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by

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Abstract:

In this paper I present a complete derivation of the Selberg supertrace formula for super Riemann surfaces and a discussion of the analytic properties of the Selberg super zeta-functions.
The Selberg supertrace formula is based on Laplace-Dirac operators $\square_m$ of weight m on super Riemann surfaces. The trace formula for all $m \in \mathbf{Z}$ is derived and it is shown that one must discriminate between even and odd m. Particularly the term in the trace formula proportional to the identity transformation is sensitive to this discrimination.
Furthermore the analytic properties of the two Selberg super zeta-functions are discussed in detail; first with, and the second without consideration of the spin structure. As it is shown the Selberg super zeta-functions have a similar zero structure as the ordinary Selberg zeta-function. Also, functional equations for the two Selberg super zeta-functions are derived.
I apply my results to discuss the spectrum of the Laplace-Dirac operators and to calculate their determinants. For the spectrum I find that the nontrivial Eigenvalues are the same for $\square_m$ and $\square_0$ up to a constant depending on m, which is analogous to the bosonic case.
The analytical properties of the determinants can be deduced from the analytical properties of the Selberg super zeta-functions, and it is shown that they are well-defined. Special cases ($m = 0, 2$) for the determinants are important in the Polyakov approach for the fermionic string. With these results it is deduced that the fermionic string integrand of the Polyakov functional integral is well-defined.
A short introduction into the theories of supermanifolds, super Riemann surfaces and the Poincaré super upper half-plane is included.

Contents

I. Introduction

TOE - Theory Of Everything!
String Theory Includes All Interactions Including Gravitation!
A String Theory Needs Just Two Constants!
There Is An End In Sight For Theoretical Physics!
These are a small selection of the most euphoric claims people have stated for the string theory in the past years (Hawking [48] or Veneziano [90]). But alas! (or fortunately), things are not so easy.

String theory started orginally as a theory, called dual model, for the strong interaction (for reviews see: Allesandrini et al. [1], Schwarz [79], Veneziano [89], Rebbi [74], Mandelstam [56] and Scherk [77], compiled by Jacob [52] and Frampton [28]). The idea was based on the observation that many particles lay empirically on straight lines if plotted as $mass^2$ over *angular momentum* - the Regge trajectories. This empirical feature was fitted into a theoretical model first by Veneziano, who introduced the four-point amplitude

$$A(s,t) = \frac{\Gamma[-\alpha(s)]\Gamma[-\alpha(t)]}{\Gamma[-\alpha(s)-\alpha(t)]} \tag{1}$$

($\alpha(s) = 1 + \alpha' s$ is the Regge-trajectory, α' the "Regge slope", s, t, u Mandelstam variables) and further on by Fubini and Veneziano, who invented characteristic oscillators explaining the Regge behaviour. The name "dual-model" stems from the fact that the four-point amplitude for a scattering process in this model remained independent, whether viewed in the s, t or u-channel. Surprisingly, this infinite set of oscillators could be explained by creation and annihilation operators coming directly from a one-dimensional object, called a string, sweeping out a two dimensional surface, called the world sheet, in space-time. The action (Nambu-Goto action) of this object was simply given by its surface, parametrized by a space-like variable σ and a time-like variable τ:

$$S = \frac{1}{2\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \sigma}\cdot\frac{\partial X_\mu}{\partial \tau}\right)^2 - \left(\frac{\partial X^\mu}{\partial \sigma}\right)^2\left(\frac{\partial X^\mu}{\partial \tau}\right)^2} \tag{2}$$

(X^μ embedding in space-time). The typical scale of the string was believed to be the scale of strong interactions: $1/length \simeq 1GeV$. The spectrum which emerged from this Veneziano model revealed a degeneracy at each mass level, asymptotically increasing exponentially with the mass. An important problem was that this spectrum seemed to contain both positive and negative norm states (ghosts), a natural consequence of Lorentz invariance. Thus it was a very strong confirmation of this model that it could be shown in the "No-Ghost-Theorem" that these ghost-states decouple; however, with the strong restriction that the quantized theory only makes sense in 26 dimensions![1] A more serious problem remained which was even worse: the appearence of tachyonic states.
Fomulated, e.g. by the action

$$S = \frac{1}{4\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \sqrt{g} g^{nm} \frac{\partial X^\mu}{\partial \sigma^n}\frac{\partial X_\mu}{\partial \sigma^m}, \tag{3}$$

[1] Alternatively, it was shown in the light-cone gauge formulation, where only the physical states remain, that the Lorentz algebra closes only in 26 dimensions.

($\sqrt{g} = \det(g_{\alpha\beta})$, $\sigma_1 = \tau$ and $\sigma_2 = \sigma$ denoting the coordinates on the world sheet) the (bosonic) string theory reveals a lot of symmetry. The action of Eq.(3) is invariant under arbitrary reparametrisations of the σ_1/σ_2-surface (diffeomorphisms), under rescalings of the metric (Weyl-transformations) and, naturally, under Lorentz-transformations.
To include also strings with fermionic degrees of freedom Ramond [72] and Neveu and Schwarz [65] introduced the spinning string which possesses even more symmetry if analysed correctly, i.e super Weyl transformations and (rigid or local) supersymmetry (see below). However, the principle problems of the pure bosonic case remain: these were the critical dimension, which was given for the fermionic string by $D = 10$, and the occurrance of tachyons. So the interest in string theory was in general fading away.

Only a few people remained in this subject. In 1974 it was shown by Scherk and Schwarz [78] that a specific interaction vertex encloses a state which possesses all the ingredients for being a graviton. This opened a quite exciting perspective, namely the view of string theory not as a theory of strong interaction but as a theory which includes gravitation. This also meant that the typical length-scale of a string theory is not of the order of the range of strong interactions (order of the Fermi length $\simeq 10^{-13}cm$) but of the order of the Planck scale $l_{Pl} \simeq 10^{-35}cm$ and, respectively the Planck mass $m_{Pl} \simeq 10^{19}GeV$. New interest emerged in view of the then developed super symmetry theories and it was quickly shown (Gliozzi, Scherk and Olive [32], GSO-projection) that a specific trunctation of the fermionic string includes a consistent supersymmetric sprectrum. The final breakthrough took place when Green and Schwarz [35] showed that this theory could be formulated in its own terms as a superstring theory and that this theory was anomaly free [36] if the gauge group is $SO(32)$ (there are two different types of superstrings, called type I and type II, respectively); this feature is also true for the heterotic string as developed by Gross, Harvey, Martinec and Rohm [44] (with $SO(32)$ and $E_8 \otimes E_8$ as anomaly free gauge groups). Reviews of the superstring theory are due to Green [34] and Schwarz [80].

Naturally, equivalent but different approaches in string theory were developed:[1]

1) The operator approach: This is the classical canonical approach to string theory and includes a great amount of literature about it (compiled in Green, Schwarz and Witten [37]). Further developments are due to Mandelstam [57] who calculated trees and loops in a functional integral approach in the light-cone gauge (not to be confused with the functional integral approach due to Polyakov). In the light-cone gauge only the physical states with positive norm are explicit. However, in this case one must prove the closure of the Lorentz algebra (which gives the critical dimensions $d = 26$ and $d = 10$ for the bosonic and fermionic string, respectively). In the light-cone gauge the scattering of strings was described by joining and splitting of strings at the endpoints. Much work has been done to get rid of the light-cone formulation which masks any general underlying gauge principle. This lacking gauge principle is one of **the** problems of all string theories. String theories possess much symmetry, but where does this symmetry come from? The need to compactify the supernumerary dimensions leads - contrary to the hope that **the** string theory could be unique - to billions upon billions of different theories ($\simeq 10^{1000}$!). Recently there has been some

[1] The idea of two and more dimensional objects like membranes is practically ruled out by the recent paper of de Wit, Lüscher and Nicolai [20]. They showed that the supermembrane has a continuous mass spectrum and no mass gap.

hope for a classification by means of a fusion algebra developed by E.Verlinde [88]. The aim of a formulation of a complete gauge invariant action for the interacting string could hopefully lead to a deeper understanding of how and why string theory works, and ultimately, what the underlying principle actually is (see e.g. West [93]).

2) The functional (or path integral) approach by Polyakov (see below).

3) BRST quantization by Siegel and Zwiebach [83]. The BRST formalism is a more general procedure for quantization of gauge theories than the Faddeev-Popov approach; not only are more general gauges allowed, but in addition the same BRST transformation which determines the action gives the condition for unitary, as well as determining the gauge-invariant part of the action and the physical states. This is a very appealing feature in view of the difficulties to project from the naive string picture onto the physical states where a "No-Ghost-Theorem" or the Lorentz covariance in the light-cone gauge must be proved. The BRST-invariant action can be written, e.g. simply as [83]:

$$S=\int L=\int \Phi O Q \Phi, \tag{4}$$

where O denotes a kinetic operator, Q the BRST-operator and Φ the string field. Here Φ includes, of course, so called ghosts. The BRST-operator has the property $Q^2=0$, which is an expression for the fact that the BRST-operator determines the physical states, by leaving the Lagrangian L invariant under BRST transformations.

4) There is still another description developed by Witten [94]. This approach removed also the light-cone formalism. In comparison to the interpretation of the interaction of strings in the light cone-gauge one has to perform a rearrangement of strings of equal and fixed lengths, where the rearrangement centers at the midpoint (for a description see e.g. Jevicki [53]). For three strings: Half of string 1 goes over into 3, while the other half overlaps with the first half of string 2 (in the light-cone gauge string 1 and 2 go over into 3). Witten suggested that this feature can be represented by the string field theory Lagrangian

$$S=\int\left(\psi * Q\psi+\frac{2}{3}\psi*\psi*\psi\right), \tag{5}$$

where the three-string-overlap corresponds to the three $*$-term (a "wedge-product") and Q is the appropriate BRST operator.

There is also the idea of "p-adic" strings developed by Freund et al.[29] and Volovich [92]. This approach is based on the conjecture that at the order of the Planck-length nonlocal properties of strings could be described by p-adic numbers (another completion of the rational numbers **Q** based on prime numbers). However, I do not consider this any further.

