Explicit calculation of multi-fold contour integrals of certain ratios of Euler gamma functions. Part 1.

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Abstract

In this paper we proceed to study properties of Mellin-Barnes (MB) transforms of Usyukina-Davydychev (UD) functions. In our previous papers [Nuclear Physics B 870 (2013) 243], [Nuclear Physics B 876 (2013) 322] we showed that multi-fold Mellin-Barnes (MB) transforms of Usyukina-Davydychev (UD) functions may be reduced to two-fold MB transforms and that the higher-order UD functions were obtained in terms of a differential operator by applying it to a slightly modified first UD function. The result is valid in $d = 4$ dimensions and its analog in $d = 4 - 2\varepsilon$ dimensions exits too [Theoretical and Mathematical Physics 177 (2013) 1515]. In [Nuclear Physics B 870 (2013) 243] the chain of recurrent relations for analytically regularized UD functions was obtained implicitly by comparing the left hand side and the right hand side of the diagrammatic relations between the diagrams with different loop orders. In turn, these diagrammatic relations were obtained due to the method of loop reductions for the triangle ladder diagrams proposed in 1983 by Belokurov and Usyukina. Here we reproduce these recurrent relations by calculating explicitly via Barnes lemmas the contour integrals produced by the left hand sides of the diagrammatic relations. In such a way we explicitly calculate a family of multi-fold contour integrals of certain ratios of Euler gamma functions. We make a conjecture that similar results for the contour integrals are valid for a wider family of smooth functions which includes the MB transforms of UD functions.

Keywords: Barnes lemmas; Mellin-Barnes transform; Usyukina-Davydychev functions

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1 Introduction

Off-shell triangle-ladder and box-ladder diagrams are the only family of the Feynman diagrams which were calculated at any loop order, for example in $d = 4$ space-time dimensions [1, 2, 3, 4] with all indices equal to 1 in the momentum space representation (m.s.r.) and in $d = 4 - 2\varepsilon$ space-time dimensions with indices equal to $1 - \varepsilon$ on the rungs of ladders in the m.s.r. too [5, 6]. For the important case of the ladder diagrams with all indices equal to 1 in the m.s.r. in $d = 4 - 2\varepsilon$ space-time dimensions the on-shell result for this family of diagrams is known only at the first three loops in the form of expansion in terms of ε [7, 8] up to a certain power of ε. The off-shell result for the whole family of the ladder diagrams in is unknown in $d = 4 - 2\varepsilon$ dimension.

The momentum integrals corresponding to the family of the ladder diagrams in $d = 4$ space-time dimensions result in UD functions [2, 3]. The order of the UD function is the loop order in the ladder diagram [2, 3, 9]. The ladder diagrams possess remarkable properties at the diagrammatic level, for example, in Refs. [10, 11] it was shown that the UD functions are invariant with respect to Fourier transformations. In Ref. [12, 9] it has been shown that such a property of Fourier invariance may be generalized to any three-point Green function via Mellin-Barnes transformation.

MB transforms of the UD functions were investigated in Refs. [13, 14]. It has been found under some analytical regularization of Ref. [1] that MB transform of n-order UD function is a linear combination of MB transforms of three UD functions of $(n-1)$-order. This means any ladder diagram of this family may be reduced via a chain of recurrent relations to the one-loop scalar massless triangle diagram, which may be expressed for any indices and in any dimensions in terms of Appell function F_4 [15, 16]. This chain of the recurrent relations for the analytically regularized UD functions in the double-uniform limit when removing this analytical regularization, is represented as a differential operator applied a to a slightly modified first UD function [14]. It has been shown there that if instead of MB transforms of UD functions we write any smooth function of the same arguments the structure of this differential operator will be maintained the same in this double uniform limit. This operator will be applied to the function of the lowest order in this chain of recurrent relations.

However, in the present paper we show that in the particular case when in the integrand of the contour integrals on the left hand sides of the diagrammatic relations the MB transforms of the UD functions stand, this chain of recurrent relations for the MB transforms of UD functions is produced by the contour integration. These contour integrals are calculated explicitly via the first and the second Barnes lemmas. Due to observation done in the previous paragraph, we make a conjecture that similar results for the contour integrals are valid for a wider family of smooth functions written instead of MB transforms of UD functions. In the next papers we describe this family of functions and also describe what kind of changes should be made for the contours of the integrals over complex variables for the case of other smooth functions different from certain ratios of Euler gamma functions. In this paper we focus on the contour integration via Barnes lemmas for the case when the integrand contains MB transforms of UD functions.

