]})^2 - \gamma\left(A^{[2]}A^* + A^*A^{[2]}\right) - \rho_0^{[2]}A^*\right] = 0,$$

(5.20)

$$\left[A^*, A^{*4}A^{[2]} - 2A^{*2}A^{[2]}A^{*2} + A^{[2]}A^{*4} + A^{*3}A^{[2]} - A^{*2}A^{[2]}A^* - A^*A^{[2]}A^{*2} + A^{[2]}A^{*3}\right.$$

$$\left. - A^{*2}A^{[2]} - 2A^*A^{[2]}A^* - A^{[2]}A^{*2} - A^*A^{[2]} - A^{[2]}A^*\right] = 0,$$

where γ and $\rho_0^{[2]}$ are scalars. We show that the complementary Bannai–Ito (5.6) and dual -1 Hahn (5.15) algebras are quotients of our non-symmetric generalization of the tridiagonal algebra, for certain values of the scalars.

Indeed, for the choice (5.12)-(5.13) the relations satisfied by the two operators $A^{[2]}$ and A^* can be derived based on the knowledge of the spectra and the action (3.3) of A^* on the eigenspaces for $A^{[2]}$. Applying similar techniques as the ones used to show Theorems 4.3 and 4.4, one finds that the two operators satisfy the relations (5.20) with

$$\gamma = 1/2 \quad \text{and} \quad \rho_0^{[2]} = g^2/4 + g/2 + 3/16.$$

In the case of the dual -1 Hahn algebra, the relations are satisfied with

$$\gamma = 0 \quad \text{and} \quad \rho_0^{[2]} = 1/4.$$

Now, observe that the defining relations of the complementary Bannai–Ito and dual -1 Hahn algebras imply[11] the relations (5.20): the second and third relations of (5.6) with $\delta_2 = 0$ and $\delta_1 = \rho_0^{[2]}$ (see (5.10)) imply (5.20) with $\gamma = 1/2$. Similarly, the second and third relations of (5.15) with $\gamma_2 = 0$ and $\gamma_1 = \rho_0^{[2]}$ (see (5.17)) imply (5.20) with $\gamma = 0$. We have thus demonstrated the following proposition.

5.5. Proposition. *We have homomorphisms from the non-symmetric generalization of the tridiagonal algebra* (5.20) *to the complementary Bannai–Ito and dual -1 Hahn algebras such as*

(5.21) $$A^{[2]} \mapsto \kappa_1, \qquad A^* \mapsto \kappa_2,$$

with the kernel described by the relations for κ_1 and κ_2 given in (5.6), *for the first case, and in* (5.15) *for the second.*

5.6. Remark. It is important to stress that the relations (5.20) hold in general for any dimension of the eigenspaces V_n and V_s^* – i.e., not necessarily one-dimensional eigenspaces – contrary to the relations (5.6) or (5.15).

5.7. Remark. According to the results of the previous Sections, it implies that the operator κ_1 can be written as a limit for $q = e^{i\pi/2}$ of a quadratic combination of a second-order $q-$difference operator with a non-degenerate spectrum of $q-$Racah type. Indeed, it is easy to check that the structure of $A^{[2]}$ with (3.12) for (5.12) or (5.18) and (5.13) agrees with the proposals [GVZ13, eq. (4.2)] and [TVZ11, eq. (5.6)], respectively.

To conclude, recall that tridiagonal pairs are certain generalizations of Leonard pairs. In this picture, the multivariable Gasper-Rahman orthogonal polynomials [GR04] find a natural interpretation within the theory of tridiagonal pairs [BM15], thanks to the construction of Iliev [Il08]. Thus, multivariable extensions of the complementary Bannai–Ito or dual -1 Hahn polynomials can be easily constructed by using the algebra (5.20). It is clear that a potential starting point for the analysis of the corresponding extended bispectral problem is given by the elements (3.12) for $N = 2$.

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[11]To obtain the first relation in (5.20), it is enough to compute the commutator of κ_1 with the left-hand side of (5.7), while for the second relation we use multiple commutators of (5.8) with κ_2.