Let us concentrate on the functional approach of Polyakov [70]. This Ansatz, very simple in its principle but very difficult in explicit calculation, starts in the following way for the closed bosonic string theory. The string pertubation theory of the g-loop contribution to a scattering amplitude is given by the functional integral over all geometries of a two-dimensional surface of genus g and over the embeddings, respectively quantum fields living on this surface. Consider the action of Eq.(3). Then the partition function is given by the functional integral

$$Z=\int \mathcal{D}g_{nm}\int \mathcal{D}X^{\mu}e^{-S} \tag{6}$$

The two-dimensional conformal field theories corresponding to string theories in the critical dimension ($D=26$ and $D=10$ for the bosonic and fermionic/super-string, respectively) are all free of both local and global anomalies (see, e.g. Friedan [30]). Due to this fact, g-loop amplitudes can be reduced to a finite-dimensional integral over the moduli space (Teichmüllerspace) $\mathcal{M}_g$.[1] More explicitly:

$$Z=\sum_{g=0}^{\infty} Z_g, \tag{7}$$

where

$$\begin{aligned} Z_g&=\int_{\mathcal{M}_g} d(WP)[\det{}'(\Delta_0^{(+)})]^{-13}\det{}'(\Delta_1^{(+)})\\ \Delta_m^{(\pm)}&=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+imy\frac{\partial}{\partial x}+m(m\pm 1). \end{aligned} \tag{8}$$

Here the Laplacian $\Delta_m^{(+)}$ is realised on the Poincaré upper half-plane $\mathcal{H}\equiv\{z=x+iy|x\in\mathbf{R},\,y>0\}$ (endowed with the hyperbolic geometry), $\mathcal{M}_g$ denote the moduli-space (Teichmüller-space) parametrising the variations of all compact surfaces of a fixed genus g and $d(WP)$ the Weil-Peterson measure as the integration measure on $\mathcal{M}_g$. This partition function in the genus g, i.e. the multiloop expansion, has been in detail discussed by D'Hoker and Phong [23], Gilbert [31] and Namazie and Rajjev [64]. Especially the determinants in Eq.(8) can be expressed with the theory of the Selberg trace formula on Riemann surfaces [49,82] by means of the Selberg zeta-function.[2] Before explaining this in some detail I want to discuss several realizations of the hyperbolic plane, of which the Poincaré upper half-plane is just one. The Poincaré upper half-plane is analytically equivalent to three further Riemannian spaces: the pseudosphere Λ^2, the Poincaré disc D and the hyperbolic strip S. I start with the
1) **pseudosphere** Λ^2 which is defined by:

$$\Lambda^2:=\{(y_1,y_2,y_3)\,|-y_1^2+y_2^2+y_3^2=-R^2\} \tag{9}$$

(in the following I set $R=1$). Λ^2 can be visualised as a hyperboloid embedded in a three dimensional Minkowski space. But be careful: Λ^2 has negative Gaussian curvature $K=-1$, as well as $\mathcal{H}$, D and S, i.e. they are everywhere saddle-shaped. A more convenient description for Λ^2 reads in pseudospherical polar coordinates (τ,ϕ)[8,85,91]:

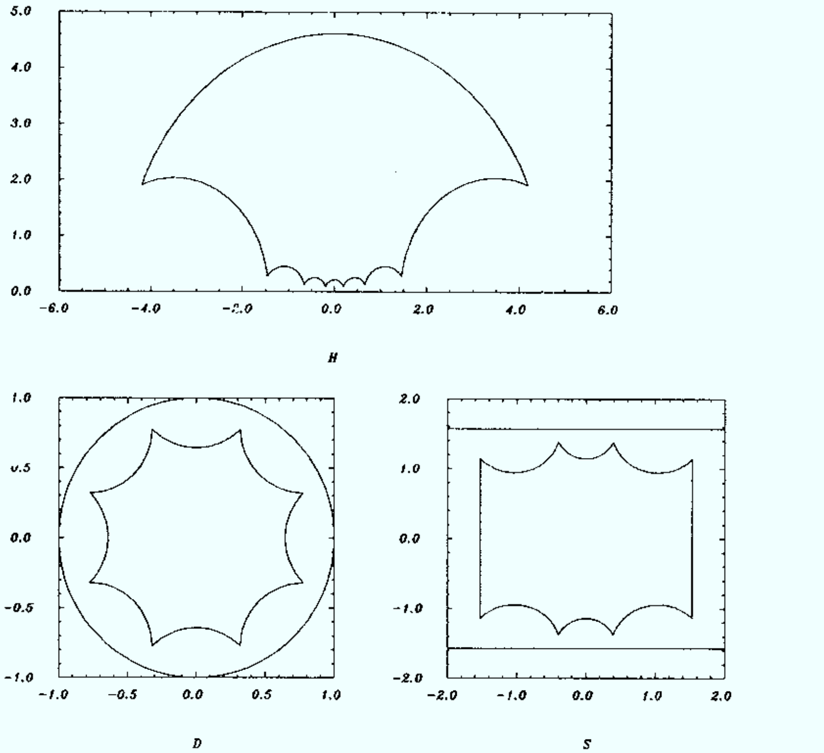
$$y_1=\cosh\tau,\quad y_2=\sinh\tau\cos\phi,\quad y_3=\sinh\tau\sin\phi,\quad(\tau\geq 0,\phi\in[0,2\pi]). \tag{10}$$

The metric g_{ab} associated with the line element $ds^2=g_{ab}dq^a dq^b$ reads $g_{ab}=\mathrm{diag}\,(1,\sinh^2\tau)$.
2) With the stereographic projection of Λ^2 onto the (x_1,x_2)-plane I get the **Poincaré disc** $D=\{z=x_1+ix_2|x_1^2+x_2^2<1\}$:

$$z=x_1+ix_2=re^{i\varphi}=\frac{y_2+iy_3}{1+y_1}=\tanh\frac{\tau}{2}(\sin\phi+i\cos\phi). \tag{11}$$

[1] Belavin and Knizhnik [12] have shown that the string amplitudes are essentially the product of an analytic and an anti-analytic function on $\mathcal{M}_g$. This factorization has a direct correspondence with the decomposition of the string modes into left and right movers.
[2] In the case of the bosonic string this integrand can also be expressed by Theta-functions [2,58]. A super analogue of the Theta-function does not yet exist. But see, however, Manin [58] for a conjecture.

Figure 1: The fundamental domains ($g = 2$) in the three Riemannian spaces $\mathcal{H}$, D, S



Here the metric reads $g_{ab} = [2/(1-r^2)]^2 \mathrm{diag}(1, r^2)$.

3) The Poincaré disc D can be mapped onto the Poincaré upper half-plane $\mathcal{H}$ by the Cayley-transformation:

$$\zeta = x + iy = \frac{-iz + i}{z+1}, \qquad z = \frac{-\zeta + i}{\zeta + i}. \tag{12}$$

The metric reads $g_{ab} = 1/y^2 \cdot \delta_{ab}$.

4) With the help of the transformation

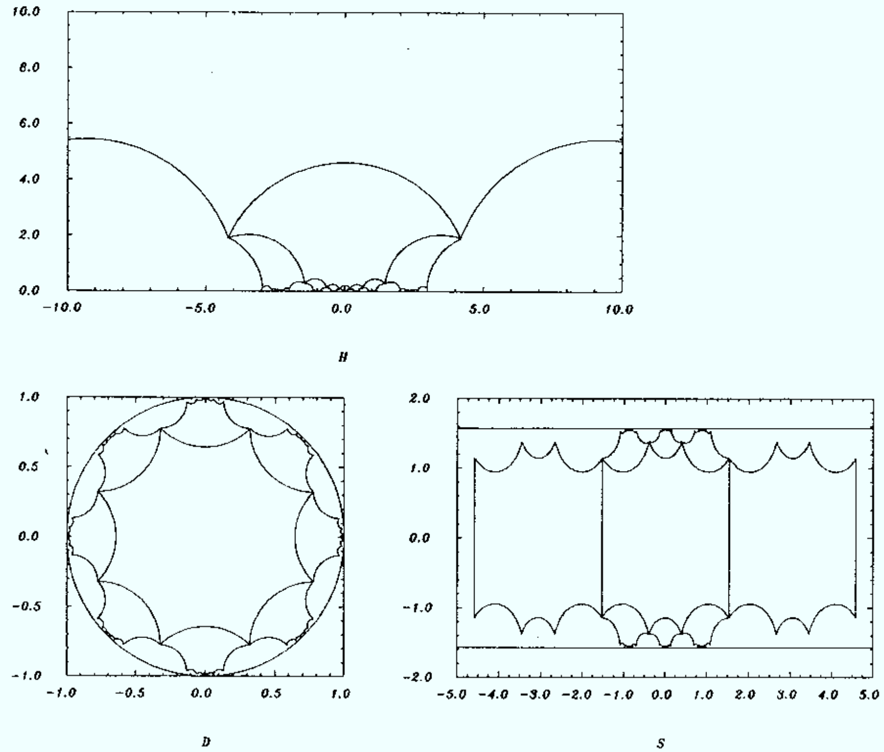
$$\eta = X + iY = -\ln(-i\zeta) \; (= 2\mathrm{artanh} z), \tag{13}$$

one can map the Poincaré upper half-plane (the Poincaré disc) onto the **hyperbolic strip** $S = \{\eta = X + iY | X \in \mathbf{R}, |Y| < \frac{\pi}{2}\}$. The metric reads $g_{ab} = 1/\cos^2 Y \cdot \delta_{ab}$. The hyperbolic distance $r = d(p'', p')$ [p - any of the coordinates (τ, ϕ), (x_1, x_2), (x, y),

(X, Y)] in these spaces is given by:

$$\left.\begin{aligned}
\cosh r &= \cosh\tau''\cosh\tau' - \sinh\tau''\sinh\tau'\cos(\phi''-\phi') && (\text{on } \Lambda^2)\\
&= 1 + \frac{2|z''-z'|^2}{(1-|z'|^2)(1-|z''|^2)} && (\text{on } D)\\
&= \frac{(x''-x')^2 + y''^2 + y'^2}{2y'y''} && (\text{on } \mathcal{H})\\
&= \frac{\cosh(X''-X')}{\cos Y'\cos Y''} - \tan Y'\tan Y'' && (\text{on } S).
\end{aligned}\right\} \tag{14}$$

Figure 2: The fundamental domains of Fig.1 boosted once in all eight directions in the three Riemannian spaces $\mathcal{H}$, D and S



The generators of the symmetrical fundamental domain for a compact Riemann surface of general genus g read in $\mathcal{H}$:

$$\gamma_k = \begin{pmatrix} \cosh\frac{l_0}{2} - \sinh\frac{l_0}{2}\cos\frac{k\pi}{2g} & \sinh\frac{l_0}{2}\sin\frac{k\pi}{2g} \\ \sinh\frac{l_0}{2}\sin\frac{k\pi}{2g} & \cosh\frac{l_0}{2} + \sinh\frac{l_0}{2}\cos\frac{k\pi}{2g} \end{pmatrix}, \qquad (k = 0, \ldots, 2g-1), \tag{15}$$

including the inverse generators $\gamma_k^{-1} = (\gamma_k)^{-1}$ and where $\cosh\frac{l_0}{2} = \cot\frac{\pi}{4g}$. In particular for $g = 2$:

$$\gamma_k = \begin{pmatrix} \cosh\frac{l_0}{2} - \sinh\frac{l_0}{2}\cos\frac{k\pi}{4} & \sinh\frac{l_0}{2}\sin\frac{k\pi}{4} \\ \sinh\frac{l_0}{2}\sin\frac{k\pi}{4} & \cosh\frac{l_0}{2} + \sinh\frac{l_0}{2}\cos\frac{k\pi}{4} \end{pmatrix}, \qquad (k = 0,1,2,3), \quad (16)$$

and $\cosh\frac{l_0}{2} = \cot\frac{\pi}{8} = 1 + \sqrt{2} = 2.41421\ldots$. These generators are also called boosts, because they correspond to explicit Lorentz transformation on the pseudosphere Λ^2. (For their action on elements $\zeta \in \mathcal{H}$ see Eq.(19) below.) They obey the important constraint

$$(\gamma_0\gamma_1^{-1}\cdots\gamma_{2g-2}\gamma_{2g-1}^{-1})(\gamma_0^{-1}\gamma_1\cdots\gamma_{2g-2}^{-1}\gamma_{2g-1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}_2. \qquad (17)$$

In figure 1 I have displayed the fundamental domain for the simplest case of genus $g = 2$ and a symmetrical fundamental domain. The various parts of the figure denote the fundamental domain if mapped onto D, S and $\mathcal{H}$, respectively. Figure 2 displays the action whereby the generators transform the fundamental domain in $\mathcal{H}$, S and D, respectively. Note that in $\mathcal{H}$ the boost in the positive y-direction produces such a large boosted domain that it is not included in the figure (in fact, a ten-times larger box would be needed).