The Barnes lemmas were introduced in science about century ago. The first Barnes lemma has been proved in Ref. [17], the second Barnes lemma has been proved in Ref. [18]. They allow to integrate a product of several Euler gamma functions in a simple manner. The Barnes lemmas will help us to demonstrate the integral relations of Refs. [13, 14] by doing complex integration along the contours typical for MB transformation. In Ref. [13] in order to obtain the results for the contour integrals we simply compared the left and the right parts of the diagrammatic relations.

2 Proof

The integral relation we need to prove via Barnes lemmas is Eq. (13) of Ref. [13],

$$\oint_C dz_2 dz_3 \; D^{(u,v)}[1+\varepsilon_1 - z_3, 1+\varepsilon_2 - z_2, 1+\varepsilon_3] D^{(z_2,z_3)}[1+\varepsilon_2, 1+\varepsilon_1, 1+\varepsilon_3] = J\left[\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2\varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1\varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1\varepsilon_3}\right] \quad (1)$$

in which the parameters $\varepsilon_1, \varepsilon_2$ and ε_3 are three complex variables of analytical regularization used in Ref. [1], subject to condition

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0,$$

the factor J is a ratio of Euler gamma functions

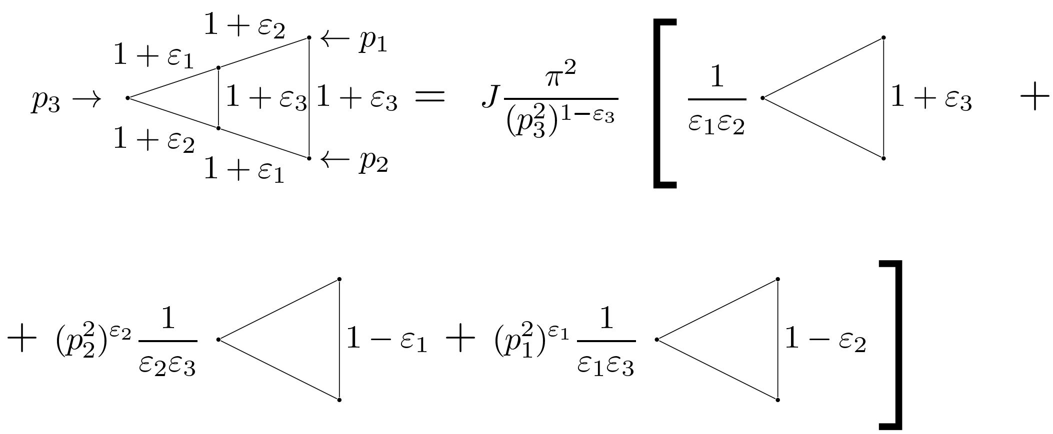

Figure 1: Equation (25) of Ref. [2] is the origin of integral relation Eq.(1).

$$J = \frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_2)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)},$$

the function $D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3]$ is the MB transform of one-loop triangle integral in the momentum space $J(\nu_1,\nu_2,\nu_3)$ taken in Ref.[13] from Refs. [16, 2, 3],

$$D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3] = \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_2-\nu_2-\nu_3+d/2)\Gamma(-z_3-\nu_1-\nu_3+d/2)}{\Pi_i\Gamma(\nu_i)} \times \frac{\Gamma(z_2+z_3+\nu_3)\Gamma(\Sigma\nu_i - d/2 + z_3 + z_2)}{\Gamma(d-\Sigma_i\nu_i)}, \quad (2)$$

and for the brevity the notation

$$D^{(u,v)}[1+\nu] \equiv D^{(u,v)}[1,1,1+\nu] \quad (3)$$

is used. The integral relations in Eq. (1) is produced by the diagrammatic relation between scalar Feynman diagrams in the momentum space given in Fig. 1 which is Eq. (25) of Ref. [2]. The derivation of this diagrammatic relation is reviewed in details in Ref. [13]. Also, in Ref. [13] the derivation of Eq. (1) from the diagrammatic relation of Fig. 1 may be found.