Appendix A. Examples of cyclic TD pairs

In this Appendix, we construct our two basic examples of cyclic TD pairs. They both come from representation theory of quantum algebras specialized at roots of unity.

A.1. **Cyclic representations.** Let $q = e^{i\pi/4}$ and $A_\pm$ denote the 4×4 matrices

$$(A.1) \qquad A_\pm \quad = \quad \begin{pmatrix} 0 & q^{\pm 1}k_+ & 0 & q^{\mp 2}\alpha^2 k_- \\ q^{\pm 1}k_- & 0 & 2k_+ & 0 \\ 0 & k_- & 0 & q^{\mp 1}k_+ \\ 0 & 0 & q^{\mp 1}k_- & 0 \end{pmatrix}$$

where $k_\pm$ and α are complex numbers. These matrices are diagonalizable with the eigenvalues

$$(A.2) \qquad \text{Spec } A_\pm : \qquad \left\{ \pm\sqrt{q^{\pm 3}\alpha k_-^2 + k_+ k_-}, \pm\sqrt{q^{\mp 1}\alpha k_-^2 + k_+ k_-} \right\}.$$

We show that $A_\pm$ form a cyclic TD pair of cyclicity 4 under certain conditions on α and $k_\pm$. Assume first that k_- and α are non-zero. The eigenstates of A_+ can be written in components as

$$v_0 = \left\{ q^3 c_- \frac{\sqrt{k_-(q^{-1}\alpha k_- + k_+)}}{k_-}, \alpha + q^2 \frac{\sqrt{2}k_+}{k_-}, -q\frac{\sqrt{k_-(q^{-1}\alpha k_- + k_+)}}{k_-}, 1 \right\},$$

$$v_1 = \left\{ q^3 c_+ \frac{\sqrt{k_-(q^3\alpha k_- + k_+)}}{k_-}, -\alpha + q^2 \frac{\sqrt{2}k_+}{k_-}, q\frac{\sqrt{k_-(q^3\alpha k_- + k_+)}}{k_-}, 1 \right\},$$

$$v_2 = \left\{ -q^3 c_- \frac{\sqrt{k_-(q^{-1}\alpha k_- + k_+)}}{k_-}, \alpha + q^2 \frac{\sqrt{2}k_+}{k_-}, q\frac{\sqrt{k_-(q^{-1}\alpha k_- + k_+)}}{k_-}, 1 \right\},$$

$$v_3 = \left\{ -q^3 c_+ \frac{\sqrt{k_-(q^3\alpha k_- + k_+)}}{k_-}, -\alpha + q^2 \frac{\sqrt{2}k_+}{k_-}, -q\frac{\sqrt{k_-(q^3\alpha k_- + k_+)}}{k_-}, 1 \right\},$$

where we set $c_\pm = (\alpha k_- \pm \sqrt{2}k_+)/k_-$. We observe that the action of A_- on any of these eigenvectors v_p, with $p \in \mathbb{Z}_4$, is tridiagonal with non-zero coefficients and corresponds to the action of $C^* = A_-$ in (1.1) if we set $V_p^{(N=4)} = \mathbb{C}v_p$. For example, the coefficients in the action of A_- on v_0 (i.e. $A_-v_0 = c_0 v_0 + c_1 v_1 + c_2 v_2 + c_3 v_3$) are $c_2 = 0$, $c_0 = -2q\frac{k_+\sqrt{k_-(q^{-1}\alpha k_- + k_+)}}{\alpha k_-}$ (recall that we assume α and k_- are non-zero), and c_1 and c_3 are also non-zero (but more complicated) expressions in $k_\pm$ and α. Similarly, the action of A_- on v_1 is a linear combination of v_0, v_1 and v_2, etc. We similarly have an eigenbasis for A_- and the cyclic tridiagonal action of A_+. This shows that the conditions (i)-(iii) in Definition 1.1 are satisfied for $N = 4$, if the spectrum (A.2) is non-degenerate. Let us call values of α and $k_\pm$ *generic* if the spectrum (A.2) is non-degenerate and the coefficients $c_{i,i}$ and $c_{i,i\pm 1}$ in the action $A_-v_i = \sum_{j=0}^3 c_{i,j}v_j$ are non-zero and similarly for $c_{i,j}^*$ for the action of A_+ in the second basis. Then at such generic values of α and $k_\pm$ our example provides a family of cyclic TD pairs which are not ordinary TD pairs.

