

Bootstrapping $\mathcal{N} = 3$ superconformal theories

Madalena Lemos,¹ Pedro Liendo,¹ Carlo Meneghelli,² Vladimir Mitev³

¹*DESY Hamburg, Theory Group, Notkestrasse 85, D22607 Hamburg, Germany*

²*Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY 11794-3636, USA*

³*PRISMA Cluster of Excellence, Institut für Physik, JGU Mainz, Staudingerweg 7, 55128 Mainz, Germany*

E-mail: madalena.lemos@desy.de, pedro.liendo@desy.de,
cmeneghelli@scgp.stonybrook.edu, vmitev@uni-mainz.de

ABSTRACT: We initiate the bootstrap program for $\mathcal{N} = 3$ superconformal field theories (SCFTs) in four dimensions. We consider the problem from two fronts: the protected subsector described by a $2d$ chiral algebra, and crossing symmetry for half-BPS operators whose superconformal primaries parametrize the Coulomb branch of $\mathcal{N} = 3$ theories. In order to describe a protected subsector of a family of $\mathcal{N} = 3$ SCFTs, we study a $2d$ chiral algebra with super Virasoro symmetry that depends on an arbitrary parameter, identified with the central charge of the theory. Turning on to crossing, we work out the superconformal block expansion of the crossing equations and apply standard numerical bootstrap techniques in order to constrain the CFT data. We obtain bounds valid for any theory but also, thanks to input from the chiral algebra results, we are able to exclude solutions with $\mathcal{N} = 4$ supersymmetry, allowing us to zoom in on a specific $\mathcal{N} = 3$ SCFT.

KEYWORDS: conformal field theory, supersymmetry, conformal bootstrap

Contents

1	Introduction	1
2	$\mathcal{N} = 3$ chiral algebras	3
2.1	Chiral algebras for $\mathcal{N} = 3$ four-dimensional SCFTs	5
3	Superblocks	13
3.1	Superspace	13
3.2	Superconformal Ward identities	15
3.3	Selection rules	17
3.4	Superconformal blocks	17
4	The bootstrap equations	21
4.1	The crossing equations	21
4.2	Contribution of Schur operators	25
4.3	Explicit form of the bootstrap equations for $R = 2, 3$	28
5	Numerical results	30
5.1	Numerical methods	31
5.2	The case $R = 2$	32
5.3	The case $R = 3$	33
6	Conclusions	42
A	Unitary representations of the $\mathcal{N} = 3$ superconformal algebra	43
A.1	Decomposition in $\mathcal{N} = 2$ multiplets	44
B	OPEs of the chiral algebra	46
C	Some blockology	48
D	Crossing equations	49
D.1	Explicit expressions for $\mathcal{F}_{\text{short}}^{(0,\pm)}$	49
D.2	Summation for H_{short}	50
E	Generalized free theory example	51

1 Introduction

The study of superconformal symmetry has given invaluable insights into quantum field theory, and in particular into the nature of strong-coupling dynamics. The presence of supersymmetry gives us additional analytical tools and allows for computations that are otherwise hard to perform. A cursory look at the superconformal literature in four dimensions shows a vast number of works on $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal field theories (SCFTs), with the intermediate case of $\mathcal{N} = 3$ almost absent.

The main reason for this is that, due to CPT invariance, the Lagrangian formulation of any $\mathcal{N} = 3$ theory becomes automatically $\mathcal{N} = 4$. By now however, there is a significant amount of evidence that superconformal theories are not restricted to just Lagrangian examples, and this has inspired recent papers that revisit the status of $\mathcal{N} = 3$ SCFTs.

Assuming these theories exist, the authors of [1] studied several of their properties. They found in particular that the a and c anomaly coefficients are always the same, that, in stark contrast with the most familiar $\mathcal{N} = 2$ theories, pure $\mathcal{N} = 3$ theories (*i.e.*, theories whose symmetry does not enhance to $\mathcal{N} = 4$) have no marginal deformations and are therefore always isolated, and also, that pure $\mathcal{N} = 3$ SCFTs cannot have a flavor symmetry that is not an R-symmetry.

Since the only possible free multiplet of an $\mathcal{N} = 3$ SCFT is a vector multiplet, the low energy theory at a generic point on the moduli space must involve vector multiplets, and the types of short multiplets whose expectation values can parametrize such branches were analyzed in [1]. When an $\mathcal{N} = 3$ vector multiplet is decomposed in $\mathcal{N} = 2$, it contains both an $\mathcal{N} = 2$ vector and hyper multiplet, which implies that the theories possess both $\mathcal{N} = 2$ Higgs and Coulomb branches that are rotated by $\mathcal{N} = 3$.

Shortly after [1], the authors of [2] presented the first evidence for $\mathcal{N} = 3$ theories by studying N D3-branes in the presence of an S-fold plane, which is a generalization of the standard orientifold construction that also includes the S-duality group. The classification of different variants of $\mathcal{N} = 3$ preserving S-folds was done in [3], leading to additional $\mathcal{N} = 3$ SCFTs, obtained by N D3-branes probing the various S-folds. In [4] yet another generalization was considered, in which in addition to including the S-duality group in the orientifold construction, one also considers T-duality. This background is known as a U-fold, and the study of M5-branes on this background leads to $\mathcal{N} = 3$ theories associated with the exceptional $(2, 0)$ theories.

The systematic study of rank one $\mathcal{N} = 2$ SCFTs (*i.e.*, with a one complex dimensional Coulomb branch) through their Coulomb branch geometries [5–8] has recovered the known $\mathcal{N} = 3$ SCFTs, as well as led to new ones [7, 9]. Some of these theories are obtained by starting from $\mathcal{N} = 4$ SYM with gauge group $U(1)$ or $SU(2)$ and gauging discrete symmetries, while others correspond to genuine $\mathcal{N} = 3$ SCFTs which are not obtained by discrete gauging. Note that, as emphasized in [3, 9], gauging by a discrete symmetry does not change the local dynamics of the theory on \mathbb{R}^4 , only the spectrum of local and non-local operators. In particular, the central charges and correlation functions remain the same.

Of the class of theories constructed in [3], labeled by the number N of D3-branes and by integers k, ℓ associated to the S-fold, some have enhanced $\mathcal{N} = 4$ supersymmetry, or arise as discretely gauged versions of $\mathcal{N} = 4$. The non-trivial $\mathcal{N} = 3$ SCFT with the smallest central charge corresponds to the theory labeled by $N = 1$ and $\ell = k = 3$ in [3], with central charge given by $\frac{15}{12}$. This corresponds to a rank one theory with Coulomb branch parameter of scaling dimension three, and it will be one of the main focus of this work. Since the Coulomb branch operators of $\mathcal{N} = 3$ theories must have integer dimensions [1], and since theories with a Coulomb branch generator of dimension one or two enhance to $\mathcal{N} = 4$, it follows that dimension three is the smallest a genuine $\mathcal{N} = 3$ theory with a Coulomb branch can have, and that this theory could indeed correspond to the “minimal” $\mathcal{N} = 3$ SCFT. By increasing the number of D3-branes, we obtain higher rank versions of this minimal theory. More generally, the rank N theories with $k = \ell$, are not obtained from others by discretely gauging, and have an N dimensional Coulomb branch.

Since pure $\mathcal{N} = 3$ SCFTs have no relevant or marginal deformations, they are hard to study by standard field theoretical approaches. Apart from the aforementioned papers, recent progress in understanding $\mathcal{N} = 3$ theories includes [10–13]. The classification of all $\mathcal{N} = 3$ SCFTs is not complete

yet, and one can wonder if there are theories not arising from the S-fold (and generalizations thereof) constructions. On the other hand, one would like to obtain more information on the spectrum of the currently known theories. In this paper we take the superconformal bootstrap approach to address these questions, and tackle $\mathcal{N} = 3$ SCFTs by studying the operators that parametrize the Coulomb branch. These operators sit in half-BPS multiplets of the $\mathcal{N} = 3$ superconformal algebra, and when decomposed in $\mathcal{N} = 2$ language contain both Higgs and Coulomb branch operators. We will mostly focus on the simplest case of Coulomb branch operators of dimension three.

The bootstrap approach does not rely on any Lagrangian or perturbative description of the theory. It depends only on the existence of an associative local operator algebra and on the symmetries of the theory in question, and is therefore very well suited to the study of $\mathcal{N} = 3$ SCFTs. Since the original work of [14] there have been many papers that study SCFTs through the lens of the numerical bootstrap, for a partial list see [15–29]. A necessary ingredient in any superconformal bootstrap analysis is the computation of the superconformal blocks relevant for the theory in question, efforts in this direction include [30–39]. Although correlation functions of half-BPS operators in various dimensions have been studied [31], the case of $\mathcal{N} = 3$ has not yet been considered, and calculating the relevant superconformal blocks will be one of the goals of this paper.

Also relevant for our work is the information encoded in the $2d$ chiral algebras associated to $4d$ SCFTs [40–54]. The original analysis of [40] implies that any four-dimensional $\mathcal{N} \geq 2$ SCFT contains a closed subsector of local operators isomorphic to a $2d$ chiral algebra. For $\mathcal{N} = 3$ theories, part of the extra supercharges of the $\mathcal{N} = 3$ make it to the chiral algebra and therefore its symmetry enhances to $\mathcal{N} = 2$ super Virasoro [10]. The authors of [10] constructed a family of chiral algebras conjectured to describe the rank one $\mathcal{N} = 3$ theories, generalizing these chiral algebras in order to accommodate the higher-rank cases will be another subject of this work.

The paper is organized as follows. In section 2 we study the two-dimensional chiral algebras associated with $\mathcal{N} = 3$ SCFTs, determining the $\mathcal{N} = 3$ superconformal multiplets they capture, and some of their general properties. We then attempt to construct chiral algebras for the higher rank $\ell = k = 3$ theories. We propose a set of generators that, under certain assumptions, describe a closed subalgebra of these theories, and write down an associative chiral algebra for them. Associativity fixes all OPE coefficients in terms of a single parameter: the central charge of the theory. In section 3.4 we use harmonic superspace techniques in order to obtain the superconformal blocks that will allow us to derive the crossing equations for half-BPS operators of section 4. We focus mostly on a dimension three operator, but also present the case of a dimension two operator as a warm up. Section 5 presents the results of the numerical bootstrap, both for generic $\mathcal{N} = 3$ SCFTs and also attempting to zoom in to the simplest known $\mathcal{N} = 3$ theory by inputting data from the chiral algebra analysis of section 2. We conclude with an overview of the paper as well as directions for future research in section 6.

2 $\mathcal{N} = 3$ chiral algebras

Every $4d$ $\mathcal{N} = 2$ SCFT contains a protected sector that is isomorphic to a $2d$ chiral algebra, obtained by passing to the cohomology of a nilpotent supercharge [40]. Because $\mathcal{N} = 3$ is a special case of $\mathcal{N} = 2$, one can also study chiral algebras associated to $\mathcal{N} = 3$ SCFTs. This program was started for rank one theories in [10], and here we explore possible modifications such that one can describe higher-rank cases as well. We will put particular emphasis on theories containing a Coulomb branch operator with scaling dimension three, since these are the correlators we will study numerically in section 5.

In order to do this we will need extensive use of the representation theory of the $\mathcal{N} = 3$ superconformal algebra; this was studied in [1, 55–59] and is briefly reviewed in appendix A. We will also leverage previous knowledge of chiral algebras for $\mathcal{N} = 2$ SCFTs, and so it will be useful to view $\mathcal{N} = 3$ theories from an $\mathcal{N} = 2$ perspective. Therefore, we will pick an $\mathcal{N} = 2$ subalgebra of $\mathcal{N} = 3$, with the $SU(3)_R \times U(1)_r$ R-symmetry of the latter decomposing in $SU(2)_{R_{\mathcal{N}=2}} \times U(1)_{r_{\mathcal{N}=2}} \times U(1)_f$. The first two factors make up the R-symmetry of the $\mathcal{N} = 2$ superconformal algebra and the last corresponds to a global symmetry. Therefore, from the $\mathcal{N} = 2$ point of view, all $\mathcal{N} = 3$ theories necessarily have a $U(1)_f$ flavor symmetry arising from the extra R-symmetry currents. The additional supercharges and the $U(1)_f$ flavor symmetry imply that the Virasoro symmetry expected in the $\mathcal{N} = 2$ chiral algebras will be enhanced to a super Virasoro symmetry in the $\mathcal{N} = 3$ case [10].

Let us start reviewing the essentials of the chiral algebra construction (we refer the reader to [40] for more details). The elements of the protected sector are given by the cohomology of a nilpotent supercharge \mathbb{Q} that is a linear combination of a Poincaré and a conformal supercharge,

$$\mathbb{Q} = \mathcal{Q}^1_- + \tilde{S}^{2\dot{-}}. \quad (2.1)$$

In order to be in the cohomology operators have to lie on a fixed plane $\mathbb{R}^2 \subset \mathbb{R}^4$. The global conformal algebra on the plane $\mathfrak{sl}(2) \times \bar{\mathfrak{sl}}(2)$ is a subalgebra of the the four-dimensional conformal algebra. While the generators of the $\mathfrak{sl}(2)$ commute with (2.1), those of $\bar{\mathfrak{sl}}(2)$ do not, and an operator in the cohomology at the origin will not remain in the cohomology if translated by the latter. However, it is possible to introduce twisted translations obtained by the diagonal subalgebra of the $\bar{\mathfrak{sl}}(2)$ and a complexification $\mathfrak{sl}(2)_R$ of the R-symmetry algebra $\mathfrak{su}(2)_R$, such that the supercharge satisfies

$$[\mathbb{Q}, \mathfrak{sl}(2)] = 0, \quad [\mathbb{Q}, \text{something}] = \text{diag}(\bar{\mathfrak{sl}}(2) \times \mathfrak{sl}(2)_R). \quad (2.2)$$

From these relations one can prove that \mathbb{Q} -closed operators restricted to the plane have meromorphic correlators. We call the operators that belong to the cohomology of \mathbb{Q} “Schur” operators. The Schur operators in $\mathcal{N} = 2$ language are local primary fields which obey the conditions

$$\Delta - (j + \bar{j}) - 2R_{\mathcal{N}=2} = 0, \quad \bar{j} - j - r_{\mathcal{N}=2} = 0. \quad (2.3)$$

The cohomology classes of the twisted translations of any such operator \mathcal{O} corresponds to a $2d$ local meromorphic operator

$$\mathcal{O}(z) = [\mathcal{O}(z, \bar{z})]_{\mathbb{Q}}. \quad (2.4)$$

The two important Schur operators that we expect to have in any $\mathcal{N} = 2$ theory with a flavor symmetry are¹

- $\hat{\mathcal{C}}_{0(0,0)}$: The highest-weight component of the $SU(2)_{R_{\mathcal{N}=2}}$ current (with charges $\Delta = 3, j = \bar{j} = \frac{1}{2}, R_{\mathcal{N}=2} = 1, r_{\mathcal{N}=2} = 0$) corresponding to the $2d$ stress tensor $T(z)$.
- $\hat{\mathcal{B}}_1$: The highest-weight component J^{11} of the moment map operator ($\Delta = 2, j = \bar{j} = 0, R_{\mathcal{N}=2} = 1$ and $r_{\mathcal{N}=2} = 0$) is mapped to the affine current $J(z)$ of the flavor group.

These two Schur operators give rise to a Virasoro and an affine symmetry in the chiral algebra [40] respectively, with the two-dimensional central charges obtained in terms of their four-dimensional

¹We follow the conventions of [60] for $\mathcal{N} = 2$ superconformal multiplets.

counterparts by

$$c_{2d} = -12c_{4d}, \quad k_{2d} = -\frac{k_{4d}}{2}. \quad (2.5)$$

Note that, since we insist on having unitarity in the four-dimensional theory, the $2d$ chiral algebra will be necessarily non-unitary.

The chiral algebra description of a protected subsector of correlators is extremely powerful. By performing the twist of [40] on a four-dimensional correlation function of Schur operators, we are left with a meromorphic $2d$ correlator that is completely determined by knowledge of its poles and residues. The poles can be understood by taking various OPE limits, thus fixing the correlator in terms of a finite number of parameters corresponding to OPE coefficients, as will be done in subsection 4.1.1. In the cases we will study in section 5, the meromorphic piece can be fixed using crossing symmetry in terms of a single parameters, which can be identified with the central charge of the theory. It is important to emphasize that this can be done without knowledge of which particular chiral algebra is relevant for the SCFT at hand.

2.1 Chiral algebras for $\mathcal{N} = 3$ four-dimensional SCFTs

Let us now study the $\mathcal{N} = 3$ case in more detail. Any local $\mathcal{N} = 3$ SCFT will necessarily contain a stress tensor multiplet, which in table 8 corresponds to $\hat{\mathcal{B}}_{[1,1]}$. After an $\mathcal{N} = 2$ decomposition of this multiplet (shown in (A.1)) one finds four terms, each contributing with a single representative to the chiral algebra. These four multiplets are related by the action of the extra supercharges enhancing $\mathcal{N} = 2$ to $\mathcal{N} = 3$, and four of these (\mathcal{Q}^3_+ and $\tilde{\mathcal{Q}}_{3\dot{+}}$ and their conjugates) commute with \mathfrak{Q} [10]. Therefore, acting on Schur operators with these supercharges produces new Schur operators, and the representatives of the four multiplets will be related by these two supercharges. The multiplets and their representatives are:

- A multiplet containing the $U(1)_f$ flavor currents ($\hat{\mathcal{B}}_1$), whose moment map M^{IJ} gives rise to a two-dimensional current $J(z) = [M(z, \bar{z})]_{\mathfrak{Q}}$ of an $U(1)_f$ affine Kac-Moody (AKM) algebra,
- Two “extra” supercurrents, responsible for the enhancement to $\mathcal{N} = 3$, which as shown in [40] contribute as operators of holomorphic dimension $\frac{3}{2}$. These are obtained from the moment map by the action of the supercharges $G(z) = [\mathcal{Q}^3_+ M(z, \bar{z})]_{\mathfrak{Q}}$ and $\tilde{G}(z) = [\tilde{\mathcal{Q}}_{3\dot{+}} M(z, \bar{z})]_{\mathfrak{Q}}$ [10].²
- The stress tensor multiplet ($\hat{\mathcal{C}}_{0,(0,0)}$) which gives rise to the stress tensor of the chiral algebra $T(z) = \frac{1}{2} \left[[\mathcal{Q}^3_+, \tilde{\mathcal{Q}}_{3\dot{+}}] M(z, \bar{z}) \right]_{\mathfrak{Q}}$ [10].

The supercharges \mathcal{Q}^3_+ and $\tilde{\mathcal{Q}}_{3\dot{+}}$ have charges ± 1 under the $U(1)_f$ flavor symmetry, where we follow the $U(1)_f$ charge normalizations of [10]. Therefore the operators $G(z)$ and $\tilde{G}(z)$ have a J charge of $+1$ and -1 respectively. This multiplet content is exactly the one we would expect from the considerations in the beginning of this section, with the extra supercharges, that commute with \mathfrak{Q} , producing a global $d = 2$, $\mathcal{N} = 2$ superconformal symmetry.³ Moreover, the operator content we just described corresponds precisely to the content of an $\mathcal{N} = 2$ stress tensor superfield which we denote by \mathcal{J} , enhancing the Virasoro algebra to an $\mathcal{N} = 2$ super Virasoro algebra [10].

²These arise from $\mathcal{N} = 2$ multiplets $\overline{\mathcal{D}}_{\frac{1}{2}(0,0)}$ and $\mathcal{D}_{\frac{1}{2}(0,0)}$ respectively in the notation of [60].

³The holomorphic $\mathfrak{sl}(2)$ that commutes with the supercharge \mathfrak{Q} , more precisely the \mathfrak{Q} -cohomology of the superconformal algebra, is enhanced to a $\mathfrak{sl}(2|1)$.

2.1.1 $\mathcal{N} = 3$ superconformal multiplets containing Schur operators

Our next task is to understand which multiplets of the $\mathcal{N} = 3$ superconformal algebra contribute to the chiral algebra, aside from the already discussed case of the stress tensor multiplet.

Instead of searching for superconformal multiplets that contain conformal primaries satisfying (2.3), we will take advantage of the fact that this was already done in [40] for $\mathcal{N} = 2$ multiplets, and simply search for $\mathcal{N} = 3$ multiplets that contain $\mathcal{N} = 2$ Schur multiplets. To accomplish this, we decompose $\mathcal{N} = 3$ multiplets in $\mathcal{N} = 2$ ones by performing the decomposition of the corresponding characters. In appendix A we present a few examples of such decompositions. Going systematically through the multiplets,⁴ we find the following list of $\mathcal{N} = 3$ Schur multiplets:

$$\hat{\mathcal{C}}_{[R_1, R_2], (j, \bar{j})} |_{\text{Schur}} = u_f^{R_2 - R_1 + 2(\bar{j} - j)} \left[\hat{\mathcal{C}}_{\frac{R_1 + R_2}{2}, (j, \bar{j})} \oplus u_f^{-1} \hat{\mathcal{C}}_{\frac{R_1 + R_2}{2}, (j, \bar{j} + \frac{1}{2})} \oplus u_f \hat{\mathcal{C}}_{\frac{R_1 + R_2}{2}, (j + \frac{1}{2}, \bar{j})} \oplus \hat{\mathcal{C}}_{\frac{R_1 + R_2}{2}, (j + \frac{1}{2}, \bar{j} + \frac{1}{2})} \right], \quad (2.6)$$

$$\hat{\mathcal{B}}_{[R_1, R_2]} |_{\text{Schur}} = u_f^{R_2 - R_1} \left[\hat{\mathcal{B}}_{\frac{R_1 + R_2}{2}} \oplus u_f^{-1} \mathcal{D}_{\frac{R_1 + R_2 - 1}{2}, (0, 0)} \oplus u_f \bar{\mathcal{D}}_{\frac{R_1 + R_2 - 1}{2}, (0, 0)} \oplus \hat{\mathcal{C}}_{\frac{R_1 + R_2 - 2}{2}, (0, 0)} \right], \quad \text{for } R_1 R_2 \neq 0, \quad (2.7)$$

$$\hat{\mathcal{B}}_{[R_1, 0]} |_{\text{Schur}} = u_f^{-R_1} \left[\hat{\mathcal{B}}_{\frac{R_1}{2}} \oplus u_f \bar{\mathcal{D}}_{\frac{R_1 - 1}{2}, (0, 0)} \right], \quad (2.8)$$

$$\hat{\mathcal{B}}_{[0, R_2]} |_{\text{Schur}} = u_f^{R_2} \left[\hat{\mathcal{B}}_{\frac{R_2}{2}} \oplus u_f^{-1} \mathcal{D}_{\frac{R_2 - 1}{2}, (0, 0)} \right], \quad (2.9)$$

$$\mathcal{D}_{[R_1, R_2], \bar{j}} |_{\text{Schur}} = u_f^{R_2 - R_1 + 2\bar{j} + 2} \left[\mathcal{D}_{\frac{R_1 + R_2}{2}, (0, \bar{j})} \oplus u_f^{-1} \mathcal{D}_{\frac{R_1 + R_2}{2}, (0, \bar{j} + \frac{1}{2})} \oplus u_f \hat{\mathcal{C}}_{\frac{R_1 + R_2 - 1}{2}, (0, \bar{j})} \oplus \hat{\mathcal{C}}_{\frac{R_1 + R_2 - 1}{2}, (0, \bar{j} + \frac{1}{2})} \right] \quad \text{for } R_1 > 0, \quad (2.10)$$

$$\bar{\mathcal{D}}_{[R_1, R_2], j} |_{\text{Schur}} = u_f^{R_2 - R_1 - 2j - 2} \left[\bar{\mathcal{D}}_{\frac{R_1 + R_2}{2}, (j, 0)} \oplus u_f^{-1} \hat{\mathcal{C}}_{\frac{R_1 + R_2 - 1}{2}, (j, 0)} \oplus u_f \bar{\mathcal{D}}_{\frac{R_1 + R_2}{2}, (j + \frac{1}{2}, 0)} \oplus \hat{\mathcal{C}}_{\frac{R_1 + R_2 - 1}{2}, (j + \frac{1}{2}, 0)} \right] \quad \text{for } R_2 > 0, \quad (2.11)$$

$$\mathcal{D}_{[0, R_2], \bar{j}} |_{\text{Schur}} = u_f^{R_2 + 2\bar{j} + 2} \left[\mathcal{D}_{\frac{R_2}{2}, (0, \bar{j})} \oplus u_f^{-1} \mathcal{D}_{\frac{R_2}{2}, (0, \bar{j} + \frac{1}{2})} \right], \quad (2.12)$$

$$\bar{\mathcal{D}}_{[R_1, 0], j} |_{\text{Schur}} = u_f^{-R_1 - 2j - 2} \left[\bar{\mathcal{D}}_{\frac{R_1}{2}, (j, 0)} \oplus u_f \bar{\mathcal{D}}_{\frac{R_1}{2}, (j + \frac{1}{2}, 0)} \right]. \quad (2.13)$$

Let us stress again that we are not showing the full decomposition in $\mathcal{N} = 2$ multiplets, but only the Schur multiplets. In performing the decompositions we kept the grading of the $\mathcal{N} = 2$ multiplets with respect to the $U(1)_f$ flavor symmetry, denoting the corresponding fugacity by u_f .

Some noteworthy multiplets in this list are the stress tensor multiplet $\hat{\mathcal{B}}_{[1, 1]}$, already discussed in the beginning of this subsection, as well as the half-BPS operators $\hat{\mathcal{B}}_{[R_1, 0]}$ (and their conjugates $\hat{\mathcal{B}}_{[0, R_1]}$) which are connected to the Coulomb branch, as discussed in section 1. Due to their physical significance we present their full decomposition in $\mathcal{N} = 2$ multiplets in A.1 and A.2. As described in

⁴One can quickly see that in table 8 multiplets that obey no $\mathcal{N} = 3$ shortening conditions on the one of the sides also obey no $\mathcal{N} = 2$ shortening condition on one of the sides, and these are known [40] not to contain Schur operators. Therefore we must go only through the multiplets that obey shortening conditions on both sides.

[58], there are no *relevant* Lorentz invariant supersymmetric deformations of $\mathcal{N} = 3$ theories, while the only such deformations that are *exactly marginal* are contained in the multiplet $\hat{\mathcal{B}}_{[2,0]}$ (and conjugate $\hat{\mathcal{B}}_{[0,2]}$). However, these multiplets also contain additional supersymmetry currents, as can be seen from their $\mathcal{N} = 2$ decomposition, that allow for the enhancement of $\mathcal{N} = 3$ to $\mathcal{N} = 4$, and thus pure $\mathcal{N} = 3$ theories are not expected to have exactly marginal operators. Let us also recall that the multiplets $\hat{\mathcal{C}}_{[0,0),(j,\bar{j})}$ contain conserved currents of spin larger than two, and therefore are expected to be absent in interacting theories [61, 62].

Quasi-primaries and Virasoro primaries

Each of the $\mathcal{N} = 2$ multiplets listed above will contribute to the chiral algebra with exactly one *global conformal primary* (also called quasi-primary), with holomorphic dimension as given in table 1 of [40] and with $U(1)_f$ charge f , under the $J(z)$ current, as can be read off from the u_f fugacity in the above decompositions. These multiplets generically will not be Virasoro primaries. As shown in [40], only the so-called Hall-Littlewood (HL) operators⁵ ($\hat{\mathcal{B}}_R, \bar{\mathcal{D}}_{R,(j,0)}$ and $\mathcal{D}_{R,(0,\bar{j})}$) are actually guaranteed to be Virasoro primaries. The remaining multiplets will appear in the chiral algebras sometimes as Virasoro primaries, sometimes as only quasi-primaries.

Super Virasoro primaries

Similarly, each $\mathcal{N} = 3$ multiplet gives rise in the chiral algebra to a global supermultiplet consisting of a global super primary and its three global superdescendants obtained by the action of \mathcal{Q}_+^3 and $\bar{\mathcal{Q}}_{3+}$.⁶ Generically however, these multiplets will not be super Virasoro primaries, even if the global super primary corresponds to a Virasoro primary. Recall that a super Virasoro primary must, in addition to being a Virasoro primary, have at most a pole of order one in its OPE with both $G(z)$ and $\tilde{G}(z)$, and have at most a singular term of order one in the OPE with $J(z)$.⁷ This last condition corresponds to being an AKM primary.

Let us consider the operators which have as a global superprimary a Virasoro primary. For the case of $\hat{\mathcal{B}}_{[R_1,R_2]}$ multiplets, we see that its two (or one in case $R_1 R_2 = 0$) level $\frac{1}{2}$ descendants are HL operators, and thus Virasoro primaries. The two-dimensional superconformal algebra then implies that the global superprimary is not only a Virasoro primary, but that it is also annihilated by all the modes $G_{n \geq +\frac{1}{2}}, \tilde{G}_{n \geq +\frac{1}{2}}$. However, this is not enough to make it a super Virasoro primary, as it is not guaranteed that these operators are AKM primaries. An obvious example is the stress tensor multiplet, where the AKM current is clearly not an AKM primary, and thus $\hat{\mathcal{B}}_{[1,1]}$ does not give rise to a super Virasoro primary.

The other guaranteed Virasoro primaries sit in $\mathcal{D}_{[R_1,R_2],\bar{j}}$ multiplets and their conjugates, for which the global superprimary is a HL operator. The authors of [10] showed that, even though one of its level $\frac{1}{2}$ descendants is not a HL operator, it is still a Virasoro primary, and the same considerations as for the $\hat{\mathcal{B}}_{[R_1,R_2]}$ multiplets apply.

In certain cases it is possible to show that the operators in question are actually super Virasoro primaries, and concrete examples will be given below. For example, if one considers a $\hat{\mathcal{B}}_{[R_1,R_2]}$ generator

⁵Following [40] we refer to operators which are $\mathcal{N} = 1$ chiral and satisfy the Schur condition as Hall-Littlewood operators.

⁶Recall that the global superprimary is annihilated only by the $G_{-\frac{1}{2}}, \tilde{G}_{-\frac{1}{2}}, L_{-\geq 0}$ modes of $G(z), \tilde{G}(z), T(z)$, and global super descendants are obtained by the action of $G_{+\frac{1}{2}}$ and $\tilde{G}_{+\frac{1}{2}}$.

⁷These conditions translate into the following modes annihilating the superprimary state: $L_{n \geq 0}, G_{n \geq +\frac{1}{2}}, \tilde{G}_{n \geq +\frac{1}{2}}$ and $J_{n \geq 0}$.

that is not the stress tensor multiplet, then the OPE selection rules for the $\mathcal{N} = 2$ $\hat{\mathcal{B}}_{\frac{R_1+R_2}{2}}$ multiplet [63] imply it is also an AKM primary [42].