As it is known from the theory of Riemann surfaces and automorphic functions, every compact Riemann surface of genus $g \geq 2$ can be mapped onto the Poincaré upper half-plane $\mathcal{H} = \{z = x + iy | y > 0\}$ (endowed with the hyperbolic geometry) into a polygon with $4g$ edges. This polygon P can be associated with a Fuchsian group Γ which tesselates the Poincaré upper half-plane in an unique way (see e.g. Fenn [26] and the above figures for $g = 2$). A specific polygon P can mapped onto another one P' by an element $\gamma \in \Gamma$ (boosts). The property of this group Γ is that it is strictly hyperbolic (i.e. the trace of its elements is larger than two). On the Poincaré upper half-plane Γ is realized in terms of discrete subgroups of the group $PSL(2, \mathbf{R})$ [on the disc D in terms of discrete subgroups of $SU(1,1)$], i.e. the special linear projections, written in matrix notation as

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (18)$$

with $a, b, c, d \in \mathbf{R}$, $ad - bc = 1$ and $\mathrm{tr}(\gamma) > 2$. Its action on $\mathcal{H}$ reads

$$\gamma\zeta = \zeta' = \frac{a\zeta + b}{c\zeta + d}. \qquad (19)$$

Further I consider closed geodesics in the polygon P which represents a fundamental domain of Γ. I denote these geodesics by l_γ where there is a direct relation between a geodesic l_γ and an element $\gamma \in \Gamma$ by $2\cosh\frac{l_\gamma}{2} = |a + d|$. For the length spectrum of P one has Huber's law $\#(x) = e^x/x$ $(x \to \infty)$, where $\#(x)$ is the number of inconjugate primitive γ's with $l_\gamma \leq x$. Here, elements $\gamma \in \Gamma$ which are not powers (greater or equal to 2) of any element in Γ are called primitive elements of Γ.

Consider the Laplace operator $\Delta_0^{(+)} = -\Delta = -y^2(\partial_x^2 + \partial_y^2)$ and a function $h(p)$ which satisfies the following conditions

i) $h(p)$ is an even function in p,
ii) $h(p) \propto (1 + |p|^2)^{-1+\epsilon}$ $(p \to \pm\infty, \epsilon > 0)$, $\epsilon > 0$.
iii) $h(p)$ is holomorph in the strip $|\mathrm{Im}(p)| \leq \frac{1}{2} + \epsilon$, $\epsilon > 0$.
Denote by $g(u)$ the Fourier transformed of $h(p)$:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(p)e^{-iup}dp. \qquad (20)$$

Now consider this Laplacian (which is invariant under the action of Γ) in a fundamental domain of Γ and the Eigenvalue problem

$$-\Delta\Psi(z) = E\Psi(z) \qquad (21)$$

with periodic boundary conditions $\Psi(\gamma z) = \Psi(z)$, $(\gamma \in \Gamma)$. Parametrize the Eigenvalues E_n by $E_n = s_n(1 - s_n)$ with $s_n = \frac{1}{2} + ip_n$. Then the Selberg trace formula reads

$$\sum_{n=0}^{\infty} h(p_n) = 2(g-1)\int_{-\infty}^{\infty} p\tanh\pi p\, h(p)dp + \sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{\sinh\frac{kl_\gamma}{2}}g(kl_\gamma), \qquad (22)$$

expressing an exact relation between the (quantum mechanical) trace of the operator valued function $\tilde{h}(-\Delta)$ with $\tilde{h}(p^2 + \frac{1}{4}) = h(p)$ and the (classical) length spectrum $\{l_\gamma\}$ [49]. On the left hand side of Eq.(22) the sum runs over the pairs p_n, $-p_n$ (corresponding to the same Eigenvalue E_n), $p = 0$ has to be counted twice if $\frac{1}{4}$ happens to be an Eigenvalue. On the right hand side the sum is taken over all primitive conjugacy classes in Γ denoted by $\{\gamma\}_p$. For the case of genus $g = 2$ and the symmetrical fundamental domain of Eq.(16) the lower part of the length spectrum was first calculated by Aurich and Steiner [5] coming in the end up to 200 million lengths [6] (including multiplicities). In a very beautiful paper by Aurich, Sieber and Steiner [7] Eq.(22) was used to calculate the energy levels E_n by an appropriate test function h - and vice versa by first solving the Eigenvalue problem and then calculate the first primitive lengths.

Note: The condition iii) on the test function h sets a growth condition on the Fourier transform g, which ensures the absolute convergence of all sums in question (note Huber's law).

Now introduce the Selberg zeta-function as

$$Z(s) := \prod_{\{\gamma\}_p}\prod_{n=0}^{\infty}\left[1 - e^{-(s+n)l_\gamma}\right], \qquad (\mathrm{Re}(s) > 1). \qquad (23)$$

Huber's law assures the convergence of Eq.(23) if $\mathrm{Re}(s) > 1$. Again the product runs over all primitive inconjugate $\gamma \in \Gamma$. With the help of Eqs.(22) and (23) one can calculate determinants of the Laplacians $\Delta_m^{(+)}$, where e.g. [16,84]:

$$\begin{aligned} \det{}'(-\Delta) &= Z'(1)e^{(g-1)[\ln 2\pi + 4\zeta'(-1) - \frac{1}{2}]}, \\ \det{}'(\Delta_1^{(+)}) &= Z(2)e^{(g-1)[3\ln 2\pi + 4\zeta'(-1) + \frac{9}{2}]} \end{aligned} \qquad (24)$$

(the prime on the determinants denote the omission of zero modes). For the calculation of $\det(\Delta_m^{(+)})$ one has to use the m-generalisation of the trace formula, i.e. the

Selberg trace formula for automorphic forms of weight m [49]. Under the same conditions on the test function h one gets (assuming that no elliptic elements appear in the group Γ) the Selberg trace formula [49]:

$$\sum_{n=0}^{\infty} h(p_n) = 2(1-g)\int_0^\infty \frac{g'(u)}{\sinh\frac{u}{2}} T_m(\cosh\tfrac{u}{2})\,du + \sum_{\{\gamma\}_p}\sum_{k=1}^{\infty} \frac{l_\gamma}{\sinh\frac{kl_\gamma}{2}} g(kl_\gamma), \tag{25}$$

where $T_m(\cosh\frac{u}{2}) = \cosh\frac{m}{2}u$ denotes the m^{th} Chebyshev-polynomial in $\cosh\frac{u}{2}$. For $m=0$ one recovers Eq.(22). Here the Eigenvalues E_n (parametrized by $E_n = \frac{1}{4}+p_n^2$) are the Eigenvalues of the Laplacian[1] $-\Delta_m = -\Delta + imy\partial_x$. The boundary conditions on the Eigenstates Ψ_n have to be appropriatly changed according to $\Psi(\gamma z) = \chi_\gamma{}^m (cz+d)^m [|cz+d|]^{-m}\Psi(z)$ $(\gamma\in\Gamma)$, where χ_γ is a possible additive character.

A procedure similar to that used for the bosonic string can be used for the fermionic string where the relevant action reads [17,19,50]:

$$S(g,X,\chi,\psi) = \frac{1}{4}\int_M d^2\sigma\sqrt{g}\big[\tfrac{1}{2}g^{mn}\partial_m X^\mu\partial_n X^\mu + i\bar\psi^\mu\gamma^m\partial_m\psi_\mu - F^\mu F_\mu - \bar\chi_a\gamma^m\gamma^a\psi^\mu\partial_m X_\mu + \tfrac{1}{8}\bar\psi^\mu\psi_\mu\bar\chi_a\gamma^a\gamma^b\chi_b\big]. \tag{26}$$

Here denote:
1) M: the two-dimensional world sheet,
2) $\mathcal{D}X^\mu$: imbeddings in space-time ($D=10$),
3) $g_{mn} = e_m{}^a e_n{}^b\delta_{ab}$: metric on the world sheet,
4) ψ^μ: real (Majorana-) spinor,
5) χ_a: spin $\frac{3}{2}$-gravitino field,
6) F^μ: nondynamical field which is needed to close the supersymmetric algebra off shell [51,77]:

$$\delta X = i\bar\epsilon\psi, \qquad \delta\Psi = \partial_a X\gamma^a\epsilon + F\epsilon, \qquad \delta F = i\bar\epsilon\gamma^a\partial_a\psi, \tag{27}$$

where ϵ is a two-dimensional spinor. One sets $F=0$, since the equation of motion just reads $F=0$.
7) γ^a $(a=0,1,5)$ denote the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{28}$$

and a bar over quantities denotes complex conjugation. The action (26) is invariant under five fundamental symmetries [24]

i) Reparametrisation invariance,

$$\begin{aligned} \delta\epsilon_m{}^a &= \delta V^n\partial_n\epsilon_m{}^a + \epsilon_n{}^a\partial_n\delta V^n, \\ \delta\chi_m &= \delta V^n\partial_n\chi_m - \chi_n\partial_m\delta V^n, \\ \delta X^\mu &= \delta V^n\partial_n X^\mu, \\ \delta\Psi^\mu &= \delta V^n\partial_n\Psi^\mu. \end{aligned} \tag{29}$$

[1] $-\Delta_m$ must not be confused with $\Delta_m^{(\pm)}$.

ii) Supersymmetry transformations,

$$\begin{aligned} &\delta\epsilon_m{}^a = i\xi\gamma^a\chi_m, \qquad \delta\chi_m = 2D_m\xi, \qquad \delta X^\mu = \xi\Psi^\mu, \\ &\delta\Psi^\mu_\alpha = -\tfrac{1}{2}i(\gamma^n)_\alpha{}^\beta\xi_\beta(\chi_n\Psi^\mu) + i(\gamma^m)_\alpha{}^\beta\xi_\beta\partial_m X^\mu. \end{aligned} \tag{30}$$

Here $D_m = \partial_m - \frac{1}{2}\omega_m\gamma^5$ are covariant derivatives taken with the connection $\omega_m := \epsilon_m{}^a\epsilon^{pq}\partial_p e_q{}^b\delta_{ab} - \frac{i}{2}\chi_m\gamma_5\gamma^n\chi_n$, where ϵ_{ab} is the totally antisymmetric tensor for the raising and lowering of spinor indices.

iii) Weyl transformations,

$$\begin{aligned} \delta\epsilon_m{}^a &= \Lambda\epsilon_m{}^a, \qquad & \delta\Psi^\mu &= -\tfrac{1}{2}\Lambda\Psi^\mu, \\ \delta\chi_m n &= \tfrac{1}{2}\Lambda\chi_m, \qquad & \delta X^\mu &= 0. \end{aligned} \tag{31}$$

iv) super-Weyl transformations

$$\delta\chi_m = \gamma_m\lambda, \qquad \delta(\text{anything else}) = 0. \tag{32}$$

v) Local Lorentz transformations,

$$\begin{aligned} \delta e_m{}^a &= l\epsilon^{ab}e_{mb}, \qquad & \delta\Psi^\mu &= \tfrac{1}{2}l\gamma_5\Psi^\mu, \\ \delta\chi_m &= \tfrac{1}{2}l\gamma_5\chi_m, \qquad & \delta X^\mu &= 0. \end{aligned} \tag{33}$$

(With δV^n an infinitesimal vector field, ξ an infinitesimal spinor, Λ and λ an infinitesimal scaling-function and l an infinitesimal Lorentz transformation, respectively.)