In this example, it is easy to see that for generic α and $k_\pm$ the condition (iv) holds as well or that the action of $A_\pm$ is irreducible. Indeed, we first note that the spectrum of A_+ is non-degenerate, therefore an invariant subspace W is a span of the (one-dimensional) A_+-eigenspaces (otherwise, W would not be invariant even with respect to the action of A_+). Then, we do not have a 1-dim invariant subspace: assume it is spanned by the A_+-eigenvector v_i for some $i = 0, 1, 2, 3$, then its image A_-v_i is in $\mathbb{C}v_{i-1} + \mathbb{C}v_i + \mathbb{C}v_{i+1}$, which is not in $\mathbb{C}v_i$. Suppose that we have a 2-dimensional invariant subspace $W = \mathbb{C}v_i + \mathbb{C}v_j$, then consider v_i – its image under the action of A_- is $c_{i,i}v_i + c_{i,i+1}v_{i+1} + c_{i,i-1}v_{i-1}$

with non zero coefficients, and it is thus not in W for any choice of v_j. Suppose then we have a 3-dimensional invariant subspace $W = \mathbb{C}v_i + \mathbb{C}v_j + \mathbb{C}v_k$, then again consider v_i – its image under the action of A_- is $c_{i,i}v_i + c_{i,i+1}v_{i+1} + c_{i,i-1}v_{i-1}$ with non zero coefficients. This action of A_- on $v_i \in W$ forces us to fix $j = i + 1$ and $k = i - 1$ or $j = i - 1$ and $k = i + 1$. For any of these choices, we have that $v_{i+1} \in W$ while $A_-v_{i+1} = c_{i+1,i+1}v_{i+1} + c_{i+1,i+2}v_{i+2} + c_{i+1,i}v_i$, again with non-zero coefficients for generic parameters α and $k_\pm$. We see here the non-zero contribution of $\mathbb{C}v_{i+2}$ which is not in W. We therefore conclude that there is only non-zero 4-dimensional invariant subspace $W = V$. Thus, there is no invariant proper subspaces under the action of the pair $A_\pm$, *i.e.*, the condition (iv) of Definition 1.1 holds indeed. We thus have the action of a cyclic TD pair with cyclicity 4. Note that we have actually a cyclic Leonard pair here (the cyclic TD pair with 'tridiagonal' instead of 'block tridiagonal' action).

We note that setting the parameter $\alpha = 0$ in (A.1), the pair $A_\pm$ becomes the (usual) TD pair of $q-$Racah type at $q = e^{i\pi/4}$, i.e. with the action from Definition 2.1.1. We have thus a continuous family of cyclic TD pairs parametrized by α that has a special point at $\alpha = 0$. We also note that for non-zero α our example is not of $q-$Racah type, i.e. it is not a specialization of a TD pair of $q-$Racah type.

We give a final remark on the origin of the construction of the $A_\pm$ operators. It is the result of the pull-back representation of the $q-$Onsager algebra (identifying $A_+ = \mathcal{W}_0$ and $A_- = \mathcal{W}_1$ in our $q-$Onsager algebra notations) under the homomorphism [B06]

$$(\text{A.3}) \qquad O_q(\widehat{sl_2}) \to U_q(sl_2): \qquad A_\pm \mapsto k_+ q^{\pm 1/2} E\sqrt{K^{\pm 1}} + k_- q^{\mp 1/2} F\sqrt{K^{\pm 1}} + \epsilon_\pm K^{\pm 1}$$

in the 4 dimensional semi-cyclic representation of $U_q(sl_2)$ at $q = e^{i\pi/4}$. Recall that the $U_q(sl_2)$ action is given in a basis $\{w_j, 0 \le j \le 3\}$ by (see e.g. [CP94, Sec. 11.1])

$$(\text{A.4}) \qquad Kw_j = q^{3-2j}w_j, \qquad Ew_j = [j]^2 w_{j-1}, \qquad Fw_j = w_{j+1}, \qquad \text{with} \quad w_4 = \alpha^2 w_0.$$