Chiral and anti-chiral operators

Finally we note that the multiplets in (2.8) and (2.9) give rise, in two dimensions, to anti-chiral and chiral operators respectively: they are killed by $\tilde{\mathcal{Q}}_{3\dot{+}}$ and \mathcal{Q}_{3+}^3 respectively. In addition these two-dimensional superfields have holomorphic dimension satisfying $h = \frac{R_2}{2} = -\frac{f}{2}$ and $h = \frac{R_1}{2} = \frac{f}{2}$ respectively.

2.1.2 [3, 0] chiral algebras

We are now in a position to describe the general features of the chiral algebras associated to the known $\mathcal{N} = 3$ theories. We will describe the chiral algebra in terms of its generators, by which we mean operators that cannot be expressed as normal-ordered products and/or (super)derivatives of other operators. In what follows we assume the chiral algebra to be finitely generated. The whole set of states of the chiral algebra is then obtained by taking normal-ordered products and derivatives of the generators. Although there is yet no complete characterization of what should be the generators of the chiral algebra of a given four-dimensional theory, it was shown in [40] that all generators of the HL chiral ring are generators of the chiral algebra. Moreover, the stress tensor is always guaranteed to be present in the chiral algebra and, with the exception of cases where a null relation identifies it with a composite operator, it must always be a generator. However this is not necessarily the complete list, and indeed examples with more generators than just the above have been given in [40, 42].⁸ The chiral algebras associated to $4d$ SCFTs do not always correspond to known examples in the literature, and in such situations one must construct a new associative two-dimensional chiral algebra. This problem can be bootstrapped by writing down the most general OPEs for the expected set of generators and then imposing associativity by solving the Jacobi identities. Chiral algebras are very rigid structures and in the cases so far considered [10, 42], the Jacobi identities are powerful enough to completely fix all OPE coefficients, including the central charges.

Rank one chiral algebras

In [10], the authors assumed that the only generators of the chiral algebras corresponding to the rank⁹ one $\mathcal{N} = 3$ SCFTs described in 1 (with $k = \ell$, $N = 1$) were the stress tensor and the generators of its Higgs branch:

$$\hat{\mathcal{B}}_{[1,1]}, \quad \hat{\mathcal{B}}_{[\ell,0]}, \quad \hat{\mathcal{B}}_{[0,\ell]}, \quad \ell = 3, 4. \quad (2.14)$$

In the above, the first gives rise in two dimensions to the stress tensor multiplet and the last two to anti-chiral and chiral operators respectively. With these assumptions they were able to write an associative chiral algebra for the cases $\ell = 3, 4$ only for a finite set of values of the central charge. This set is further restricted to the correct value expected for the known $\mathcal{N} = 3$ theories

$$c_{4d} = a_{4d} = \frac{2\ell - 1}{4}, \quad (2.15)$$

⁸ One possible way to determine which generators a given chiral algebra should have is through a Schur index [57, 64–66] analysis, as done in [40, 42]. Similarly if one believes to have the correct chiral algebra associated with a given theory then a further test is to compare the corresponding graded partition function with the superconformal index. In the case at hand, however, the superconformal index remains elusive.

⁹Defined as the dimension of the Coulomb branch of the theory when viewed as an $\mathcal{N} = 2$ SCFT.

by imposing the expected Higgs branch relation

$$\hat{\mathcal{B}}_{[\ell,0]}\hat{\mathcal{B}}_{[0,\ell]} \sim \left(\hat{\mathcal{B}}_{[1,1]}\right)^\ell, \quad (2.16)$$

where by an abuse of notation we represented the highest weight of the multiplet by the same symbol as the multiplet itself. We recall that relations on the Higgs branch appear as null states in the chiral algebra. Associativity also fixes all other OPE coefficients. The authors of [10] were also able to construct an associative chiral algebra for $\ell = 5$ and $\ell = 6$ satisfying the Higgs branch relation if the central charge is given by (2.15). However, as they point out, $\ell = 5$ does not correspond to an allowed value for an $\mathcal{N} = 3$ SCFTs, as five is not an allowed scaling dimension for the Coulomb branch of a rank one theory, following from Kodaria's classification of elliptic surfaces (see, *e.g.*, [5, 10]). The case $\ell = 6$ is in principle allowed, however no such $\mathcal{N} = 3$ theory was obtained in the S-fold constructions of [3].¹⁰

Higher rank theories

We will now attempt to generalize the chiral algebras of [10] to the higher-rank case (with $k = \ell$, $N > 1$). In particular, we focus on the theories whose lowest dimensional generator corresponds to a $\hat{\mathcal{B}}_{[3,0]}$ and its conjugate, since these are the ones relevant for the following sections. To compute OPEs and Jacobi identities we will make extensive use of the Mathematica package [67]. Following its conventions, we use the two-dimensional $\mathcal{N} = 2$ holomorphic superspace with bosonic coordinate z and fermionic coordinates θ and $\bar{\theta}$, and define the superderivatives as

$$\mathcal{D} = \partial_\theta - \frac{1}{2}\bar{\theta}\partial_z, \quad \bar{\mathcal{D}} = \partial_{\bar{\theta}} - \frac{1}{2}\theta\partial_z. \quad (2.17)$$

We will denote the two-dimensional generators arising from the half-BPS Higgs branch generators $\hat{\mathcal{B}}_{[0,3]}$ ($\hat{\mathcal{B}}_{[3,0]}$) by \mathcal{W} ($\bar{\mathcal{W}}$).¹¹ Furthermore, we denote the two-dimensional superfield arising from the stress tensor ($\hat{\mathcal{B}}_{[1,1]}$) by \mathcal{J} . The OPE of \mathcal{J} with itself is fixed by superconformal symmetry,

$$\mathcal{J}(Z_1)\mathcal{J}(Z_2) \sim \frac{c_{2d}/3 + \theta_{12}\bar{\theta}_{12}\mathcal{J}}{Z_{12}^2} + \frac{-\theta_{12}\mathcal{D}\mathcal{J} + \bar{\theta}_{12}\bar{\mathcal{D}}\mathcal{J} + \theta_{12}\bar{\theta}_{12}\partial\mathcal{J}}{Z_{12}}, \quad (2.18)$$

where we defined

$$Z_{ij} = z_1 - z_2 + \frac{1}{2}(\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1), \quad \theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2. \quad (2.19)$$

The OPEs of \mathcal{J} with \mathcal{W} and $\bar{\mathcal{W}}$, given in (B.1), are fixed by demanding that these two operators be super Virasoro primaries. As discussed in the previous subsection, \mathcal{W} and $\bar{\mathcal{W}}$ could fail to be super Virasoro primaries only if their global superprimary (arising from an $\mathcal{N} = 2$ $\hat{\mathcal{B}}_{3/2}$) fails to be an AKM primary. However, since we are assuming the $\hat{\mathcal{B}}_{3/2}$ multiplet to be a generator, and since the AKM current comes from a $\hat{\mathcal{B}}_1$ $\mathcal{N} = 2$ multiplet, it is clear from the selection rules of $\mathcal{N} = 2$ $\hat{\mathcal{B}}_R$ operators [63] that these must be AKM primaries.

The self OPEs of the chiral (anti-chiral) \mathcal{W} ($\bar{\mathcal{W}}$) superfields are regular, as is also clear from the $\mathcal{N} = 3$ OPE selection rules derived in 3.22. Finally, all that is left is to write down the most general

¹⁰We emphasize that the existence of a two-dimensional chiral algebra does not imply that there exists a four-dimensional theory that gives rise to it. In fact it is still not clear what are the sufficient conditions for a chiral algebra to correspond to a physical four-dimensional theory.

¹¹Note that in [67] what is called chiral primary is what we call anti-chiral primary, *e.g.*, $\bar{\mathcal{W}}$ which obeys $\mathcal{D}\bar{\mathcal{W}} = 0$.

OPE for $\mathcal{W}\bar{\mathcal{W}}$ in terms of all of the existing generators [10]

$$\mathcal{W}(Z_1)\bar{\mathcal{W}}(Z_2) \sim \sum_{h=0}^2 \frac{1}{Z_{12}^{3-h}} \left(\frac{3-h}{2} \frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}} + 1 + \theta_{12}\mathcal{D} \right) \lambda_{\mathcal{O}_h} \mathcal{O}_h, \quad (2.20)$$

where the sum runs over all uncharged operators, including composites and (super)derivatives.

The authors of [10] showed that, considering just these three fields as generators, one finds an associative chiral algebra only if $c_{2d} = -15$, which indeed corresponds to the correct value for the simplest known non-trivial $\mathcal{N} = 3$ SCFT ($k = \ell = 3$ and $N = 1$ in the notation of [3]). However, there are higher rank versions of this theory ($k = \ell = 3$ and $N > 1$), that contain these half-BPS operators plus higher dimensional ones. From [3], we see that a list of half-BPS operators would correspond to

$$\hat{\mathcal{B}}_{[0,R]}, \quad \hat{\mathcal{B}}_{[R,0]}, \quad \text{with } R = 3, 6, \dots, 3N, \quad (2.21)$$

giving rise in two dimensions to additional chiral and antichiral operators with charges $f = \pm 6, \dots, \pm 6N$, and holomorphic dimension $h = |f|/2$. One can quickly see that the extra generators never appear in the OPEs of $\mathcal{W}, \bar{\mathcal{W}}, \mathcal{J}$, as the only OPE not fixed by symmetry is the $\mathcal{W}\bar{\mathcal{W}}$, and $U(1)_f$ charge conservation forbids any of the $\hat{\mathcal{B}}_{[R,0]}$ with $R \geq 6$ to appear. If the generators of the chiral algebras of higher rank theories corresponded *only* to the half-BPS operators plus the stress tensor, then we would reach a contradiction: $\mathcal{W}, \bar{\mathcal{W}}, \mathcal{J}$ would form a closed subalgebra of the full chiral algebra, but the central charge would be frozen at $c_{2d} = -15$, which is not the correct value for rank greater than one.

Therefore, to resolve this contradiction we must allow for more generators in the higher-rank case, and at least one of these must be exchanged in the $\mathcal{W}\bar{\mathcal{W}}$ OPE. The *only* freedom in this OPE is to add an uncharged dimension two generator. From the OPE selection rules we derive in 3.21 one can see that this operator must correspond to a $\hat{\mathcal{B}}_{[2,2]}$. There is another possibility, namely a $\hat{\mathcal{C}}_{[0,0],0}$ multiplet, but in four dimensions it contains conserved currents of spin greater than two, which should be absent [61, 62] in interacting theories such as the ones we are interested in. The minimal resolution is to add the generator corresponding to $\hat{\mathcal{B}}_{[2,2]}$. We then make the following *assumption*:

The generators of the chiral algebra associated with the $\ell = k = 3$ theories with $N > 1$ are given by

- The stress tensor \mathcal{J} ,
- (Anti-)chiral operators arising from the generators of the Coulomb branch operators $\hat{\mathcal{B}}_{[0,R]}$ ($\hat{\mathcal{B}}_{[R,0]}$) with $R = 3, 6, \dots, 3N$,
- A generator corresponding to $\hat{\mathcal{B}}_{[2,2]}$ which we denote by \mathcal{U} .

and as before we denote by \mathcal{W} and $\bar{\mathcal{W}}$ the generators arising from $\hat{\mathcal{B}}_{[0,3]}$ and conjugate.¹² Of course nothing forbids the existence of additional generators, but the minimal modification that unfreezes the value of the central charge is the addition of \mathcal{U} . In fact, examples are known where the number of generators not arising from generators of the HL ring grows with the number of HL generators [42].

¹²The fact that we do not allow for any other operator of dimension one (or smaller) prevents the symmetry of the chiral algebra from enhancing to the small $\mathcal{N} = 4$ superconformal algebra one gets from $4d$ $\mathcal{N} = 4$ theories, thereby excluding $\mathcal{N} = 4$ solutions from our analysis. And by not allowing for additional dimension $3/2$ generators we also exclude discretely gauged versions of $\mathcal{N} = 4$.

We can then proceed to write down the most general OPEs. It is easy to check that in the OPEs involving

$$\mathcal{J}, \quad \mathcal{W}, \quad \bar{\mathcal{W}}, \quad \text{and} \quad \mathcal{U}, \quad (2.22)$$

the operators in (2.21) with $R \geq 6$ cannot be exchanged. This implies that if our assumptions above are correct, the generators in (2.22) form a closed subalgebra.

In what follows we will check if it is possible to write an associative algebra for (2.22) without needing to fix the central charge. We write down the most general ansatz for the OPEs of these operators which, as explained above, are all super Virasoro primaries with the exception of \mathcal{J} . The regularity of the self OPEs of \mathcal{W} and $\bar{\mathcal{W}}$ follows simply from OPE selection rules, while the OPE between \mathcal{W} and $\bar{\mathcal{W}}$ is given by (2.20), allowing for the exchange of \mathcal{U} as well. The OPEs involving \mathcal{U} are quite long and therefore we collect them in appendix B. Imposing Jacobi identities we were able to fix all the OPE coefficients in terms of the central charge, as can be seen from the results in appendix B. We should point out that our goal was to obtain a one-parameter family of associative algebras labeled by the central charge, and as such we did not take into consideration null states that could arise for specific values of c_{2d} .

2.1.3 Fixing the OPE coefficients from the chiral algebra

In the next sections we will study numerically the *complete* four-point function of two $\hat{\mathcal{B}}_{[3,0]}$ and two $\hat{\mathcal{B}}_{[0,3]}$ operators, but first we want to fix as much as possible from knowledge of the chiral algebra. Thanks to our solution we can compute the OPE coefficients of all operators appearing in the right hand side of the $\mathcal{W}\bar{\mathcal{W}}$ OPE. However, we still need to identify the four-dimensional superconformal multiplet that each two-dimensional operator corresponds to. Let us start by examining the low dimensional operators appearing in this OPE: we can write all possible operators with a given dimension that can be made out of the generators by normal ordered products and (super) derivatives. Furthermore, they must be uncharged, since the product $\mathcal{W}\bar{\mathcal{W}}$ is. All in all we find the following list:

dimension	operators
0	Identity
1	\mathcal{J}
2	$\mathcal{U}, \mathcal{J}\mathcal{J}, \mathcal{D}\bar{\mathcal{D}}\mathcal{J}, \mathcal{J}'$
3	$\mathcal{W}\bar{\mathcal{W}}, \mathcal{J}\mathcal{D}\bar{\mathcal{D}}\mathcal{J}, \mathcal{J}'', \mathcal{J}'\mathcal{J}, \mathcal{J}\mathcal{J}\mathcal{J}, \mathcal{D}\bar{\mathcal{D}}\mathcal{J}', \mathcal{D}\mathcal{J}\bar{\mathcal{D}}\mathcal{J}, \mathcal{D}\bar{\mathcal{D}}\mathcal{U}, \mathcal{J}\mathcal{U}, \mathcal{U}'$
...	...

From these operators we are only interested in the combinations that are global superprimary fields, as the contributions of descendants will be fixed from them.¹³ Note also that, if we are interested in the four point function of $\langle \mathcal{W}\bar{\mathcal{W}}\mathcal{W}\bar{\mathcal{W}} \rangle$, we only see, for the exchange of an operator of a given dimension, a sum of the contributions of all global primaries, and we cannot distinguish between individual fields.

At dimension $h = 1$ there is only one operator – the super primary of the stress-tensor multiplet – and its OPE coefficient squared can be computed to be (after normalizing the identity operator to appear with coefficient one in the four-point function decomposition, and normalizing the \mathcal{J} two-point function to match the normalization for the blocks ($g^{2d, \mathcal{N}=2}$, see (C.5)) that we use in the following sections)

$$|\lambda_{\mathcal{W}\bar{\mathcal{W}}\mathcal{J}}|^2 = -\frac{27}{c_{2d}}. \quad (2.23)$$

¹³Note that this is only true because \mathcal{W} and $\bar{\mathcal{W}}$ are chiral and anti-chiral, and therefore their three-point function with an arbitrary superfield has a unique structure, being totally determined by a single number.

This does not depend on the particular chiral algebra at hand, as the OPE coefficient with which the current \mathcal{J} is exchanged, is totally fixed in terms of their charge f and the central charge. As we will show in 4.1.1, the two-dimensional correlation function of the two \mathcal{W} and two $\bar{\mathcal{W}}$, is fixed in terms of one parameter which we take to be the OPE coefficient of \mathcal{J} , and thus related to c_{2d} . This implies that, for the exchange of operators of dimension larger than one, any sum of OPE coefficients corresponds to a universal function of c_{2d} that does not depend on the particular chiral algebra under consideration.

At dimension $h = 2$ we find two global superprimaries, one corresponding to \mathcal{U} itself, and the other containing $\mathcal{J}\mathcal{J}$. From the four-dimensional OPE selection rules, shown in (3.21), it follows that both superprimaries must correspond to $\hat{\mathcal{B}}_{[2,2]}$ supermultiplets in four dimensions, as the only other option is $\hat{\mathcal{C}}_{[0,0],0}$ which should be absent in interacting theories [61, 62]. This means that, even from the point of view of the four-dimensional correlation function, these two operators are indistinguishable. Thus, all we can fix is the sum of two OPE coefficients squared:

$$|\lambda_{\mathcal{W}\bar{\mathcal{W}}\mathcal{U}}|^2 + |\lambda_{\mathcal{W}\bar{\mathcal{W}}(\mathcal{J}\mathcal{J})}|^2 = -\frac{18}{c_{2d}}, \quad (2.24)$$

where we used the same normalizations as before, and fixed an orthonormal basis for the operators. This number is independent of the particular details of the chiral algebra: it only requires the existence of \mathcal{W} , $\bar{\mathcal{W}}$ and \mathcal{J} .

At dimension $h = 3$, we find four global super primaries made out of the fields listed above, three of which are Virasoro primaries. In this case, however, these three different operators must belong to two different types of four-dimensional multiplets (once again we are excluding the multiplet containing higher spin currents). Namely, they must correspond to $\hat{\mathcal{B}}_{[3,3]}$ and $\hat{\mathcal{C}}_{[1,1],\ell=0}$, and distinguishing them from the point of view of the chiral algebra is hard. The two-dimensional operators arising from $\hat{\mathcal{B}}_{[3,3]}$ are guaranteed to be Virasoro primaries, while those of $\hat{\mathcal{C}}_{[1,1],0}$ could be or not. *Assuming* that all Virasoro primaries come exclusively from $\hat{\mathcal{B}}_{[3,3]}$ we can compute the OPE coefficient with which this multiplet is exchanged by summing the squared OPE coefficients of all Virasoro primaries

$$\sum_{i=1}^3 |\lambda_{\mathcal{J}\mathcal{J}\text{Vir}_i, h=3}|^2 = \frac{2(c_{2d}(5c_{2d} + 127) + 945)}{5c_{2d}(2c_{2d} + 13)}. \quad (2.25)$$

We can take the large $c_{2d} = -12c_{4d}$ limit, where the solution should correspond to generalized free field theory. In this case we can find from the four-point function given in appendix E that the OPE coefficient above should go to 1, and indeed this is the case. We could also have assumed that different subsets of the three Virasoro primaries correspond to $\hat{\mathcal{B}}_{[3,3]}$. Not counting the possibility used in (2.25), there is one possibility which does not have the correct behavior as $c_{4d} \rightarrow \infty$, and two that have:

$$\sum_{i=1}^2 |\lambda_{\mathcal{J}\mathcal{J}\text{Vir}_i, h=3}|^2 = \frac{351378 - 10c_{2d}(c_{2d}(c_{2d} + 22) - 260) - 8430}{(c_{2d} - 1)c_{2d}(2c_{2d} + 13)(12 - 5c_{2d})}, \quad (2.26)$$

$$\sum_{i=2}^3 |\lambda_{\mathcal{J}\mathcal{J}\text{Vir}_i, h=3}|^2 = \frac{2(c_{2d} + 15)(c_{2d}(c_{2d}(5c_{2d} + 37) + 39) + 4482)}{5(c_{2d} - 1)c_{2d}(c_{2d} + 6)(2c_{2d} - 3)}. \quad (2.27)$$

We can now also compute for each of the above cases the OPE coefficient of the $\hat{\mathcal{C}}_{[1,1],\ell=0}$ multiplet, and we find that only (2.25) and (2.27) are compatible with $4d$ unitarity (the precise relation between $2d$ and $4d$ OPE coefficients is given by (4.20)).

Rank one case

Let us now comment on what happens for the case of the rank one theory, where $c_{2d} = -15$ and the extra generator \mathcal{U} is absent. In this case we find a single (non-null) Virasoro primary at dimension three.¹⁴ This implies that either there is no $\hat{\mathcal{B}}_{[3,3]}$ multiplet and that the OPE coefficient is zero, or, which seems like a more natural option, that the Virasoro primary corresponds to this multiplet, with OPE coefficient

$$|\lambda_{\mathcal{J}\mathcal{J}\text{Vir}_{h=3}}|^2 = \frac{22}{85}. \quad (2.28)$$

The above corresponds to setting $c_{2d} = -15$ in both (2.25) and (2.26), as expected since for this value the extra generator is not needed and decouples. The possibility that there is no $\hat{\mathcal{B}}_{[3,3]}$ multiplet in the rank one theory and thus that the OPE coefficient is zero corresponds to the $c_{2d} = -15$ case of (2.27). If this last possibility were true, then we would have that the operator $\mathcal{W}\bar{\mathcal{W}} \sim \mathcal{J}^3$ is not in the Higgs branch, since Higgs branch operators correspond to $\hat{\mathcal{B}}_R$ multiplets in $\mathcal{N} = 2$ language. Hence, there would be an extra Higgs branch relation setting $\mathcal{J}^3 = 0$, which does not seem plausible. Nevertheless, we will allow for (2.27) for generic values of the central charge. It is possible that one can select among the two options ((2.25) and (2.27)) for generic central charge precisely by using knowledge of the Higgs branches of the higher rank theories, and making use of the considerations in [68] about recovering the Higgs branch out of the chiral algebra, but we leave this for future work.

If we now go to higher dimension, the list of operators keeps on growing, and their four-dimensional interpretation is always ambiguous. A dimension h global superprimary can either be a $\hat{\mathcal{C}}_{[2,2],h-4}$ or a $\hat{\mathcal{C}}_{[1,1],h-3}$ four-dimensional multiplet, and in this case there does not seem to be an easy way to resolve the ambiguity.¹⁵

3 Superblocks

In this section we will use harmonic superspace techniques in order to study correlation functions of half-BPS operators. We will follow closely [39, 69], where a similar approach was used to study correlation functions in several superconformal setups.

3.1 Superspace

Coset superspace. We introduce the superspace \mathcal{M} as a coset $\mathcal{M} \simeq \text{SL}(4|3)/G_{\leq 0}$. Here, the factor $G_{\leq 0}$ corresponds to lower triangular block matrices with respect to the decomposition given in (3.1) below. We take $E(\mathbf{p}) \in G_{>0}$ as coset representative explicitly given by

$$E(\mathbf{p}) := \exp \begin{pmatrix} 0_{(2|1)} & V & X \\ 0 & 0 & \bar{V} \\ 0 & 0 & 0_{(2|1)} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{(2|1)} & V & X_+ \\ 0 & 1 & \bar{V} \\ 0 & 0 & \mathbf{1}_{(2|1)} \end{pmatrix}, \quad (3.1)$$

where

$$X = \begin{pmatrix} x^{\alpha\dot{\alpha}} & \lambda^\alpha \\ \pi^{\dot{\alpha}} & y \end{pmatrix}, \quad V = \begin{pmatrix} \theta^\alpha \\ v \end{pmatrix}, \quad \bar{V} = (\bar{\theta}^{\dot{\alpha}} \bar{v}). \quad (3.2)$$

In the above, $\alpha \in \{1, 2\}$, $\dot{\alpha} \in \{1, 2\}$ are the familiar Lorentz indices and the coordinates $\{\lambda^\alpha, \pi^{\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\}$ are fermionic, while the y, v, \bar{v} are bosonic R-symmetry coordinates. The action of $\text{SL}(4|3)$ on this

¹⁴There is another Virasoro primary, which is a composite operator that is null for this central charge. This null corresponds precisely to the Higgs branch relation of the form $\mathcal{W}\bar{\mathcal{W}} \sim \mathcal{J}\mathcal{J}^3$ described in [10].

¹⁵One possibility would be to find two sets of OPEs such that in each set of the above multiplets is forbidden to appear by selection rules.

	$G_{(+,0)}$	$G_{(0,+)}$	$G_{(0,0)}$	$G_{(-,0)}$	$G_{(0,-)}$
$g \circ (X_+, \bar{V})$	$(X_+ + b\bar{V}, \bar{V})$	$(X_+, \bar{V} + \bar{b})$	$(AX_+, \bar{V})D^{-1}$	$(X_+, \bar{V} + cX_+)$	$(X_+, \bar{V})\bar{h}$
$g \circ (X_-, V)$	$(X_-, V + b)$	$(X_- - V\bar{b}, V)$	$A(X_- D^{-1}, V)$	$h(X_-, V)$	$(X_-, V - X_- c)$

Table 1. We used the definitions $h := (\mathbf{1}_3 + V\bar{c})^{-1}$ and $\bar{h} := (\mathbf{1}_3 + c\bar{V})^{-1}$. The transformations corresponding to $G_{(+,+)}$, $G_{(-,-)}$ are generated by the ones above. For convenience we give the explicit form of super-special-conformal transformations $G_{(-,-)}: (X_+, \bar{V}) \mapsto (X_+, \bar{V})(1 + CX_+)^{-1}$ and $(X_-, V) \mapsto (1 + X_-C)^{-1}(X_-, V)$.

superspace follows from the coset construction and is summarized in table 1. Notice that $\text{SL}(4|3)$ acts invariantly within the superspaces \mathcal{M}_+ , \mathcal{M}_- with coordinates $\{X_+, \bar{V}\}$, $\{X_-, V\}$ respectively, where we have defined $X_{\pm} = X \pm \frac{1}{2}V\bar{V}$. The basic covariant objects extracted from the invariant product $E(\mathbf{p}_2)^{-1}E(\mathbf{p}_1)$ are

$$X_{\bar{1}2} := X_{+,1} - X_{-,2} - V_2\bar{V}_1, \quad V_{12} := V_1 - V_2, \quad \bar{V}_{12} := \bar{V}_1 - \bar{V}_2. \quad (3.3)$$

We also define $X_{2\bar{1}} := -X_{\bar{1}2}$.

Superfields for superconformal multiplets . The supermultiplets $\hat{\mathcal{B}}_{[R_1, R_2]}$ correspond to ‘‘scalar’’ superfields on \mathcal{M} . Among them, as discussed in the previous section, the ones with $R_1 R_2 = 0$ are special in the sense that they satisfy certain chirality conditions. We call *chiral (anti-chiral)* a superfield¹⁶ that depends only on the coordinates $\{X_-, V\}$ ($\{X_+, \bar{V}\}$). Within this terminology, the operators $\hat{\mathcal{B}}_{[0, R]}$ are chiral while the $\hat{\mathcal{B}}_{[R, 0]}$ are antichiral. More general supermultiplets can be described as superfields on \mathcal{M} with $\text{SL}(2|1) \times \text{SL}(2|1)$ indices which extend the familiar Lorentz indices. We will not need to develop the dictionary between $\mathcal{N} = 3$ superconformal representations and $\text{SL}(2|1) \times \text{SL}(2|1) \times \text{GL}(1) \times \text{GL}(1)$ induced representations in this work and thus leave it for the future.

Remark 1. The subspace $\mathcal{M}_{\mathcal{N}=2}$ corresponding to setting $V = \bar{V} = 0$ is acted upon by the $\mathcal{N} = 2$ superconformal group $\text{SL}(4|2)$. The corresponding superspace is well known, see e.g. [37]. The superfields corresponding to the $\mathcal{N} = 3$ supermultiplets $\hat{\mathcal{B}}_{[R_1, R_2]}$ reduce to the $\mathcal{N} = 2$ supermultiplet $\hat{\mathcal{B}}_{\frac{1}{2}(R_1+R_2)}$ when restricted to the superspace $\mathcal{M}_{\mathcal{N}=2}$. The other operators in the decomposition of $\hat{\mathcal{B}}_{[R_1, R_2]}$ in $\mathcal{N} = 2$ supermultiplets, see (A.1), (A.2), roughly corresponds to the expansion of the superfield in V and \bar{V} . There is also a $\mathcal{N} = 1$ subspace $\mathcal{M}_{\mathcal{N}=1}$, which is not a subspace of $\mathcal{M}_{\mathcal{N}=2}$, defined by setting $\lambda^\alpha, \pi^{\dot{\alpha}}, v, \bar{v}$ to zero. A $\text{SL}(4|1) \times \text{SL}(2)$ subgroup of $\text{SL}(4|3)$ acts on $\mathcal{M}_{\mathcal{N}=1}$. This observation will be useful in the derivation of the superconformal blocks in section 3.4.

Examples of two- and three-point functions. We denote superfields and supermultiplets in the same way. Let us list some relevant examples of two- and three-point functions of $\hat{\mathcal{B}}$ -operators of increasing complexity:

$$\langle \hat{\mathcal{B}}_{[R_1, R_2]}(1) \hat{\mathcal{B}}_{[R_3, R_4]}(2) \rangle = \delta_{R_1, R_4} \delta_{R_2, R_3} (\bar{2}1)^{R_1} (\bar{1}2)^{R_2}, \quad (3.4)$$

$$\langle \hat{\mathcal{B}}_{[0, R]}(1) \hat{\mathcal{B}}_{[R, 0]}(2) \hat{\mathcal{B}}_{[S, S]}(3) \rangle = (\bar{2}1)^{R-S} ((\bar{2}3)(\bar{3}1))^S, \quad (3.5)$$

$$\langle \hat{\mathcal{B}}_{[0, R]}(1) \hat{\mathcal{B}}_{[0, R]}(2) \hat{\mathcal{B}}_{[R_1, R_2]}(3) \rangle = \delta_{R_1, 2R} \delta_{R_2, 0} ((\bar{3}1)(\bar{2}1))^R, \quad (3.6)$$

$$\langle \hat{\mathcal{B}}_{[R, R]}(1) \hat{\mathcal{B}}_{[R, R]}(2) \hat{\mathcal{B}}_{[R, R]}(3) \rangle = ((\bar{2}1)(\bar{3}2)(\bar{1}3))^R P_R(C), \quad (3.7)$$

¹⁶ This is not the standard terminology for chiral superfields in \mathcal{N} -extended superspace. We hope this will not cause any confusion to the reader.

where we have defined

$$(\bar{1}2) := \frac{1}{\text{sdet}(X_{\bar{1}2})}, \quad C := \frac{(\bar{3}1)(\bar{1}2)(\bar{2}3)}{(\bar{2}1)(\bar{3}2)(\bar{1}3)}. \quad (3.8)$$

In (3.5) superspace analyticity implies that $S \leq R$ and that the correlation function vanishes otherwise. Similarly, in (3.7), C is a superconformal invariant and superspace analyticity implies that $P_R(C)$ is a polynomial of degree R in C . Since the three operators are identical, one further imposes Bose symmetry which translates to $P_R(x) = x^R P_R(x^{-1})$. Equation (3.7) specialized to the case $R = 1$ corresponds to the three-point function of the stress-tensor supermultiplet $\hat{\mathcal{B}}_{[1,1]}$ and the argument above implies that $P_1(x) = \text{const} \times (1 + x)$. This provides a quick proof of the fact that for $\mathcal{N} = 3$ superconformal theories one has the relation $a = c$ as first derived in [1].