It is very important that this action can be cast into a compact form if a superspace notion is used [50]. Consider a $(2+2)$-dimensional superspace with coordinates $Z^M := (x^\mu, \theta^m)$ with commuting x^μ and anticommuting θ^m (for more details see [50] and chapter II). Introduce the Vierbein $E_M{}^A$:

$$\begin{aligned} E_\mu{}^\alpha &= \epsilon_\mu{}^\alpha + i\theta^\dagger\gamma^0\gamma^\alpha\chi_\mu \qquad & E_m{}^\alpha &= i(\theta^\dagger\gamma^0\gamma^\alpha)_m \\ E_\mu{}^a &= \tfrac{1}{2}[\chi_\mu{}^a - \omega_\mu(\gamma_5\theta)^a] \qquad & E_m{}^a &= \delta_m{}^a. \end{aligned} \tag{34}$$

Here, $\epsilon_\mu{}^\alpha$ is the x-space Vierbein and $e = \det(\epsilon_\mu{}^\alpha)$. Now define

$$\Phi(Z) = X(x) + i\theta^\dagger\gamma^0\psi(x) + \frac{i}{2}\theta^\dagger\gamma^0\theta F(x). \tag{35}$$

Then the action (26) can be rewritten as

$$\begin{aligned} S &= \frac{1}{4}\int dx^1 dx^2 d\theta_1 d\theta_2\, \epsilon L \\ L &= E_a{}^M\partial_m\Phi(Z)E^{aN}\partial_N\Phi(Z) \end{aligned} \tag{36}$$

The equations of motion read

$$D_\alpha\bar D^\alpha\Phi(Z) = 0. \tag{37}$$

where $D_\alpha = E_\alpha{}^M \partial_m$. Mapping (in the sense that I want to study partition functions) a closed compact world sheet onto a fundamental domain (of a super Fuchsian group) on the Poincaré super upper half-plane $S\mathcal{H}$ I get the Laplace-Dirac operator $\Box_0 = 2YD\bar{D}$ which I have to study ($Y = \mathrm{Im}(z) + \frac{\theta\bar\theta}{2} = y + \theta\bar\theta/2$, $\theta = \theta_1 + i\theta_2$, $\bar\theta = \theta_1 - i\theta_2$). The action S rewritten in terms of $\Box_0$ reads

$$S = -\frac{1}{4}\int dV(Z)\Phi(Z)\Box_0\Phi(Z), \tag{38}$$

where $dV(Z) = dzd\bar{z}d\theta d\bar\theta/2Y$ is the invariant measure on $S\mathcal{H}$. The partition function is calculated as follows:

$$\left.\begin{aligned} Z &= \sum_{g=0}^{\infty} Z_g \\ Z_g &= \int \mathcal{D}g_{ab}\int \mathcal{D}\chi_a \int \mathcal{D}X^\mu \int \mathcal{D}\psi^\mu e^{-S(g,X,\chi,\psi)}. \end{aligned}\right\} \tag{39}$$

Analogous considerations as in the bosonic case yield [9]:

$$Z_g = \int_{S\mathcal{M}_g} d(SWP)[\mathrm{sdet}(-\Box_0^2)]^{-\frac{5}{2}}[\mathrm{sdet}(-\Box_2^2)]^{\frac{1}{2}}, \tag{40}$$

where (in the notation I am using):
1) $S\mathcal{M}_g$: super moduli space,
2) $d(SWP)$: super Weil-Peterson measure,
3) $\Box_m = 2YD\bar{D} - m(i\theta - \bar\theta)\bar{D}$,
4) $D = \partial_\theta + \theta\partial_z$, $\bar{D} = -\partial_{\bar\theta} + \bar\theta\partial_{\bar z}$.
In the fermionic string, the spinors on the Riemann surface are defined with some spinor structure, which can be independently choosen for left- and right movers. In type II superstring theory[1], these spinor structures must be summed over to project (GSO-projection) onto the correct sector of the Neveu-Schwarz-Ramond theory [24]. It turns out that at the one-loop level, for type II superstrings the resulting sum for the partition function vanishes by the use of a famous Jacobi identity on theta-functions ("aequatio identica satis abstrusa"), indicating the presence of 10-dimensional space-time supersymmetry, i.e. the equality at each mass level of the numbers of bosonic and fermionic states (for details see e.g. [24]).
Throughout this paper I work with type II theories in flat space-time having critical dimension $d = 10$.

The contents of the chapters will be as follows:

The second chapter will deal with the theory of Grassmann numbers, superanalysis and super Riemann surfaces. Grassmann numbers or odd numbers are introduced, respectively, in an elementary way; supernumbers as combinations of ordinary numbers and Grassmann numbers are defined; furthermore, super analytic functions as functions of super variables with super coefficients are defined. Differentiation and integration are introduced which are very simple for Grassmann numbers because functions of Grassmann numbers can be at most linear due to their nilpotent property. Supervector spaces, supertraces and superdeterminants are considered. Having

[1] Type I theories contain open strings, whereas type II theories only closed strings.

this machinery it is an easy task to proceed to super manifolds and super Riemann surfaces. The main application will be to consider and derive the explicit form of superconformal automorphisms on super Riemann surfaces, which includes the statement of an super uniformisation theorem.

The third chapter is devoted to the study of physics on the super Poincaré upper half-plane $S\mathcal{H}$, which is a special super Riemann surface and can be tesselated by a generalisation of $SL(2, \mathbf{R})$ into compact super Riemann surfaces whith fixed genus g. Classical and quantum physics will be considered which will give a Lagrangian and Hamiltonian formalism. Furthermore the important Laplacians and Dirac operators on $S\mathcal{H}$ are introduced, which take on an important role in the Polyakov approach to the fermionic string theory. Also I note the path integral approach on $S\mathcal{H}$.

With the fourth chapter starts the main part of my paper. I discuss and derive the Selberg supertrace formula for automorphic forms of weight m on compact super Riemann surfaces, which are visualized as bounded domains on the super Poincaré upper half-plane $S\mathcal{H}$. This trace formula was already derived by Baranov et al. [10], but in their discussion the term corresponding to the unit transformation (except $m = 0$) was missing. This term is explicitly derived and thus their work completed. However, I do not claim to be mathematically rigorous.

The fifth chapter is devoted to the discussion of the two Selberg super zeta-functions and contains entirely new results. Zeta-functions were originally introduced by Selberg [82] in order to study spectra of Laplacians on compact Riemann surfaces of genus g. The super Selberg zeta-functions are similarly defined as the usual Selberg zeta-function. I find similarities but also important differences for Z_0 and Z_1 in comparison with the usual Selberg zeta-function. I derive functional relations for Z_0, Z_1 and a relation linking these two functions.

In the sixth chapter I apply my results to the fermionic string theory. This includes first the discussion of the spectra of the Laplace-Dirac operators, and second the calculation of their determinants. It is shown that the relevant determinants which have to be considered in the Polyakov functional integral exist and are finite. Discussions of the superdeterminants are due to Baranov et al. [10] and Aoki [3]. In Ref.[10] ratios of superdeterminants corresponding to different copies of super Fuchsian groups were considered (due to lack of knowledge of the analytic behaviour of super zeta-functions). In Ref.[3] attempts have been made to express the superdeterminants by the super zeta-functions, where the functional equation for the usual Selberg zeta-function has been used, which is, however, questionable. Furthermore the behaviour of the superdeterminants of the operators $\Box_m$ in the case of degenerate super Riemann surfaces is discussed.

Chapter VII contains a summary and concluding remarks.

In Appendix A the invariance properties of some important quantities is discussed. In Appendix B the validity of the path integral formulation on $S\mathcal{H}$ is shown and in Appendix C some results of Aoki are summarized, who calculated the heat-kernel on $S\mathcal{H}$. Appendix D deals with the result of path integration on the Poincaré upper-half plane $\mathcal{H}$.

1. Survey of Superanalysis

In this chapter I want to give a short survey of superanalysis and supermanifolds to make the material available for the next sections. No new results are presented and I do not claim completeness. Most of this section was acquired in collaboration with Holger Ninnemann and already presented in a seminar talk in the summersemester 1988 at the University of Hamburg [66]. The text is based on DeWitt [22], Rabin and Crane [71] and Rogers [76]. A compiled version of superanalysis can be found in the book of Berezin [14].

Let us start with the essentials.
*Def.*1: Let Λ_N be the **Grassmann-algebra** over $\mathbf{C}$ which is generated by the elements ζ_a $(a = 1, \ldots, N)$. Λ_N forms a 2^N-dimensional vector space with basis

$$1, \zeta^a, \zeta^a\zeta^b, \ldots \qquad (a < b) \tag{1}$$

and the anticommutation relation

$$\zeta^a\zeta^b = -\zeta^b\zeta^a \qquad \forall a, b. \tag{2}$$

Often one considers the formal limit $N \to \infty$ and the infinite dimensional vector space Λ_∞. Following Rogers one should not be concerned about introducing as much of elements ζ^a as one needs: "To those physicists who use supermanifolds, but do not often lie awake at night worrying about the finer points of analysis, the message of this paper is simple - if you need more generators for your Grassmann algebra, help yourself!"
The anticommutation relation (2) gives at once for $a = b$: $(\zeta^a)^2 = 0$. Anticommuting variables were first introduced by Schwinger [81], Martin [59] and Berezin [13]. However, really concluding $\zeta^2 = 0$ is due to the latter two authors.

*Def.*2: The elements of $z \in \Lambda_\infty$ are called **supernumbers** and can be decomposed as

$$\begin{aligned} z &= z_B + z_S, \qquad z_B \in \mathbf{C} \\ z_S &= \sum_{n=1}^{\infty} \frac{1}{n!} c_{a_1,\ldots,a_n} \zeta^{a_n} \ldots \zeta^{a_1} \end{aligned} \tag{3}$$

with $c_{a_1,\ldots,a_n} \in \mathbf{C}$ totaly antisymmetric. z_B and z_S are called the **body** and the **soul** of the supernumber z, respectively.

A supernumber has an **inverse** only iff $z_B \neq 0$. It is given by

$$z^{-1} = z_B^{-1} \sum_{n=0}^{\infty} (-z_B^{-1} z_S)^n. \tag{4}$$

Every supernumber $z \in \Lambda_\infty$ can be decomposed into an **even** and **odd** contribution:

$$\begin{aligned} z &= u + v \\ u &:= u_B + u_S = z_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{a_1,\ldots,a_{2n}} \zeta^{a_{2n}} \ldots \zeta^{a_1} \\ v &:= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} c_{a_1,\ldots,a_{2n+1}} \zeta^{a_{2n+1}} \ldots \zeta^{a_1}. \end{aligned} \tag{5}$$

The number v is pure soul and can not be inverted. Furthermore $v^2 = 0$.

*Def.*3: The even supernumbers u are called **(type-) c-numbers** and commute with all other numbers. The set of all c-numbers is denoted by $\mathbf{C}_c$.
The odd supernumbers are called **(type-) a-numbers** and the set of all a-numbers is denoted by $\mathbf{C}_a$.

Let us pass to superanalytic functions.
*Def.*4: Let $z = u + v$ $(u \in \mathbf{C}_c, v \in \mathbf{C}_a)$, $f : \Lambda_\infty \to \Lambda_\infty$. The function f is called **superanalytic** if:

(i) f is a linear function in the odd argument

$$f(z) = g_1(u) + v g_2(u); \tag{6}$$

(ii) there exist analytical functions $h_k : \mathbf{C} \to \Lambda_\infty$ $(k = 1, 2)$, so that the functions $g_k : \mathbf{C}_c \to \Lambda_\infty$ $(k = 1, 2)$ can be written as a power series expansion in u_S

$$g_k(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n h_k}{dx^n}\bigg|_{x=u_B} u_S^n. \tag{7}$$

The condition (i) is a natural generalisation of the definition of an "ordinary" analytic function. The power series expansion in the odd arguments can be at most linear and a superanalytic function $f : \mathbf{C}_a \to \Lambda_\infty$ is superanalytic on whole $\mathbf{C}_a$.