The two matrices $A_\pm$ given in (A.1) are derived from (A.3) with (A.4) where, for simplicity, we have chosen $\epsilon_\pm = 0$. By construction [B06], $A_\pm$ satisfy the defining relations of the $q-$Onsager algebra (2.11) with $\rho = (q + q^{-1})^2 k_+ k_-$, as can be checked explicitly.

A.2. $q-$Racah type representations.

We consider now a $q-$Racah type representation of the $q-$Onsager algebra and show that it satisfies Assumption 3.2. We thus construct a cyclic TD pair of the $q-$Racah type. The simplest and non-trivial one we were able to find is on an 8-dimensional vector space. We construct it as follows. Recall the homomorphism of the $q-$Onsager algebra $O_q(\widehat{sl_2})$ to the quantum affine algebra $U_q(\widehat{sl_2})$ defined on generators as [B06]

$$(\text{A.5}) \qquad \begin{aligned} \mathcal{W}_0 &\mapsto (k_+ E_1 + k_- F_1)\sqrt{K_1} + \epsilon_+ K_1, \\ \mathcal{W}_1 &\mapsto (k_+ F_0 + k_- E_0)\sqrt{K_0} + \epsilon_- K_0, \end{aligned}$$

where E_i, F_i, and[12] $\sqrt{K_i} = q^{H_i/2}$ are the standard Chevalley generators of $U_q(\widehat{sl_2})$. The image of this homomorphism is a coideal subalgebra and one can use the iterated coproduct of $U_q(\widehat{sl_2})$ to define an evaluation representation of $O_q(\widehat{sl_2})$ on the 8-dimensional vector space (for any $u_i \in \mathbb{C}^\times$)

$$(\text{A.6}) \qquad V = \mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2) \otimes \mathbb{C}^2(u_3) .$$

---

[12]Note that we use here q instead of $q^{1/2}$ in [B06].

This representation is obtained by the recursive formula (applied for $n = 3$ here) [B06]

$$(A.7) \qquad \begin{aligned} \mathcal{W}_0^{(n)} &= \left(k_+ u_n q^{1/2}\sigma_+ + k_- u_n^{-1} q^{-1/2}\sigma_-\right)\sqrt{K} \otimes \mathbf{1}^{(n-1)} + K \otimes \mathcal{W}_0^{(n-1)}, \\ \mathcal{W}_1^{(n)} &= \left(k_+ u_n^{-1} q^{-1/2}\sigma_+ + k_- u_n q^{1/2}\sigma_-\right)\sqrt{K^{-1}} \otimes \mathbf{1}^{(n-1)} + K^{-1} \otimes \mathcal{W}_1^{(n-1)}, \end{aligned}$$

with $\mathcal{W}_0^{(0)} = \epsilon_+$ and $\mathcal{W}_1^{(0)} = \epsilon_-$, $\sigma_\pm$ are Pauli matrices[13] and $\sqrt{K}$ is the diagonal 2×2 matrix with entries $q^{1/2}$ and $q^{-1/2}$.