Let us consider the three-point functions relevant for the non-chiral OPE $\hat{\mathcal{B}}_{[R,0]} \times \hat{\mathcal{B}}_{[0,R]}$. A little superspace analysis reveals that the three-point function of a chiral and an anti-chiral operator with a generic operator takes the form

$$\langle \hat{\mathcal{B}}_{[0,R]}(X_{-,1}, V_1) \hat{\mathcal{B}}_{[R,0]}(X_{+,2}, \bar{V}_2) \mathcal{O}(X_3, V_3, \bar{V}_3) \rangle = (\bar{2}1)^R \rho_{\mathcal{O}}(X_{\bar{3}1} X_{21}^{-1} X_{\bar{2}3}). \quad (3.9)$$

The quantity $\rho_{\mathcal{O}}$ is determined uniquely up to a multiplicative constant by the requirement that (3.9) is superconformally covariant. It is not hard to verify that one can set the coordinates $V_1, \bar{V}_2, V_3, \bar{V}_3$ to zero by an $\text{SL}(4|3)$ transformation which is not part of the $\mathcal{N} = 2$ superconformal group $\text{SL}(4|2)$ (with the embedding specified in the remark 1 above). This means that (3.9) is zero if its $\mathcal{N} = 2$ reduction (*i.e.*, the result obtained after setting $V_i = \bar{V}_i = 0$) is zero, as confirmed by the selection rules result (3.21) that we derive later in section 3.3.

Turning to the three-point functions relevant for the chiral OPE $\hat{\mathcal{B}}_{[R,0]} \times \hat{\mathcal{B}}_{[R,0]}$, it is not hard to convince oneself that they take the form

$$\langle \hat{\mathcal{B}}_{[0,R]}(X_{-,1}, V_1) \hat{\mathcal{B}}_{[R,0]}(X_{-,2}, V_2) \tilde{\mathcal{O}}(X_3, V_3, \bar{V}_3) \rangle = ((\bar{3}1)(\bar{3}2))^R \tilde{\rho}_{\tilde{\mathcal{O}}}(\hat{X}, \hat{V}), \quad (3.10)$$

where

$$\hat{X} = (X_{2\bar{3}}^{-1} - X_{1\bar{3}}^{-1})^{-1}, \quad \hat{V} = X_{2\bar{3}}^{-1} V_{23} - X_{1\bar{3}}^{-1} V_{13}, \quad (3.11)$$

and $\tilde{\rho}_{\tilde{\mathcal{O}}}$ is fixed by requiring superconformal covariance of (3.10). It is important to remark that, as opposed to (3.9), in this case one cannot set the coordinates V_1, V_2, V_3, \bar{V}_3 to zero using superconformal transformations. However, they can be set to the values

$$\{(X_{-,1}, V_1), (X_{-,2}, V_2), (X_{-,3}, V_3, \bar{V}_3)\} \mapsto \{(\infty, 0), \hat{X}(1, \hat{V}), (0, 0, 0)\}. \quad (3.12)$$

The combinations \hat{X} and \hat{V} carry non trivial superconformal weights only with respect to the third point corresponding to the operator $\tilde{\mathcal{O}}$.

3.2 Superconformal Ward identities

We will now derive, along the same lines as [31, 39, 69], the superconformal Ward identities for the four point correlation function $\langle \hat{\mathcal{B}}_{[0,R]} \hat{\mathcal{B}}_{[R,0]} \hat{\mathcal{B}}_{[0,R]} \hat{\mathcal{B}}_{[R,0]} \rangle$. Let us first introduce super-cross-ratios for this four point function. The eigenvalues of the graded matrix

$$\mathcal{Z} := X_{1\bar{2}} X_{3\bar{2}}^{-1} X_{3\bar{4}} X_{1\bar{4}}^{-1}, \quad (3.13)$$

are invariant and will be denoted by x_1, x_2, y . It is easy to convince oneself that these are the only invariants by noticing that all fermionic coordinates in this four point function can be set to zero by a superconformal transformation. It follows that

$$\langle \hat{\mathcal{B}}_{[0,R]}(1) \hat{\mathcal{B}}_{[R,0]}(\bar{2}) \hat{\mathcal{B}}_{[0,R]}(3) \hat{\mathcal{B}}_{[R,0]}(\bar{4}) \rangle = (1\bar{2})^R (3\bar{4})^R G_R(x_1, x_2, y), \quad (3.14)$$

where $G_R(x_1, x_2, y) = G_R(x_2, x_1, y)$. The form of $G_R(x_1, x_2, y)$ is strongly restricted by the requirement of superspace analyticity. Firstly, after setting all fermionic variables to zero

$$\mathcal{Z}|_{\text{ferm}=0} = \begin{pmatrix} x_{12} x_{32}^{-1} x_{34} x_{14}^{-1} & 0 \\ 0 & \frac{y_{12} y_{34}}{y_{32} y_{14}} \end{pmatrix}, \quad y_{\bar{i}j} := y_i - y_j - v_j \bar{v}_i, \quad (3.15)$$

polynomiality in the R-symmetry variables implies that $G_R(x_1, x_2, y)$ is a polynomial of degree R in y^{-1} . Secondly, one has to make sure that the fermionic coordinates can be turned on without introducing extra singularities in the R-symmetry variables. By looking at the expansion of the eigenvalues of (3.13) in fermions, one concludes that the absence of spurious singularities is equivalent to the conditions

$$(\partial_{x_1} + \partial_y) G_R(x_1, x_2, y)|_{x_1=y} = 0, \quad (\partial_{x_2} + \partial_y) G_R(x_1, x_2, y)|_{x_2=y} = 0. \quad (3.16)$$

These equations imply in particular that $G_R(x, x_2, x) = f_R(x_2)$ and $G_R(x_1, x, x) = f_R(x_1)$. The general solution of the Ward identities can be parametrized as

$$G_R(x_1, x_2, y) = \frac{(x_1^{-1} - y^{-1})f_R(x_1) - (x_2^{-1} - y^{-1})f_R(x_2)}{x_1^{-1} - x_2^{-1}} + (x_1^{-1} - y^{-1})(x_2^{-1} - y^{-1}) H_R(x_1, x_2, y), \quad (3.17)$$

where $H_R(x_1, x_2, y)$ is a polynomial of degree $R - 2$ in y^{-1} . In particular, it is zero for the case $R = 1$ corresponding to a free theory. For the following analysis it is useful to introduce the variables z, \bar{z}, w as

$$x_1 = \frac{z}{z-1}, \quad x_2 = \frac{\bar{z}}{\bar{z}-1}, \quad y = \frac{w}{w-1}. \quad (3.18)$$

This change of variable is an involution in the sense that $z = \frac{x_1}{x_1-1}$ and so on. They are related to the more familiar cross ratios as

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}|_{\text{ferm}=0}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})|_{\text{ferm}=0}. \quad (3.19)$$

Notice that the WI (3.16) take the same form in the new variables and that moreover

$$(z^{-1} - w^{-1})(\bar{z}^{-1} - w^{-1}) = (x_1^{-1} - y^{-1})(x_2^{-1} - y^{-1}), \quad (3.20a)$$

$$\frac{(x_1^{-1} - y^{-1})f(x_1) - (x_2^{-1} - y^{-1})f(x_2)}{x_1^{-1} - x_2^{-1}} = \frac{(z^{-1} - w^{-1})f(x_1) - (\bar{z}^{-1} - w^{-1})f(x_2)}{z^{-1} - \bar{z}^{-1}}, \quad (3.20b)$$

for any function $f(x)$. We refer to appendix E for an analysis of the generalized free theory solution of the Ward identities (3.16).

3.3 Selection rules

Non-chiral channel. The OPE in the non-chiral channel $\hat{\mathcal{B}}_{[R,0]} \times \hat{\mathcal{B}}_{[0,R]}$ can be obtained by using the superconformal Ward identities just derived, together with the fact that the three-point function $\langle \hat{\mathcal{B}}_{[R,0]} \hat{\mathcal{B}}_{[0,R]} \mathcal{O} \rangle$, where \mathcal{O} is a generic operator, is non-zero only if the three-point function of the corresponding primary states¹⁷ is non-zero by conformal and R-symmetry. The latter condition can be derived by recalling that the fermionic coordinates in this three-point function can be set to zero by a super-conformal transformation. A little analysis shows that

$$\hat{\mathcal{B}}_{[R,0]} \times \hat{\mathcal{B}}_{[0,R]} = \mathcal{I} + \sum_{a=1}^R \hat{\mathcal{B}}_{[a,a]} + \sum_{\ell=0}^{\infty} \left[\sum_{a=0}^{R-1} \hat{\mathcal{C}}_{[a,a],\ell} + \sum_{a=0}^{R-2} \mathcal{A}_{[a,a],r=0,\ell}^{\Delta} \right]. \quad (3.21)$$

Notice that these relations are remarkably similar to the $\hat{\mathcal{B}}_{R/2} \times \hat{\mathcal{B}}_{R/2}$ OPE in the $\mathcal{N} = 2$ case, see [63]. The three upper bounds on the finite summations $R, R-1, R-2$ could be derived by imposing that the three point function $\langle \hat{\mathcal{B}}_{[R,0]} \hat{\mathcal{B}}_{[0,R]} \mathcal{O} \rangle$ is free of superspace singularities. Equivalently, it can be derived by requiring that the associated superconformal blocks takes the form (3.17). We followed the latter strategy as it seems more economical.

Chiral channel. The chiral channel selection rules are obtained by requiring that a given multiplet can only contribute if it contains an operator annihilated by all the supercharges that annihilate the highest weight of $\hat{\mathcal{B}}_{[R,0]}$, and said operator transforms in one of the representations appearing in the tensor product of the R-symmetry representations $[R, 0] \times [R, 0]$ and with the appropriate spin to appear in the OPE of the scalars fields we consider. We have performed this calculation for $R = 2, 3$ and on the base of it propose that the expression form for general R is

$$\begin{aligned} \hat{\mathcal{B}}_{[R,0]} \times \hat{\mathcal{B}}_{[R,0]} &= \hat{\mathcal{B}}_{[2R,0]} + \sum_{a=2}^R \bar{\mathcal{B}}_{[2(R-a),a],r=4R,0} + \\ &+ \sum_{\ell=0}^{\infty} \left[\hat{\mathcal{C}}_{[2R-2,0],(\frac{\ell+1}{2},\frac{\ell}{2})} + \sum_{a=2}^R \left(\bar{\mathcal{C}}_{[2(R-a),a-1],(\frac{\ell+1}{2},\frac{\ell}{2})}^{r=4R-1} + \mathcal{A}_{[2(R-a),a-2],(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta,r=4R-2} \right) \right]. \end{aligned} \quad (3.22)$$

We have checked the above in several cases for $R > 3$ and superspace arguments suggest it is indeed the correct selection rule. Note that in (3.22) the \mathcal{B} -type multiplets have $r = 4R$, the \mathcal{C} -type multiplets $r = 4R - 1$ and the \mathcal{A} -type multiplets $r = 4R - 2$. Moreover, if we are considering identical $\hat{\mathcal{B}}_{[R,0]}$ then Bose symmetry further constraints the spin of the operators appearing on the right-hand-side according to their $SU(3)_R$ representation.

3.4 Superconformal blocks

We will now derive the super-conformal blocks relevant for the expansion of the four point function (3.14). The superconformal Ward identities alone turn out not to be sufficient to uniquely determine all the superblocks. We resolve the leftover ambiguity by requiring that they are linear combinations of $SL(4|1) \times SU(2)$ ($\mathcal{N} = 1$) superblocks. There are two types of blocks corresponding to the two channels (3.21) for the non-chiral OPE and (3.22) for the chiral one. The two kinds of block are closely connected to $\mathcal{N} = 2$ superconformal blocks relevant for the four-point function of $\hat{\mathcal{B}}$ -type operators and are collected in tables 2 and 3. When the kinematics is restricted to $(z, \bar{z}, w) = (z, w, w)$, only

¹⁷With the terminology primary state we mean the conformal \times R-symmetry multiplet with the lowest value of Δ in the decomposition of a superconformal multiplet in conformal \times R-symmetry multiplets.

superconformal blocks corresponding to the exchange of Schur operators, defined in section 2.1.1, are non-vanishing. Moreover they reduce to (global) superblocks for the $\mathcal{N} = 2$ superconformal algebra $\mathfrak{sl}(2|1) \simeq \mathfrak{osp}(2|2)$.

3.4.1 Superconformal blocks for the non-chiral channel.

On general grounds, the $\mathcal{N} = 3$ superconformal blocks contributing to the four point function (3.14) in the non-chiral channel can be written as an expansion in terms of conformal times $SU(3)$ R-symmetry blocks:

$$\mathcal{G}_\chi(z, \bar{z}, w) = \sum_{\alpha \in S_\chi} c_\alpha(\chi) g_{\Delta_\alpha, \ell_\alpha}(z, \bar{z}) h_{[R_\alpha, R_\alpha]}(w). \quad (3.23)$$

The explicit form of the conformal blocks $g_{\Delta, \ell}$ is given in Appendix C. The $SU(3)$ R-symmetry blocks take the form

$$h_{[a, a]}(w) = \binom{2a+1}{a+1}^{-1} {}_2F_1(-a, a+2, 1, y^{-1}), \quad y = \frac{w}{w-1}, \quad (3.24)$$

see Appendix C. The normalization in (3.24) is chosen so that $h_{[a, a]}(w) = w^{-a} + \dots$ for $w \rightarrow 0$. The set S_χ is determined by considering the decomposition of the $\mathcal{N} = 3$ representation being exchanged into representations of the bosonic subalgebra. This is done using supercharacters. The normalization can be fixed by taking for instance $c_\alpha(\chi) = 1$ for the label α corresponding to the minimum value of Δ_α in the supermultiplet.

Consider the superblocks corresponding to the non-chiral OPE channel of (3.21). Concerning the superblocks for the $\hat{\mathcal{B}}_{[R, R]}$ exchange it turns out that they are uniquely fixed by imposing the superconformal WI on (3.23). The superblocks corresponding to the exchange of a $\hat{\mathcal{C}}_{[R, R], \ell}$ on the other hand are not uniquely fixed by the this procedure. The remaining ambiguity can be resolved by requiring that they reduce to $OSP(2|2)$ (this is the global part of the chiral half of the $d = 2, \mathcal{N} = 2$ superconformal symmetry) superblocks when restricted to $(z, \bar{z}, w) = (z, w, w)$. Specifically, this amounts to requiring

$$f_{\hat{\mathcal{C}}_{[R, R], \ell}}(z) = \mathcal{G}_{\hat{\mathcal{C}}_{[R, R], \ell}}(z, w, w) = (-1)^{\ell+1} g_{R+\ell+2}^{2d, \mathcal{N}=2}\left(\frac{z}{z-1}\right) \quad \text{with} \quad g_h^{2d, \mathcal{N}=2}(x) = x^h {}_2F_1(h, h, 2h+1, x), \quad (3.25)$$

where $f(z)$ corresponds to the parametrization (3.17).¹⁸ Finally the superblocks for the exchange of long operators $\mathcal{A}_{[a, a], r=0, \ell}^\Delta$ are not uniquely determined by the two conditions given above. The leftover ambiguity can be resolved by studying the Casimir equations. However, we decide to take a shortcut and use the knowledge of the $\mathcal{N} = 1$ superblocks. The relevant superblocks, which are the ones derived in [15, 34], are given by

$$\mathcal{G}_{\Delta, \ell}^{\mathcal{N}=1}(z, \bar{z}) = (z\bar{z})^{-\frac{1}{2}} g_{\Delta+1, \ell}^{\Delta_{12}=\Delta_{34}=1}(z, \bar{z}). \quad (3.26)$$

It follows from the remark 1, that the $\mathcal{N} = 3$ superblocks can be expanded in $\mathcal{N} = 1$ times $SU(2)$ “flavor symmetry” blocks as

$$\mathcal{G}_{\mathcal{A}_{[m, m], r=0, \ell}^\Delta}^{\mathcal{N}=3}(z, \bar{z}, w) = \tilde{d}_{\Delta, \ell}^{(0,0)}(w) \mathcal{G}_{\Delta, \ell}^{\mathcal{N}=1}(z, \bar{z}) + \tilde{d}_{\Delta, \ell}^{(1,1)}(w) \mathcal{G}_{\Delta+1, \ell+1}^{\mathcal{N}=1}(z, \bar{z}) + \dots + \tilde{d}_{\Delta, \ell}^{(4,0)}(w) \mathcal{G}_{\Delta+4, \ell}^{\mathcal{N}=1}(z, \bar{z}). \quad (3.27)$$

¹⁸To each superblock \mathcal{G}_χ corresponds a function f_χ and a function H_χ by using the parametrization (3.17).

χ	f_χ	H_χ
identity	1	0
$\hat{\mathcal{B}}_{[m,m]}$	$g_m^{2d, \mathcal{N}=2}$	$(-1)^m \sum_{k=0}^{m-2} \mathcal{G}_{m+k+2, m-k-2}^{\mathcal{N}=1} h_{[k,k]}$
$\hat{\mathcal{C}}_{[m,m], \ell}$	$(-1)^{\ell+1} g_{m+\ell+2}^{2d, \mathcal{N}=2}$	$(-1)^{m+1} \sum_{k=0}^{m-1} \mathcal{G}_{m+\ell+k+4, m+\ell-k}^{\mathcal{N}=1} h_{[k,k]}$
$\mathcal{A}_{[m,m], \ell}^\Delta$	0	$(-1)^m \mathcal{G}_{\Delta+2, \ell}^{\mathcal{N}=1} h_{[m,m]}$

Table 2. List of super-conformal blocks contributing to (3.21) in the parametrization (3.17). These expressions are consistent with the decompositions of superblocks at unitarity bounds, see (3.30). We recall that the explicit expressions for the auxiliary block entering the table are given in (3.25), (3.26) and (3.24). Notice that for the stress tensor supermultiplets $\hat{\mathcal{B}}_{[1,1]}$, the function $H_{\hat{\mathcal{B}}_{[1,1]}}$ is zero.

On the right hand side, the sum runs over the terms

$$(\Delta, \ell), \quad (\Delta + 1, \ell \pm 1), \quad (\Delta + 2, \ell \pm 2), (\Delta + 2, \ell), \quad (\Delta + 3, \ell \pm 1), \quad (\Delta + 4, \ell). \quad (3.28)$$

Imposing that the form (3.23), subject to the WI, can be expanded as in (3.27), fixes the leftover ambiguity in the $\mathcal{N} = 3$ superblocks and the coefficient functions $\tilde{d}_{\Delta, \ell}^{(a,b)}(w)$ up to an overall normalization. The solution can then be rewritten in the compact form

$$\mathcal{G}_{\mathcal{A}_{[m,m], r=0, \ell}^{\mathcal{N}=3}}(z, \bar{z}, w) = (-1)^m (z^{-1} - w^{-1})(\bar{z}^{-1} - w^{-1}) \mathcal{G}_{\Delta+2, \ell}^{\mathcal{N}=1}(z, \bar{z}) h_{[m,m]}(w). \quad (3.29)$$

The simplicity of this expression will be justified in remark 2 below. This concludes the derivation of superconformal blocks relevant for the non-chiral channel. The results are summarized in table 2.

Before turning to the discussion of the superblocks relevant for the chiral channel, we perform a consistency check on the blocks just derived. As can be seen in table 2, short blocks can be obtained from the long ones (3.29) at the unitarity bounds by using

$$\mathcal{G}_{\mathcal{A}_{[m,m], r=0, \ell}^{\Delta=\ell+2+2m}} = \mathcal{G}_{\hat{\mathcal{C}}_{[m,m], \ell}} + \mathcal{G}_{\hat{\mathcal{C}}_{[m+1, m+1], \ell-1}}, \quad (3.30)$$

where we identify $\hat{\mathcal{C}}_{[m,m], -1} \equiv \hat{\mathcal{B}}_{[m+1, m+1]}$. This is consistent with the multiplet decomposition at the unitarity bound: $\mathcal{A}_{[m,m], r=0, \ell}^{\Delta=\ell+2+2m} \rightarrow \hat{\mathcal{C}}_{[m,m], \ell} \oplus \hat{\mathcal{C}}_{[m+1, m+1], \ell-1} \oplus \text{"extra"}$, where "extra" does not contribute to the block.

3.4.2 Superconformal blocks for the chiral channel.

We denote the superconformal blocks contributing to this channel as $\tilde{\mathcal{G}}_\chi(z, \bar{z}, w)$, where χ labels the representations being exchanged from the list (3.22). As in the case of the non-chiral channel, we start with an expansion of the superblocks in conformal times SU(3) blocks and impose the superconformal Ward identities, (3.16). Specifically we take

$$\tilde{\mathcal{G}}_\chi(z, \bar{z}, w) = \sum_{\alpha \in \tilde{\mathcal{S}}_\chi} \tilde{c}_\alpha(\chi) g_{\Delta_\alpha, \ell_\alpha}(z, \bar{z}) \tilde{h}_{[2(R-n_\alpha), n_\alpha]}(w). \quad (3.31)$$

χ	\tilde{f}_χ	\tilde{H}_χ
$\hat{\mathcal{B}}_{[2m,0]}$	g_m^{2d}	$\sum_{a=0}^{m-2} g_{m+a+2,m-a-2} h_a^{\text{SU}(2)}$
$\hat{\mathcal{C}}_{[2m-2,0],(\frac{\ell+1}{2},\frac{\ell}{2})}$	$g_{m+\ell+2}^{2d}$	$\sum_{a=0}^{m-2} g_{m+\ell+a+4,m+\ell-a} h_a^{\text{SU}(2)}$
$\overline{\mathcal{B}}_{[2(m-a),a],0}^{r=4m}$	0	$g_{2m+2,0} h_{m-a}^{\text{SU}(2)}$
$\overline{\mathcal{C}}_{[2(m-a),a-1],(\frac{\ell+1}{2},\frac{\ell}{2})}^{r=4m-1}$	0	$g_{2m+\ell+3,\ell+1} h_{m-a}^{\text{SU}(2)}$
$\mathcal{A}_{[2(m-a),a-2],(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta,r=4m-2}$	0	$g_{\Delta+3,\ell} h_{m-a}^{\text{SU}(2)}$

Table 3. List of super-conformal blocks contributing to (3.22) in the parametrization (3.17). These expressions are consistent with the decompositions of superblocks at unitarity bounds, see (3.35). We recall that the explicit expression of the SL(2) and R-symmetry blocks is given in (C.4) and (3.32) respectively.

It appears, perhaps not too surprisingly, that the SU(3) R-symmetry blocks $\tilde{h}_{[2m,n]}(w)$ in this channel coincide with SU(2) blocks. They take the form¹⁹

$$\tilde{h}_{[2m,n]}(w) = h_m^{\text{SU}(2)}(w) = (-1)^m \binom{2m}{m}^{-1} {}_2F_1(-m, m+1, 1, w^{-1}), \quad (3.32)$$

where the normalization is chosen so that $\tilde{h}_{[2m,n]}(w) \sim w^m$ for $w \sim 0$. The set \tilde{S}_χ is determined by looking at the content of the representation χ using supercharacters. Using this information, all the coefficients $\tilde{c}_\alpha(\chi)$ are then fixed by the requirement that (3.31) satisfies the superconformal WI (3.16).

With a little inspection on the solutions, one recognizes that the superblocks in this channel are the $\mathcal{N} = 2$ superconformal blocks that contribute to the four-point function of $\hat{\mathcal{B}}_{\mathcal{N}=2}$ supermultiplets [30–32]. The identification is given by

$$\tilde{\mathcal{G}}_\chi^{\mathcal{N}=3}(z, \bar{z}, w) = \mathcal{G}_{p(\chi)}^{\mathcal{N}=2}(z, \bar{z}, w), \quad (3.33)$$

where p maps the $\mathcal{N} = 3$ representations being exchanged in the chiral channel, see (3.22), to $\mathcal{N} = 2$ representations as follows

$$p \left(\begin{array}{c} \hat{\mathcal{B}}_{[2R,0]} \\ \hat{\mathcal{C}}_{[2(R-1),0],(\frac{\ell+1}{2},\frac{\ell}{2})} \\ \overline{\mathcal{B}}_{[2(R-a),a],0}^{r=4R} \\ \overline{\mathcal{C}}_{[2(R-a),a-1],(\frac{\ell+1}{2},\frac{\ell}{2})}^{r=4R-1} \\ \mathcal{A}_{[2(R-a),a-2],(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta,r=4R-2} \end{array} \right) = \left(\begin{array}{c} \hat{\mathcal{B}}_R \\ \hat{\mathcal{C}}_{R-1,\ell+1} \\ \mathcal{A}_{R-a,0}^{2R} \\ \mathcal{A}_{R-a,\ell+1}^{2R+\ell+1} \\ \mathcal{A}_{R-a,\ell}^{\Delta+1} \end{array} \right). \quad (3.34)$$

The resulting superblocks in the parametrization (3.17) are given in Table 3. The blocks g_R^{2d} are given in (C.4). The equality (3.33) is of course not accidental, we will comment on its origin in the remark below.

The unitarity bound relevant for the chiral channel is

$$\mathcal{A}_{[2(m-a),a-2],(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta,r=4m-2} \xrightarrow{\Delta=\ell+2m-1} \overline{\mathcal{C}}_{[2(m-a),a-2],(\frac{\ell}{2},\frac{\ell}{2})}^{r=4m-2} \oplus \overline{\mathcal{C}}_{[2(m-a),a-1],(\frac{\ell}{2},\frac{\ell-1}{2})}^{r=4m-1}, \quad (3.35)$$

¹⁹ One can recognize the appearance of Legendre polynomials as $(-1)^m {}_2F_1(-m, m+1, 1, w^{-1}) = P_m(\frac{2}{w} - 1)$.

where $\overline{\mathcal{C}}_{[2(m-a), a-1], (0, -\frac{1}{2})}^{r=4m-1} = \overline{\mathcal{B}}_{[2(m-a), a], 0}^{r=4m-1}$. Only the underlined term contributes to the superblocks $\tilde{\mathcal{G}}$, as can be seen in table 3.

Remark 2. In [37], the authors derived superconformal partial waves for scalar four-point functions on a super Grassmannian space $Gr(m|n, 2m|2n)$. It is an interesting problem to generalize the analysis of [37] to the more general case of $Gr(m|n, M|N)$. The example we just studied corresponds to $Gr(2|1, 4|3)$. The example of chiral superfields (in the traditional sense) for \mathcal{N} -extended supersymmetry corresponds to the super Grassmannian $Gr(2|0, 4|N)$ and the corresponding superblocks are given in [34]. The simplicity of the results (3.29) and (3.33) and the one presented in [34] suggests a simple unified picture.

4 The bootstrap equations

The main purpose of this section is to derive the bootstrap equations for the four-point function (3.14). The result is given in equation (4.14) and its numerical analysis is the subject of section 5. As first done in [20] and systematized in [19], it is convenient to divide the sum over operators being exchanged in the OPE into two groups: the ones that survive the cohomological reduction (referred to as *Schur operators*) described in section 2, and the rest. In some cases the contribution from the exchange of Schur operators can be determined entirely in terms of the cohomologically reduced correlators, but this is not the generic situation. In the specific case of the $\mathcal{N} = 3$ four-point function (3.14), the chiral channel contributions of Schur operators can be extracted from the reduced correlator $f_R(x)$, while for the non-chiral channel only a component (corresponding to the R-symmetry singlet part of the function H_R in the parametrization (3.17)) can be extracted unambiguously from $f_R(x)$.

4.1 The crossing equations

As discussed in section 3.2, it follows from the superconformal WI (3.16) that the correlation function G_R defined in (3.14) can be parametrized as (3.17). We will now derive crossing relations for the functions $f_R(x)$, $H_R(x_1, x_2, y)$ entering (3.17) from the crossing symmetry of G_R . As expected, $f_R(x)$ and $\tilde{f}_R(z)$ satisfy crossing equations by themselves that are solved in terms of a small number of parameters. Once this is done, $f_R(x)$ and $\tilde{f}_R(z)$ play the role of source terms in the crossing relations for $H_R(x_1, x_2, y)$, $\tilde{H}_R(x_1, x_2, y)$.

First equation. Consider the four-point function (3.14). Imposing that it is invariant upon the exchange of points $1 \leftrightarrow 3$ implies the crossing equations

$$G_R(x_1, x_2, y) = \left(\frac{x_1 x_2}{y} \right)^R G_R(x_1^{-1}, x_2^{-1}, y^{-1}). \quad (4.1)$$

This is due to the fact that the matrix \mathcal{Z} , given in (3.13), transforms to its inverse up to a similarity transform if points one and three are exchanged. In terms of the solution of the WI (3.17), the crossing equation (4.1) implies that the single variable function $f_R(x)$ is crossing invariant by itself:

$$f_R(x) = x^R f_R(x^{-1}). \quad (4.2)$$

The above is a specialization of (4.1) to $(x_1, x_2, y) = (x, y, y)$. Furthermore, it follows that the function $H_R(x_1, x_2, y)$ satisfies the relation

$$\frac{(x_1 x_2)^{R+1}}{y^{R-2}} H_R(x_1^{-1}, x_2^{-1}, y^{-1}) - H_R(x_1, x_2, y) = \frac{y^{2-R}}{x_1^{-1} - x_2^{-1}} (x_2 A_R(x_2, y) f_R(x_1) - x_1 \leftrightarrow x_2), \quad (4.3)$$

where we have defined

$$A_R(x, y) := \frac{x^{R-1} - y^{R-1}}{x - y}, \quad (4.4)$$

which is a polynomial in x, y . The relation (4.3) has been obtained after imposing (4.2). Note that in the special case $R = 2$, all the y dependence disappears from the last equation.

The function $f_R(x)$ is easily argued to be a polynomial of degree R in x as we shall see in section 4.1.1. The crossing relation (4.2), together with the normalization $f_R(0) = 1$, reduces the number of independent parameters entering this function to $\frac{R}{2}$ (respectively $\frac{R-1}{2}$) for R even (respectively odd).