*Def.*5: **Differentiation** with respect to the c-coordinate is the same as for ordinary functions over $\mathbf{C}$. **Differentiation** with respect to the a-coordinates has to be done by commuting or anticommuting the a-variable to the differential operator. Example (differentiation from the left):

$$\frac{d}{dv}(a + vb) = b = b_c + b_a,$$

but

$$\frac{d}{dv}(a + bv) = \frac{d}{dv}[a + v(b_c - b_a)] = b_c - b_a \tag{8}$$

$(a, b \in \Lambda_\infty,\ b_c \in \mathbf{C}_c,\ b_a \in \mathbf{C}_a)$.
*Def.*6: **Integration over $\mathbf{R}_a$** is defined by

$$\int_{\mathbf{R}_a} dx = 0, \qquad \int_{\mathbf{R}_a} dx\, x = 1 = -\int_{\mathbf{R}_a} x\, dx. \tag{9}$$

Integration and differentiation with respect to Grassmann variables is a formal procedure and was introduced by Berezin [13]. "dx" does not describe any infinitesimal. In the following the index $\mathbf{R}_a$ in the Grassmann integration will be omitted.
A comparison of Eqs.(8) and (9) yields that differentiation and integration with respect to Grassmann variables are identical. Let $f(x) = a + xb$ with $a, b \in \Lambda_\infty$, $x \in \mathbf{R}_a$. Then

$$\begin{aligned} \frac{d}{dx} f(x) &= b \\ \int dx\, f(x) &= b. \end{aligned} \tag{10}$$

This curious result shows again that differentiation and integration are only formal procedures.

*Def.*7: Let f as before in Eq.(10). The δ-**function** on $\mathbf{R}_a$ is defined by

$$\delta(x) = x, \qquad x \in \mathbf{R}_a$$
$$\int dx\, \delta(x) f(x) = \int dx\, x \cdot a = a = f(0). \tag{11}$$

*Def.*8: **Complex conjugation** is given by

$$\overline{(z+z')} = \bar{z} + \bar{z}', \qquad \overline{(zz')} = \bar{z}'\bar{z}, \qquad \forall z, z' \in \Lambda_\infty. \tag{12}$$

*Def.*9: A supernumber z is called

$$\textbf{real} : \Leftrightarrow \bar{z} = z$$
$$\textbf{imaginary} : \Leftrightarrow \bar{z} = -z. \tag{13}$$

Note:(i) The generators of Λ_∞ are choosen as real Grassmannians

$$\bar{\zeta}^a = \zeta^a, \qquad \forall a. \tag{14}$$

(ii) The product of two real a-numbers is an imaginary c-number:

$$\overline{(v_1 v_2)} = \bar{v}_2 \bar{v}_1 = v_2 v_1 = -v_1 v_2, \qquad \forall v_1, v_2 \in \mathbf{R}_a. \tag{15}$$

Let us introduce the concept of a supervector space.
*Def.*10: The set S of mappings is called a **supervector space** if

(i) S is under the action $+ : S \times S \to S$ an abelian group;

(ii) For all $\alpha \in \Lambda_\infty$ and $\vec{x} \in S$ there exist two mappings
multiplication from the left $\alpha_L : \quad S \to S, \vec{x} \to \alpha_L \vec{x}$,
multiplication from the right $\alpha_R : \quad S \to S, \vec{x} \to \vec{x}\alpha_R$.
Furthermore one has for all $\alpha_L, \beta_L \in \Lambda_\infty, \quad \vec{x}, \vec{y} \in S$:

$$(\alpha_L + \beta_L)\vec{x} = \alpha_L \vec{x} + \beta_L \vec{x}$$
$$\alpha_L(\vec{x} + \vec{y}) = \alpha_L \vec{x} + \alpha_L \vec{y}$$
$$(\alpha_L \beta_L)\vec{x} = \alpha_L(\beta_L \vec{x}) \equiv \alpha_L \beta_L \vec{x} \tag{17}$$
$$1 \cdot \vec{x} = \vec{x}$$

and analogously for the multiplication from the right.
In the following (if not explicetly noted otherwise) multiplication from the left is implicetely assumed and I omit furtheron the index L.

(iii)

$$(\alpha\vec{x})\beta = \alpha(\vec{x}\beta) = \alpha\vec{x}\beta, \qquad \forall \alpha, \beta \in \Lambda_\infty, \vec{x} \in S$$
$$\alpha\vec{x} = \vec{x}\alpha, \qquad \forall \alpha \in \mathbf{C}_c, \vec{x} \in S \tag{18}$$
$$\alpha\vec{u} = \vec{u}\alpha,\ \alpha\vec{v} = -\vec{v}\alpha, \qquad \forall \alpha \in \mathbf{C}, \vec{x} = \vec{u} + \vec{v} \in S,$$

where $\vec{u}$ and $\vec{v}$ are the even and odd components of $\vec{x}$, respectively. In anology to the supernumbers one defines pure supervectors of type c and type a, respectively. For pure supernumbers and supervectors Eq.(18) can be rewritten as

$$\alpha\vec{x} = (-1)^{\alpha\vec{x}}\vec{x}\alpha. \tag{19}$$

The expressions in the exponent of Eq.(19) take on the values 0(1), depending on whether the corresponding expression is of type c(a).

(iv) One has **complex conjugation** with

$$\bar{\bar{\vec{x}}} = \vec{x}$$
$$\overline{(\vec{x} + \vec{y})} = \bar{\vec{x}} + \bar{\vec{y}} \tag{20}$$
$$\overline{\alpha\vec{x}} = \bar{\vec{x}}\,\bar{\alpha}$$

for all $\vec{x}, \vec{y} \in S$ and all $\alpha \in \Lambda_\infty$.

A supervector $\vec{x} \in S$ can be represented with respect to a **basis** $\{{}_i\vec{e}\}$:

$$\vec{x} = x^i{}_i\vec{e} \qquad x^i \in \Lambda_\infty, \tag{21}$$

where the contraction between two indices can be done according to the rule "↗" or "↘", respectively. A supervector space has total **dimension d**, iff it has a basis containing d supervectors. If d is finite, one can always find a basis which contains m real c-type supervectors and n real a-type supervectors. Such a basis is called a pure real basis and the ordered pair (m, n) is called the dimension of the superspace. Both, d and (m, n) are independent from the choice of a basis.

Let $\{{}_i\vec{e}\}$ and $\{{}_j\vec{e}\}$ two real bases. The basis-changing matrix has the block structure

$$({}_iK^j) = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \tag{22}$$

with $A \in \mathrm{Mat}\,(m \times m; \mathbf{C}_c)$, $\quad B \in \mathrm{Mat}\,(n \times n; \mathbf{C}_c)$, $\quad \mathbf{C} \in \mathrm{Mat}\,(m \times n; \mathbf{C}_a)$ and $D \in \mathrm{Mat}\,(n \times m; \mathbf{C}_a)$. Then for $\vec{x} \in S$:

$$x'^j = x^i{}_iK^j. \tag{23}$$

A matrix K with the block structure (22) will be called a standard matrix.
*Def.*11: The **shifting of super vector indices** is described by

$${}^ix := (-1)^{\bar{x}i} x^i. \tag{24}$$

*Def.*12: Let K be a standard matrix. Then

$${}^jK_i^{\sim} := (-1)^{i(i+j)}{}_iK^j \tag{25}$$

is called the **supertranspose of K**.

With these definitions one has the properties ($\vec{x} \in S$)

$$x^i{}_iK^j = {}^jK_i^{\sim}\,{}^ix$$
$$(KL)^{\sim} = L^{\sim}K^{\sim}. \tag{26}$$

Explicitly

$$\begin{pmatrix} A & C \\ B & B \end{pmatrix}^{\sim} = \begin{pmatrix} A^T & -D^T \\ C^T & B^T \end{pmatrix} \tag{27}$$

where T denotes the ordinary transposition of matrices.
*Def.*13: The **shifting of matrix indices** is described by

$$K_i{}^j = (-1)^i{}_i K^j. \tag{28}$$

*Def.*14: The **supertrace** of a standard matrix is defined by

$$\mathrm{str} K := K_i{}^i$$

explicitly:

$$\mathrm{str} K = \mathrm{tr} A - \mathrm{tr} B. \tag{29}$$

One has the properties

$$\begin{aligned} \mathrm{str}\mathbf{1}_{(m,n)} &= m - n \\ \mathrm{str} K^{\sim} &= \mathrm{str} K \\ \mathrm{str}(MN) &= \mathrm{str}(NM). \end{aligned} \tag{30}$$

*Def.*15: The **superdeterminant** of a standard matrix M is defined by integration of the relation

$$\delta(\ln \mathrm{sdet} M) = \mathrm{str}(M^{-1}\delta M) \tag{31}$$

with the boundary condition

$$\mathrm{sdet}\mathbf{1}_{(m,n)} \equiv 1. \tag{32}$$

Explicitly:

$$\mathrm{sdet}\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \frac{\det(A - CB^{-1}D)}{\det B} = \frac{\det A}{\det(B - DA^{-1}C)}. \tag{33}$$

The superdeterminant of a matrix exists only iff the matrices A and B have non-vanishing body.

Without proof I state the inverse of a standard matrix

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{1}_m - A^{-1}CB^{-1}D)^{-1}A^{-1} & -(\mathbf{1}_m - A^{-1}CB^{-1}D)^{-1}A^{-1}CB^{-1} \\ -(\mathbf{1}_n - B^{-1}DA^{-1}C)^{-1}B^{-1}DA^{-1} & (\mathbf{1}_n - B^{-1}DA^{-1}C)^{-1}B^{-1} \end{pmatrix}. \tag{34}$$

In the next section I discuss supermanifolds and superconformal automorphisms.

2. Supermanifolds

To discuss global properties of superspaces I need the notion of supermanifolds. Ordinary manifolds are locally homöomorph to $\mathbf{R}^n$, whereas supermanifolds have locally the structure of $\mathbf{R}_c^m \times \mathbf{R}_a^n$.
*Def.*16: The mapping $\pi : \mathbf{R}_c^m \times \mathbf{R}_a^n \to \mathbf{R}^m$, which maps every coordinate $p \in \mathbf{R}_c^m \times \mathbf{R}_a^n$ onto its body is called a **canonical projection**. The set $\pi^{-1}(\pi(x))$ with $x \in \mathbf{R}_c^m \times \mathbf{R}_a^n$ is called the **soulfiber** of x.
*Def.*17: Let $U \subset \mathbf{R}^m$ an open set. $V \subset \mathbf{R}_c^m \times \mathbf{R}_a^n$ is an **open set** only iff it can be represented by $\pi^{-1}(U)$.

$\mathbf{R}_c^m \times \mathbf{R}_a^n$ with this topology is called a **projective Hausdorff space**. Two points $x, y \in \mathbf{R}_c^m \times \mathbf{R}_a^n$ have only disjunct neighbourhoods iff $x_B \neq y_B$.
*Def.*18: Let M be a set, $U_A \subset M$ an open subset and A an index set. Let Φ_A be a bijective mapping from U_A to an open subset of $\mathbf{R}_c^m \times \mathbf{R}_a^n$. M is called a **supermanifold of dimension (m, n)**, together with a collection of ordered pairs (U_A, Φ_A), if each U_A is a subset of M and its associated Φ_A is one-to-one mapping of U_A onto an open subset in $\mathbf{R}_c^m \times \mathbf{R}_a^n$. The collection of ordered pairs is required to have the following properties:

(i) $\bigcup_A U_A = M$

(ii) $\Phi_A \circ \Phi_B^{-1}$ is differentiable over all nonempty intersections $U_A \cap U_B$

The ordered pair (U_A, Φ_a) is called a chart, or a local coordinate patch or simply a coordinate system.

Let $p \in M$. The set $\Phi^{-1}\{\pi^{-1}[\pi \circ \Phi(p)]\}$ is called the **soulfiber of p**. The soulfibers of all $p \in M$ together with the mapping $\pi \circ \Phi$ form an ordinary manifold M_B with dimension m, the body of M.
Supermanifolds of this kind are called **DeWitt-supermanifolds**. They have a trivial topology in the direction of the soul coordinates, they are fiber bundles over their body M_B. The reason for this property is the fact that an open set in M is always the cartesian product of an open set of $\mathbf{R}^m$ with the entire space of the soul coordinates. If this restriction is omitted one gets Rogers-supermanifolds which allow a more complicated structure in the soul coordinates [76]. From the point of view of physics only the DeWitt-supermanifolds are of interest, see Rabin and Crane [71].