We fix the three evaluation parameters for simplicity as $u_1 = u_2 = u_3 = 1$, abbreviate $\mathcal{W}_i = \mathcal{W}_i^{(3)}$ and set $\epsilon_\pm = 0$ to make the exposition below clearer. In a basis in V, we have then the matrices

$$(A.8) \quad \mathcal{W}_{0(1)} \equiv A_\pm = \begin{pmatrix} 0 & q^{\pm 2}k_+ & q^{\pm 1}k_+ & 0 & k_+ & 0 & 0 & 0 \\ q^{\pm 2}k_- & 0 & 0 & q^{\pm 1}k_+ & 0 & k_+ & 0 & 0 \\ q^{\pm 1}k_- & 0 & 0 & k_+ & 0 & 0 & k_+ & 0 \\ 0 & q^{\pm 1}k_- & k_- & 0 & 0 & 0 & 0 & k_+ \\ k_- & 0 & 0 & 0 & 0 & k_+ & q^{\mp 1}k_+ & 0 \\ 0 & k_- & 0 & 0 & k_- & 0 & 0 & q^{\mp 1}k_+ \\ 0 & 0 & k_- & 0 & q^{\mp 1}k_- & 0 & 0 & q^{\mp 2}k_+ \\ 0 & 0 & 0 & k_- & 0 & q^{\mp 1}k_- & q^{\mp 2}k_- & 0 \end{pmatrix},$$

where we set A_+ for $\mathcal{W}_0$ and A_- for $\mathcal{W}_1$. With respect to the $\mathcal{W}_0$ action thus defined, the vector space $V = \left(\mathbb{C}^2\right)^{\otimes 3}$ is decomposed onto a direct sum of four $\mathcal{W}_0$-eigenspaces

$$(A.9) \qquad V = V_{[+1]} \oplus V_{[-1]} \oplus V_{[q^2+1+q^{-2}]} \oplus V_{[-(q^2+1+q^{-2})]},$$

where each eigenspace $V_{[\lambda]}$ corresponds to the eigenvalue $\lambda\sqrt{k_+ k_-}$, and $V_{[\pm 1]}$ are 3-dimensional and $V_{[\pm(q^2+1+q^{-2})]}$ are 1-dimensional eigenspaces. Note that the spectrum of $\mathcal{W}_0$ is of the q−Racah type, i.e. of the form (2.3) with the identification:

$$(A.10) \qquad a = 0, \quad b = \frac{\sqrt{k_+ k_-}}{q - q^{-1}}q^{-3}, \quad c = -\frac{\sqrt{k_+ k_-}}{q - q^{-1}}q^3$$

and $p = 0, 1, 2, 3$. Introducing $\varkappa = \sqrt{k_+/k_-}$, the corresponding $\mathcal{W}_0$-eigenvectors for $V_{[\mp 1]}$ are

$$(A.11) \qquad v_1^{(\mp)} = \pm\frac{1}{q}\left(\varkappa^3, 0, 0, -q\varkappa, \mp\varkappa^2, 0, 0, \pm q\right),$$

$$(A.12) \qquad v_2^{(\mp)} = \pm\frac{1}{q}\left(0, \varkappa, (q^{-1} - q)\varkappa, \mp q^{-1}, -\varkappa, 0, \pm q, 0\right),$$

$$(A.13) \qquad v_3^{(\mp)} = \pm\frac{1}{q}\left(0, 0, \varkappa, \mp 1, \mp q\varkappa, \pm q, 0, 0\right)$$

and for $V_{[\mp(q^2+1+q^{-2})]}$ are

$$(A.14) \qquad v_\pm^{(0)} = \left(\mp q^3\varkappa^3, q^3\varkappa^2, q^2\varkappa^2, \mp q^2\varkappa, q\varkappa^2, \mp q\varkappa, \mp\varkappa, 1\right).$$

We next study the action of $\mathcal{W}_1$ on these 8 vectors (a direct calculation using (A.8)) and check that $\mathcal{W}_1$ acts as a block tridiagonal matrix: the standard ordering of $\mathcal{W}_0$-eigenspaces here is

$$(A.15) \qquad V_0 = V_{[-(q^2+1+q^{-2})]}, \qquad V_1 = V_{[-1]}, \qquad V_2 = V_{[+1]}, \qquad V_3 = V_{[q^2+1+q^{-2}]},$$