Second equation. Next, we take again the four-point function (3.14) and after some relabeling of the points we obtain

$$\begin{aligned} \langle \hat{\mathcal{B}}_{[R,0]}(1) \hat{\mathcal{B}}_{[R,0]}(2) \hat{\mathcal{B}}_{[0,R]}(\bar{3}) \hat{\mathcal{B}}_{[0,R]}(\bar{4}) \rangle &= \left[(1\bar{3})(2\bar{4}) \left(\frac{\tilde{w}}{\tilde{z}\tilde{z}} \right) \right]^R \tilde{G}_R(\tilde{z}, \tilde{z}, \tilde{w}) \\ &= (2\bar{3})^R (1\bar{4})^R G_R(\hat{z}_1, \hat{z}_2, \hat{y}), \end{aligned} \quad (4.5)$$

where we have defined

$$\hat{\mathcal{Z}} := X_{2\bar{3}} X_{1\bar{3}}^{-1} X_{1\bar{4}} X_{2\bar{4}}^{-1} \sim \text{diag}(\hat{z}_1, \hat{z}_2, \hat{y}), \quad (4.6)$$

and $(\tilde{z}, \tilde{z}, \tilde{w}) := (1 - \hat{z}_1, 1 - \hat{z}_2, 1 - \hat{y}_1)$. It is useful to observe that if the superspace coordinates are $V = \bar{V} = 0$, the cross ratios above are related to the one entering (3.14) as $(\tilde{z}, \tilde{z}, \tilde{w}) = (z, \bar{z}, w)$ and $(\hat{z}_1, \hat{z}_2, \hat{y}) = ((1-x_1)^{-1}, (1-x_2)^{-1}, (1-y)^{-1})$. The first equality in (4.5) is to be understood as defining the function²⁰ \tilde{G}_R , while the second one is a rewriting of (3.14). The function \tilde{G}_R satisfies the same superconformal Ward identities as G_R . We thus parametrize it as in (3.17) with the functions f_R and H_R replaced by \tilde{f} and \tilde{H}_R and the variables x_1, x_2, y replaced by $\tilde{z}, \tilde{z}, \tilde{w}$. An immediate consequence of (4.5) is

$$\tilde{G}_R(z, \bar{z}, w) = \left(\frac{\left(\frac{z}{1-z} \right) \left(\frac{\bar{z}}{1-\bar{z}} \right)}{\left(\frac{w}{1-w} \right)} \right)^R G_R(1-z, 1-\bar{z}, 1-w). \quad (4.7)$$

Note, since the $\tilde{z}, \tilde{z}, \tilde{w}$ are dummy variables, we have renamed them in (4.7) as $(\tilde{z}, \tilde{z}, \tilde{w}) \rightarrow (z, \bar{z}, w)$. As for the crossing equation (4.1), the relation (4.7) implies a relation for the single variable function \tilde{f}_R :

$$\tilde{f}_R(z) = \left(\frac{z}{1-z} \right)^R f_R(1-z), \quad (4.8)$$

while for the \tilde{H} function we get

$$(-1)^R \frac{(x_1 x_2)^{R+1}}{y^{R-2}} H_R(1-z, 1-\bar{z}, 1-w) - \tilde{H}_R(z, \bar{z}, w) = \frac{x_1^{R-1} A_R(x_1, y) \tilde{f}_R(\tilde{z}) - (z \leftrightarrow \bar{z})}{z^{-1} - \bar{z}^{-1}} \quad (4.9)$$

²⁰ The strange prefactor is the natural supersymmetric completion of $(x_{12}^2 x_{34}^2)^{-R}$.

where $A_R(x, y)$ was defined in (4.4) and we remind that in (3.18) we set $(x_1, x_2, y) = (\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}, \frac{w}{w-1})$. As for the first crossing equation (4.3), the dependence on w disappears from (4.9) for $R = 2$.

Third equation. Bose symmetry under the exchange $1 \leftrightarrow 2$ for \tilde{G}_R implies the relation

$$\tilde{G}_R(z, \bar{z}, w) = (-1)^R \tilde{G}_R(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}, \frac{w}{w-1}). \quad (4.10)$$

Using the identities (3.20), one recognizes that (4.10) implies that

$$\tilde{f}_R(z) = (-1)^R \tilde{f}_R(\frac{z}{z-1}), \quad \tilde{H}_R(z, \bar{z}, w) = (-1)^R \tilde{H}_R(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}, \frac{w}{w-1}), \quad (4.11)$$

It should be noticed that the first equation in (4.11) follows from (4.2) and (4.8).

4.1.1 Determination of the function $f_R(x)$

The cohomological reduction of the correlator (3.14), which in superspace corresponds to a specialization of the superspace coordinates in (3.2) to $X = \text{diag}(z, y, y)$, $V = (\theta, 0, 0)^T$ and $\bar{V} = (\bar{\theta}, 0, 0)$, gives the holomorphic correlator

$$\langle \mathcal{W}(z_{-,1}, \theta_1) \bar{\mathcal{W}}(z_{+,2}, \bar{\theta}_2) \mathcal{W}(z_{-,3}, \theta_3) \bar{\mathcal{W}}(z_{+,4}, \bar{\theta}_4) \rangle = \frac{f_R(x)}{(z_{1\bar{2}} z_{3\bar{4}})^R}, \quad x = \frac{z_{1\bar{2}} z_{3\bar{4}}}{z_{3\bar{2}} z_{1\bar{4}}}, \quad (4.12)$$

where $z_{\pm} = z \pm \frac{1}{2}\theta\bar{\theta}$ and $z_{1\bar{2}} = z_{1,-} - z_{2,+} + \theta_1\bar{\theta}_2$. For the following discussion we set the fermionic coordinates $\theta = \bar{\theta} = 0$. We can view the correlator above as a meromorphic function of z_1 , whose poles correspond to singular terms in the OPE of $\mathcal{W}(1)$ with the remaining operators. The chiral OPE is non-singular, so there is no pole when $z_1 \sim z_3$ (corresponding to $x \sim 1$). The singularity for $z_1 \sim z_2$ (corresponding to $x \sim 0$), on the other hand, is already taken care of by the prefactor in the right hand side of (4.12). Finally, for $z_1 \sim z_4$ (corresponding to $x \sim \infty$) we have $f_R(x) \sim x^R$. There is no other singularity so $f_R(x)$ is a polynomial of degree R in x , that we normalize as $f_R(1) = 1$, subject to the crossing relation (4.2). It is thus fixed in terms of $\lfloor \frac{R}{2} \rfloor$ constants. The small x expansion of the correlator takes the form $f_R(x) = 1 + \frac{R^2}{4c_{4d}}x + \dots$ where $c_{4d} \equiv c$ is the central charge²¹ of the four dimensional theory. For $R = 1$ the crossing relation (4.2) implies that $f_1(x) = 1 + x$, this forces the central charge to take the value $c_{4d} = \frac{1}{4}$ which corresponds to $\mathcal{N} = 4$ SYM with gauge group $U(1)$. The case $R = 3$ was already given in (2.23) with the identification (2.5). The derivation of this result is standard, see *e.g.*, [70].

4.1.2 The bootstrap equations

We will now turn the relations (4.3), (4.9), (4.11) into a system of bootstrap equations ready for the numerical analysis. We assume that the function $f_R(x)$ is known. We proceed as follows

- The first step is to expand the correlation functions G_R and \tilde{G}_R entering (4.1) (4.7), (4.10) in the superconformal blocks derived in section 3.4. From this, the block decomposition for the functions H_R, \tilde{H}_R appearing in (4.3), (4.9), (4.11) follows using the results collected in tables 2 and 3.
- The next step is to split the sum over the exchanged operators into a summation over the Schur operators

$$\hat{\mathcal{B}}_{[R,R]}, \quad \hat{\mathcal{C}}_{[R,R],\ell}, \quad \hat{\mathcal{B}}_{[2R,0]}, \quad \hat{\mathcal{C}}_{[2R-2,0](\frac{\ell+1}{2}, \frac{\ell}{2})}, \quad (4.13)$$

²¹If the subscript is omitted, it is understood that $c = c_{4d}$.

and the remaining operators.²² We remind that they are the only ones in the OPEs (3.21) and (3.22) that contribute to the chiral algebra, see (2.6) and the following.

- The third step is to re-sum the contribution of Schur operators to the H_R and \tilde{H}_R functions, denoted as $H_{R,\text{short}}$ and $\tilde{H}_{R,\text{short}}$ in the following. (We sometimes omit the index R and write just H_{short} and \tilde{H}_{short} .) We deal with these functions in section 4.2 and our final results are given in (4.24) and (4.29).

It is important to keep in mind that, in general, knowledge of the function $f_R(x)$ alone is not sufficient to determine $H_{R,\text{short}}$ unambiguously, in contrast with the chiral channel function $\tilde{H}_{R,\text{short}}$, which is fixed in terms of $f_R(x)$. This is due to the fact that different $\mathcal{N} = 3$ supermultiplets give the same contribution, in the sense of holomorphic blocks, to the functions $f_R(x)$ and $\tilde{f}_R(x)$. As we will see in section 4.2, *assuming the absence of supermultiplets that contains conserved currents*,²³ the function $\tilde{H}_{R,\text{short}}$ and the component of $H_{R,\text{short}}$ in the R-symmetry singlet channel can be extracted *unambiguously* from the knowledge of $f_R(x)$.

Summary of the result. Following the procedure that we just outlined one arrives at the bootstrap equations:

$$\sum_{\chi \in \hat{\mathcal{B}}_{[R,0]} \times \hat{\mathcal{B}}_{[0,R]} |_{\mathcal{A}}} |\lambda_\chi|^2 \begin{bmatrix} \mathcal{F}_\chi \\ +\mathcal{F}_{-, \chi}^{\text{b}} \\ -\mathcal{F}_{+, \chi}^{\text{b}} \end{bmatrix} + \sum'_{\chi \in \hat{\mathcal{B}}_{[R,0]} \times \hat{\mathcal{B}}_{[0,R]} |_{\mathcal{A}, \bar{\mathcal{C}}, \bar{\mathcal{B}}}} |\tilde{\lambda}_\chi|^2 \begin{bmatrix} 0 \\ \tilde{\mathcal{F}}_{-, \chi} \\ \tilde{\mathcal{F}}_{+, \chi} \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{\text{short}}^{(0)} \\ \mathcal{F}_{\text{short}}^{(-)} \\ \mathcal{F}_{\text{short}}^{(+)} \end{bmatrix}. \quad (4.14)$$

We now have to make several remarks in order to explain our notation.

- a) We have defined the functions

$$\mathcal{F}_\chi = \frac{[(1-z)(1-\bar{z})]^{R+1}}{(1-w)^{R-2}} H_\chi(z, \bar{z}, w) - (-1)^R \left[(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w) \right], \quad (4.15)$$

$$\mathcal{F}_{\pm, \chi}^{\text{b}} = \frac{[(1-z)(1-\bar{z})]^{R+1}}{(1-w)^{R-2}} H_\chi\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}, \frac{w}{w-1}\right) \pm \left[(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w) \right], \quad (4.16)$$

$$\tilde{\mathcal{F}}_{\pm, \chi} = \frac{[(1-z)(1-\bar{z})]^{R+1}}{(1-w)^{R-2}} \tilde{H}_\chi(z, \bar{z}, w) \pm \left[(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w) \right]. \quad (4.17)$$

The explicit form of the functions H_χ, \tilde{H}_χ is given in tables 2 and 3 for each representation χ .

- b) The functions $\mathcal{F}_{\text{short}}^{(0, \pm)}$ receive contributions from two sources. The first one comes from the right hand side of (4.3), (4.9) and contains the function f_R explicitly. The second one corresponds to the contribution of Schur operators to the left hand side of (4.3), (4.9). Specifically, we have

$$\mathcal{F}_{\text{short}}^{(0, \pm)} = \mathcal{F}_{\text{short}}^{(0, \pm)}[f] - \mathcal{F}_{\text{short}}^{(0, \pm)}[H_{\text{short}}, \tilde{H}_{\text{short}}], \quad (4.18)$$

with the explicit form of $\mathcal{F}_{\text{short}}^{(0, \pm)}[f]$ and $\mathcal{F}_{\text{short}}^{(0, \pm)}[H_{\text{short}}, \tilde{H}_{\text{short}}]$ given in appendix D.1.

- c) The precise range of summation in (4.14) is specified by the selection rules (3.21) and (3.22), where we only take the operators that are *not* of Schur type, *i.e.*, \mathcal{A} in the non-chiral channel

²² This bisection does not coincide in general with the separation between long and short operators, as can be seen in the chiral channel.

²³ These are expected to be absent in interaction theories [61, 62].

and $\mathcal{A}, \overline{\mathcal{B}}, \overline{\mathcal{C}}$ in the chiral one. The prime in the second sum \sum' indicates that the parity of the spin label ℓ of the exchanged operator is fixed in terms of its R-symmetry representation. Specifically even spins appear only for irreps in the $\text{sym}([R, 0] \otimes [R, 0])$ while odd spins for irreps in $\text{antisym}([R, 0] \otimes [R, 0])$. This follows from (4.11) and the braiding relations (C.8) of individual blocks.

- d) The second and third equations in (4.14) are obtained respectively from the antisymmetrization and symmetrization of the superconformal block expansion of (4.9) with respect to the exchange $(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w)$. An important remark, relevant for the numerical implementation, is that the arguments of four dimensional superconformal and R-symmetry blocks entering (4.16), namely $(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}, \frac{w}{w-1})$ and their inverses, can be traded for (z, \bar{z}, w) and $(1-z, 1-\bar{z}, 1-w)$ using the braiding properties of the conformal blocks (C.8). This fact justifies the use of the suffix b to denote “braided” in (4.16).
- e) Finally, as customary, the identification $|\lambda_\chi|^2 = \sum_{\mathcal{O}|\chi(\mathcal{O})=\chi} |\lambda_{\mathcal{O}}|^2$ is understood. By $\chi(\mathcal{O})$ we mean the representation χ in which the operator \mathcal{O} transforms.

4.2 Contribution of Schur operators

We have defined in the above section the functions $H_{R,\text{short}}$ and $\tilde{H}_{R,\text{short}}$ as the contribution from the exchange of Schur operators (in the relevant channel) to the H_R and \tilde{H}_R functions entering (4.3) and (4.9). We will now discuss to which extent the functions $H_{R,\text{short}}$ and $\tilde{H}_{R,\text{short}}$ can be extracted from the knowledge of $f_R(x)$, or more generally, from the knowledge of the chiral algebra.

Non-chiral channel. Consider the expansion of the function $f_R(x)$ in holomorphic $\mathcal{N} = 2$ (global $\text{OSP}(2|2)$) blocks as

$$f_R(x) = 1 + \sum_{h=1}^{\infty} b_h^{(R)} g_h^{2d \mathcal{N}=2}(x). \quad (4.19)$$

The function $f_R(x)$ is a polynomial in x and the monomials $x^n, n > 0$ can be expanded in holomorphic $\mathcal{N} = 2$ blocks as in (D.4). Using the result given in table 2 and the selection rules (3.21), it is clear that in general one cannot reconstruct the four-dimensional OPE coefficients corresponding to Schur operators (4.13) from the knowledge of the expansion (4.19). This is best illustrated by looking at the examples

$$\begin{aligned} R = 1 : \quad & b_1^{(1)} = |\lambda_1^{\hat{\mathcal{B}}}|^2, \quad b_{h>1}^{(1)} = (-1)^{h+1} |\lambda_{0,h-2}^{\hat{\mathcal{C}}}|^2, \\ R = 2 : \quad & b_1^{(2)} = |\lambda_1^{\hat{\mathcal{B}}}|^2, \quad b_2^{(2)} = |\lambda_2^{\hat{\mathcal{B}}}|^2, \quad b_{h>2}^{(2)} = (-1)^h |\lambda_{1,h-3}^{\hat{\mathcal{C}}}|^2, \\ R = 3 : \quad & b_1^{(3)} = |\lambda_1^{\hat{\mathcal{B}}}|^2, \quad b_2^{(3)} = |\lambda_2^{\hat{\mathcal{B}}}|^2, \quad b_3^{(3)} = |\lambda_3^{\hat{\mathcal{B}}}|^2 - |\lambda_{1,0}^{\hat{\mathcal{C}}}|^2, \quad b_{h>3}^{(3)} = (-1)^h (|\lambda_{1,h-3}^{\hat{\mathcal{C}}}|^2 - |\lambda_{2,h-4}^{\hat{\mathcal{C}}}|^2), \end{aligned} \quad (4.20)$$

and so on. Above we used the compact notation $\lambda_m^{\hat{\mathcal{B}}} \equiv \lambda_{\hat{\mathcal{B}}[m,m]}^{(R)}$ and $\lambda_{m,\ell}^{\hat{\mathcal{C}}} \equiv \lambda_{\hat{\mathcal{C}}[m,m],\ell}^{(R)}$. Of course, λ 's from different rows (*i.e.*, for different values of R) in (4.20) are not the same, even though this is not captured by the notation. The general pattern is quite simple and one finds

$$b_{1 \leq h \leq R}^{(R)} = |\lambda_h^{\hat{\mathcal{B}}}|^2 - \sum_{a=1}^{h-2} (-1)^{h-a} |\lambda_{a,h-a-2}^{\hat{\mathcal{C}}}|^2, \quad b_{h>R}^{(R)} = - \sum_{a=1}^{R-1} (-1)^{h-a} |\lambda_{a,h-a-2}^{\hat{\mathcal{C}}}|^2. \quad (4.21)$$

Notice that compared to the results that can be obtained from table 2 and the selection rules (3.21), we *omitted by hand* the supermultiplets $\hat{\mathcal{C}}_{[0,0],\ell}$ for external fields with $R \geq 2$, because they are

the supermultiplets that contain higher spin conserved currents. The latter are included only in the free field case $R = 1$. For $R \geq 2$ they are allowed by the selection rules (3.21), but we want to demand that they are absent since we want to focus on interacting theories. We remark further that the OPE coefficient

$$b_1^{(R)} = |\lambda_{\hat{\mathcal{B}}_{[1,1]}}^{(R)}|^2 = \frac{R^2}{4c_{4d}}, \quad (4.22)$$

see discussion in section 4.1.1, corresponding to the exchange of the stress-tensor supermultiplet $\hat{\mathcal{B}}_{[1,1]}$ can be extracted unambiguously, as can be seen in (4.20).

It follows from the above considerations, and under the assumption that no higher spin conserved current is present, that also $|\lambda_2^{\hat{\mathcal{B}}}|^2$ can be extracted without ambiguity. In general the four dimensional OPE coefficients cannot be extracted uniquely from the expansion (4.19). As discussed in section 2.1.3 and section 5.3.2 using the knowledge of the chiral algebra and some extra assumptions one can find, in the case $R = 3$, only two allowed values for $\lambda_{\hat{\mathcal{B}}_{[3,3]}}^{(3)}$ and $\lambda_{\hat{\mathcal{C}}_{[1,1],0}}^{(3)}$.

Let us now investigate the structure of $H_{R,\text{short}}$. By definition, we have

$$H_{R,\text{short}} := \sum_{a=2}^R |\lambda_{\hat{\mathcal{B}}_{[a,a]}}|^2 H_{\hat{\mathcal{B}}_{[a,a]}} + \sum_{a=1}^{R-1} \sum_{\ell=0}^{\infty} |\lambda_{\hat{\mathcal{C}}_{[a,a],\ell}}|^2 H_{\hat{\mathcal{C}}_{[a,a],\ell}} \quad (4.23)$$

which we can express in terms of the blocks $h_{[a,a]}(w)$ and $\mathcal{G}_{\Delta,\ell}^{\mathcal{N}=1}(z, \bar{z})$ given in (3.24), (3.26) as

$$H_{R,\text{short}}(x_1, x_2, y) = \sum_{h=2}^{\infty} (-1)^h b_h^{(R)} \mathcal{G}_{h+2, h-2}^{\mathcal{N}=1}(z, \bar{z}) - \begin{cases} -h_{[0,0]}(w) \sum_{n=0}^{\infty} |\lambda_{\hat{\mathcal{C}}_{[0,0],n}}^{(R)}|^2 \mathcal{G}_{n+4,n}^{\mathcal{N}=1}, & R = 2 \\ h_{[1,1]}(w) \sum_{n=0}^{\infty} |\lambda_{\hat{\mathcal{C}}_{[2,2],n-1}}^{(R)}|^2 \mathcal{G}_{n+6,n}^{\mathcal{N}=1}(z, \bar{z}) - h_{[0,0]}(w) \sum_{n=0}^{\infty} |\lambda_{\hat{\mathcal{C}}_{[0,0],n}}^{(R)}|^2 \mathcal{G}_{n+4,n}^{\mathcal{N}=1}, & R = 3 \\ \sum_{t=0}^{R-2} h_{[t,t]}(w) C_t^{(R)}(z, \bar{z}), & \text{general } R \end{cases} \quad (4.24)$$

In (4.23) the first summation starts from $h = 2$, since $H_{\hat{\mathcal{B}}_{[1,1]}} = 0$. In writing this equation we allowed for higher spin currents to have a non-vanishing OPE coefficient, such that it becomes clear how they would contribute to the crossing equations. Looking at table 2 we see that if higher spin currents are present they contribute exactly the same way as the R-symmetry singlet long multiplet at the unitarity bound $\Delta = \ell + 2$. Of course for an interacting theory we want to set $|\lambda_{\hat{\mathcal{C}}_{[0,0],n}}^{(R)}|^2 = 0$. As claimed above, after setting the higher spin currents to zero, only the part of $H_{R,\text{short}}$ in the R-symmetry singlet channel is completely fixed in terms of the function $f_R(x)$. The explicit expression for the function $C_t^{(R)}(z, \bar{z})$ is easily worked out, but will not be relevant here. We finally remark that the summation of the first term in (4.24) can be done explicitly for any R as first done in [20]. See appendix D.2 for details.

Example: For $R = 2$, we find

$$f_2(x) = 1 + c^{-1}x + x^2 = 1 + \sum_{h=1}^{\infty} b_h^{(2)} g_h^{2d} \mathcal{N}=2(x), \quad (4.25)$$

where from (4.20) we take, $|\lambda_{\hat{\mathcal{B}}_{[1,1]}}|^2 = b_1^{(2)} = c^{-1}$, $|\lambda_{\hat{\mathcal{B}}_{[2,2]}}|^2 = b_2^{(2)} = 1 - \frac{1}{3c}$ and

$$|\lambda_{\hat{\mathcal{C}}_{[1,1],\ell}}|^2 = (-1)^{\ell+1} b_{\ell+3}^{(2)} = \frac{(\ell+2)(4)_{\ell+1}}{2^{2\ell+2} \left(\frac{5}{2}\right)_{\ell+1}} - \frac{\Gamma(\ell+4)}{2^{2\ell+5} \left(\frac{1}{2}\right)_{\ell+3}} c^{-1}. \quad (4.26)$$

Note that if higher spin currents are present the above identification of OPE coefficients with b_h cannot be made for $h > 1$.

Chiral channel. In the chiral channel, we expand the function \tilde{f}_R , related to f_R by (4.8), in $\mathcal{N} = 2$ holomorphic blocks, which in this channel coincide with ordinary $\text{SL}(2)$ blocks, see (C.4). Specifically

$$\tilde{f}_R(z) = \sum_{\substack{h=R \\ h+R \text{ even}}}^{\infty} \tilde{b}_h^{(R)} g_h^{2d}(z), \quad (4.27)$$

where we note that the sum starts from $h = R$, which is due to the fact that the relevant OPE is non singular. Moreover, the index h has the same parity as R as follows from the braiding relations of individual blocks (C.10) together with (4.11). By looking at the selection rules in this channel given in (3.22), and after a quick look at table 3, one concludes that

$$\tilde{b}_R^{(R)} = |\tilde{\lambda}_{\hat{\mathcal{B}}_{[2R,0]}}|^2, \quad \tilde{b}_{R+1}^{(R)} = 0, \quad \tilde{b}_{\ell+R+2}^{(R)} = |\tilde{\lambda}_{\hat{\mathcal{C}}_{[2R-2,0],((\ell+1)/2,\ell/2)}}|^2, \quad (4.28)$$

where $\ell \geq 0$. Notice that in this channel we can reconstruct the four dimensional OPE coefficients of Schur operator completely in terms of the OPE coefficients of the cohomologically reduced problem. We can thus uniquely determine the contribution of these operators to \tilde{H}_R :

$$\tilde{H}_{R,\text{short}} = \tilde{b}_R^{(R)} \tilde{H}_{\hat{\mathcal{B}}_{[2R,0]}} + \sum_{n=0}^{\infty} \tilde{b}_{R+2n+2}^{(R)} \tilde{H}_{\hat{\mathcal{C}}_{[2(R-1),0],(n+\frac{1}{2},n)}}. \quad (4.29)$$

The summation of this expression is straightforward and similar to the one done in appendix D.2. The final result is given by

$$\tilde{H}_{R,\text{short}}(z, \bar{z}, w) = \frac{\tilde{f}_R(\bar{z}) \gamma_R(z, w) - (z \leftrightarrow \bar{z})}{z^{-1} - \bar{z}^{-1}}, \quad (4.30)$$

where we have defined the kinematic factor.

$$\gamma_R(z, w) = \sum_{a=0}^{R-2} k_{2a+2}(z) h_a^{\text{SU}(2)}(w), \quad (4.31)$$

with $k_\beta(z)$ and $h_a^{\text{SU}(2)}(w)$ given in (C.2) and (3.32) respectively.

Having thus obtained in this section explicit expressions for $H_{R,\text{short}}$ and $\tilde{H}_{R,\text{short}}$, we can explicitly compute the functions $\mathcal{F}_{\text{short}}^{(0,\pm)}$ entering the crossing equations (4.14). See appendix D.1 for more details.

4.3 Explicit form of the bootstrap equations for $R = 2, 3$

We will now show the explicit form of the (4.14) in the cases $R = 2, 3$. In order to do so, it is convenient to define the combinations of conformal blocks

$$\begin{aligned}
\mathbb{F}_{\pm, \Delta, \ell} &:= [(1-z)(1-\bar{z})]^{R+1} (z\bar{z})^{-\frac{1}{2}} g_{\Delta+3, \ell}^{1,1}(z, \bar{z}) \pm [(z, \bar{z}) \leftrightarrow (1-z, 1-\bar{z})] , \\
\mathbb{F}_{\pm, \Delta, \ell}^{\text{b}} &:= (-1)^\ell [(1-z)(1-\bar{z})]^{R+2} (z\bar{z})^{-\frac{1}{2}} g_{\Delta+3, \ell}^{-1,1}(z, \bar{z}) \pm [(z, \bar{z}) \leftrightarrow (1-z, 1-\bar{z})] , \\
\tilde{\mathbb{F}}_{\pm, \Delta, \ell} &:= [(1-z)(1-\bar{z})]^{R+1} g_{\Delta+3, \ell}^{0,0}(z, \bar{z}) \pm [(z, \bar{z}) \leftrightarrow (1-z, 1-\bar{z})] .
\end{aligned} \tag{4.32}$$

compare to (4.15), (4.16), (4.17). As before, we suppressed the index R from the notation. Its value should be clear from the context.

Multiplet χ	$f_\chi(x)$	$H_\chi(z, \bar{z})$
Identity	$g_0^{2d, \mathcal{N}=2}(x) = 1$	0
$\hat{\mathcal{B}}_{[1,1]}$	$g_1^{2d, \mathcal{N}=2}(x)$	0
$\hat{\mathcal{C}}_{[0,0], \ell}$	$(-1)^{\ell+1} g_{\ell+2}^{2d, \mathcal{N}=2}(x)$	0
$\hat{\mathcal{B}}_{[2,2]}$	$g_2^{2d, \mathcal{N}=2}(x)$	$\mathcal{G}_{4,0}^{\mathcal{N}=1} = (z\bar{z})^{-\frac{1}{2}} g_{5,0}^{1,1}(z, \bar{z})$
$\hat{\mathcal{C}}_{[1,1], \ell}$	$(-1)^{\ell+1} g_{\ell+3}^{2d, \mathcal{N}=2}(x)$	$\mathcal{G}_{\ell+5, \ell+1}^{\mathcal{N}=1} = (z\bar{z})^{-\frac{1}{2}} g_{\ell+6, \ell+1}^{1,1}(z, \bar{z})$
$\mathcal{A}_{[0,0], \ell}^{\Delta > \ell+2}$	0	$\mathcal{G}_{\Delta+2, \ell}^{\mathcal{N}=1} = (z\bar{z})^{-\frac{1}{2}} g_{\Delta+3, \ell}^{1,1}(z, \bar{z})$

Table 4. This table shows the contributions of the various $\mathcal{N} = 3$ multiplets appearing in the non-chiral OPE (3.21) for $R = 2$ to the functions $f_\chi(x_1)$ and $H_\chi(z, \bar{z})$. The multiplets $\hat{\mathcal{C}}_{[0,0], \ell}$ contain conserved currents of spin larger than two, and must be set to zero if we want an interacting theory [61, 62]. We remind that $\hat{\mathcal{B}}_{[1,1]}$ is the stress-tensor multiplet. When the long multiplet $\mathcal{A}_{[0,0], \ell}^{\Delta > \ell+2}$ hits the unitarity bound $\Delta = \ell + 2$ it decomposes in a $\hat{\mathcal{C}}_{[0,0], \ell}$ and a $\hat{\mathcal{C}}_{[1,1], \ell-1}$, where $\hat{\mathcal{C}}_{[1,1], -1} = \hat{\mathcal{B}}_{[2,2]}$. Note that while long multiplets arbitrarily close to the unitarity bound mimic higher spin conserved currents, they do not mimic the stress tensor. In particular, this ensures that we can fix the central charge completely.