According to Eqs. (2) and (3), we write

$$D^{(z_2,z_3)}[1+\varepsilon_2,1+\varepsilon_1,1+\varepsilon_3] = \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_2+\varepsilon_2)\Gamma(-z_3+\varepsilon_1)\Gamma(1+z_2+z_3)\Gamma(1+z_2+z_3+\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)},$$

$$D^{(u,v)}[1+\varepsilon_1-z_3,1+\varepsilon_2-z_2,1+\varepsilon_3] = \frac{\Gamma(-u)\Gamma(-v)\Gamma(-u+\varepsilon_1+z_2)\Gamma(-v+\varepsilon_2+z_3)\Gamma(1+u+v+\varepsilon_3)\Gamma(1-z_2-z_3+u+v)}{\Gamma(1+\varepsilon_1-z_3)\Gamma(1+\varepsilon_2-z_2)\Gamma(1+\varepsilon_3)\Gamma(1+z_2+z_3)},$$

and the integrand on the left hand side of Eq.(1) is

$$D^{(u,v)}[1+\varepsilon_1-z_3,1+\varepsilon_2-z_2,1+\varepsilon_3]D^{(z_2,z_3)}[1+\varepsilon_2,1+\varepsilon_1,1+\varepsilon_3] = \frac{\Gamma(-u)\Gamma(-v)\Gamma(1+u+v+\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma^2(1+\varepsilon_3)}\times$$
$$\times\frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v)}{(\varepsilon_1-z_3)(\varepsilon_2-z_2)}. \quad (4)$$

On the right hand side of Eq.(1) we should obtain

$$\frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1\varepsilon_2} = \frac{1}{\varepsilon_1\varepsilon_2}\frac{\Gamma(-u)\Gamma(-v)\Gamma(-\varepsilon_3-u)\Gamma(-\varepsilon_3-v)\Gamma^2(1+\varepsilon_3+u+v)}{\Gamma(1-\varepsilon_3)\Gamma(1+\varepsilon_3)}, \quad (5)$$

$$\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2\varepsilon_3} = \frac{1}{\varepsilon_2\varepsilon_3}\frac{\Gamma(-u)\Gamma(\varepsilon_2-v)\Gamma(\varepsilon_1-u)\Gamma(-\varepsilon_3-v)\Gamma^2(1+\varepsilon_3+u+v)}{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_1)}, \quad (6)$$

$$\frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1\varepsilon_3} = \frac{1}{\varepsilon_1\varepsilon_3}\frac{\Gamma(\varepsilon_1-u)\Gamma(-v)\Gamma(-\varepsilon_3-u)\Gamma(\varepsilon_2-v)\Gamma^2(1+\varepsilon_3+u+v)}{\Gamma(1-\varepsilon_2)\Gamma(1+\varepsilon_2)}. \quad (7)$$

The poles at the points $z_2=\varepsilon_2$ and $z_3=\varepsilon_1$ were originally "right" since they come from the Euler gamma functions with negative signs of the integration variables of their arguments. The contribution of the corresponding residues at the points $z_2=\varepsilon_2$ and $z_3=\varepsilon_1$ in the integrand of Eq.(1) which is Eq.(4) reproduces term (5) on the right hand side of Eq.(1).

To obtain terms (6) and (7) on the right hand side of Eq.(1) we need to use the Barnes lemmas. The first lemma has been published in 1908 in Ref. [17]

$$\oint_C dz\,\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3-z)\Gamma(\lambda_4-z) = \frac{\Gamma(\lambda_1+\lambda_3)\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_3)\Gamma(\lambda_2+\lambda_4)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}, \quad (8)$$

in which $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are complex numbers, chosen in a such a way that on the right hand side of Eq. (8) there are no singularities, while the second Barnes lemma has been published in 1910 in

Ref. [18],

$$\oint_C dz \, \frac{\Gamma(\lambda_1+z)\Gamma(\lambda_2+z)\Gamma(\lambda_3+z)\Gamma(\lambda_4-z)\Gamma(\lambda_5-z)}{\Gamma(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+z)} =$$
$$\frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \tag{9}$$

in which $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are complex numbers, chosen in a such a way that on the right hand side of Eq. (9) there are no singularities.

The integrand of Eq. (4) may be represented as

$$\begin{aligned}
&\frac{1}{z_3-\varepsilon_1}\frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\times \\
&\qquad\times\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v) = \\
&= \frac{z_2+z_2+\varepsilon_3}{(z_3-\varepsilon_1)(z_2-\varepsilon_2)}\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\times \\
&\qquad\times\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v) = \\
&= \left(\frac{1}{z_3-\varepsilon_1}+\frac{1}{z_2-\varepsilon_2}\right)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\times \\
&\qquad\times\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v),
\end{aligned} \tag{10}$$

and this is a sum of two terms. We consider the second term,

$$\oint_C dz_2 \frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-u+\varepsilon_1+z_2)\oint_C dz_3\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\times$$
$$\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v),$$