---

[13]$\sigma_\pm = (\sigma_x \pm \mathrm{i}\sigma_y)/2$ and $\sigma_{x,y,z}$ are the usual Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

i.e., the action is $\mathcal{W}_1 V_0 \subseteq V_0 \oplus V_1$, $\mathcal{W}_1 V_1 \subseteq V_0 \oplus V_1 \oplus V_2$, etc., and the coefficients in the action are certain rational functions in q where the denominator is a power of q or $(1+q^2)$, or $(1+q^4)$, or the product of them. An explicit block tridiagonal matrix for $\mathcal{W}_1$ can be obtained as $S^{-1} \mathcal{W}_1^{\mathrm{diag}} S$, where the matrix S is the transition matrix defined below in (A.16) and $\mathcal{W}_1^{\mathrm{diag}}$ is the diagonal matrix of $\mathcal{W}_1$ in its eigenbasis that we discuss now.

We have also a decomposition onto $\mathcal{W}_1$-eigenspaces as in (A.9) (i.e., $\mathcal{W}_1$ has the same spectrum), and similarly we construct the $\mathcal{W}_1$-eigenvectors as in (A.11)-(A.14) by replacing $q \to q^{-1}$. In this basis, the action of $\mathcal{W}_0$ is block tridiagonal where the coefficients are again rational functions in q with the denominators as described just above.

Setting $t = (1+q^2)(1+q^4)$, we give also the transition matrix S from the $\mathcal{W}_0$-eigenbasis to the $\mathcal{W}_1$-eigenbasis:

(A.16)

$$
S = \frac{1}{t}
\begin{pmatrix}
t & 0 & 0 & \frac{1-q^8}{q}\varkappa^2 & 0 & \frac{1-q^8}{q^2}\varkappa^2 & \frac{1-q^8}{q^3}\varkappa^2 & 0 \\
0 & t & \frac{1-q^8}{q} & 0 & \frac{1-q^8}{q^2} & 0 & 0 & \frac{1-q^8}{q^3}\varkappa^2 \\
0 & q^7 - q & t+q^6-q^8 & 0 & \frac{1}{q}+q^5-2q^7 & 0 & 0 & \frac{1+q^6-2q^8}{q^2}\varkappa^2 \\
\frac{q^5-q^3}{\varkappa^2} & 0 & 0 & 1+2q^4+q^6 & 0 & \frac{1-q^2+q^4-q^8}{q} & \frac{1+q^4-q^8-q^{10}}{q^2} & 0 \\
0 & 1-q^6 & q^7-q^5 & 0 & 1+q^2+2q^6 & 0 & 0 & \frac{1-q^4+q^6-q^8}{q}\varkappa^2 \\
\frac{q^2-q^4}{\varkappa^2} & 0 & 0 & q-q^3 & 0 & 2+q^4+q^6 & \frac{1}{q}-q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t
\end{pmatrix}
$$

and note that its determinant does not depend on q and $k_\pm$: $\det S = 1$.

According to [IT10, Theorem 1.17], the tensor product representation V of $O_q(\widehat{sl_2})$ in (A.6) is irreducible and thus there are no proper invariant subspaces for the action of $\mathcal{W}_0$ and $\mathcal{W}_1$. We have thus shown explicitly that $\mathcal{W}_0$ and $\mathcal{W}_1$ form a TD pair of the $q-$Racah type, as they should for an indeterminate q.

A.2.1. *Specialization to $q = e^{i\pi/3}$.* We specialize now q to the root of unity $e^{i\pi/3}$ and consider the representation of $\mathcal{W}_i = \mathcal{W}_i^{(3)}$ defined by (A.7) (or equivalently by (A.8)) at this value of q. We notice that the points (i)-(iii) of Assumption 3.2 hold: for the point (i), the common denominator of the diagonal and block tridiagonal matrices for $\mathcal{W}_0$ and $\mathcal{W}_1$ is some power of q multiplied by $t = (1+q^2)(1+q^4)$ that has a finite value at $q = e^{i\pi/3}$; for the point (ii), the entries of the transition matrix S from (A.16) have denominators with finite values at the specialization; and for the point (iii) the ratio $b/c = q^3$. The vector space $V = \left(\mathbb{C}^2\right)^{\otimes 3}$ is then decomposed onto a direct sum of three $\mathcal{W}_0$-eigenspaces (in agreement with Prop. 3.4)