The case $R = 2$. The bootstrap equations (4.14) in this case are independent of the R-symmetry variables w . Using the $R = 2$ specializations of the tables 2 and 3, namely table 4 and 5, we obtain

$$\sum_{\Delta \geq \ell+2} |\lambda_{\Delta, \ell}|^2 \begin{bmatrix} \mathbb{F}_{-, \Delta, \ell} \\ +\mathbb{F}_{-, \Delta, \ell}^{\text{b}} \\ -\mathbb{F}_{+, \Delta, \ell}^{\text{b}} \end{bmatrix} + \sum_{\substack{\Delta \geq \ell+3 \\ \ell \text{ even}}} |\tilde{\lambda}_{\Delta, \ell}|^2 \begin{bmatrix} 0 \\ \tilde{\mathbb{F}}_{-, \Delta, \ell} \\ \tilde{\mathbb{F}}_{+, \Delta, \ell} \end{bmatrix} = \begin{bmatrix} \mathbb{F}_{\text{short}}^{(0)} \\ \mathbb{F}_{\text{short}}^{(-)} \\ \mathbb{F}_{\text{short}}^{(+)} \end{bmatrix} , \tag{4.33}$$

The case $R = 3$. In this situation, in order to write down the bootstrap equations (4.14) in components we need to fix a basis in the space of R-symmetry polynomials. There is a natural choice which

Multiplet χ	$\tilde{f}_\chi(z)$	$\tilde{H}_\chi(z, \bar{z})$
$\hat{\mathcal{B}}_{[4,0]}$	$g_2^{2d}(z)$	$g_{4,0}(z, \bar{z})$
$\hat{\mathcal{C}}_{[2,0],(\frac{\ell+1}{2}, \frac{\ell}{2})}$	$g_{4+\ell}^{2d}(z)$	$g_{\ell+6,\ell+2}(z, \bar{z})$
$\bar{\mathcal{B}}_{[0,2],r=8,(0,0)}$	0	$g_{6,0}(z, \bar{z})$
$\bar{\mathcal{C}}_{[0,1],r=7,(\frac{\ell+1}{2}, \frac{\ell}{2})}$	0	$g_{\ell+7,\ell+1}(z, \bar{z})$
$\mathcal{A}_{[0,0],r=6,\ell}^{\Delta>3+\ell}$	0	$g_{\Delta+2,\ell}(z, \bar{z})$

Table 5. This table shows the contributions of the various $\mathcal{N} = 3$ multiplets appearing in the chiral OPE (3.22) for $R = 2$ to the functions $\tilde{f}_\chi(z)$ and $\tilde{H}_\chi(z, \bar{z})$. Note that at the unitarity bound (see (3.35)) of the long multiplet we find two types (for $\ell = 0$ and $\ell \neq 0$) of short multiplets which do not contribute to the chiral algebra, namely $\bar{\mathcal{B}}_{[0,2],r=8,(0,0)}$ and $\bar{\mathcal{C}}_{[0,1],r=7,(\frac{\ell+1}{2}, \frac{\ell}{2})}$. When considering identical $\hat{\mathcal{B}}_{[2,0]}$ operators Bose symmetry requires ℓ to be even for $\mathcal{A}_{[0,0],r=6,\ell}^{\Delta>3+\ell}$ and odd for $\bar{\mathcal{C}}_{[0,1],r=7,(\frac{\ell+1}{2}, \frac{\ell}{2})}$.

follows by noticing that

$$\begin{aligned}
(1-w) \mathcal{F}_{\mathcal{A}_{[0,0],\ell}^{\Delta,r=0}} &= +\frac{1}{2}(1-y^{-1}) F_{+,\Delta,\ell} + \frac{1}{2}(1+y^{-1}) F_{-,\Delta,\ell}, \\
(1-w) \mathcal{F}_{\mathcal{A}_{[1,1],\ell}^{\Delta,r=0}} &= -\frac{2}{3}(1-y^{-1}) F_{+,\Delta,\ell} + \frac{1}{3}(1+y^{-1}) F_{-,\Delta,\ell}, \\
\mathcal{F}_{\pm, \mathcal{A}_{[0,0],\ell}^{\Delta,r=0}}^b &= +\frac{1}{2} \left(\frac{1}{1-w} + \frac{1}{w} \right) F_{\pm,\Delta,\ell}^b + \frac{1}{2} \left(\frac{1}{1-w} - \frac{1}{w} \right) F_{\mp,\Delta,\ell}^b, \\
\mathcal{F}_{\pm, \mathcal{A}_{[1,1],\ell}^{\Delta,r=0}}^b &= +\frac{5}{6} \left(\frac{1}{1-w} + \frac{1}{w} \right) F_{\pm,\Delta,\ell}^b - \frac{1}{6} \left(\frac{1}{1-w} - \frac{1}{w} \right) F_{\mp,\Delta,\ell}^b, \\
\tilde{\mathcal{F}}_{\pm, \mathcal{A}_{[0,1],\ell}^{\Delta,r=10}} &= +\frac{1}{2} \left(\frac{1}{1-w} + \frac{1}{w} \right) \tilde{F}_{\pm,\Delta,\ell} + \frac{1}{2} \left(\frac{1}{1-w} - \frac{1}{w} \right) \tilde{F}_{\mp,\Delta,\ell}, \\
\tilde{\mathcal{F}}_{\pm, \mathcal{A}_{[2,0],\ell}^{\Delta,r=10}} &= +\frac{3}{4} \left(\frac{1}{1-w} + \frac{1}{w} \right) \tilde{F}_{\pm,\Delta,\ell} - \frac{1}{4} \left(\frac{1}{1-w} - \frac{1}{w} \right) \tilde{F}_{\mp,\Delta,\ell}.
\end{aligned} \tag{4.34}$$

The equations for $\bar{\mathcal{B}}$ and $\bar{\mathcal{C}}$ in the chiral channel follow from the last two at the unitarity bound. Let us go back to the bootstrap equations (4.14) specialized to the case $R = 3$. Using the relations above,

the equations (4.14) are easily recognized to be equivalent to

$$\begin{aligned}
& \sum_{\Delta \geq \ell+2} |\lambda_{0,\Delta,\ell}|^2 \begin{bmatrix} +\frac{1}{2} F_{-,\Delta,\ell} \\ +\frac{1}{2} F_{+,\Delta,\ell} \\ +\frac{1}{2} F_{-,\Delta,\ell}^b \\ +\frac{1}{2} F_{+,\Delta,\ell}^b \\ -\frac{1}{2} F_{+,\Delta,\ell}^b \\ -\frac{1}{2} F_{-,\Delta,\ell}^b \end{bmatrix} + \sum_{\Delta \geq \ell+4} |\lambda_{1,\Delta,\ell}|^2 \begin{bmatrix} +\frac{1}{3} F_{-,\Delta,\ell} \\ -\frac{1}{3} F_{+,\Delta,\ell} \\ +\frac{5}{6} F_{-,\Delta,\ell}^b \\ -\frac{1}{6} F_{+,\Delta,\ell}^b \\ -\frac{5}{6} F_{+,\Delta,\ell}^b \\ +\frac{1}{6} F_{-,\Delta,\ell}^b \end{bmatrix} + \\
& + \sum_{\substack{\Delta \geq \ell+5 \\ \ell \text{ even}}} |\tilde{\lambda}_{0,\Delta,\ell}|^2 \begin{bmatrix} 0 \\ 0 \\ +\frac{1}{2} \tilde{F}_{-,\Delta,\ell} \\ +\frac{1}{2} \tilde{F}_{+,\Delta,\ell} \\ +\frac{1}{2} \tilde{F}_{+,\Delta,\ell} \\ +\frac{1}{2} \tilde{F}_{-,\Delta,\ell} \end{bmatrix} + \sum_{\substack{\Delta \geq \ell+5 \\ \ell \text{ odd}}} |\tilde{\lambda}_{1,\Delta,\ell}|^2 \begin{bmatrix} 0 \\ 0 \\ +\frac{3}{4} \tilde{F}_{-,\Delta,\ell} \\ -\frac{1}{4} \tilde{F}_{+,\Delta,\ell} \\ +\frac{3}{4} \tilde{F}_{+,\Delta,\ell} \\ -\frac{1}{4} \tilde{F}_{-,\Delta,\ell} \end{bmatrix} = \vec{F}_{\text{short}}.
\end{aligned} \tag{4.35}$$

The explicit expression for \vec{F}_{short} follows from the expressions collected in appendix D.1. Since it is rather long, we will not present it here. As a consistency check, we verified that the bootstrap equations above are satisfied with positive coefficients for the cases of free $U(1)$ $\mathcal{N} = 4$ SYM (considered a special $\mathcal{N} = 3$ theory) and for the generalized free theory discussed in appendix E.

Multiplet χ	$f_\chi(x)$	$H_\chi(z, \bar{z}, w)$
Identity	$g_0^{2d} \mathcal{N}=2(z) = 1$	0
$\hat{\mathcal{B}}_{[1,1]}$	$g_1^{2d} \mathcal{N}=2(z)$	0
$\hat{\mathcal{B}}_{[2,2]}$	$g_2^{2d} \mathcal{N}=2(z)$	$\mathcal{G}_{4,0}^{\mathcal{N}=1}(z, \bar{z})$
$\hat{\mathcal{B}}_{[3,3]}$	$g_3^{2d} \mathcal{N}=2(z)$	$-\mathcal{G}_{5,1}^{\mathcal{N}=1}(z, \bar{z}) - \mathcal{G}_{4,0}^{\mathcal{N}=1}(z, \bar{z})h_{[1,1]}(w)$
$\hat{\mathcal{C}}_{[0,0],\ell}$	$(-1)^{\ell+1} g_{\ell+2}^{2d} \mathcal{N}=2(z)$	0
$\hat{\mathcal{C}}_{[1,1],\ell}$	$(-1)^{\ell+1} g_{\ell+3}^{2d} \mathcal{N}=2(z)$	$\mathcal{G}_{\ell+5,\ell+1}^{\mathcal{N}=1}(z, \bar{z})$
$\hat{\mathcal{C}}_{[2,2],\ell}$	$(-1)^{\ell+1} g_{\ell+4}^{2d} \mathcal{N}=2(z)$	$-\mathcal{G}_{\ell+6,\ell+2}^{\mathcal{N}=1}(z, \bar{z}) - \mathcal{G}_{\ell+7,\ell+1}^{\mathcal{N}=1}(z, \bar{z})h_{[1,1]}(w)$
$\mathcal{A}_{[0,0],\ell}^{\Delta > \ell+2}$	0	$\mathcal{G}_{\Delta+2,\ell}^{\mathcal{N}=1}(z, \bar{z})$
$\mathcal{A}_{[1,1],\ell}^{\Delta > \ell+4}$	0	$-\mathcal{G}_{\Delta+2,\ell}^{\mathcal{N}=1}(z, \bar{z})h_{[1,1]}(w)$

Table 6. This table shows the contributions of the various $\mathcal{N} = 3$ multiplets appearing in the non-chiral OPE (3.21) for $R = 3$ to the functions $f_\chi(x)$ and $H_\chi(z, \bar{z}, w)$. Note that we can make the following identification $\hat{\mathcal{C}}_{[k,k],\ell=-1} = \hat{\mathcal{B}}_{[k+1,k+1]}$. In the text we will take the latter to be a special case of the first.

5 Numerical results

Our goal is to chart out the allowed parameter space of $\mathcal{N} = 3$ theories, and also to “zoom in” to particular solutions of the crossing equations that correspond to pure $\mathcal{N} = 3$ SCFTs. Since theories with $\mathcal{N} = 4$ are also $\mathcal{N} = 3$ theories, we expect solutions corresponding to these theories to make an

Multiplet χ	$\tilde{f}_\chi(z)$	$\tilde{H}_\chi(z, \bar{z}, w)$
$\hat{\mathcal{B}}_{[6,0]}$	$g_3^{2d}(z)$	$g_{5,1}(z, \bar{z}) + g_{6,0}(z, \bar{z})h_1^{\text{SU}(2)}(w)$
$\hat{\mathcal{C}}_{[4,0],(\frac{\ell+1}{2}, \frac{\ell}{2})}$	$g_{\ell+5}^{2d}(z)$	$g_{\ell+7,\ell+3}(z, \bar{z}) + g_{8+\ell,2+\ell}(z, \bar{z})h_1^{\text{SU}(2)}(w)$
$\overline{\mathcal{B}}_{[2,2],0}^{r=12}$	0	$g_{8,0}(z, \bar{z})h_1^{\text{SU}(2)}(w)$
$\overline{\mathcal{B}}_{[0,3],0}^{r=12}$	0	$g_{8,0}(z, \bar{z})$
$\overline{\mathcal{C}}_{[2,1],(\frac{\ell+1}{2}, \frac{\ell}{2})}^{r=11}$	0	$g_{\ell+9,\ell+1}(z, \bar{z})h_1^{\text{SU}(2)}(w)$
$\overline{\mathcal{C}}_{[0,2],(\frac{\ell+1}{2}, \frac{\ell}{2})}^{r=11}$	0	$g_{\ell+9,\ell+1}(z, \bar{z})$
$\mathcal{A}_{[2,0],\ell}^{\Delta>\ell+5,r=10}$	0	$g_{\Delta+3,\ell}(z, \bar{z})h_1^{\text{SU}(2)}(w)$
$\mathcal{A}_{[0,1],\ell}^{\Delta>\ell+5,r=10}$	0	$g_{\Delta+3,\ell}(z, \bar{z})$

Table 7. This table shows the contributions of the various $\mathcal{N} = 3$ multiplets appearing in the chiral OPE (3.22) for $R = 3$ to the functions $\tilde{f}_\chi(z)$ and $\tilde{H}_\chi(z, \bar{z}, w)$. Since we are interested in the correlation functions of identical operators, Bose symmetry under the exchange of the two identical operators forbids the multiplet $\overline{\mathcal{B}}_{[0,3],0}^{r=12}$ from appearing and restricts the ℓ to be even for $\mathcal{A}_{[2,0],\ell}^{r=10}$, $\overline{\mathcal{C}}_{[0,2],(\frac{\ell+1}{2}, \frac{\ell}{2})}^{r=11}$ and $\hat{\mathcal{C}}_{[4,0],(\frac{\ell+1}{2}, \frac{\ell}{2})}$, and odd for $\mathcal{A}_{[0,1],\ell}^{r=10}$ and $\overline{\mathcal{C}}_{[2,1],(\frac{\ell+1}{2}, \frac{\ell}{2})}^{r=11}$.

appearance, and we must find ways to exclude them from the analysis. In this section we take a first step towards this, focusing most of our attention on $\mathcal{N} = 3$ SCFTs with a Coulomb branch operator of dimension three. We should point out however that, just as we cannot impose the symmetry to be only $\mathcal{N} = 3$, we can also not impose this Coulomb branch operator to be a generator.²⁴

We start by considering the multiplet containing a Coulomb branch operator of dimension two, which we recall also contains extra supercharges. In the remainder of the section we then focus on obtaining numerical bounds valid for any theory with a Coulomb branch operator of dimension three, and various values of the central charge. Towards the end we will input knowledge of the specific chiral algebra that is conjectured [10] to correspond to the simplest known $\mathcal{N} = 3$ SCFT in an attempt to zoom in to this particular solution.

5.1 Numerical methods

The crossing equations written in (4.14) are too complicated to study exactly, beyond focusing on special limits, or protected subsectors, like we did in section 2. Therefore we proceed to analyze these equations using the, by now standard, numerical techniques pioneered in [14] (see *e.g.* [71, 72] for reviews).

Very schematically, we have a system of crossing equations (three (4.33) and six (4.34) for the $\hat{\mathcal{B}}_{[2,0]}$ and $\hat{\mathcal{B}}_{[3,0]}$ respectively) of the form

$$\sum_{\mathcal{O}_i} |\lambda_{\mathcal{O}_i}|^2 \vec{V}_{\mathcal{O}_i}(z, \bar{z}) = \vec{V}_{\text{fixed}}(z, \bar{z}). \quad (5.1)$$

²⁴One could imagine setting up a mixed correlator system where the multiplets containing the extra supercharges, or the candidate generators for which our operator could be a composite are exchanged.

We use the SDPB solver of [73], and rule out assumptions on the spectrum $\{\mathcal{O}_i\}$ of local operators and their OPE coefficients $|\lambda_{\mathcal{O}_i}|^2$ (CFT data), by considering linear functionals

$$\vec{\Phi} = \sum_{n,m=0}^{n+m \leq \Lambda} \vec{\Phi}_{m,n} \partial_z^m \partial_{\bar{z}}^n \Big|_{z=\bar{z}=\frac{1}{2}}, \quad (5.2)$$

acting on the crossing equations. In the crossing equation (4.33) and (4.34) we will be taking derivatives $\partial_z^m \partial_{\bar{z}}^n$ of $F_{\pm}, F_{\pm}^b, \tilde{F}_{\pm}$ and from their symmetry properties under $z \rightarrow 1-z, \bar{z} \rightarrow 1-\bar{z}$ we see that only even (odd) derivatives of F_+, F_+^b, \tilde{F}_+ (F_-^b, F_-, \tilde{F}_-) survive.²⁵

The numerical bounds will be obtained for different values of the cutoff Λ , which effectively means we are considering a truncation of the Taylor series expansion of the crossing equations around $z = \bar{z} = \frac{1}{2}$. We rule out assumptions on the CFT data by proving that they are inconsistent with the truncated system of crossing equations at order Λ . Therefore, for each cutoff we find valid bounds, that will improve as we send $\Lambda \rightarrow \infty$. We refer to reader to the by now extensive literature on these numerical techniques, *e.g.* [16, 73], for all the other technical details and approximations needed for the numerical bootstrap.

5.2 The case $R = 2$

As a warm up, we begin by considering external operators $\hat{\mathcal{B}}_{[2,0]}, \hat{\mathcal{B}}_{[0,2]}$, which contain the extra supercharges allowing for an enhancement to $\mathcal{N} = 4$. Because we expect to recover $\mathcal{N} = 4$ SYM, for this correlator we will only bound the minimal allowed central charge c as a consistency check. We recall that the OPE selection rules in this case are given by

$$\hat{\mathcal{B}}_{[2,0]} \times \hat{\mathcal{B}}_{[0,2]} = \mathcal{I} + \hat{\mathcal{B}}_{[1,1]} + \hat{\mathcal{B}}_{[2,2]} + \sum_{\ell=0}^{\infty} \left[\hat{\mathcal{C}}_{[0,0],\ell} + \hat{\mathcal{C}}_{[1,1],\ell} + \mathcal{A}_{[0,0],r=0,\ell}^{\Delta} \right], \quad (5.3)$$

$$\hat{\mathcal{B}}_{[2,0]} \times \hat{\mathcal{B}}_{[2,0]} = \hat{\mathcal{B}}_{[4,0]} + \bar{\mathcal{B}}_{[0,2],r=8,0} + \sum_{\ell=0}^{\infty} \left[\hat{\mathcal{C}}_{[2,0],(\frac{\ell+1}{2},\frac{\ell}{2})} + \bar{\mathcal{C}}_{[0,1],(\frac{\ell+1}{2},\frac{\ell}{2})}^{r=7} + \mathcal{A}_{[0,0],(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta,r=6} \right], \quad (5.4)$$

with each multiplet contributing with a superblock as given in tables 4 and 5, with a positive OPE coefficient squared, and the crossing equations are given in (4.33). To obtain central charge bounds, we allow for all operators consistent with unitarity that have not been fixed by the chiral algebra. In the chiral channel this amounts to allowing all long operators consistent with unitarity, together with the short multiplets which sit at the long unitarity bound (which are not Schur operators). In the non-chiral channel the OPE coefficient of $\hat{\mathcal{B}}_{[1,1]}$ is fixed unambiguously from the chiral algebra, and brings in the dependence on the central charge. For the remaining Schur operators one is left with ambiguities in solving for this operators from the chiral algebra four-point function. We were able to fix universally the OPE coefficients of $\hat{\mathcal{C}}_{[1,1],\ell}$ and $\hat{\mathcal{B}}_{[2,2]}$ in terms of those of the $\hat{\mathcal{C}}_{[0,0],\ell}$ multiplets. These last multiplets contain conserved currents of spin greater than two, and are expected to be absent in interacting theories [61, 62] thereby resolving the ambiguity. Nevertheless, as shown in (4.24), the contribution of multiplets containing higher spin currents is identical to that of long multiplets at the unitarity bound, and thus, by allowing for long multiplets to have a dimension arbitrarily close to the unitarity bound, we allow for these currents to appear with arbitrary coefficient. Therefore, we do not truly exclude free theories in the bootstrap, and we should expect to recover the solution corresponding to U(1) $\mathcal{N} = 4$ SYM theory.

²⁵As usual the equations are antisymmetric in $z \leftrightarrow \bar{z}$ and so we only need derivatives with $m < n$.

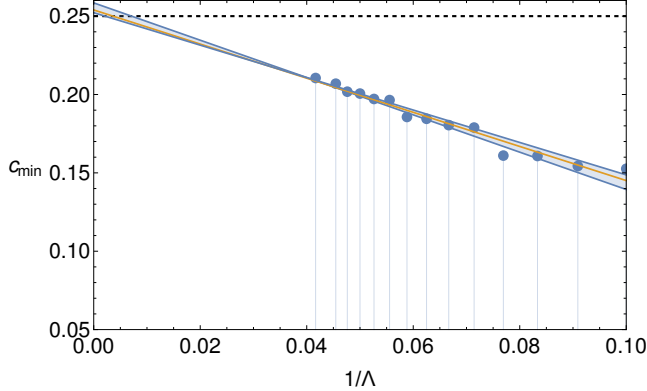


Figure 1. Numerically minimum allowed central charge for the $\hat{\mathcal{B}}_{[2,0]}$, $\hat{\mathcal{B}}_{[0,2]}$ four point function as a function of the inverse of the number of derivatives Λ . The dashed horizontal line marks the central charge of the U(1) $\mathcal{N} = 4$ SYM theory. The middle orange line shows a linear fit to all the data points, while the top and bottom blue lines show fits to different subsets of the points.

The numerical c bound is shown in figure 1 as a function of Λ^{-1} , where Λ is the number of derivatives used in the numerical bootstrap (5.2). The solid yellow and blue lines correspond to various linear fits to subsets of points, and attempt to give a rough estimate of the $\Lambda = \infty$ bound. It seems plausible that the bound is converging to $c = \frac{3}{12}$ which corresponds to the central charge of U(1) $\mathcal{N} = 4$ SYM. Recall that for this value of the central charge the coefficient $b_2^{(2)} = 1 - \frac{1}{3c}$ in (4.25) is negative, which means that it cannot be interpreted as arising only from a $\hat{\mathcal{B}}_{[2,2]}$ multiplet, and that the conserved current multiplet $\hat{\mathcal{C}}_{[0,0],0}$ has to be present. But this is exactly what our crossing equations are allowing for, as when we solve for the OPE coefficient of $\hat{\mathcal{B}}_{[2,2]}$ in terms of b_2 and let the OPE coefficient of $\hat{\mathcal{C}}_{[0,0],0}$ be arbitrary we find it contributes just as long at the unitarity bound. Naturally, if one wanted to obtain dimension bounds on the long operators for $c = \frac{3}{12}$ we would have to allow for the multiplets $\hat{\mathcal{C}}_{[0,0],0}$ to be present by adding their explicit contribution, but if no gap is imposed, then allowing for long multiplets of arbitrary dimension automatically allows for these currents.

5.3 The case $R = 3$

We now turn our attention to the correlation function of $\hat{\mathcal{B}}_{[3,0]}$, $\hat{\mathcal{B}}_{[0,3]}$ multiplet, whose crossing equations are given in section 4.3. We recall that in the chiral channel the OPE coefficients of the Schur multiplets $\hat{\mathcal{C}}_{[4,0],(\frac{\ell+1}{2},\frac{\ell}{2})}$ and $\hat{\mathcal{B}}_{[6,0]}$ were fixed universally from the chiral algebra correlation function. Therefore, the undetermined CFT data in this channel amounts to

- Scaling dimensions and OPE coefficients of long multiplets $\mathcal{A}_{[2,0],10,\ell}^{\Delta>\ell+5}$ and $\mathcal{A}_{[0,1],10,\ell}^{\Delta>\ell+5}$,
- OPE coefficients of short multiplets $\bar{\mathcal{B}}_{[2,2],0}^{r=12}$, $\bar{\mathcal{C}}_{[2,1],(\frac{\ell+1}{2},\frac{\ell}{2})}^{r=11}$ and $\bar{\mathcal{C}}_{[0,2],(\frac{\ell+1}{2},\frac{\ell}{2})}^{r=11}$,

where the last multiplets contribute the same way as the longs at the unitarity bound as seen in (3.35) and table 7.

In the non-chiral channel, various Schur multiplets were indistinguishable at the level of the chiral algebra, as manifest in table 6. To take into account the constraints on the OPE coefficients coming

from the knowledge of the chiral algebra correlator, we solved for the OPE coefficients of $\hat{\mathcal{C}}_{[1,1],\ell}$ and $\hat{\mathcal{B}}_{[2,2]}$ in terms of the remaining ones, such that we were left with the following unfixed CFT data

- Scaling dimensions and OPE coefficients of long multiplets $\mathcal{A}_{[0,0],\ell}^{\Delta>\ell+2}$ and $\mathcal{A}_{[1,1],\ell}^{\Delta>\ell+4}$,
- OPE coefficients of the Schur multiplets $\hat{\mathcal{C}}_{[2,2],\ell}$, $\hat{\mathcal{B}}_{[3,3]}$, and $\hat{\mathcal{C}}_{[0,0],\ell}$,

where the Schur multiplets in the last line end up contributing to the crossing equations in the same way as the long multiplets in the line above at the unitarity bound (see (3.30)). This implies that, unless we impose a gap in the spectrum of the corresponding long multiplets, we can never truly fix the OPE coefficients of these Schur operators. The multiplets $\hat{\mathcal{C}}_{[0,0],\ell}$ contain conserved currents of spin greater than two, and as such should be set to zero for interacting theories. However this is not enough to resolve all the ambiguities, and we must resort to numerics in order to study the OPE coefficient of the remaining operators. In the last part of this section we will see how these ambiguities turn out to be useful to exclude $\mathcal{N} = 4$ solutions to the crossing equations by inputting the OPE coefficient of $\hat{\mathcal{B}}_{[3,3]}$ computed from the chiral algebra of an $\mathcal{N} = 3$ SCFT.

5.3.1 Central charge bounds

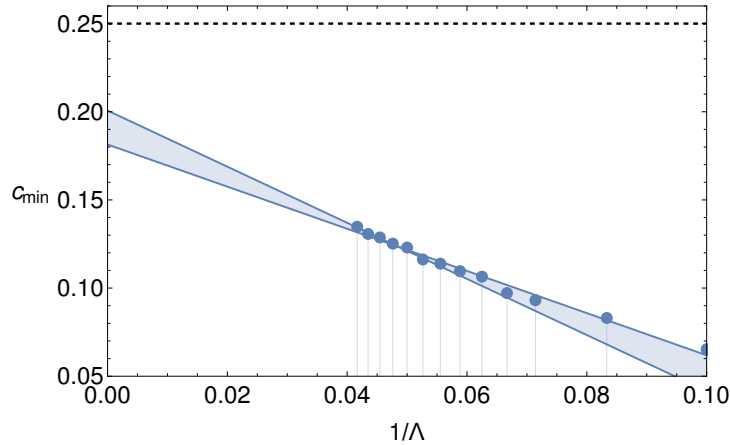


Figure 2. Minimum allowed central charge from the correlation function of $\hat{\mathcal{B}}_{[3,0]}$ and conjugate, for varying Λ , the number of derivatives. The dashed horizontal line marks the central charge of the U(1) $\mathcal{N} = 4$ SYM theory. The two blue lines show a linear fits to different subsets of points, in order to give very rough idea of where the bound is converging to with $\Lambda \rightarrow \infty$.

Once more the first question we should ask is what range of central charges is allowed. Thus we start by placing a lower bound on the central charge, allowing again for the presence of all operators consistent with unitarity. In particular we recall that long multiplets $\mathcal{A}_{[0,0],\ell}^{\Delta}$ of arbitrary dimension, allow for conserved currents of spin larger than two, and thus not ruling out free theories.

The central charge of the smallest known nontrivial theory is $c = \frac{15}{12}$.²⁶ Naturally the known solution of the U(1) $\mathcal{N} = 4$ SYM theory is also a solution of the crossing equations, since we cannot impose the solutions to the crossing equations to correspond to interacting theories. Therefore, the

²⁶By nontrivial we mean it cannot be obtained by $\mathcal{N} = 4$ SYM by a discrete gauging which does not change the correlation functions.

strongest bound one could possibly hope to find corresponds to the central charge U(1) $\mathcal{N} = 4$ SYM ($c = \frac{1}{4}$).

In figure 2 we show the minimal allowed central charge as a function of Λ^{-1} , the inverse of the number of derivatives. However, extrapolation for infinitely many derivatives this time does not seem to converge to the value of the U(1) $\mathcal{N} = 4$, which is $c = \frac{1}{4} = 0.25$.²⁷ Since the central charge is smaller than that of the free $\mathcal{N} = 4$ theory one might suspect the solution to this set of crossing equations saturating the central charge bound does not correspond to a physical SCFT, and could imagine a mixed correlator system, *e.g.*, adding the stress tensor multiplet, would improve on this.

5.3.2 Bounding OPE coefficients

Apart from the central charge, there are other OPE coefficients of physical interest, which were not fixed analytically and can be bounded numerically. An important point to emphasize in what follows is that the $\mathcal{N} = 3$ stress-tensor multiplet $\hat{\mathcal{B}}_{[1,1]}$ cannot recombine to form a long multiplet, unlike the $\mathcal{N} = 2$ stress-tensor multiplet. This has the important consequence that, when we add the stress tensor multiplet with a particular coefficient, we are truly *fixing the central charge* to a particular value. In comparison, in $\mathcal{N} = 2$ theories this was only accomplished when one imposed a gap in a particular channel, preventing those long multiplets to hit the unitarity bound and mimic the stress tensor. Therefore, in what follows we will bound the OPE coefficients as a function of the central charge, which we allow to vary in the range $\frac{1}{4} \leq c \leq \infty$. The lower end of the interval corresponds to the central charge of U(1) $\mathcal{N} = 4$ SYM, although interacting theories should have higher central charges. In particular there is an analytic lower bound for interacting $\mathcal{N} = 2$ SCFTs of $c \geq \frac{11}{30} \approx 0.37$ [46], and since any $\mathcal{N} = 3$ SCFT is also an $\mathcal{N} = 2$ theory this provides a lower bound on $\mathcal{N} = 3$ interacting SCFTs. Furthermore it can be shown, by considering the $\mathcal{N} = 3$ stress tensor four-point function in the chiral algebra, that any interacting $\mathcal{N} \geq 3$ SCFT must obey $c \geq \frac{13}{24} \approx 0.54$ [75]. These two bounds will be depicted as vertical dashed lines in all the numerical results. In the limit $c \rightarrow \infty$ the stress tensor decouples and we expect, as happened in all known cases, that the numerical bounds converge to the values of generalized free field theory (*i.e.* the four-point function is given just by a sum of disconnected pieces).

The Schur operator $\hat{\mathcal{B}}_{[3,3]}$

A particularly interesting operator to consider is the $\hat{\mathcal{B}}_{[3,3]}$ appearing in the non-chiral channel. Despite being captured by the two-dimensional chiral algebra of [40], is not possible to fix it universally from the chiral algebra the four-point function of $\langle \mathcal{W}\bar{\mathcal{W}}\mathcal{W}\bar{\mathcal{W}} \rangle$, due to the ambiguities described in 4.2. Making assumptions about what particular chiral algebra corresponds to the $4d$ theory we are interested in studying, one can try to resolve this ambiguity, as done in section 2.1.3, which gave two seemingly consistent possibilities to the value of this OPE coefficient as a function of the central charge. However, we will first take an agnostic viewpoint, and ask what numerical constraints crossing symmetry and unitarity place on the squared OPE coefficient of this operator ($|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$). These are shown in figure 3 as a function of the inverse of the central charge.