in which the integral over z_3 may be calculated via the first Barnes lemma,

$$\begin{aligned}
&\oint_C dz_2 \frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-u+\varepsilon_1+z_2)\oint_C dz_3\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\times \\
&\qquad\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v) = \\
&\qquad\oint_C dz_2 \frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-u+\varepsilon_1+z_2)\times \\
&\times\frac{\Gamma(z_2+\varepsilon_3)\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)\Gamma(1+\varepsilon_2+u-z_2)}{\Gamma(1+u-\varepsilon_1)} = \\
&= \frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)}{\Gamma(1+u-\varepsilon_1)}\oint_C dz_2 \frac{1}{z_2-\varepsilon_2}\Gamma(z_2+\varepsilon_3)\Gamma(-z_2)\times \\
&\qquad\times\Gamma(-u+\varepsilon_1+z_2)\Gamma(1+\varepsilon_2+u-z_2).
\end{aligned}$$

Now we do reflection of the complex variable z_2 of contour integration, $z_2 \longrightarrow -z_2$, and apply the second Barnes lemma,

$$
\begin{aligned}
&\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)}{\Gamma(1+u-\varepsilon_1)}\oint_C dz_2\frac{1}{z_2-\varepsilon_2}\Gamma(z_2+\varepsilon_3)\Gamma(-z_2)\times\\
&\times\Gamma(-u+\varepsilon_1+z_2)\Gamma(1+\varepsilon_2+u-z_2)=\\
&-\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)}{\Gamma(1+u-\varepsilon_1)}\oint_C dz_2\frac{1}{z_2+\varepsilon_2}\Gamma(-z_2+\varepsilon_3)\Gamma(z_2)\times\\
&\times\Gamma(-u+\varepsilon_1-z_2)\Gamma(1+\varepsilon_2+u+z_2)=\\
&-\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)}{\Gamma(1+u-\varepsilon_1)}\oint_C dz_2\frac{\Gamma(z_2+\varepsilon_2)}{\Gamma(1+z_2+\varepsilon_2)}\Gamma(-z_2+\varepsilon_3)\Gamma(z_2)\times\\
&\times\Gamma(-u+\varepsilon_1-z_2)\Gamma(1+\varepsilon_2+u+z_2)=\\
&-\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)}{\Gamma(1+u-\varepsilon_1)}\Gamma(\varepsilon_3)\Gamma(-\varepsilon_1)\Gamma(1+u-\varepsilon_1)\times\\
&\frac{\Gamma(-u-\varepsilon_3)\Gamma(-u+\varepsilon_1)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)\Gamma(-u)}=\\
&\frac{1}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_3)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)}\times\\
&\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)\Gamma(-u-\varepsilon_3)\Gamma(-u+\varepsilon_1)}{\Gamma(-u)}.
\end{aligned}
$$

Taking into account the factor from Eq. (4), we obtain

$$
\begin{aligned}
&\frac{\Gamma(-u)\Gamma(-v)\Gamma(1+u+v+\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma^2(1+\varepsilon_3)}\times\frac{1}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_3)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)}\times\\
&\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)\Gamma(-u-\varepsilon_3)\Gamma(-u+\varepsilon_1)}{\Gamma(-u)}=\frac{J}{\varepsilon_1\varepsilon_3}D^{(u-\varepsilon_1,v)}[1-\varepsilon_2].
\end{aligned}
$$

The first term in Eq. (10) analogously reproduces term $\frac{J}{\varepsilon_2\varepsilon_3}D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]$ on the right hand side of Eq. (1). We need to comment that there is no double counting residues at the points $z_2=\varepsilon_2$ and $z_3=\varepsilon_1$ because after the reflection these points become “left” poles, that is, they come from Euler gamma functions with positive signs of the integration variable in the arguments of gamma functions, while we calculate the “right” residues only, that is, the residues which come from Euler gamma functions with negative signs of the integration variable in the arguments of gamma functions.

3 Conclusion

We showed in Ref.[14] [Nuclear Physics B 876 (2013) 322] that structure of the chain of recurrent relations for the Mellin-Barnes transforms of the analytically regularized UD functions guarantees the finiteness of the double-uniform limit when removing the analytical regularization. The limit was expressed in terms of a differential operator. This operator is the same for any smooth function written instead of the MB transforms of the UD functions and has nothing to do with explicit form of these MB transforms. The present paper shows that the first and the second Barnes lemmas permit to work out the contour integration only in a particular case of MB transforms of UD functions to produce this chain of the recurrent relations. For a wider family of smooth functions the Barnes lemmas should be replaced with another integration trick by using more complicate contour of integration.

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