*Def.*19: Let $\mathcal{F}(M)$ the set of all superanalytic functions $f : M \to \Lambda_\infty$. The mapping

$$\vec{X} : \mathcal{F}(M) \to \mathcal{F}(M), \quad f \mapsto \vec{X} f \tag{35}$$

is called a **contravariant** vectorfield on M if the following chain rule is valid:

$$(\vec{X} f)(p) = (\vec{X}\Phi^i)(p)\left[\frac{\partial}{\partial y^i} f(\Phi^{-1}(y_1, \ldots, y_{m+n}))\right]_{y=\Phi(p)} \tag{36}$$

$\forall p \in M$, $f \in \mathcal{F}(M)$, $y \equiv (y_1, \ldots, y_{m+n}) \in \mathbf{R}_c^m \times \mathbf{R}_a^n$, (U, Φ) chart around p.

The set of all contravariant vectorfields is denoted by $\mathcal{X}(M)$.
Since the chain rule is valid one can represent every $\vec{X} \in \mathcal{X}(M)$ by partial differentiation

$$\vec{X} f = X^i \frac{\partial}{\partial y^i} \hat{f}, \qquad \hat{f} \equiv f \circ \Phi^{-1}. \tag{37}$$

$X^i \equiv \vec{X}y^i$ are called the components of $\vec{X}$ in the coordinate system, which is defined by $y^i \equiv \Phi^i$.

*Def.*20: Let $\vec{X} \in \mathcal{X}(M)$ and $p \in M$. Then

$$\vec{X}_p : \mathcal{F}(M) \to \Lambda_\infty, \quad f \mapsto (\vec{X}f)(p) \tag{38}$$

is called the contravariant vector at p. The set of all contravariant vectors is called the **tangent space** T_pM.

The space T_pM is a supervector space of the same dimension (m,n) as the supermanifold M. Let y_i be the coordinate functions of the chart at p, then ${}_i\vec{e} = (\partial/\partial y^i)|_p$ is a coordinate basis of T_pM.
Every complete set of linearly independent elements ${}_a\vec{e} \in T_pM$ can be used as a basis of the tangent space. It can be expressed as

$${}_a\vec{e} = {}_a\vec{e}^{\,i}\left(\frac{\partial}{\partial y^i}\right)_p . \tag{39}$$

Such a basis is denoted as a **Vielbein**.

Let us turn to the study of special supermanifolds of real dimension (2,2) or of complex dimension (1,1). Let the coordinates be $z \in \mathbf{C}_c$, $\theta \in \mathbf{C}_a$. In the fermionic string theory one is interested in **superconformal symmetry**. The notion of superspaces and supermanifolds enables one to represent these symmetry transformations as pure "geometrical" transformations in the coordinates $(z,\theta) \in \mathbf{C}_c \times \mathbf{C}_a$. I consider the transformation ($\epsilon \in \mathbf{C}_a$):

$$\begin{aligned} \tilde{z} &= z + \theta\epsilon \\ \tilde{\theta} &= \theta + \epsilon. \end{aligned} \tag{40}$$

Lagrangians are constructed from fields and their derivatives. Therefore one is lead to use supersymmetric differential opertators. This is nothing but to choose a Vielbein of complex dimension (1,1), which is invariant under the transformation (40). One rewrites Eq.(40) in homogeneous coordinates

$$\begin{pmatrix} \tilde{z} \\ \tilde{\theta} \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -\epsilon & 0 \\ 0 & 1 & \epsilon \\ 0 & 0 & 1 \end{pmatrix}}_{=1_3+\epsilon X = \exp(\epsilon X)} \begin{pmatrix} z \\ \theta \\ 1 \end{pmatrix} \tag{41}$$

and realises the infinitesimal generator

$$X = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{42}$$

as a differential operator

$$\begin{aligned} \mathcal{L}_X : \mathcal{F}(\mathbf{C}_c \times \mathbf{C}_a) \to \mathcal{F}(\mathbf{C}_c \times \mathbf{C}_a), \quad & f(z,\theta) \mapsto \left[\frac{d}{d\epsilon} f(\exp(\epsilon X)(z,\theta)\right]_{\epsilon=0} \\ \Longrightarrow \mathcal{L}_X = -\theta\partial_z + \partial_\theta . & \end{aligned} \tag{43}$$

This operator obviously has odd parity. Invariance of an operator $D \in T_pM$ is now equivalent to the restriction

$$[D, \mathcal{L}_X] = 0, \tag{44}$$

where

$$[A,B] = AB - (-1)^{AB}BA \tag{45}$$

is the supercommutator of two operators A and B. For the operator D one makes the Ansatz

$$D = a(z,\theta)\partial_z + b(z,\theta)\partial_\theta, \qquad a,b \in \mathcal{F}(\mathbf{C}_c \times \mathbf{C}_a). \tag{46}$$

In addition one has the constraints $D^2 \neq 0$ and $D^2 \in T_pM$. For the supercommutator of D and $\mathcal{L}_X$ one gets:

$$[D, \mathcal{L}_X] = (-b - \theta(\partial_z a) + \partial_\theta a)\partial_z + (-\theta(\partial_z b) + \partial_\theta b)\partial_\theta, \tag{47}$$

which yields for D

$$D = b[(a+\theta)\partial_z + \partial_\theta], \qquad a \in \mathbf{C}_a,\, b \in \mathbf{C}_c. \tag{48}$$

We choose the normalization $a = 0$ and $b = 1$, i.e.

$$D = \theta\partial_z + \partial_\theta. \tag{49}$$

Therefore the operator D is something like the square root of ∂_z since $D^2 = \partial_z$.
Let us consider a general superanalytic coordinate transformation

$$\begin{aligned} \tilde{z} &= \tilde{z}(z,\theta) \\ \tilde{\theta} &= \tilde{\theta}(z,\theta). \end{aligned} \tag{50}$$

Then D transforms as follows

$$\begin{aligned} D &= (\partial\tilde{z})\tilde{\partial}_z + (\partial_\theta\tilde{\theta})\tilde{\partial}_\theta + \theta(\partial_z\tilde{z})\tilde{\partial}_z + \theta(\partial_z\tilde{\theta})\tilde{\partial}_\theta \\ &= [(\partial_\theta\tilde{\theta}) - \theta(\partial_z\tilde{\theta})]\tilde{\partial}_\theta + [(\partial_\theta\tilde{z}) + \theta(\partial_z\tilde{z})]\tilde{\partial}_z \\ &= (D\tilde{\theta})(\tilde{D} - \tilde{\theta}\tilde{D}^2) + (D\tilde{z})\tilde{D}^2 \\ &= (D\tilde{\theta})\tilde{D} + (D\tilde{z} - \tilde{\theta}D\tilde{\theta})\tilde{D}^2. \end{aligned} \tag{51}$$

*Def.*21: A superanalytic coordinate transformation is called **superconformal**, iff the (0,1)-dimensional subspace of the tangential space generated by the action of D is invariant under the coordinate transformation, i.e.

$$D = (D\tilde{\theta})\tilde{D}. \tag{52}$$

*Def.*22: A **Super Riemann Surface** is a complex (1,1)-dimensional supermanifold, whose coordinate transformations are superconformal mappings.

Example: The **Super Riemann sphere** $S\mathbf{C}$ is generated by two coordinate systems $Z = (z,\theta)$ and $\tilde{Z} = (\tilde{z},\tilde{\theta})$ which are connected by the superconformal transformation

$$(\tilde{z},\tilde{\theta}) = \left(\frac{1}{z}, \frac{\theta}{z}\right). \tag{53}$$

One introduces homogeneous coordinates and can represent $S\hat{\mathbf{C}}$ as a $(1,1)$-dimensional projective space $(\xi, z_1, z_2) \in \mathbf{C}_a \times \mathbf{C}_c^2 \setminus \{0\} \equiv \mathbf{P}_{(1,1)}(\Lambda_\infty)$. The local coordinate systems read

$$(z,\theta) = \left(\frac{z_1}{z_2}, \frac{\xi}{z_2}\right), \qquad (\tilde{z},\tilde{\theta}) = \left(\frac{z_2}{z_1}, \frac{\xi}{z_1}\right). \tag{54}$$

The group $SPL(2,\mathbf{C})$ of **superconformal automorphisms** of $S\hat{\mathbf{C}}$ is the natural super-generalisation of the Möbius transformations.[1] On $\mathbf{P}_{(1,1)}(\Lambda_\infty)$ these transformations can be realised as linear transformations, which are superconformal in the local coordinates of $S\hat{\mathbf{C}}$. Also one has the constraint that $SPL(2,\mathbf{C})_{Body} = SL(2,\mathbf{C})$. A reasonable Ansatz reads:

$$SPL(1,2;\mathbf{C}_a \times \mathbf{C}_c^2) := \left\{ \gamma = \begin{pmatrix} e & \alpha & \beta \\ \delta & a & b \\ \gamma & c & d \end{pmatrix} : a,b,c,d,e \in \mathbf{C}_c; \right.$$
$$\left. \alpha,\beta,\gamma,\delta \in \mathbf{C}_a;\ ad - bc = 1;\ \mathrm{sdet}\gamma = 1 \right\}. \tag{55}$$

Locally the transformation

$$x' = \gamma x, \qquad x, x' \in \mathbf{P}_{(1,1)}(\Lambda_\infty), \qquad \gamma \in SPL(1,2;\mathbf{C}_a \times \mathbf{C}_c^2) \tag{56}$$

reads as

$$\begin{aligned} z' &= \frac{\delta\theta + az + b}{\gamma\theta + cz + d} \equiv \frac{A}{B} \\ \theta' &= \frac{e\theta + \alpha z + \beta}{\gamma\theta + cz + d} \equiv \frac{\Gamma}{B}. \end{aligned} \tag{57}$$

Superconformal invariance gives the constraint:

$$\begin{aligned} &Dz' = \theta' D\theta' \\ \Rightarrow\ &\frac{(DA)B - A(DB)}{B^2} = \frac{\Gamma}{B}\frac{(D\Gamma)B + \Gamma(DB)}{B^2} \\ &\Rightarrow (DA)B - A(DB) = \Gamma(D\Gamma). \end{aligned} \tag{58}$$

Comparison of the coefficients yields

$$\begin{aligned} e^2 &= 1 + \beta\alpha + 2\gamma\delta, & & e = 1 + (3/2)\beta\alpha, \\ e\alpha + c\delta - a\gamma &= 0, & \Rightarrow\quad & \gamma = d\alpha - c\beta, \\ e\beta + d\delta - b\gamma &= 0; & & \delta = b\alpha - a\beta. \end{aligned} \tag{59}$$

Inserting (59) into (57) gives finally

$$\begin{aligned} z' &= \frac{az + b}{cz + d} + \theta\frac{\alpha z + \beta}{(cz + d)^2}, \\ \theta' &= \frac{\alpha z \beta}{cz + d} + \frac{\theta}{cz + d}\left(1 + \frac{\beta\alpha}{2}\right). \end{aligned} \tag{60}$$

[1] Its generators are the operators L_0, $L_{\frac{1}{2}}$, $L_{-\frac{1}{2}}$, $G_{\frac{1}{2}}$ and $G_{-\frac{1}{2}}$ of the Neveu-Schwarz section of the Virasoro super algebra of the fermionic string. In Ref.[3] $SPL(2,\mathbf{C})$ is denoted as $OSp(2|1,\mathbf{C})$.