Since this operator is protected, we can start by comparing the value of the bound to the well known $\mathcal{N} = 4$ solutions. We extracted the OPE coefficient of this multiplet from the four-point function of $\mathcal{N} = 4$ half-BPS operators in the $[0, 3, 0]$ representation of $SU(4)_R$ given in [37], after projecting the $\mathcal{N} = 4$ multiplets to the particular $\mathcal{N} = 3$ multiplet we are considering. It turns out to have a constant value of one, irrespective of the central charge of the theory. We depicted this as a red

²⁷Similar results were also observed in the case of chiral correlators in $\mathcal{N} = 2$ theories [19, 74].

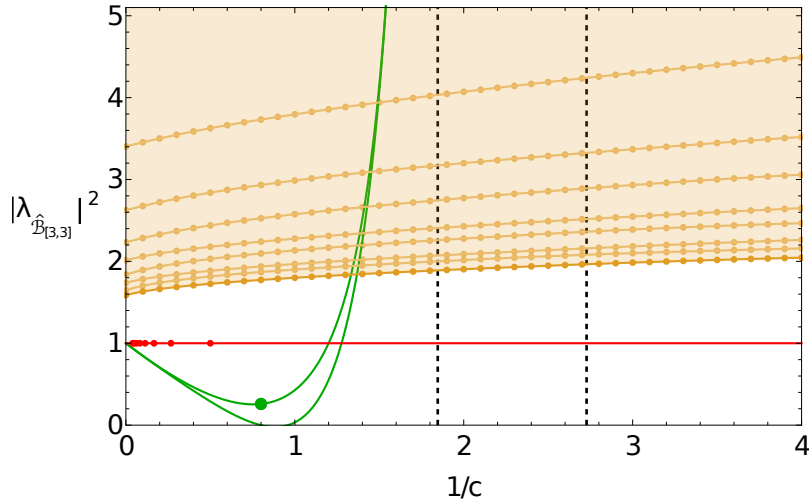


Figure 3. Upper bound on the OPE coefficient squared of $\hat{\mathcal{B}}_{[3,3]}$ versus the inverse central charge $1/c$. The shaded region is excluded and the number of derivatives is increased from 10 to 24 in steps of two. The two green curves show the possible value of the OPE coefficient computed by the chiral algebra in section 2.1.3, while the green dot shows the expected value for the $\mathcal{N} = 3$ theory of $1/c = 0.8$, extracted from the chiral algebra of [10]. The red line and dots corresponds to the solution of $\mathcal{N} = 4$ SYM theories. The two dashed lines correspond to the minimum central charges for an interacting $\mathcal{N} = 2$ [46] and $\mathcal{N} = 3$ SCFTs [75] ($c^{-1} = \frac{30}{11} \approx 2.73$ and $c^{-1} = \frac{24}{13} \approx 1.84$ respectively).

line in figure 3 and, to give an idea of where the physical $\mathcal{N} = 4$ theories sit, we also added red dots in the positions corresponding to the central charge of $\mathcal{N} = 4$ SYM with gauge group $SU(n)$ ($c = \frac{n^2-1}{4}$) for $n \in \{3, 4, \dots\}$.

The value expected from the block decomposition of both the $U(1)$ $\mathcal{N} = 4$ SYM ($c = \frac{1}{4}$) and the generalized free field theory ($c = \infty$, given in appendix E) is also one, and is marked by red dots as well. A rough extrapolation of our results for infinite central charge and for $c = \frac{1}{4}$ suggests the numerical bounds could converge to the values expected for these theories.

Finally, we compare the numerical bounds with the results that can be extracted from a particular chiral algebra. Let us first consider the chiral algebra of [10] (in the notation of that article, this corresponds to $\ell = 3$, where of course this ℓ has no relation to the spin), that is conjectured to correspond to the simplest known $\mathcal{N} = 3$ SCFT with $c = \frac{15}{12}$. As discussed in section 2.1.2 we can construct the candidate operators in the chiral algebra that could correspond to a $\hat{\mathcal{B}}_{[3,3]}$. In this case there is only one candidate, and if one assumes it to be in fact a $\hat{\mathcal{B}}_{[3,3]}$ we find

$$|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2 = \frac{22}{85}, \quad (5.5)$$

which is shown as a green dot in figure 3.²⁸ Note that this value lies well inside the numerical bounds, and in particular it is also smaller than the continuation to arbitrary c of the value corresponding to $SU(N)$ $\mathcal{N} = 4$ SYM. Since the correlation function of [37] which we decomposed in blocks is a solution of the crossing equations for any value of c , the best numerical bound one can hope to obtain

²⁸The other possibility, that the $\hat{\mathcal{B}}_{[3,3]}$ multiplet is absent in the chiral algebra, does not appear plausible from the Higgs branch perspective.

is $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2 \leq 1$. In fact, our numerical results appear consistent with the upper bound converging to one for $c = \frac{15}{12}$. Therefore, to be able to reach the known nontrivial $\mathcal{N} = 3$ with $c = \frac{15}{12}$ we must go inside these bounds, and fix the OPE coefficient of $\hat{\mathcal{B}}_{[3,3]}$ to a value that is incompatible with the $\mathcal{N} = 4$ solution to the crossing equation; we will do this at the end of this section.

We now turn to the chiral algebra constructed in 2.1.2, with the goal of understanding the higher rank versions of the aforementioned theory. Recall that in doing so, we assumed the chiral algebra of the higher rank theories to be generated solely by the Higgs branch generators, the stress tensor, and an additional dimension three operator. Under this assumption, we were able to construct a closed subalgebra of all of these chiral algebras, which is associative for generic values of c . In that setting we can attempt to compute $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$, and there were two options consistent with the large central charge behavior of the generalized free field theory and unitarity, given in (2.25) and (2.27), which are plotted as green curves in figure 3. Equation (2.27) is the one that does not go through the expected value for $c = \frac{15}{12}$, but that we kept for arbitrary values of c . If our assumptions are correct, then we see that the value of $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$ lies well inside the numerical bounds, and is weaker than that of $\mathcal{N} = 4$ SYM. This is not necessarily a downside, as one of our goals must be to determine ways to exclude the $\mathcal{N} = 4$ solutions to our crossing equations, and this provides such a way. *By imposing the value of the OPE coefficient corresponding to (2.25) or (2.27) we are sure to exclude $\mathcal{N} = 4$ from our analysis.* We will come back to this point at the end of this section.

Note that both (2.25) and (2.27) diverge at $c = \frac{13}{24}$, which corresponds to the analytic central charge bound obtained in [75], following from the fact that the norm of one of the candidate $\hat{\mathcal{B}}_{[3,3]}$ operators is going to zero. We note that the chiral algebra in 2.1.2 was constructed with a generic central charge in mind and care was not given to possible null states arising at specific values of the central charge. It is not clear that the solution we have is consistent for $c = \frac{13}{24}$, as null states are expected to decouple.

It is also worth noting the interesting interplay between analytical and numerical results. The analytical OPE coefficient is only consistent with the (current) numerical bounds for $1/c \lesssim 1.33 - 1.36$ depending on which curve one takes. This provides a lower bound $c \gtrsim 0.74 - 0.75$ on the central charge of any $\mathcal{N} = 3$ SCFT with a dimension three Coulomb branch operator ($\hat{\mathcal{B}}_{[3,0]}$) of which the chiral algebra presented in 2.1.2 is a closed subalgebra, improving over the analytical bound $c \geq \frac{13}{24} \approx 0.54$ of [75]. On the other hand, this bound is lower than the one obtained using the sum-rule of [76, 77] for a rank one theory with a generator of dimension three, namely $c \geq \frac{15}{12}$. Although there are known cases where this sum rule does not hold (see [3, 9]), they correspond to theories obtained by gauging discrete symmetries, so this bound could be valid for theories which are not of this type.²⁹

The multiplets $\bar{\mathcal{B}}_{[2,2]}$ and $\bar{\mathcal{C}}_{[0,2],(\frac{1}{2},0)}$

Next we turn our attention to the short multiplets in the chiral OPE that sit at the unitarity bound of the long multiplets, and are not captured by the chiral algebra. It follows that the best we can do is to bootstrap them numerically. As representatives, we show the upper bounds on the OPE coefficients squared of the multiplets $\bar{\mathcal{B}}_{[2,2]}$ and $\bar{\mathcal{C}}_{[0,2],(\frac{1}{2},0)}$ in figure 4. Again we focus on the region of central charges larger than that of $U(1)$ $\mathcal{N} = 4$ SYM.

We show in figure 4 the value of these OPE coefficients in the case of the generalized free field theory, and of the $U(1)$ $\mathcal{N} = 4$ SYM as the two red dots at $c^{-1} = 0$ and $c^{-1} = 4$ respectively. The convergence of our numerical results is rather slow and one cannot conclude if they will converge for

²⁹See [9] for a proposed correction of this formula to hold also in the case of discretely gauged theories.

these central charges to the known solutions, although they are not incompatible with this possibility. The green lines in the plots mark the central charge of the “minimal” $\mathcal{N} = 3$ SCFT ($c = \frac{15}{12}$) with the

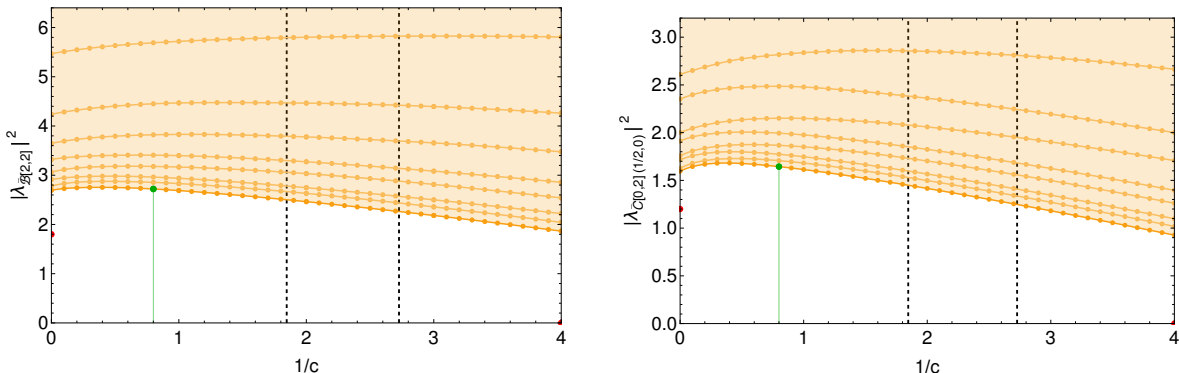


Figure 4. Upper bound on the OPE coefficient squared of $\tilde{\mathcal{B}}_{[2,2]}$ ($|\tilde{\lambda}_{\tilde{\mathcal{B}}_{[2,2]}}|^2$, depicted on the left) and of $\tilde{\mathcal{C}}_{[0,2],(\frac{1}{2},0)}$ ($|\tilde{\lambda}_{\tilde{\mathcal{C}}_{[0,2],(\frac{1}{2},0)}}|^2$, shown on the right) versus the inverse central charge $1/c$. The first vertical dashed line marks $c = \frac{13}{24}$ and the second $c = \frac{11}{30}$ (the minimal central charges for $\mathcal{N} = 3$ and $\mathcal{N} = 2$ interacting theories respectively [46, 75]). The number of derivatives Λ is increased from 10 to 24 in steps of two. The red dots mark the value of this OPE coefficient for generalized free field theory and U(1) $\mathcal{N} = 4$ SYM, while the green line marks the central charge $c = \frac{15}{12}$ of the simplest known $\mathcal{N} = 3$ SCFT, with the green dot providing an upper bound for the OPE coefficients of this theory.

green dot providing a valid upper bound for the OPE coefficients of this theory.

Finally, to better understand what is failing in the crossing symmetry equations if one tries to go below the minimal numerically allowed central charge (c_{\min} in figure 2), it is instructive to look at the OPE coefficient bounds near those central charges. One finds (not shown), that while the bounds on the squared OPE coefficients of both $\hat{\mathcal{B}}_{[3,3]}$ and $\tilde{\mathcal{B}}_{[2,2]}$ have a very sharp drop near c_{\min} , the upper bound on the squared $\tilde{\mathcal{C}}_{[0,2],(\frac{1}{2},0)}$ OPE coefficient has as smooth drop and becomes negative right after c_{\min} , which is inconsistent with unitarity. This suggests it is the last multiplet that is responsible for the lower bound on the central charge, and that the solution at c_{\min} would have the other two short operators present. Note that both $\tilde{\mathcal{B}}_{[2,2]}$ and $\tilde{\mathcal{C}}_{[0,2],(\frac{1}{2},0)}$ have zero OPE coefficient for the U(1) $\mathcal{N} = 4$ SYM theory.

5.3.3 Dimension bounds

Next we turn to the dimensions of the lowest lying scalar long operators in the various channels. In doing so we must worry about the short multiplets whose OPE coefficients we bootstrapped in the previous subsection, as they all sit at the unitarity bound of the different long multiplets we study (see the tables 7 and 6). By allowing for long multiplets with arbitrary dimension, these short multiplets can appear with any coefficient. Even if we were to explicitly add by hand the short multiplets with a given OPE coefficient, the long multiplet at the bound would mimic a short, and in practice we would only be imposing the OPE coefficient of the short multiplet to be greater or equal than a given value. However, once we impose a gap in the spectrum of the long operator, then we can truly fix the OPE coefficient of the corresponding short multiplet.

In the non-chiral channel, we focus on the dimension of the first scalar long of each type

$$\mathcal{A}_{[0,0],0}^{\Delta>2} \quad \text{and} \quad \mathcal{A}_{[1,1],0}^{\Delta>4}, \quad (5.6)$$

while in the chiral channel we focus on the first scalar long multiplet³⁰

$$\mathcal{A}_{[2,0],10,0}^{\Delta>5}. \quad (5.7)$$

Non-chiral channel

The upper bounds on the dimensions $\Delta_{[0,0]}$ and $\Delta_{[1,1]}$, of the first long multiplets $\mathcal{A}_{[0,0],0}^{\Delta>2}$ and $\mathcal{A}_{[1,1],0}^{\Delta>4}$ respectively, as functions of the inverse central charge are depicted in figures 5. Once again red dots mark the dimension of the lowest dimensional operator in the generalized free field theory and the U(1) $\mathcal{N} = 4$ SYM solutions. In both cases the green vertical line ending on a dot marks the central charge of the simplest known nontrivial $\mathcal{N} = 3$ SCFT, and provides an upper bound for the dimension of these two operators in this theory. We will improve on the latter bound at the end of this section.

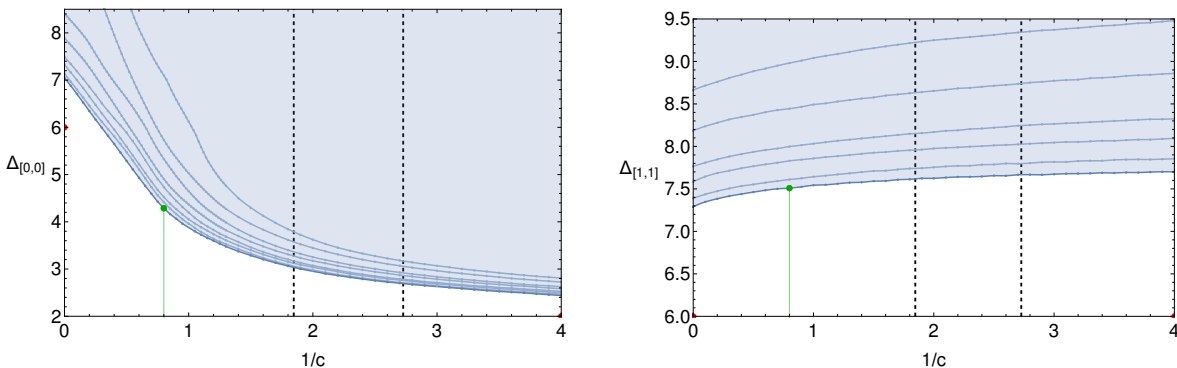


Figure 5. Upper bound on the dimensions of long multiplets $\mathcal{A}_{[0,0],0}^{\Delta>2}$ (left) and $\mathcal{A}_{[1,1],0}^{\Delta>4}$ (right) for different values of the inverse of the central charge c . The maximum number of derivatives is $\Lambda = 24$, and the weaker bounds correspond to decreasing the number of derivatives by two. The red dots mark the dimension of the first long operators for generalized free field theory and U(1) $\mathcal{N} = 4$ SYM, while the green line marks the central charge $c = \frac{15}{12}$ of the simplest known $\mathcal{N} = 3$ SCFT, with the green dot providing an upper bound for this theory. The two dashed lines correspond to the minimum central charges for an interacting $\mathcal{N} = 2$ [46] and $\mathcal{N} = 3$ SCFTs [75].

At the unitarity bound, the long multiplet of type $\mathcal{A}_{[0,0],0}^{\Delta>2}$ mimics a higher spin conserved current multiplet ($\hat{\mathcal{C}}_{[0,0]\ell=0}$), expected to be absent in an interacting theory, and therefore when obtaining the bound on the left side of 5 we do not allow for such a multiplet to be present. This explains why the upper bound is presumably converging to the unitarity bound $\Delta = 2$ for $c^{-1} = 4$, since such currents should be present in the U(1) $\mathcal{N} = 4$ solution, as indicated by the red dot. For larger central charges the upper bound is far away from unitarity, and thus theories saturating the upper bound do not contain the $\hat{\mathcal{C}}_{[0,0]\ell=0}$ multiplet, although they could have the higher spin versions of this multiplet which also contain higher spin conserved currents.

On the other hand the multiplet that sits at the unitarity bound of $\mathcal{A}_{[1,1],0}^{\Delta>4}$ is the $\hat{\mathcal{B}}_{[3,3]}$ discussed in the previous subsection, and in obtaining the bounds for $\Delta_{[1,1]}$, we allowed the short multiplet to

³⁰Table 7 contains also the long multiplets $\mathcal{A}_{[0,1],10,\ell}^{\Delta>5+\ell}$ but for those the spin ℓ must be odd by Bose symmetry.

be present with arbitrary OPE coefficient. We can obtain a stronger bound for specific $\mathcal{N} = 3$ SCFTs by fixing the short OPE coefficient according to section 2.1.3, as we shall do later in figure 7 for the case of $c = \frac{15}{12} \Rightarrow c^{-1} = 0.8$.

Chiral channel

Tuning to the scalar long operator appearing in the chiral OPE, we obtain an upper bound for the first $\mathcal{A}_{[2,0],10,0}^{\Delta > 5}$ multiplet. In imposing a gap in this channel we must decide on whether the short multiplet $\bar{\mathcal{B}}_{[2,2]}$ is present or not. Recall that, unlike the shorts at the bound of operators appearing in the non-chiral channel, this short is not captured by the chiral algebra and therefore we have no reason to expect it to be present or absent. Therefore, we show a bound on the dimension $\Delta_{[2,0]}$ of this long multiplet both allowing for (left plot in 6) and disallowing for (right plot in 6) the presence of $\bar{\mathcal{B}}_{[2,2]}$. Once again the red dots depict the value of each of these dimensions expected for the generalized free field theory and the U(1) $\mathcal{N} = 4$ SYM. We observe that for $c \rightarrow \infty$, the right hand side of figure

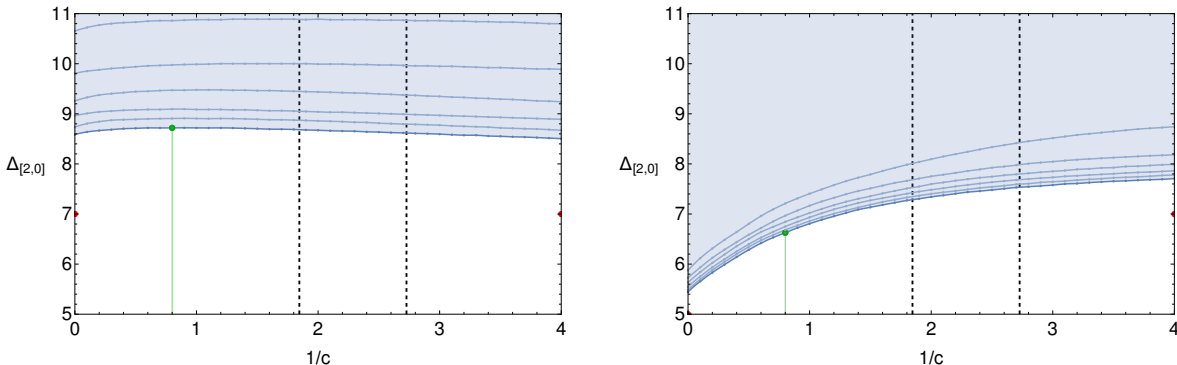


Figure 6. Upper bound on the dimensions of the long multiplet $\mathcal{A}_{[2,0],10,0}^{\Delta > 5}$, allowing for (left) and disallowing for (right) the short multiplet $\bar{\mathcal{B}}_{[2,2]}$. The strongest bound corresponds to 24 derivatives, and they are reduced in steps of two. The red dots mark the dimension of the first long operators for generalized free field theory and U(1) $\mathcal{N} = 4$ SYM, in the right plot the red dot of generalized free field theory is at the unitarity bound, meaning that the short multiplet is present in this solution. The green line marks the central charge $c = \frac{15}{12}$ of the simplest known $\mathcal{N} = 3$ SCFT, with the green dot providing an upper bound for this theory. The two dashed lines correspond to the minimum central charges for an interacting $\mathcal{N} = 2$ [46] and $\mathcal{N} = 3$ SCFTs [75].

6 comes close to the unitarity bound $\Delta = 5$. In fact, a simple extrapolation seems to suggest that for $\Lambda \rightarrow \infty$ the bound will converge to around 5. This is consistent with the fact that this multiplet is *present* in the generalized free theory solution (see appendix E), *i.e.* the bounds force the long multiplet to “become short” for $c \rightarrow \infty$. (Said multiplet is absent in the U(1) $\mathcal{N} = 4$ SYM solution.) For values of c around the value relevant for the “minimal” $\mathcal{N} = 3$ SCFT, marked as green lines in the plots, there seems to be a solution of the crossing equations with this multiplet absent.

Carving out solutions inside the bounds

As the final point of this section we come back to the issue of distinguishing $\mathcal{N} = 4$ solutions to the crossing equations from pure $\mathcal{N} = 3$ ones. One possibility is to extract the spectrum of the extremal solution [78] saturating each of the above bounds and check if it is consistent or inconsistent with $\mathcal{N} = 4$ supersymmetry. However, we would like to do better, and to be able to exclude the $\mathcal{N} = 4$ solution altogether. Our explorations in the first part of this section provide such a way, namely by

fixing the OPE coefficient of $\hat{\mathcal{B}}_{[3,3]}$ to the value expected to correspond to the $\mathcal{N} = 3$ theories of interest (see section 2.1.3). This value is smaller than the one of $SU(N)$ $\mathcal{N} = 4$ SYM and in its derivation we did not allow in the chiral algebra for the currents enhancing the supersymmetry to $\mathcal{N} = 4$. One subtle point is that, because a long multiplet at the unitarity bound ($\mathcal{A}_{[1,1],0}^{\Delta=4}$) mimics the contribution of this short multiplet to the crossing equations, we cannot really fix its OPE coefficient unless we impose a gap in the spectrum of said longs.

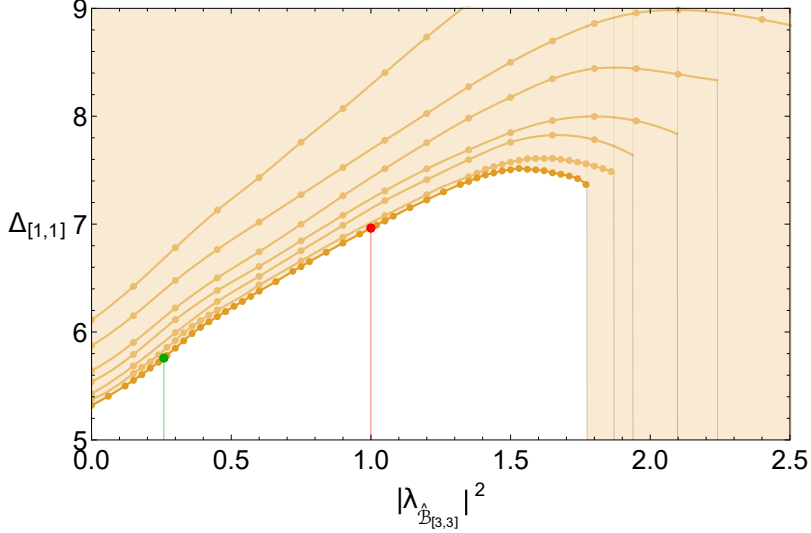


Figure 7. Upper bound on the dimension of the long $\mathcal{A}_{[1,1],0}^{\Delta>4}$ as a function of the OPE coefficient squared of $\hat{\mathcal{B}}_{[3,3]}$ for $c^{-1} = 0.8$. For each cutoff Λ the bounds end abruptly at the value corresponding to the maximum value $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$ can have, as read off from figure 3 at $c^{-1} = 0.8$. The green line marks the expected value for the OPE coefficient for the $c^{-1} = 0.8$ $\mathcal{N} = 3$ SCFT (5.5) with the green dot providing an upper bound for this OPE coefficient, while the red line marks the value for $\mathcal{N} = 4$ SYM. The strongest bound corresponds to 24 derivatives, and they are reduced in steps of two.

This is what is done when bounding the lowest dimensional $\mathcal{A}_{[1,1],0}^{\Delta>4}$, and so we repeat the analysis leading to the right side of figure 5, but now fixing the OPE coefficient of $\hat{\mathcal{B}}_{[3,3]}$. The result is shown in figure 7, where we plot the upper bound on the dimension as a function of the OPE coefficient for fixed $c = \frac{15}{12}$. The red line marks the value of the OPE coefficient for the $\mathcal{N} = 4$ solution with this particular value of c . While the green line marks the value of the OPE coefficient expected for the $\mathcal{N} = 3$ SCFT we are interested in (5.5), and provides an upper bound for the dimension $\Delta_{[1,1]}$ in this theory, which improves significantly on the one obtained from figure 4. This shows that, at least in figure 4, the theory saturating the bound does not correspond to the $\mathcal{N} = 3$ SCFT we were after, and thus, to zoom in to this specific theory we must carve further inside the bounds as done here. This however does not guarantee the theory now sits at the bound.³¹

³¹We seem to observe a small bump for $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2 \approx 0.33$, and preliminary functional analysis suggest this is correlated to the fact that to the left of the bump a conserved current $\hat{\mathcal{C}}_{[0,0]\ell=0}$ is allowed, and to the right disallowed. This does not necessarily imply that the conserved currents are present for the $c = \frac{15}{12}$ extremal solution, but could mean that to get closer to an interacting $\mathcal{N} = 3$ SCFT we should simultaneously impose a gap in the $\mathcal{A}_{[0,0],0}^{\Delta>2}$ long channel.

Similarly, we can repeat this analysis for the central charges of the higher rank theories and we find that, at fixed $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$, if the central charge is (increased) decreased the bound seems to get (stronger) weaker (not shown). Due to the dependence of (2.25) and (2.27) on c the upper bound on $\Delta_{[1,1]}$ does not change that significantly.

6 Conclusions

In this paper we have initiated the $\mathcal{N} = 3$ superconformal bootstrap program with two goals in mind. First, to constrain the space of four-dimensional $\mathcal{N} = 3$ SCFTs, and second, to focus on specific examples of $\mathcal{N} = 3$ theories with the hope of obtaining information about their spectrum. In order to zoom in on the known $\mathcal{N} = 3$ SCFTs we relied on a combination of numerical bootstrap results and analytical results from two-dimensional chiral algebras, with particular emphasis on the “minimal” $\mathcal{N} = 3$ SCFT, and its higher-rank versions. We approached these theories from the point of view of the Coulomb branch, focusing mostly on a half-BPS operator of dimension three, which is the only Coulomb branch generator of the “minimal” $\mathcal{N} = 3$ SCFT, and which is also present in its higher-rank versions.

A basic requirement for any bootstrap study is the conformal block expansion of the four-point function. Writing the four-point function of $\mathcal{N} = 3$ half-BPS operators in superspace we noticed that there are no nilpotent invariants, allowing us to concentrate on superconformal primaries without any loss of information. Demanding the absence of singularities when turning on the fermionic coordinates places strong restrictions on the form of the four-point function, giving rise to the Ward identities. However, the WI are not enough to completely fix the superblock (unlike the cases of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ half-BPS superblocks [31, 32]). In order to fix the superblocks associated to short multiplets, we used information coming from the $2d$ chiral algebra, while for long blocks we leveraged knowledge of $\mathcal{N} = 1$ blocks. In the end, we wrote the solution for the long blocks in an elegant way, in terms of a *single* $\mathcal{N} = 1$ conformal block with shifted arguments.

The existence of a protected subsector captured by the $2d$ chiral algebra allowed us to solve the crossing equations for this subsector exactly. In turn, we were able to fix the OPE coefficients of certain short operators universally, *i.e.*, without needing to specify a particular four-dimensional theory. However, some operators appear indistinguishable at the level of the chiral algebra, leading to ambiguities in fixing the corresponding OPE coefficients. Some of these ambiguities can be resolved by knowledge of the specific chiral algebra associated to the $\mathcal{N} = 3$ theory in question, but this is not always the case.

An important question is what can be said about the chiral algebras associated to a given $\mathcal{N} = 3$ SCFT. To that end, we determined which $\mathcal{N} = 3$ superconformal multiplets are captured by the $2d$ chiral algebra, and some of their general properties, which could allow distinguishing between the aforementioned operators. Taking advantage of the chiral algebra conjectured to correspond to the “minimal” SCFT [10], we were able to compute the OPE coefficient $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$. Moreover, we proposed, under certain assumptions, a closed subsector for the higher-rank versions of this theory, and used it to compute $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$ in this case.

To go beyond this protected subsector, or even to distinguish between operators appearing identically in the chiral algebra, we had to resort to numerical techniques. The numerical bootstrap then provided constraints on the spectrum of unprotected long operators, and on the OPE coefficients of various short operators. For the particular OPE coefficient $|\lambda_{\hat{\mathcal{B}}_{[3,3]}}|^2$ that we were able to fix from the chiral algebra, we compared the general numerical results valid for *any* $\mathcal{N} = 3$ SCFT, with the ones

of the *specific* theories captured by the chiral algebra that we constructed. This comparison provided a numerical lower bound on the central charge for theories captured by our chiral algebra.