$$V = V_{[+1]} \oplus V_{[-1]} \oplus V_{[0]},$$

where 3-dimensional $V_{[\pm 1]}$ corresponds to the eigenvalue $\pm\sqrt{k_+ k_-}$ and $V_{[0]}$ is to the zero eigenvalue, it is two-dimensional. Note that the two 1-dimensional eigenspaces $V_0(q) \equiv V_0$ and $V_3(q) \equiv V_3$ from (A.15) at generic q have a well defined specialization at $q = e^{i\pi/3}$ (in the basis (A.14) stated above) and contribute to the same 0-eigenvalue eigenspace $V_{[0]}$. The eigenvectors for $V_{[\mp 1]}$ are the specialization (the limit) of the vectors in (A.11)-(A.13), while an eigenbasis for $V_{[0]}$ is given by $v_\pm^{(0)}$ at $q = e^{i\pi/3}$ defined in (A.14). It is then straightforward to check that the action of $\mathcal{W}_1$ (under the

specialization of (A.8)) in this basis is cyclic:

$$\mathcal{W}_1 V_{[\mp 1]} \subset V_{[-1]} \oplus V_{[0]} \oplus V_{[+1]} \ ,$$
$$\mathcal{W}_1 V_{[0]} \subset V_{[-1]} \oplus V_{[0]} \oplus V_{[+1]} \ ,$$

with non-trivial contribution of each of the three direct summands in the action.

We have also eigenvectors for $\mathcal{W}_1$ similar to (the specialization of) (A.11)-(A.14) and the cyclic tridiagonal action of $\mathcal{W}_0$. We have thus demonstrated that the first three conditions in Definition 1.1 hold for $N = 3$. We were not however able to neither find non-trivial invariant subspaces of dimension less than 8 neither prove that there are no such subspaces. We thus can only conjecture[14] that the fourth (irreducibility) condition for $V = \mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2) \otimes \mathbb{C}^2(u_3)$ and for the action of $\mathcal{W}_0$, $\mathcal{W}_1$ in (A.7) in the definition of a cyclic TD pair holds, at least for generic values of the evaluation parameters u_i and generic $\epsilon_\pm$.

A.2.2. *Divided polynomials.* In this section, we compute the divided polynomials (3.12) associated with (A.7) at $N = 3$ or $q = e^{i\pi/3}$. To this end, one needs first to compute powers $\mathcal{W}_i^m$, $i = 0, 1$ for $m = 2, 3$. Define $w = k_+\sigma_+ + k_-\sigma_-$. From (3.12) with (3.11) and (A.10), by straightforward calculation one finds:

$$\mathcal{W}_0^{[3]} \ = \ w \otimes w \otimes w + k_+ k_- (w \otimes \mathbf{1} \otimes \mathbf{1} + w \otimes \ q^{2\sigma_z} \otimes \mathbf{1} - \mathbf{1} \otimes w \otimes \mathbf{1}) \ ,$$

and similarly for $\mathcal{W}_1^{[3]}$ replacing $q \to q^{-1}$ in the above expression. Note that the action of the divided polynomials on V_p, $p = 0, 1, 2, 3$ is given by Lemma 3.7.2 and Proposition 3.9 for $N = 3$.

Setting $k_\pm$ such that $\sqrt{k_+ k_-} = q - q^{-1}$, the coefficients a, b, and c (together with a^*, b^*, and c^*) satisfy the conditions (4.10), with $\beta = -3$. Therefore, the four operators $\mathcal{W}_i$ and $\mathcal{W}_i^{[3]}$, for $i = 1, 2$, satisfy the relations in Theorem 4.3 and Theorem 4.4.

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[14]One can certainly introduce the notion of *mock cyclic TD pairs* in analogy with [IT08] by weakening conditions – removing the "irreducibility" condition (iv) in Definition 1.1. We have thus demonstrated that our example is a mock cyclic TD pair.

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