I further define the quantities N_γ and χ_γ by

$$\chi_\gamma [N_\gamma^{\frac{1}{2}} + N_\gamma^{-\frac{1}{2}}] = (a + d)(1 - \frac{\alpha\beta}{2}) - \alpha\beta. \tag{61}$$

N_γ is called the **norm** of an hyperbolic $\gamma \in \Gamma$ and χ_γ describes the corresponding spin structure. χ_γ can take on the values ± 1 and has to be chosen as $\chi_\gamma = \mathrm{sign}(a + d)$. N_{γ_0} will denote the norm of a primitive $\gamma_0 \in \Gamma$, where elements $\gamma \in \Gamma$ which are not powers (greater or equal to 2) of any element in Γ are called primitive elements of Γ in analogy to the usual bosonic case. Γ is called a super Fuchsian group, the subgroup $SPL(2,\mathbf{R})$ of $SPL(2,\mathbf{C})$, thus the group of superconformal automorphisms of $S\mathcal{H}$. Its body is the corresponding norm of an element $\gamma_B \in PSL(2,\mathbf{R})$, the group of hyperbolic transformations on the Poincaré upper half-plane. In analogy to the classical bosonic case I denote by $l_\gamma = \ln N_\gamma$ the length of a closed geodesic corresponding to a hyperbolic $\gamma \in \Gamma$. Of course, l_{γ_0} is the length corresponding to a primitive γ_0. Furthermore, a hyperbolic transformation is always conjugate to the transformation

$$z' = N_\gamma z, \qquad \theta' = \chi_\gamma \sqrt{N_\gamma}\theta, \tag{62}$$

or in matrix representation:

$$\text{hyperbolic } \gamma \in \Gamma \text{ conjugate to} \quad \begin{pmatrix} \chi_\gamma & 0 & 0 \\ 0 & N_\gamma^{\frac{1}{2}} & 0 \\ 0 & 0 & N_\gamma^{-\frac{1}{2}} \end{pmatrix}. \tag{63}$$

To normalize γ correctly by $\mathrm{sdet}\gamma = 1$ one has to multiply all matrix-entries of Eq.(55) by $K = 1 - \frac{1}{2}\beta\alpha = 1 + \frac{1}{2}\alpha\beta$. Therefore:

$$\gamma = K \begin{pmatrix} 1 + \frac{3}{2}\beta\alpha & \alpha & \beta \\ b\alpha - a\beta & a & b \\ d\alpha - c\beta & c & d \end{pmatrix} = \begin{pmatrix} 1 + \beta\alpha & \alpha & \beta \\ b\alpha - a\beta & a(1 - \frac{1}{2}\beta\alpha) & b(1 - \frac{1}{2}\beta\alpha) \\ d\alpha - c\beta & c(1 - \frac{1}{2}\beta\alpha) & d(1 - \frac{1}{2}\beta\alpha) \end{pmatrix} \tag{64}$$

and with Eq.(34):

$$\gamma^{-1} = \begin{pmatrix} 1 + \beta\alpha & c\beta - d\alpha & b\alpha - a\beta \\ -\beta & d(1 - \frac{1}{2}\beta\alpha) & -b(1 - \frac{1}{2}\beta\alpha) \\ -\alpha & -c(1 - \frac{1}{2}\beta\alpha) & a(1 - \frac{1}{2}\beta\alpha) \end{pmatrix}. \tag{65}$$

To formulate **super uniformisation** let us first remember the uniformisation theorem for Riemann surfaces (e.g.[15]):

Theorem:
Every compact Riemann surface is conformaly equivalent to M/Γ, where $M = \hat{\mathbf{C}}$ (Riemann sphere), $M = \mathbf{C}$ (for the torus) or $M = \mathcal{H}$ (upper half-plane) where Γ is a discrete, fix-point free subgroup of the conformal automorphisms of M.

Since $\hat{\mathbf{C}}, \mathbf{C}$ and $\mathcal{H}$ are simply connected, and super Riemann surfaces are fiber bundles over their body, there exist generalisations $S\hat{\mathbf{C}}$, $S\mathbf{C}$ and $S\mathcal{H}$. The conformal automorphisms of $\mathbf{C}$ and $\mathcal{H}$ are subgroups of $SL(2,\mathbf{C})$. This is not true in general for the superconformal automorphisms of $S\hat{\mathbf{C}}$ and $S\mathcal{H}$. But for application in physics we

need in general a metric and we can restrict ourselves to "metrizable" super Riemann surfaces. Superconformal automorphisms of a "metrizable" super Riemann surface, which leave the metric invariant, are always subgroups of $SPL(2,\mathbf{C})$.
With the DeWitt definition of open sets, a subgroup $\Gamma \in SPL(2,\mathbf{R})$ acts discrete and without fix-points, iff $\Gamma_{Red} \equiv \Gamma_{Body} \subset SL(2,\mathbf{R})$.

Theorem [71]:
Every "metrizable" super Riemann surface Σ is superconformally equivalent to M/Γ with $M = S\hat{\mathbf{C}}$, $S\mathbf{C}$ or $S\mathcal{H}$ and Γ is a discrete fix-point free subgroup of the superconformal automorphisms on M.
The coefficients in Eq.(55) are specified by $a,b,c,d \in \mathbf{R}_c$ and $\alpha,\beta \in \mathbf{C}_a$, $\bar{\alpha} = i\alpha$, $\bar{\beta} = i\beta$.

As is well known, the fundamental group of a compact Riemann surface of genus g can be defined by $2g$ generators satisfying the relation (I.17). In the super case one has analogously:

$$(\gamma_0\gamma_1^{-1}\cdots\gamma_{2g-2}\gamma_{2g-1}^{-1})(\gamma_0^{-1}\gamma_1\cdots\gamma_{2g-2}^{-1}\gamma_{2g-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{66}$$

The real Teichmüller space $\mathcal{T}$ of a compact Riemann surface with genus g has dimension $d_{\mathcal{T}} = 6g-6$, whereas super Teichmüller space $S\mathcal{T}$ has dimension $d_{S\mathcal{T}} = (6g-6, 4g-4)$.

The simplest generalisation of the Poincaré metric on $\mathcal{H}$ which is $SPL(2,\mathbf{R})$ invariant reads on $S\mathcal{H}$

$$ds^2 = \frac{|dz+\theta d\theta|^2}{(\mathrm{Im}z + \frac{1}{2}\theta\bar{\theta})^2}, \tag{67}$$

but this metric turns out not to be invertible!

1. Classical Motion

To construct the metric on $S\mathcal{H}$ let us consider the Vierbein E^A. The general method for constructing the Vierbein in a curved 2 + 2-dimensional super space was given by Howe [51]. Because a 2 + 2-dimensional super space is conformally flat, if there exists a coordinate system in which the metric is proportional to the flat metric, one starts with the Vierbein $\hat{E}_M{}^A$ in flat superspace

$$\hat{E}_M{}^A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\theta & 0 & 1 & 0 \\ 0 & \bar{\theta} & 0 & 1 \end{pmatrix}, \qquad \hat{E}_A{}^M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \theta & 0 & 1 & 0 \\ 0 & -\bar{\theta} & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\hat{E}_A{}^M = (\hat{E}_M{}^A)^{-1}$ is the inverse Vierbein. This gives for the quantities $\hat{E}^A = dz^M\hat{E}_M{}^A$:

$$\begin{aligned} \hat{E}^z &= dz + \theta d\theta, & \hat{E}^\theta &= d\theta, \\ \hat{E}^{\bar{z}} &= d\bar{z} - \bar{\theta}d\bar{\theta}, & \hat{E}^{\bar{\theta}} &= d\bar{\theta}. \end{aligned} \tag{2}$$

Under a super Weyl transformation the Vierbein $\hat{E}_M{}^A$ changes as

$$\hat{E}_M{}^A \to E_M{}^A = \begin{cases} E_M{}^a = \Lambda(Z)\hat{E}_M{}^a, & (a = z,\bar{z}), \\ E_M{}^\alpha = \Lambda^{\frac{1}{2}}(Z)\hat{E}_M{}^\alpha - i\hat{E}_M^a(\gamma_a)^{\alpha\beta}D_\beta\Lambda^{\frac{1}{2}}(Z), & (\alpha = \theta,\bar{\theta}), \end{cases} \tag{3}$$

where $D_\alpha = E_\alpha{}^M\partial_M$, $\Lambda(Z)$ the scaling function and (γ_a) the γ-matrices which in my notation read[1]

$$(\gamma_z)^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad (\gamma_{\bar{z}})^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \tag{4}$$

Since the Vierbein E^A should be (up to phase factors) invariant under the action of $SPL(2,\mathbf{R})$ the appropriate scaling function reads $\Lambda(Z) = Y^{-1}$, where Y is given by $Y := \mathrm{Im}z + \frac{\theta\bar{\theta}}{2} = y + i\theta_1\theta_2$, if I further set $\theta = \theta_1 + i\theta_2$ and $\bar{\theta} = \theta_1 - i\theta_2$, where θ_1 and θ_2 are real Grassmannians. Note: $\bar{Y} = Y$. The E^A are now given as (see [71,86]):

$$\begin{aligned} E^z &= \frac{dz+\theta d\theta}{Y}, \qquad E^{\bar{z}} = \frac{d\bar{z}+\bar{\theta}d\bar{\theta}}{Y}, \\ E^\theta &= \frac{d\theta}{2Y^{\frac{3}{2}}} + \frac{i\theta-\bar{\theta}}{Y^{\frac{3}{2}}}(dz+\theta d\theta), \\ E^{\bar{\theta}} &= \frac{d\bar{\theta}}{2Y^{\frac{3}{2}}} - \frac{i\theta+\bar{\theta}}{2Y^{\frac{3}{2}}}(d\bar{z} - \bar{\theta}d\bar{\theta}). \end{aligned} \tag{5}$$

In Appendix A the $SPL(2,\mathbf{R})$ invariance of the E^A is shown. The $SPL(2,\mathbf{R})$ invariant line element can now be constructed by [86]:[2]

$$\begin{aligned} ds^2 &= E^zE^{\bar{z}} - 2E^\theta E^{\bar{\theta}} \\ &= \frac{1}{Y^2}\big(|dz - i\bar{\theta}d\theta|^2 - 2Y|d\theta|^2\big) \\ &= \frac{1}{Y^2}\big[d\bar{z}dz - i\bar{\theta}d\bar{z}d\theta - i\theta d\bar{\theta}dz - (2Y + \theta\bar{\theta})d\theta d\bar{\theta}\big]. \end{aligned} \tag{6}$$

[1] Raising and lowering of spin-indices are performed by the totally antisymmetric $\epsilon_{\alpha\beta}$-tensor.
[2] See the remark following Eq.(44) below.