A natural limitation of any $\mathcal{N} = 3$ bootstrap program, as it was also for the $\mathcal{N} = 2$ bootstrap, is that theories with more supersymmetry will generically be solutions of the crossing equations we consider. In order to restrict to pure $\mathcal{N} = 3$ SCFTs, one would have to exclude the presence of superconformal multiplets containing the currents allowing for this enhancement. However, the multiplets that are physically relevant for the study of these theories (for example the ones considered in [19, 46, 74] in the $\mathcal{N} = 2$ case) usually do not allow for the multiplets containing the extra supercurrents to be exchanged in their OPEs, and therefore we cannot set them to zero. To overcome this limitation we fed, into the numerical bootstrap, information arising from the chiral algebras of pure $\mathcal{N} = 3$ SCFTs, namely, the OPE coefficient $|\lambda_{\mathcal{B}_{[3,3]}}|^2$. This allowed us explore inside the numerical bounds, and zoom in on the $\mathcal{N} = 3$ solutions with this particular value of the OPE coefficient. By fixing the central charge to that of the “minimal” $\mathcal{N} = 3$ theory, and fixing the OPE coefficient accordingly, it is plausible that this theory sits at the bound of figure 7, although currently there is no evidence this has to be the case, and we would have to provide more information (such as adding stress tensors as external operators). Nevertheless, the ambiguity in fixing OPE coefficients turned out to be crucial in excluding the $\mathcal{N} = 4$ solution to the crossing equations. For the higher rank versions, one would have to also consider the four-point functions of the additional Coulomb branch operator, which is a natural next step in the $\mathcal{N} = 3$ bootstrap, along with the study of stress tensor four-point functions.

Acknowledgments

We have greatly benefited from discussions with P. Argyres, M. Martone, L. Rastelli, D. Regalado, and B. van Rees. The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Unions Seventh Framework Programme FP7/2007-2013/ under REA Grant Agreement No 317089 (GATIS). M. L., P. L., and C. M. thank the Galileo Galilei Institute for Theoretical Physics for hospitality and the INFN for partial support during the completion of this work during the workshop “Conformal field theories and renormalization group flows in dimensions $d > 2$ ”. P. L. acknowledges the hospitality of Universidad de Santiago de Chile during the last stages of this work. The authors gratefully acknowledge the computing time granted on the DESY Theory and BIRD clusters and on the supercomputer Mogon at Johannes Gutenberg University Mainz (hpc.uni-mainz.de).

A Unitary representations of the $\mathcal{N} = 3$ superconformal algebra

We summarize the unitary representations of the four-dimensional $\mathcal{N} = 3$ superconformal algebra, which fall in the classifications of [55–57] and which were recently discussed with emphasis on $\mathcal{N} = 3$ theories in [1, 58, 59]. We list the possible representations in table 8. The first column lists the name the we give to the representation, inspired by the conventions of [60], while the second one uses the notation of [58]. The third column list the quantum numbers of the highest weight conformal primary, denoted by $(j, \bar{j})_{[R_1, R_2], r}^\Delta$, where $(j, \bar{j}) \in \frac{\mathbb{N}_0}{2} \times \frac{\mathbb{N}_0}{2}$ are the double of the left/right spins,³² $\Delta \in \mathbb{R}$ is the conformal dimension, $(R_1, R_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ are the Dynkin labels of SU(3) and $r \in \mathbb{R}$ is the U(1) R-charge. We follow the $r_{\mathcal{N}=3}$ charge conventions of [58], while for the $\mathcal{N} = 2$ R-charges we follow the conventions of Dolan and Osborn [60]. Lastly, we make two remarks:

³²An irreducible representation of label (j, \bar{j}) has dimension $(2j + 1)(2\bar{j} + 1)$.

Name	Name in [58]	Primary	Conditions
$\mathcal{A}_{[R_1, R_2], r, (j, \bar{j})}^\Delta$	$L\bar{L}$	$(j, \bar{j})_{[R_1, R_2], r}^\Delta$	$\Delta > 2 + 2j + \frac{2}{3}(2R_1 + R_2) - \frac{r}{6}$ $\Delta > 2 + 2\bar{j} + \frac{2}{3}(R_1 + 2R_2) + \frac{r}{6}$
$\mathcal{B}_{[R_1, R_2], r, \bar{j}}$	$B_1\bar{L}$	$(0, \bar{j})_{[R_1, R_2], r}^{\frac{2}{3}(2R_1 + R_2) - \frac{r}{6}}$	$-6\bar{j} + 2(R_1 - R_2) - 6 > r$
$\bar{\mathcal{B}}_{[R_1, R_2], r, j}$	$L\bar{B}_1$	$(j, 0)_{[R_1, R_2], r}^{\frac{2}{3}(R_1 + 2R_2) + \frac{r}{6}}$	$6j + 2(R_1 - R_2) + 6 < r$
$\hat{\mathcal{B}}_{[R_1, R_2]}$	$B_1\bar{B}_1$	$(0, 0)_{[R_1, R_2], 2(R_1 - R_2)}^{R_1 + R_2}$	
$\mathcal{C}_{[R_1, R_2], r, (j, \bar{j})}$	$A_1\bar{L}$	$(j, \bar{j})_{[R_1, R_2], r}^{2+2j+\frac{2}{3}(2R_1+R_2)-\frac{r}{6}}$	$6(j - \bar{j}) + 2(R_1 - R_2) > r$
$\bar{\mathcal{C}}_{[R_1, R_2], r, (j, \bar{j})}$	$L\bar{A}_1$	$(j, \bar{j})_{[R_1, R_2], r}^{2+2\bar{j}+\frac{2}{3}(R_1+2R_2)+\frac{r}{6}}$	$6(j - \bar{j}) + 2(R_1 - R_2) < r$
$\hat{\mathcal{C}}_{[R_1, R_2], (j, \bar{j})}$	$A_1\bar{A}_1$	$(j, \bar{j})_{[R_1, R_2], 6(j - \bar{j}) + 2(R_1 - R_2)}^{2+j+\bar{j}+R_1+R_2}$	
$\mathcal{D}_{[R_1, R_2], \bar{j}}$	$B_1\bar{A}_1$	$(0, \bar{j})_{[R_1, R_2], 2(R_1 - R_2) - 6 - 6\bar{j}}^{1+\bar{j}+R_1+R_2}$	
$\bar{\mathcal{D}}_{[R_1, R_2], j}$	$A_1\bar{B}_1$	$(j, 0)_{[R_1, R_2], 2(R_1 - R_2) + 6 + 6j}^{1+j+R_1+R_2}$	

Table 8. We list here the unitary representations of $\mathcal{N} = 3$ with the name that we give them in the present article accompanied by the one that they have in [58], which was based on the type of shortening condition that they obey. The third column shows the charges of the superconformal primary in the representation, while the fourth one lists the conditions that the charges have to obey. The A_2 , respectively \bar{A}_2 shortening cases are obtained by putting $j = 0$, respectively $\bar{j} = 0$. This changes the null states drastically, but not our labels.

- When dealing with symmetric-traceless representations, we shall label the spins by $j = \bar{j} = \frac{\ell}{2}$, and by an abuse of notation we will replace the two spin labels (j, \bar{j}) by ℓ in these cases. For example, we have $\mathcal{A}_{[R_1, R_2], r, \ell}^\Delta \equiv \mathcal{A}_{[R_1, R_2], r, (\frac{\ell}{2}, \frac{\ell}{2})}^\Delta$.
- If the r label is zero, we will often omit it. Furthermore, in order to keep some equations compact, we will freely write it up or down, *e.g.* $\mathcal{A}_{[R_1, R_2], r, \ell}^\Delta \equiv \mathcal{A}_{[R_1, R_2], \ell}^{\Delta, r}$.

A.1 Decomposition in $\mathcal{N} = 2$ multiplets

Since $\mathcal{N} = 3$ representations are probably less familiar to most readers than $\mathcal{N} = 2$ representations, we give a few examples of how $\mathcal{N} = 3$ multiplets decompose in $\mathcal{N} = 2$ multiplets. In doing so we pick an $\mathcal{N} = 2$ subalgebra of the $\mathcal{N} = 3$, and therefore the $SU(3)_R \times U(1)_r$ R-symmetry of the latter decomposes in $SU(2)_{R_{\mathcal{N}=2}} \times U(1)_{r_{\mathcal{N}=2}} \times U(1)_f$, where the first two factors are the R-symmetry of the $\mathcal{N} = 2$ superconformal algebra, and the last corresponds, from the $\mathcal{N} = 2$ point of view, to a global symmetry. Therefore when viewed as $\mathcal{N} = 2$ theories, all $\mathcal{N} = 3$ theories have a $U(1)_f$ flavor symmetry, and we will keep this flavor grading when decomposing $\mathcal{N} = 3$ representations in $\mathcal{N} = 2$. We follow the conventions of [10] for the definition of the flavor charges. We note that we follow the naming conventions of Dolan and Osborn [60] for the representations of $\mathcal{N} = 2$, which are summarized for instance in Appendix A of [19]. While the interpretation of most of these multiplets might be obscure, the following have a natural physical interpretation³³

- $\hat{\mathcal{C}}_{0, (0, 0)}$ is the stress tensor multiplet of an $\mathcal{N} = 2$ SCFT, containing in addition to the stress tensor, the $SU(2)_{R_{\mathcal{N}=2}}$ and $U(1)_{r_{\mathcal{N}=2}}$ currents,

³³For a more detailed description see, *e.g.*, section 2 of [19]

- \mathcal{B}_R are closely related to the Higgs branch of the theory, in particular the $\hat{\mathcal{B}}_1$ multiplet contains conserved currents of spin one, associated to flavor currents of the theory,
- $\mathcal{E}_{r,(0,0)}$ are $\mathcal{N} = 2$ chiral operators, and are related to the Coulomb branch of the theory,
- $\mathcal{D}_{\frac{1}{2},(0,0)}$ (and conjugate) which are additional supercurrent multiplets,
- $\hat{\mathcal{C}}_{0,(j>0,\bar{j}>0)}$ contain conserved currents of spin greater than two, which signal free theories [61, 62].

In addition, the multiplets dubbed ‘‘Schur’’ operators in [40], that is the ones captured by the two-dimensional chiral algebra reviewed in section 2, also play an important role. These are $\hat{\mathcal{B}}_R$, $\mathcal{D}_{R(0,\bar{j})}$, $\overline{\mathcal{D}}_{R(j,0)}$ and $\hat{\mathcal{C}}_{R(j,\bar{j})}$, giving rise to two-dimensional $\mathfrak{sl}(2)$ primaries of scaling dimension R , $R + \bar{j} + 1$, $R + j + 1$ and $R + j + \bar{j} + 2$ respectively. The $\mathcal{N} = 3$ multiplets that contain such operators are listed in equations (2.6)-(2.13), together with their decomposition in $\mathcal{N} = 2$, but where we omitted any $\mathcal{N} = 2$ multiplets not containing Schur operators. Below we present a few examples of the complete $\mathcal{N} = 2$ decomposition. These decompositions are obtained by computing the characters of the $\mathcal{N} = 3$ multiplets of table 8, following the method described in appendix C of [79], and re-writing it in terms of characters of $\mathcal{N} = 2$ representations, which can be obtained from the tables of [60].

The stress tensor multiplet decomposes in the expected way, containing only Schur multiplets

$$\hat{\mathcal{B}}_{[1,1]} = \hat{\mathcal{B}}_1 \oplus u_f^{-1} \mathcal{D}_{\frac{1}{2},(0,0)} \oplus u_f \overline{\mathcal{D}}_{\frac{1}{2},(0,0)} \oplus \hat{\mathcal{C}}_{0,(0,0)}. \quad (\text{A.1})$$

Also of particular importance are the half-BPS multiplets, related to the Coulomb branch of $\mathcal{N} = 3$ theories. Their full decomposition is given by

$$\hat{\mathcal{B}}_{[R_1,0]} = u_f^{-R_1} \hat{\mathcal{B}}_{\frac{R_1}{2}} \oplus u_f^{-R_1+1} \overline{\mathcal{D}}_{\frac{R_1-1}{2},(0,0)} \oplus \left(\bigoplus_{a=1}^{R_1-2} u_f^{-R_1+a+1} \overline{\mathcal{B}}_{\frac{R_1-a-1}{2},-a-1,(0,0)} \right) \oplus \overline{\mathcal{E}}_{-R_1,(0,0)} \quad (\text{A.2})$$

and similarly for the conjugate multiplet. An interesting question to ask is, apart from the above $\hat{\mathcal{B}}_{[R_1,0]}$ and conjugate, which $\mathcal{N} = 3$ multiplets contain $\mathcal{N} = 2$ Coulomb branch operators. An obvious place to look would be to consider $\mathcal{N} = 3$ chiral operators, which decompose as

$$\overline{\mathcal{B}}_{[0,0],r,0} = \bigoplus_{a=0}^2 u_f^{a-\frac{r}{3}} \overline{\mathcal{E}}_{-\frac{1}{2}(a+\frac{r}{3}),(\frac{1}{2}a(2-a),0)}, \quad (\text{A.3})$$

and their conjugates. Note that the above decomposition contains ‘‘exotic’’ $\mathcal{N} = 2$ $\overline{\mathcal{E}}_{r,(j,0)}$ operators with spin $j > 0$, which do not seem to occur in known $\mathcal{N} = 2$ SCFTs (see [80] for a discussion). Similarly in [1] the question of which $\mathcal{N} = 3$ operators could contain operators whose vevs parametrized the Coulomb branch was addressed. The authors of [1] argue that the only type of such multiplets are $\hat{\mathcal{B}}_{[R_1,0]}$ and conjugates, since the $\overline{\mathcal{B}}_{[0,0],r,0}$ multiplet would not be consistent with the three different $\mathcal{N} = 2$ subalgebras $\mathcal{N} = 3$ contains.

We finish this appendix with the the example of the decomposition of a generic long $\mathcal{N} = 3$ multiplet. Considering a multiplet whose highest weight transforms in the symmetric traceless representation for simplicity, $\mathcal{A}_{[R_1,R_2],r,\ell}^\Delta$, there appears to be a simple prescription for the decomposition into $\mathcal{N} = 2$ multiplets, which we have checked in a variety of cases. Namely, we first decompose the $\text{SU}(3) \times \text{U}(1)$ representation $([R_1, R_2], r)$ of the superconformal primary of the $\mathcal{N} = 3$ multiplet in representations of $\text{SU}(2)_{R_{\mathcal{N}=2}} \times \text{U}(1)_{r_{\mathcal{N}=2}} \times \text{U}(1)_f$. Let $\{(R', r', F)\}$ be the list of representations appearing in that decomposition. To each such representation, we associate an $\mathcal{N} = 2$ multiplet

$\mathcal{A}_{R',r',\ell}^\Delta$, graded by the corresponding $U(1)_f$ charge $(u_f)^F$. Finally, in the decomposition of the $\mathcal{N} = 3$ multiplet, each of these $\mathcal{N} = 2$ multiplets will be accompanied by the following list of long multiplets:

$$\begin{aligned}
& (u_f)^F \left(\mathcal{A}_{R',r'-1,(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta+1} u_f^2 + u_f^{-2} \mathcal{A}_{R',r'+1,(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta+1} + u_f^{-1} \mathcal{A}_{R',r'+\frac{1}{2},(\frac{\ell}{2},\frac{\ell+1}{2})}^{\Delta+\frac{1}{2}} + u_f^{-1} \mathcal{A}_{R',r'+\frac{1}{2},(\frac{\ell}{2},\frac{\ell-1}{2})}^{\Delta+\frac{1}{2}} \right. \\
& + u_f^{-1} \mathcal{A}_{R',r'+\frac{1}{2},(\frac{\ell+1}{2},\frac{\ell}{2})}^{\Delta+\frac{3}{2}} + u_f^{-1} \mathcal{A}_{R',r'+\frac{1}{2},(\frac{\ell-1}{2},\frac{\ell}{2})}^{\Delta+\frac{3}{2}} + \mathcal{A}_{R',r'-\frac{1}{2},(\frac{\ell}{2},\frac{\ell+1}{2})}^{\Delta+\frac{3}{2}} u_f + \mathcal{A}_{R',r'-\frac{1}{2},(\frac{\ell}{2},\frac{\ell-1}{2})}^{\Delta+\frac{3}{2}} u_f \\
& + \mathcal{A}_{R',r'-\frac{1}{2},(\frac{\ell+1}{2},\frac{\ell}{2})}^{\Delta+\frac{1}{2}} u_f + \mathcal{A}_{R',r'-\frac{1}{2},(\frac{\ell-1}{2},\frac{\ell}{2})}^{\Delta+\frac{1}{2}} u_f + \mathcal{A}_{R',r',(\frac{\ell}{2},\frac{\ell}{2})}^{\Delta+2} + \mathcal{A}_{R',r',(\frac{\ell+1}{2},\frac{\ell+1}{2})}^{\Delta+1} + \mathcal{A}_{R',r',(\frac{\ell+1}{2},\frac{\ell-1}{2})}^{\Delta+1} \\
& \left. + \mathcal{A}_{R',r',(\frac{\ell-1}{2},\frac{\ell+1}{2})}^{\Delta+1} + \mathcal{A}_{R',r',(\frac{\ell-1}{2},\frac{\ell-1}{2})}^{\Delta+1} \right). \tag{A.4}
\end{aligned}$$

B OPEs of the chiral algebra

In this appendix we collect the OPEs corresponding to the chiral algebra constructed in section 2.1.2, with generators given by (2.22). Here we show all the OPE coefficients already fixed to the values dictated by the Jacobi identities. These computations were performed using the Mathematica package SOPEN2defs of [67] and we follow their conventions. In what follows we take a product of operators $\mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_{n-1} \mathcal{O}_n$ to mean the normal ordered product $(\mathcal{O}_1(\mathcal{O}_2(\cdots(\mathcal{O}_{n-1}\mathcal{O}_n))))$.

Since all generators, with the exception of the stress tensor multiplet, are super Virasoro primaries, the OPE of a generator \mathcal{O} of dimension $\Delta_{\mathcal{O}}$ and $U(1)_f$ charge $f_{\mathcal{O}}$ with the stress-tensor current \mathcal{J} is fixed to be

$$\mathcal{J}(Z_1)\mathcal{O}(Z_2) \sim \frac{\Delta_{\mathcal{O}}\theta_{12}\bar{\theta}_{12}\mathcal{O}}{Z_{12}^2} + \frac{-f_{\mathcal{O}}\mathcal{O} - \theta_{12}\mathcal{D}\mathcal{O} + \bar{\theta}_{12}\bar{\mathcal{D}}\mathcal{O} + \theta_{12}\bar{\theta}_{12}\partial\mathcal{O}}{Z_{12}}. \tag{B.1}$$

The stress tensor multiplet has the standard self-OPE given in (2.18), while the OPEs $\mathcal{W}(Z_1)\mathcal{W}(Z_2)$ and $\bar{\mathcal{W}}(Z_1)\bar{\mathcal{W}}(Z_2)$ are regular. The $\mathcal{W}(Z_1)\bar{\mathcal{W}}(Z_2)$ OPE is given in a general form in (2.20) where the sum is taken to run over all uncharged generators, composites and/or (super)derivatives thereof. The coefficients $\lambda_{\mathcal{O}_n}$ in (2.20) are completely fixed by the Jacobi identities to

$$\begin{aligned}
\lambda_1 &= -\frac{c_{2d}}{9}, \quad \lambda_{\mathcal{J}} = 1, \quad \lambda_{\mathcal{J}\mathcal{J}} = -\frac{4}{c_{2d}-1}, \quad \lambda_{\mathcal{D}\bar{\mathcal{D}}\mathcal{J}} = \frac{c_{2d}-9}{6(c_{2d}-1)}, \quad \lambda_{\mathcal{J}'} = \frac{1}{2}, \\
\lambda_{\mathcal{U}} &= -\frac{4(5c_{2d}+27)}{\beta(c_{2d}-9)(c_{2d}-1)}, \tag{B.2}
\end{aligned}$$

where β is related to the norm of \mathcal{U} . The remaining non-trivial OPEs were found to be

$$\begin{aligned}
\mathcal{W}(Z_1)\mathcal{U}(Z_2) &\sim -\frac{\beta(c_{2d}-9)(c_{2d}+15)\theta_{12}\bar{\theta}_{12}\mathcal{W}}{2(5c_{2d}+27)Z_{12}^3} \\
&+ \frac{\beta(c_{2d}+15)}{12(5c_{2d}+27)} \frac{18\theta_{12}\bar{\theta}_{12}\mathcal{J}\mathcal{W} - 2(c_{2d}-9)\theta_{12}\mathcal{D}\mathcal{W} - (c_{2d}-27)\theta_{12}\bar{\theta}_{12}\mathcal{W}' - 6(c_{2d}-9)\mathcal{W}}{Z_{12}^2} \\
&+ \frac{\beta}{12(5c_{2d}+27)} \frac{6(c_{2d}+63)\theta_{12}\mathcal{J}\mathcal{D}\mathcal{W} + 54(c_{2d}-1)\theta_{12}\mathcal{W}\mathcal{D}\mathcal{J} - (c_{2d}-9)(c_{2d}+39)\theta_{12}\mathcal{D}\mathcal{W}'}{Z_{12}} \\
&+ \frac{\beta(c_{2d}+15)}{6(5c_{2d}+27)} \frac{18\mathcal{J}\mathcal{W} - (c_{2d}-27)\mathcal{W}'}{Z_{12}}, \tag{B.3}
\end{aligned}$$

and

$$\begin{aligned}
\bar{W}(Z_1)\mathcal{U}(Z_2) &\sim \frac{\beta(c_{2d}-9)(c_{2d}+15)}{2(5c_{2d}+27)} \frac{\theta_{12}\bar{\theta}_{12}\bar{W}}{Z_{12}^3} \\
&+ \frac{\beta(c_{2d}+15)}{12(5c_{2d}+27)} \frac{18\theta_{12}\bar{\theta}_{12}\mathcal{J}\bar{W} - 2(c_{2d}-9)\bar{\theta}_{12}\bar{D}\bar{W} + (c_{2d}-27)\theta_{12}\bar{\theta}_{12}\bar{W}' - 6(c_{2d}-9)\bar{W}}{Z_{12}^2} \\
&- \frac{\beta}{12(5c_{2d}+27)} \frac{6(c_{2d}+63)\bar{\theta}_{12}\mathcal{J}\bar{D}\bar{W} + 54(c_{2d}-1)\bar{\theta}_{12}\bar{W}\bar{D}\mathcal{J}(c_{2d}-9)(c_{2d}+39)\bar{\theta}_{12}\bar{D}\bar{W}'}{Z_{12}} \\
&- \frac{\beta(c_{2d}+15)}{6(5c_{2d}+27)} \frac{18\mathcal{J}\bar{W} + (c_{2d}-27)\bar{W}'}{Z_{12}}, \tag{B.4}
\end{aligned}$$

with the most complicated one being

$$\begin{aligned}
\mathcal{U}(Z_1)\mathcal{U}(Z_2) &\sim -\frac{(c_{2d}-9)^2(c_{2d}-1)(c_{2d}+15)\beta^2}{72(5c_{2d}+27)^2} \frac{c_{2d}+6\mathcal{J}\theta_{12}\bar{\theta}_{12}}{Z_{12}^4} \\
&+ \frac{\beta^2(c_{2d}-9)^2(c_{2d}-1)(c_{2d}+15)}{12(5c_{2d}+27)^2} \frac{-\bar{\theta}_{12}\bar{D}\mathcal{J} + \theta_{12}\mathcal{D}\mathcal{J} - \theta_{12}\bar{\theta}_{12}\mathcal{J}'}{Z_{12}^3} \\
&+ \frac{1}{Z_{12}^2} \left(\frac{\beta^2\mathcal{D}\bar{D}\mathcal{J}'\theta_{12}\bar{\theta}_{12}(c_{2d}-9)^3}{8(5c_{2d}+27)^2} + \frac{c_{2d}(c_{2d}+15)\beta^2\mathcal{D}\bar{D}\mathcal{J}(c_{2d}-9)^2}{36(5c_{2d}+27)^2} + \frac{(c_{2d}+15)\beta^2\mathcal{J}\mathcal{J}(c_{2d}-9)^2}{12(5c_{2d}+27)^2} \right. \\
&+ \frac{(c_{2d}+15)(2c_{2d}-3)\beta^2\mathcal{D}\mathcal{J}'\theta_{12}(c_{2d}-9)^2}{36(5c_{2d}+27)^2} + \frac{(c_{2d}+15)\beta^2\mathcal{J}\mathcal{D}\mathcal{J}\theta_{12}(c_{2d}-9)^2}{12(5c_{2d}+27)^2} + \frac{(c_{2d}+15)\beta^2\mathcal{J}\bar{D}\mathcal{J}\bar{\theta}_{12}(c_{2d}-9)^2}{12(5c_{2d}+27)^2} \\
&+ \frac{(c_{2d}-21)\beta^2\mathcal{J}\mathcal{D}\bar{D}\mathcal{J}\theta_{12}\bar{\theta}_{12}(c_{2d}-9)^2}{6(5c_{2d}+27)^2} - \frac{\beta^2\mathcal{D}\mathcal{J}\bar{D}\mathcal{J}\theta_{12}\bar{\theta}_{12}(c_{2d}-9)^2}{8(5c_{2d}+27)} - \frac{(c_{2d}+15)(2c_{2d}-3)\beta^2\bar{D}\mathcal{J}'\bar{\theta}_{12}(c_{2d}-9)^2}{36(5c_{2d}+27)^2} \\
&+ \frac{(c_{2d}+63)\beta^2\mathcal{J}\mathcal{J}\theta_{12}\bar{\theta}_{12}(c_{2d}-9)}{2(5c_{2d}+27)^2} - \frac{27(c_{2d}-1)\beta^2\mathcal{J}\mathcal{J}'\theta_{12}\bar{\theta}_{12}(c_{2d}-9)}{2(5c_{2d}+27)^2} - \frac{27(c_{2d}-1)^2\beta^2\mathcal{W}\bar{W}\theta_{12}\bar{\theta}_{12}(c_{2d}-9)}{8(5c_{2d}+27)^2} \\
&- \frac{(c_{2d}^3-11c_{2d}^2-105c_{2d}+243)\beta^2\theta_{12}\bar{\theta}_{12}\mathcal{J}''(c_{2d}-9)}{24(5c_{2d}+27)^2} - \frac{1}{6}(c_{2d}+3)\mathcal{U}\beta - \frac{1}{12}(c_{2d}+3)\beta\mathcal{D}\mathcal{U}\theta_{12} \\
&+ \frac{(c_{2d}^2-8c_{2d}+135)\beta\mathcal{D}\bar{D}\mathcal{U}\theta_{12}\bar{\theta}_{12}}{8(5c_{2d}+27)} - \frac{(7c_{2d}-135)\beta\mathcal{J}\mathcal{U}\theta_{12}\bar{\theta}_{12}}{2(5c_{2d}+27)} - \frac{27(c_{2d}-1)\beta\theta_{12}\bar{\theta}_{12}\mathcal{U}'}{4(5c_{2d}+27)} - \frac{1}{12}(c_{2d}+3)\beta\bar{D}\mathcal{U}\bar{\theta}_{12} \Big) \\
&+ \frac{1}{Z_{12}} \left(\frac{\beta^2\mathcal{J}'\mathcal{D}\bar{D}\mathcal{J}\theta_{12}\bar{\theta}_{12}(c_{2d}-9)^3}{6(5c_{2d}+27)^2} + \frac{c_{2d}(c_{2d}+15)\beta^2\mathcal{D}\bar{D}\mathcal{J}'(c_{2d}-9)^2}{72(5c_{2d}+27)^2} + \frac{(c_{2d}+15)\beta^2\mathcal{J}\mathcal{J}'(c_{2d}-9)^2}{12(5c_{2d}+27)^2} \right. \\
&+ \frac{(c_{2d}-33)\beta^2\mathcal{J}\mathcal{D}\mathcal{J}'\theta_{12}(c_{2d}-9)^2}{24(5c_{2d}+27)^2} + \frac{(c_{2d}-33)\beta^2\mathcal{J}\bar{D}\mathcal{J}'\bar{\theta}_{12}(c_{2d}-9)^2}{24(5c_{2d}+27)^2} + \frac{3(c_{2d}-1)\beta^2\bar{D}\mathcal{J}\mathcal{D}\bar{D}\mathcal{J}\bar{\theta}_{12}(c_{2d}-9)^2}{16(5c_{2d}+27)^2} \\
&+ \frac{(c_{2d}-33)\beta^2\mathcal{J}\mathcal{D}\bar{D}\mathcal{J}'\theta_{12}\bar{\theta}_{12}(c_{2d}-9)^2}{12(5c_{2d}+27)^2} + \frac{\beta^2\bar{D}\mathcal{J}'\mathcal{D}\mathcal{J}\theta_{12}\bar{\theta}_{12}(c_{2d}-9)^2}{12(5c_{2d}+27)} \\
&+ \frac{(c_{2d}-1)(2c_{2d}+15)\beta^2\theta_{12}(\mathcal{D}\mathcal{J}')'(c_{2d}-9)^2}{96(5c_{2d}+27)^2} - \frac{\beta^2\mathcal{D}\mathcal{J}'\bar{D}\mathcal{J}\theta_{12}\bar{\theta}_{12}(c_{2d}-9)^2}{12(5c_{2d}+27)} \\
&- \frac{3(c_{2d}-1)\beta^2\mathcal{D}\mathcal{J}\mathcal{D}\bar{D}\mathcal{J}\theta_{12}(c_{2d}-9)^2}{16(5c_{2d}+27)^2} - \frac{(c_{2d}-1)(2c_{2d}+15)\beta^2\bar{\theta}_{12}(\bar{D}\mathcal{J}')'(c_{2d}-9)^2}{96(5c_{2d}+27)^2} \\
&+ \frac{9(c_{2d}-1)^2\beta^2\bar{W}\mathcal{D}\mathcal{W}\theta_{12}(c_{2d}-9)}{8(5c_{2d}+27)^2} + \frac{(c_{2d}-81)(c_{2d}-1)\beta^2\mathcal{J}'\mathcal{D}\mathcal{J}\theta_{12}(c_{2d}-9)}{16(5c_{2d}+27)^2} \\
&+ \frac{(c_{2d}+63)\beta^2\mathcal{J}\mathcal{J}\bar{D}\mathcal{J}\bar{\theta}_{12}(c_{2d}-9)}{2(5c_{2d}+27)^2} + \frac{(c_{2d}-81)(c_{2d}-1)\beta^2\mathcal{J}'\bar{D}\mathcal{J}\bar{\theta}_{12}(c_{2d}-9)}{16(5c_{2d}+27)^2} \\
&+ \frac{(c_{2d}+63)\beta^2\mathcal{J}\mathcal{J}'\theta_{12}\bar{\theta}_{12}(c_{2d}-9)}{(5c_{2d}+27)^2} - \frac{4\beta\mathcal{U}\mathcal{J}'\theta_{12}\bar{\theta}_{12}(c_{2d}-9)}{5c_{2d}+27}
\end{aligned}$$

$$\begin{aligned}
& - \frac{(c_{2d} + 63)\beta^2 \mathcal{J} \mathcal{D} \mathcal{J} \theta_{12}(c_{2d} - 9)}{2(5c_{2d} + 27)^2} - \frac{9(c_{2d} - 1)^2 \beta^2 \mathcal{W} \bar{\mathcal{W}}' \theta_{12} \bar{\theta}_{12}(c_{2d} - 9)}{4(5c_{2d} + 27)^2} \\
& - \frac{9(c_{2d} - 1)^2 \beta^2 \bar{\mathcal{W}} \mathcal{W}' \theta_{12} \bar{\theta}_{12}(c_{2d} - 9)}{4(5c_{2d} + 27)^2} - \frac{9(c_{2d} - 1)^2 \beta^2 \mathcal{W} \bar{\mathcal{D}} \bar{\mathcal{W}} \theta_{12}(c_{2d} - 9)}{8(5c_{2d} + 27)^2} \\
& - \frac{(c_{2d} + 3)(c_{2d}^2 - 41c_{2d} + 72) \beta^2 \theta_{12} \bar{\theta}_{12} \mathcal{J}^{(3)}(c_{2d} - 9)}{72(5c_{2d} + 27)^2} + \frac{9(c_{2d} - 1) \beta \mathcal{U} \mathcal{D} \mathcal{J} \theta_{12}}{2(5c_{2d} + 27)} \\
& + \frac{(c_{2d} + 63) \beta \mathcal{J} \bar{\mathcal{D}} \mathcal{U} \bar{\theta}_{12}}{2(5c_{2d} + 27)} + \frac{(c_{2d}^2 - 3c_{2d} + 162) \beta \mathcal{D} \bar{\mathcal{D}} \mathcal{U}' \theta_{12} \bar{\theta}_{12}}{12(5c_{2d} + 27)} - \frac{(c_{2d} - 1)(c_{2d} + 27) \beta \bar{\mathcal{D}} \mathcal{U}' \bar{\theta}_{12}}{4(5c_{2d} + 27)} \\
& - \frac{1}{2} \beta \mathcal{D} \mathcal{J} \bar{\mathcal{D}} \mathcal{U} \theta_{12} \bar{\theta}_{12} + \frac{1}{2} \beta \mathcal{D} \mathcal{U} \bar{\mathcal{D}} \mathcal{J} \theta_{12} \bar{\theta}_{12} - \frac{1}{12} (c_{2d} + 3) \beta \mathcal{U}' - \frac{1}{4} \beta \theta_{12} \bar{\theta}_{12} \mathcal{U}'' - \frac{(c_{2d} + 63) \beta \mathcal{J} \mathcal{D} \mathcal{U} \theta_{12}}{2(5c_{2d} + 27)} \\
& - \frac{9(c_{2d} - 1) \beta \mathcal{U} \bar{\mathcal{D}} \mathcal{J} \bar{\theta}_{12}}{2(5c_{2d} + 27)} - \frac{3(c_{2d} - 33) \beta \mathcal{J} \mathcal{U}' \theta_{12} \bar{\theta}_{12}}{2(5c_{2d} + 27)} - \frac{(c_{2d} - 1)(c_{2d} + 27) \beta \mathcal{D} \mathcal{U}' \theta_{12}}{4(5c_{2d} + 27)} \Big). \tag{B.5}
\end{aligned}$$

C Some blockology

In this appendix we gather our conventions for the conformal and R-symmetry blocks as well as some of their most important properties.

Conformal blocks. Regarding the four dimensional (bosonic) conformal blocks, we adopt the conventions

$$g_{\Delta, \ell}^{\Delta_{12}, \Delta_{34}}(z, \bar{z}) = \frac{z\bar{z}}{z - \bar{z}} \left(k_{\Delta+\ell}^{\Delta_{12}, \Delta_{34}}(z) k_{\Delta-\ell-2}^{\Delta_{12}, \Delta_{34}}(\bar{z}) - (z \leftrightarrow \bar{z}) \right), \tag{C.1}$$

$$k_{\beta}^{a,b}(x) = x^{\frac{\beta}{2}} {}_2F_1\left(\frac{\beta-a}{2}, \frac{\beta+b}{2}, \beta, x\right). \tag{C.2}$$

We also set $k_{\beta}(x) := k_{\beta}^{0,0}(x)$ and $g_{\Delta, \ell}(z, \bar{z}) := g_{\Delta, \ell}^{0,0}(z, \bar{z})$.

The $4d$, $\mathcal{N} = 1$ superblocks derived in [15, 34] can be expressed very simply through the usual conformal blocks. Specifically, one has

$$\mathcal{G}_{\Delta, \ell}^{\mathcal{N}=1}(z, \bar{z}) = (z\bar{z})^{-\frac{1}{2}} g_{\Delta+1, \ell}^{\Delta_{12}=\Delta_{34}=1}(z, \bar{z}). \tag{C.3}$$

In two dimensions, we have the (bosonic) conformal blocks

$$g_h^{2d}(z) = z^h {}_2F_1(h, h, 2h, z), \tag{C.4}$$

while the $\text{OSP}(2|2)$ superblocks are

$$g_h^{2d, \mathcal{N}=2}(x) = x^h {}_2F_1(h, h, 2h + 1, x). \tag{C.5}$$

We remind that as usual the variables are related via $x = \frac{z}{z-1}$.

R-symmetry blocks. The $\text{SU}(3)$ blocks for the **non-chiral channel** of section 3.4.1 are

$$h_{[m,m]}(w) = \binom{2m+1}{m+1}^{-1} {}_2F_1(-m, m+2, 1, y^{-1}), \quad y = \frac{w}{w-1}. \tag{C.6}$$

For the **chiral channel** of section 3.4.2, we find that in our normalization, the SU(3) blocks are identical to SU(2) R-symmetry blocks:

$$\tilde{h}_{[2m,n]}(w) = h_m^{\text{SU}(2)}(w) = (-1)^m \binom{2m}{m}^{-1} {}_2F_1(-m, m+1, 1, w^{-1}). \quad (\text{C.7})$$

In particular, they are independent of the label n .

Braiding. We recall the conformal/ R-symmetry blocks obey various useful braiding relations. The most important ones for us are

$$((1-z)(1-\bar{z}))^{-\frac{\Delta_{34}}{2}} g_{\Delta,\ell}^{-\Delta_{12},\Delta_{34}}\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right) = (-1)^\ell g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z, \bar{z}). \quad (\text{C.8})$$

for the $4d$ conformal blocks and

$$h_m^{\text{SU}(2)}(w) = (-1)^m h_m^{\text{SU}(2)}\left(\frac{w}{w-1}\right), \quad (\text{C.9})$$

for the SU(2) R-symmetry ones. In particular (C.8) implies that $g_{\Delta,\ell}\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}\right) = (-1)^\ell g_{\Delta,\ell}(z, \bar{z})$. The holomorphic blocks (C.4) satisfy

$$g_h^{2d}(z) = (-1)^h g_h^{2d}\left(\frac{z}{z-1}\right). \quad (\text{C.10})$$

D Crossing equations

In this appendix we collect some bulky equations used in the crossing equations (4.14) and summarize the computation used in section 4.2 for the function $H_{R,\text{short}}$.

D.1 Explicit expressions for $\mathcal{F}_{\text{short}}^{(0,\pm)}$

Here we collect the expressions for $\mathcal{F}_{\text{short}}^{(0,\pm)}$ that we need in the crossing equations (4.14). Using the definition (4.4) for the function A_R , we write

$$\begin{aligned} \mathcal{F}_{\text{short}}^{(0)}[f] &= - \frac{[(1-z)(1-\bar{z})]^{R+1}}{(1-w)^{R-2}} \frac{y^{2-R}}{x_1^{-1} - x_2^{-1}} (x_2 A_R(x_2, y) f_R(x_1) - x_1 \leftrightarrow x_2), \\ \mathcal{F}_{\text{short}}^{(0)}[H_{\text{short}}, \tilde{H}_{\text{short}}] &= \frac{[(1-z)(1-\bar{z})]^{R+1}}{(1-w)^{R-2}} H_{R,\text{short}}(z, \bar{z}, w) \\ &\quad - (-1)^R \left[(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w) \right], \end{aligned} \quad (\text{D.1})$$

for $\mathcal{F}_{\text{short}}^{(0)}$ and

$$\begin{aligned}
\mathcal{F}_{\text{short}}^{(\pm)}[f] &= - \left\{ \frac{[(z-1)(\bar{z}-1)]^{R+1}}{(w-1)^{R-2}} \frac{x_1^{R-1} A_R(x_1, y) \tilde{f}_R(\bar{z}) - (z \leftrightarrow \bar{z})}{z^{-1} - \bar{z}^{-1}} \right. \\
&\quad \left. \pm [(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w)] \right\}, \\
\mathcal{F}_{\text{short}}^{(\pm)}[H_{\text{short}}, \tilde{H}_{\text{short}}] &= (-1)^R \left\{ \frac{[(1-z)(1-\bar{z})]^{R+1}}{(1-w)^{R-2}} \tilde{H}_{R,\text{short}}(z, \bar{z}, w) \right. \\
&\quad \left. \pm [(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w)] \right\} \\
&\quad \mp \left\{ \frac{[(1-z)(1-\bar{z})]^{R+1}}{(1-w)^{R-2}} H_{R,\text{short}}\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1}, \frac{w}{w-1}\right) \right. \\
&\quad \left. \pm [(z, \bar{z}, w) \leftrightarrow (1-z, 1-\bar{z}, 1-w)] \right\},
\end{aligned} \tag{D.2}$$

for $\mathcal{F}_{\text{short}}^{(\pm)}$.

D.2 Summation for H_{short}

Given the function $f_R(x)$ in the parametrization (3.17) of a four point function (3.14), one can associate a contribution to the function H , called $H_{\text{short}}[f]$, corresponding to the exchange of the short operators which survive the cohomological truncation. The goal of this appendix is to explicitly perform the summations in the first term in (4.24). This can be done for two reasons:

1. The coefficients $b_h^{(R)}$ in the expansion (4.19) can be easily determined in terms of a finite number of parameters. This follows from the fact that $f_R(x)$ is a polynomial of degree R that satisfy the crossing property $f_R(x) = x^R f_R(x^{-1})$.
2. Each block entering the first sum in (4.24) has the form

$$\mathcal{G}_{h+4,h}^{d=4,\mathcal{N}=1} = \frac{t_h(z)s(\bar{z}) - t_h(\bar{z})s(z)}{z - \bar{z}}, \quad s(t) = -2(t + \log(1-t)), \tag{D.3}$$

where $t_h(z) = t^{h+3} {}_2F_1(h+2, h+3, 2h+5, t)$.

Each monomial term in $f_R(x)$, except for $x^0 = 1$, can be expanded in superblocks as

$$x^n = \sum_{h=n}^{\infty} \hat{b}_{n,h} g_h^{2d,\mathcal{N}=2}\left(\frac{x}{x-1}\right), \quad \hat{b}_{n,h} = -4^{n-h} h \frac{(2n)_{h-n} (1-h)_{n-1}}{\Gamma(n+1)(n+\frac{1}{2})_{h-n}}. \tag{D.4}$$

It follows that the part of $H_{R,\text{short}}$ in the R-symmetry singlet channel is

$$H_{R,\text{short}}^{\text{singlet}}[x^n] := \sum_{h=n}^{\infty} \hat{b}_{n,h} \mathcal{G}_{h+4,h}^{d=4,\mathcal{N}=1}(z, \bar{z}) = \frac{z^{n+1}s(\bar{z}) - \bar{z}^{n+1}s(z)}{z - \bar{z}}, \quad n \geq 0, \tag{D.5}$$

and $H_{R,\text{short}}^{\text{singlet}}[1] = 0$. Above $H_{R,\text{short}}^{\text{singlet}}$ is considered as a linear map. In the cases relevant for $R = 2, 3$ (recall that $H_{R=1} = 0$), the only solution of the crossing symmetry conditions are

$$\begin{aligned} f_2(x) &= 1 + c^{-1}x + x^2, & f_3(x) &= (1+x)(1+\gamma(c)x+x^2) \\ & & &= 1 + (1+\gamma(c))x + (1+\gamma(c))x^2 + x^3, \end{aligned} \quad (\text{D.6})$$

with $\gamma(c) = \frac{9}{4c} - 1$ and $c = c_{4d}$. It follows from (D.5) that

$$\begin{aligned} H_{R,\text{short}}^{\text{singlet}}(z, \bar{z}) &= \frac{\hat{t}^{(R)}(x_1)\hat{s}(x_2) - \hat{t}^{(R)}(x_2)\hat{s}(x_1)}{x_1 - x_2}, \\ \hat{t}^{(R)}(x) &= x(f_R(x) - 1), \quad \hat{s}\left(\frac{z}{z-1}\right) = \frac{s(z)}{1-z}, \end{aligned} \quad (\text{D.7})$$

with a now familiar identification $(x_1, x_2) = (\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1})$. In the cases $R = 2, 3$, the function $\hat{t}^{(R)}(x)$ takes the form

$$\hat{t}^{(R=2)}(x) = x^2 \left(x + \frac{1}{c} \right), \quad \hat{t}^{(R=3)}(x) = x^2 \left(x^2 + \frac{9}{4c}(1+x) \right). \quad (\text{D.8})$$

E Generalized free theory example

In this appendix we present a solution to the Ward identities (3.16) and to the crossing equations of section 4.1. This solution is commonly referred to as generalized free theory. It reads

$$G_R^{\text{gft}}(x_1, x_2, y) = 1 + \left(\frac{x_1 x_2}{y} \right)^R, \quad (\text{E.1})$$

from which we can extract by setting $x_2 = y$ the expression $f_R^{\text{gft}}(x) = 1 + x^R$. Furthermore, via (3.17), we get

$$H_R^{\text{gft}}(x_1, x_2, y) = \frac{x_1 x_2 [x_1^R x_2^R y^{2-R}(x_1 - x_2) - y x_1^R x_2(x_1 - y) + y x_1 x_2^R(x_2 - y)]}{(x_1 - x_2)(x_1 - y)(x_2 - y)}. \quad (\text{E.2})$$

In particular, we have for small R the expressions

$$\begin{aligned} H_2^{\text{gft}}(x_1, x_2, y) &= (x_1 x_2)^2, \\ H_3^{\text{gft}}(x_1, x_2, y) &= (x_1 x_2)^2(x_1 + x_2) + \frac{(x_1 x_2)^3}{y} \\ &= (x_1 x_2)^2 \left[(x_1 + x_2 + \frac{1}{3}x_1 x_2) - x_1 x_2 h_{[1,1]}(w) \right], \end{aligned} \quad (\text{E.3})$$

with $h_{[1,1]}(w)$ given in (C.6) with $y = \frac{w}{w-1}$. The block expansion of $f_R(x)$ for the generalized free theory is explicitly given by

$$f_R(x) = 1 + \sum_{h=R}^{\infty} b_{R,h}^{\text{gft}} g_h^{2d} \mathcal{N}^{=2}(x), \quad b_{R,h}^{\text{gft}} = -4^{R-h} h \frac{(2R)_{h-R}(1-h)_{R-1}}{\Gamma(R+1)(R+\frac{1}{2})_{h-R}}. \quad (\text{E.4})$$

Note (and compare with the discussion around (4.20)) that in the generalized free theory example we have $b_{h<R}^{(R)} = 0$. In particular, there is no stress-tensor being exchanged.

References

- [1] O. Aharony and M. Evtikhiev, *On four dimensional $N = 3$ superconformal theories*, *JHEP* **04** (2016) 040, [[arXiv:1512.03524](#)].
- [2] I. Garca-Etxebarria and D. Regalado, *$\mathcal{N} = 3$ four dimensional field theories*, *JHEP* **03** (2016) 083, [[arXiv:1512.06434](#)].
- [3] O. Aharony and Y. Tachikawa, *S -folds and $4d$ $N=3$ superconformal field theories*, *JHEP* **06** (2016) 044, [[arXiv:1602.08638](#)].
- [4] I. Garca-Etxebarria and D. Regalado, *Exceptional $\mathcal{N} = 3$ theories*, [arXiv:1611.05769](#).
- [5] P. Argyres, M. Lotito, Y. Lü, and M. Martone, *Geometric constraints on the space of $N=2$ SCFTs I: physical constraints on relevant deformations*, [arXiv:1505.04814](#).
- [6] P. C. Argyres, M. Lotito, Y. Lü, and M. Martone, *Geometric constraints on the space of $N=2$ SCFTs II: Construction of special Kähler geometries and RG flows*, [arXiv:1601.00011](#).
- [7] P. C. Argyres, M. Lotito, Y. Lü, and M. Martone, *Expanding the landscape of $\mathcal{N} = 2$ rank 1 SCFTs*, *JHEP* **05** (2016) 088, [[arXiv:1602.02764](#)].
- [8] P. Argyres, M. Lotito, Y. Lü, and M. Martone, *Geometric constraints on the space of $N=2$ SCFTs III: enhanced Coulomb branches and central charges*, [arXiv:1609.04404](#).
- [9] P. C. Argyres and M. Martone, *$4d$ $\mathcal{N}=2$ theories with disconnected gauge groups*, [arXiv:1611.08602](#).
- [10] T. Nishinaka and Y. Tachikawa, *On $4d$ rank-one $\mathcal{N} = 3$ superconformal field theories*, *JHEP* **09** (2016) 116, [[arXiv:1602.01503](#)].
- [11] Y. Imamura and S. Yokoyama, *Superconformal index of $\mathcal{N} = 3$ orientifold theories*, *J. Phys.* **A49** (2016), no. 43 435401, [[arXiv:1603.00851](#)].
- [12] Y. Imamura, H. Kato, and D. Yokoyama, *Supersymmetry Enhancement and Junctions in S -folds*, *JHEP* **10** (2016) 150, [[arXiv:1606.07186](#)].
- [13] P. Agarwal and A. Amariti, *Notes on S -folds and $\mathcal{N} = 3$ theories*, *JHEP* **09** (2016) 032, [[arXiv:1607.00313](#)].
- [14] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in $4D$ CFT*, *JHEP* **0812** (2008) 031, [[arXiv:0807.0004](#)].
- [15] D. Poland and D. Simmons-Duffin, *Bounds on $4D$ Conformal and Superconformal Field Theories*, *JHEP* **1105** (2011) 017, [[arXiv:1009.2087](#)].
- [16] D. Poland, D. Simmons-Duffin, and A. Vichi, *Carving Out the Space of $4D$ CFTs*, *JHEP* **1205** (2012) 110, [[arXiv:1109.5176](#)].
- [17] M. Berkooz, R. Yacoby, and A. Zait, *Bounds on $\mathcal{N} = 1$ superconformal theories with global symmetries*, *JHEP* **1408** (2014) 008, [[arXiv:1402.6068](#)].
- [18] D. Poland and A. Stergiou, *Exploring the Minimal $4D$ $\mathcal{N} = 1$ SCFT*, [arXiv:1509.06368](#).
- [19] C. Beem, M. Lemos, P. Liendo, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 2$ superconformal bootstrap*, *JHEP* **03** (2016) 183, [[arXiv:1412.7541](#)].
- [20] C. Beem, L. Rastelli, and B. C. van Rees, *The $N=4$ Superconformal Bootstrap*, *Phys.Rev.Lett.* **111** (2013) 071601, [[arXiv:1304.1803](#)].
- [21] L. F. Alday and A. Bissi, *The superconformal bootstrap for structure constants*, *JHEP* **1409** (2014) 144, [[arXiv:1310.3757](#)].

- [22] L. F. Alday and A. Bissi, *Generalized bootstrap equations for $\mathcal{N} = 4$ SCFT*, *JHEP* **02** (2015) 101, [[arXiv:1404.5864](#)].
- [23] S. M. Chester, J. Lee, S. S. Pufu, and R. Yacoby, *The $\mathcal{N} = 8$ superconformal bootstrap in three dimensions*, *JHEP* **1409** (2014) 143, [[arXiv:1406.4814](#)].
- [24] S. M. Chester, J. Lee, S. S. Pufu, and R. Yacoby, *Exact Correlators of BPS Operators from the 3d Superconformal Bootstrap*, *JHEP* **03** (2015) 130, [[arXiv:1412.0334](#)].
- [25] S. M. Chester, S. Giombi, L. V. Iliesiu, I. R. Klebanov, S. S. Pufu, and R. Yacoby, *Accidental Symmetries and the Conformal Bootstrap*, [[arXiv:1507.04424](#)].
- [26] D. Bashkirov, *Bootstrapping the $\mathcal{N} = 1$ SCFT in three dimensions*, [[arXiv:1310.8255](#)].
- [27] N. Bobev, S. El-Showk, D. Mazac, and M. F. Paulos, *Bootstrapping the Three-Dimensional Supersymmetric Ising Model*, *Phys. Rev. Lett.* **115** (2015), no. 5 051601, [[arXiv:1502.04124](#)].
- [28] N. Bobev, S. El-Showk, D. Mazac, and M. F. Paulos, *Bootstrapping SCFTs with Four Supercharges*, *JHEP* **08** (2015) 142, [[arXiv:1503.02081](#)].
- [29] C. Beem, M. Lemos, L. Rastelli, and B. C. van Rees, *The $(2, 0)$ superconformal bootstrap*, *Phys. Rev. D* **93** (2016), no. 2 025016, [[arXiv:1507.05637](#)].
- [30] F. Dolan and H. Osborn, *Superconformal symmetry, correlation functions and the operator product expansion*, *Nucl.Phys.* **B629** (2002) 3–73, [[hep-th/0112251](#)].
- [31] F. A. Dolan, L. Gallot, and E. Sokatchev, *On four-point functions of 1/2-BPS operators in general dimensions*, *JHEP* **0409** (2004) 056, [[hep-th/0405180](#)].
- [32] M. Nirschl and H. Osborn, *Superconformal Ward identities and their solution*, *Nucl.Phys.* **B711** (2005) 409–479, [[hep-th/0407060](#)].
- [33] J.-F. Fortin, K. Intriligator, and A. Stergiou, *Current OPEs in Superconformal Theories*, *JHEP* **09** (2011) 071, [[arXiv:1107.1721](#)].
- [34] A. L. Fitzpatrick, J. Kaplan, Z. U. Khandker, D. Li, D. Poland, and D. Simmons-Duffin, *Covariant Approaches to Superconformal Blocks*, *JHEP* **1408** (2014) 129, [[arXiv:1402.1167](#)].
- [35] Z. U. Khandker, D. Li, D. Poland, and D. Simmons-Duffin, *$\mathcal{N} = 1$ superconformal blocks for general scalar operators*, *JHEP* **1408** (2014) 049, [[arXiv:1404.5300](#)].
- [36] A. Bissi and T. Lukowski, *Revisiting $\mathcal{N} = 4$ superconformal blocks*, *JHEP* **02** (2016) 115, [[arXiv:1508.02391](#)].
- [37] R. Doobary and P. Heslop, *Superconformal partial waves in Grassmannian field theories*, *JHEP* **12** (2015) 159, [[arXiv:1508.03611](#)].
- [38] Z. Li and N. Su, *The Most General 4D $\mathcal{N} = 1$ Superconformal Blocks for Scalar Operators*, *JHEP* **05** (2016) 163, [[arXiv:1602.07097](#)].
- [39] P. Liendo and C. Meneghelli, *Bootstrap equations for $\mathcal{N} = 4$ SYM with defects*, [[arXiv:1608.05126](#)].
- [40] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, *Infinite Chiral Symmetry in Four Dimensions*, *Communications in Mathematical Physics* **336** (2015), no. 3 1359–1433, [[arXiv:1312.5344](#)].
- [41] C. Beem, W. Peelaers, L. Rastelli, and B. C. van Rees, *Chiral algebras of class S*, *JHEP* **1505** (2015) 020, [[arXiv:1408.6522](#)].
- [42] M. Lemos and W. Peelaers, *Chiral Algebras for Trinion Theories*, *JHEP* **1502** (2015) 113, [[arXiv:1411.3252](#)].

- [43] O. Chacaltana, J. Distler, and A. Trimm, *A Family of 4D $\mathcal{N} = 2$ Interacting SCFTs from the Twisted A_{2N} Series*, [arXiv:1412.8129](#).
- [44] M. Buican and T. Nishinaka, *On the Superconformal Index of Argyres-Douglas Theories*, [arXiv:1505.05884](#).
- [45] C. Cordova and S.-H. Shao, *Schur Indices, BPS Particles, and Argyres-Douglas Theories*, [arXiv:1506.00265](#).
- [46] P. Liendo, I. Ramirez, and J. Seo, *Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories*, *JHEP* **02** (2016) 019, [[arXiv:1509.00033](#)].
- [47] M. Buican and T. Nishinaka, *Argyres-Douglas Theories, the Macdonald Index, and an RG Inequality*, *JHEP* **02** (2016) 159, [[arXiv:1509.05402](#)].
- [48] J. Song, *Superconformal indices of generalized Argyres-Douglas theories from 2d TQFT*, *JHEP* **02** (2016) 045, [[arXiv:1509.06730](#)].
- [49] S. Cecotti, J. Song, C. Vafa, and W. Yan, *Superconformal Index, BPS Monodromy and Chiral Algebras*, [arXiv:1511.01516](#).
- [50] M. Lemos and P. Liendo, *$\mathcal{N} = 2$ central charge bounds from 2d chiral algebras*, *JHEP* **04** (2016) 004, [[arXiv:1511.07449](#)].
- [51] T. Arakawa and K. Kawasetsu, *Quasi-lisse vertex algebras and modular linear differential equations*, [arXiv:1610.05865](#).
- [52] M. Buican and T. Nishinaka, *Conformal Manifolds in Four Dimensions and Chiral Algebras*, *J. Phys.* **A49** (2016), no. 46 465401, [[arXiv:1603.00887](#)].
- [53] C. Cordova, D. Gaiotto, and S.-H. Shao, *Infrared Computations of Defect Schur Indices*, [arXiv:1606.08429](#).
- [54] D. Xie, W. Yan, and S.-T. Yau, *Chiral algebra of Argyres-Douglas theory from M5 brane*, [arXiv:1604.02155](#).
- [55] V. Dobrev and V. Petkova, *All Positive Energy Unitary Irreducible Representations of Extended Conformal Supersymmetry*, *Phys.Lett.* **B162** (1985) 127–132.
- [56] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, *Adv.Theor.Math.Phys.* **2** (1998) 781–846, [[hep-th/9712074](#)].
- [57] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun.Math.Phys.* **275** (2007) 209–254, [[hep-th/0510251](#)].
- [58] C. Cordova, T. T. Dumitrescu, and K. Intriligator, *Deformations of Superconformal Theories*, [arXiv:1602.01217](#).
- [59] C. Cordova, T. T. Dumitrescu, and K. Intriligator, *Multiplets of Superconformal Symmetry in Diverse Dimensions*, [arXiv:1612.00809](#).
- [60] F. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41–89, [[hep-th/0209056](#)].
- [61] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry*, *J.Phys.* **A46** (2013) 214011, [[arXiv:1112.1016](#)].
- [62] V. Alba and K. Diab, *Constraining conformal field theories with a higher spin symmetry in $d=4$* , [arXiv:1307.8092](#).
- [63] G. Arutyunov, B. Eden, and E. Sokatchev, *On nonrenormalization and OPE in superconformal field theories*, *Nucl.Phys.* **B619** (2001) 359–372, [[hep-th/0105254](#)].

- [64] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *The 4d Superconformal Index from q-deformed 2d Yang-Mills*, *Phys.Rev.Lett.* **106** (2011) 241602, [[arXiv:1104.3850](#)].
- [65] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *Gauge Theories and Macdonald Polynomials*, *Commun.Math.Phys.* **319** (2013) 147–193, [[arXiv:1110.3740](#)].
- [66] L. Rastelli and S. S. Razamat, *The superconformal index of theories of class S*, [arXiv:1412.7131](#).
- [67] S. Krivonos and K. Thielemans, *A Mathematica package for computing N=2 superfield operator product expansions*, *Class. Quant. Grav.* **13** (1996) 2899–2910, [[hep-th/9512029](#)].
- [68] C. Beem and L. Rastelli, *Vertex operator algebras, Higgs branches, and modular differential equations*, *to appear*.
- [69] P. Liendo, C. Meneghelli, and V. Mitev, *On correlation functions of BPS operators in 3d $\mathcal{N} = 6$ superconformal theories*, *Commun. Math. Phys.* (2016) 1–33, [[arXiv:1512.06072](#)].
- [70] H. Osborn and A. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311–362, [[hep-th/9307010](#)].
- [71] S. Rychkov, *EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions*, [arXiv:1601.05000](#).
- [72] D. Simmons-Duffin, *TASI Lectures on the Conformal Bootstrap*, [arXiv:1602.07982](#).
- [73] D. Simmons-Duffin, *A Semidefinite Program Solver for the Conformal Bootstrap*, *JHEP* **06** (2015) 174, [[arXiv:1502.02033](#)].
- [74] M. Lemos and P. Liendo, *Bootstrapping $\mathcal{N} = 2$ chiral correlators*, *JHEP* **01** (2016) 025, [[arXiv:1510.03866](#)].
- [75] M. Cornagliotto, M. Lemos, and V. Schomerus. *To appear*.
- [76] P. C. Argyres and J. R. Wittig, *Infinite coupling duals of N=2 gauge theories and new rank 1 superconformal field theories*, *JHEP* **0801** (2008) 074, [[arXiv:0712.2028](#)].
- [77] A. D. Shapere and Y. Tachikawa, *Central charges of N=2 superconformal field theories in four dimensions*, *JHEP* **0809** (2008) 109, [[arXiv:0804.1957](#)].
- [78] S. El-Showk and M. F. Paulos, *Bootstrapping Conformal Field Theories with the Extremal Functional Method*, *Phys.Rev.Lett.* **111** (2013), no. 24 241601, [[arXiv:1211.2810](#)].
- [79] C. Beem, L. Rastelli, and B. C. van Rees, *\mathcal{W} symmetry in six dimensions*, *JHEP* **1505** (2015) 017, [[arXiv:1404.1079](#)].
- [80] M. Buican, T. Nishinaka, and C. Papageorgakis, *Constraints on Chiral Operators in N=2 SCFTs*, [arXiv:1407.2835](#).