Rewriting $ds^2 = dq^a{}_a g_b dq^b$ one gets the metric tensor on $S\mathcal{H}$

$$({}_a g_b) = \frac{1}{2Y^2}\begin{pmatrix} 0 & 1 & 0 & -i\theta \\ 1 & 0 & -i\bar\theta & 0 \\ 0 & i\bar\theta & 0 & -(2Y+\theta\bar\theta) \\ i\theta & 0 & 2Y+\theta\bar\theta & 0 \end{pmatrix}. \tag{7}$$

The metric tensor is obviously of the form

$$({}_a g_b) = \begin{pmatrix} A & \Gamma \\ -\Gamma^T & B \end{pmatrix}, \qquad A = A^T,\, B = -B^T. \tag{8}$$

Equation (II.33) gives the superdeterminant of $({}_a g_b)$

$$\mathrm{sdet}({}_a g_b) = \frac{\det A}{\det(B + \Gamma^T A^{-1}\Gamma)} = -\frac{1}{4Y^2}. \tag{9}$$

Note that $({}_a g_b)$ is a "super" Kähler metric

$$\left.\begin{aligned} {}_z g_{\bar z} &= \frac{1}{2Y^2} = \partial_z\partial_{\bar z}\ln\frac{1}{Y^2} \\ {}_{\bar\theta} g_z &= \frac{-i\theta}{2Y^2} = \partial_z\partial_{\bar\theta}\ln\frac{1}{Y^2} \\ {}_\theta g_{\bar z} &= \frac{i\bar\theta}{2Y^2} = \partial_{\bar z}\partial_\theta\ln\frac{1}{Y^2} \\ {}_\theta g_{\bar\theta} &= -\frac{2Y+\theta\bar\theta}{2Y^2} = \partial_{\bar\theta}\partial_\theta\ln\frac{1}{Y^2}. \end{aligned}\right\} \tag{10}$$

One constructs the $SPL(2,\mathbf{R})$ invariant volume element on $S\mathcal{H}$ as

$$\begin{aligned} dV(Z) &= \sqrt{|\mathrm{sdet}({}_a g_b)|}\,dz d\bar z d\theta d\bar\theta \\ &= \frac{dz d\bar z d\theta d\bar\theta}{2Y}. \end{aligned} \tag{11}$$

Note the difference in the power of Y to the $PSL(2,\mathbf{R})$ invariant volume element on $\mathcal{H}$: $dV(z) = dxdy/y^2$. In Appendix A it is shown that $dV(Z)$ is indeed invariant under the action of $SPL(2,\mathbf{R})$. The $SPL(2,\mathbf{R})$ invariant Lagrangian is constructed as (following Refs.[86,87]):

$$L = \frac{m}{2}\left(\frac{ds}{dt}\right)^2 = \frac{m}{2Y^2}\left[\dot{\bar z}\dot z - i\bar\theta\dot{\bar z}\dot\theta - i\theta\dot{\bar\theta}\dot z - (2Y+\theta\bar\theta)\dot\theta\dot{\bar\theta}\right]. \tag{12}$$

The Euler-Lagrange equations derived from this Lagrangian read

$$\begin{aligned} \ddot z + i\frac{\dot z^2}{Y} + \frac{\theta\dot z\dot\theta}{Y} &= 0, \qquad \ddot\theta + i\frac{\dot z\dot\theta}{Y} = 0, \\ \ddot{\bar z} + i\frac{\dot{\bar z}^2}{Y} + \frac{\theta\dot{\bar z}\dot{\bar\theta}}{Y} &= 0, \qquad \ddot{\bar\theta} + i\frac{\dot{\bar z}\dot{\bar\theta}}{Y} = 0. \end{aligned} \tag{13}$$

These are geodesic equations

$${}^a\ddot q + {}^a\Gamma_{bc}{}^c\dot q^b\dot q = 0, \tag{14}$$

where

$${}^a\Gamma_{bc} = \frac{1}{2}{}^a g^d\left[(-1)^{(d+1)b}\frac{\partial_d g_c}{\partial q^b} + (-1)^{(d+b+1)c}\frac{\partial_d g_b}{\partial q^c} - (-1)^b\frac{\partial_b g_c}{\partial q^d}\right]. \tag{15}$$

Here $({}^a g^b) = ({}_a g_b)^{-1}$ denotes the inverse of the metric tensor and with the help of Eq.(II.33) it is given by

$$({}^a g^b) = Y\begin{pmatrix} 0 & 2Y-\theta\bar\theta & 0 & i\bar\theta \\ 2Y-\theta\bar\theta & 0 & -i\theta & 0 \\ 0 & -i\theta & 0 & 1 \\ i\bar\theta & 0 & -1 & 0 \end{pmatrix}. \tag{16}$$

Let us assume that the classical solutions of the "pure bosonic" equations

$$\ddot z + i\frac{\dot z^2}{y} = \ddot{\bar z} - i\frac{\dot{\bar z}}{y} = 0 \tag{17}$$

have been found, $[z(t), \bar z(t)] = [z_0(t), \bar z_0(t)]$, then

$$[z(t), \bar z(t), \theta(t), \bar\theta(t)] = [z_0(t), \bar z_0(t), \epsilon_1 z_0(t) + \epsilon_2, \omega(\epsilon_1\bar z_0(t) + \epsilon_2)] \tag{18}$$

are classical solutions of the Euler-Lagrange equations, where ϵ_1 and ϵ_2 are Grassmann constants and ω is a constant phase factor with $|\omega| = 1$. Explicitly

$$\begin{aligned} z(t) &= c_1\tanh\omega(t+t_0) + c_2 + ic_1\frac{1}{\cosh\omega(t+t_0)}, \\ \theta(t) &= \epsilon_1 z(t) + \epsilon_2, \end{aligned} \tag{19}$$

where c_1, c_2 and t_0 are real even numbers and ϵ_1 and ϵ_2 are odd ones (for more details see [87]). It turns out that there exist periodic orbits on bounded domains on $S\mathcal{H}$ and that this motion is chaotic [60]. The corresponding geodesics also obey Huber's law. The **super hyperbolic distance** between two points $q^{(1)}$ and $q^{(2)}$ on $S\mathcal{H}$ is now defined as

$$d(q^{(1)}, q^{(2)}) = \int_{q^{(1)}}^{q^{(2)}} ds = \int_{t_1}^{t_2}\sqrt{\left(\frac{ds}{dt}\right)^2}\,dt = \omega(t_2 - t_1). \tag{20}$$

This can be rewritten as

$$\cosh d(q^{(1)}, q^{(2)}) = 1 + \tfrac{1}{2}R(q^{(1)}, q^{(2)}) - 2r(q^{(1)}, q^{(2)}), \tag{21}$$

where [set $q^{(1)} \equiv Z$, $q^{(2)} \equiv W = (u + iv, \nu_1 + \nu_2)$, $V = v + \nu\bar\nu/2$]:

$$R(Z,W) = \frac{|z - w - \theta\nu|^2}{YV} = \frac{|z - w - \theta\nu|^2}{(y + \theta\bar\theta/2)(v + \nu\bar\nu/2)}, \tag{22}$$

$$r(Z,W) = \frac{2\theta\bar\theta + i(\nu - i\bar\nu)(\theta + i\bar\theta)}{4Y} + \frac{2\nu\bar\nu + i(\theta - i\bar\theta)(\nu + i\bar\nu)}{4V} + \frac{(\nu + i\bar\nu)(\theta + i\bar\theta)\mathrm{Re}(z - w - \theta\nu)}{4YV}. \tag{23}$$

All these two-point quantities enjoy the following property

$$\bullet(\gamma Z, \gamma W) = \bullet(Z,W) = \bullet(W,Z). \tag{24}$$

In Appendix A the $SPL(2,\mathbf{R})$ invariance of these quantities is discussed.

Let us note the parity of the matrixelements and the rules for index shifting for the metric tensor and its inverse

$$\begin{aligned} \mathcal{P}({}_a g_b) &= (-1)^{a+b}, \qquad {}_b g_a = (-1)^{a+b+ab}\,{}_a g_b \\ \mathcal{P}({}^a g^b) &= (-1)^{a+b}, \qquad {}^b g^a = (-1)^{ab}\,{}^a g^b. \end{aligned} \tag{25}$$

Let us turn to the Hamiltonian formulation. We have for the conjugate momenta

$${}_a p = \frac{\partial L}{\partial \dot q^a} = \frac{m}{2}\left[{}_a g_b{}^b\dot q + (-1)^{a+(b+a)b}{}_b g_a{}^b \dot q\right] = m_a g_b{}^b \dot q. \tag{26}$$

Explicitly

$$\begin{aligned} p_z &= \frac{m}{2Y^2}(\dot{\bar z} - i\theta\dot{\bar\theta}) \\ p_{\bar z} &= \frac{m}{2Y^2}(\dot z + i\dot\theta\bar\theta) \\ p_\theta &= \frac{im}{2Y^2}\bar\theta(\dot{\bar z} - i\theta\dot{\bar\theta}) - m\frac{\dot{\bar\theta}}{Y} \\ p_{\bar\theta} &= -\frac{im}{2Y^2}\theta(\dot z + i\dot\theta\bar\theta) + m\frac{\dot\theta}{Y}. \end{aligned} \tag{27}$$

Note the important relations $\overline{p_z} = p_{\bar z}$ and $\overline{p_\theta} = -p_{\bar\theta}$. This yields for the velocities

$$\dot q^a = \frac{1}{m}{}^a g^b{}_b p, \tag{28}$$

and therefore for the Hamiltonian:

$$\begin{aligned} \mathcal{H}(p,q) &= \dot q^a{}_a p - \mathcal{L}(q,\dot q)\big|_{\dot q = \dot q(q,p)} \\ &= \frac{1}{2m}{}^a g^b{}_b p\,{}_a p = \frac{(-1)^a}{2m} p_a{}^a g^b{}_b p = \mathcal{L}(q,\dot q). \end{aligned} \tag{29}$$

2. Quantization

Let us consider the quantum mechanics on $S\mathcal{H}$. Since our Lagrangian is a non-linear one, i.e. the ${}_a g_b$ in Eq.(III.7) are functions of the coordinates, the ordering of operators cannot be ignored. Let us start with general considerations. To define position and momentum operators canonical **super commutation relations** are assumed:

$$[\underline{q}^a, \underline{q}^b] = 0 = [\underline{p}_a, \underline{p}_b], \qquad [\underline{p}_a, \underline{q}^b] = -i\hbar_a\delta^b. \tag{30}$$

In the q-representation $\underline{q}^a$ and $\underline{p}_a$ are given by

$$\underline{q}_a = q_a, \qquad \underline{p}_a = -i\hbar g^{-\frac14}\partial_a g^{-\frac14} = -i\hbar(\partial_a + \tfrac12\Gamma_a), \tag{31}$$

where $g = |\mathrm{sdet}({}_a g_b)|$ and $\Gamma_a = \frac12\partial_a \ln g$ (In the following I omit the bar on the operators and set $\hbar = 1$.). Following Ref.[86] the super Laplace-Beltrani operator can be constructed as

$$\Delta_{SLB} = (-1)^a g^{-\frac14} p_a g^{\frac12}\,{}^a g^b{}_b p\, g^{-\frac14}. \tag{32}$$

This super Laplacian is a straightforward generalisation of the classical bosonic one and is in general the simplest one which is invariant under general point canonical transformations (see [42,67,68,86] for a discussion of the classical bosonic case). The quantum Hamiltonian on a super Riemann manifold is then given by

$$H = -\frac{1}{2m}\Delta_{SLB}. \tag{33}$$

More explicitly (do not worry about the position of the indices)

$$\begin{aligned} H &= -\frac{1}{2m}\left[{}^a g^b\partial_b\partial_a + \frac{(-1)^a}{\sqrt g}(\partial_a\sqrt g\,{}^a g^b)\partial_b\right] \\ &= -\frac{1}{2m}\left[{}^a g^b\partial_b\partial_a + (-1)^a(\Gamma_a{}^a g^b + (\partial_a{}^a g^b))\partial_b\right]. \end{aligned} \tag{34}$$

The ordering prescription (32) is a special one. Another possibility is **super Weyl ordering**

$$H = \frac{1}{8m}\left[{}^a g^b p_b p_a + 2(-1)^a p_a{}^a g^b p_b + (-1)^{a+b} p_b p_a{}^a g^b\right] + \Delta V, \tag{35}$$

where the **quantum correction** ΔV (of order $\hbar^2$) is given by

$$\Delta V = \frac{1}{8m}\left[{}^a g^b\Gamma_b\Gamma_a + (-1)^a(\partial_a{}^a g^b\Gamma_b) + (-1)^{a+b}(\partial_b\partial_a{}^a g^b)\right]. \tag{36}$$

All these formulas are straightforward generalisations of the classical bosonic case [42]. Let us apply these prescriptions to $S\mathcal{H}$. The momenta are given by

$$\begin{aligned} p_z &= \frac1i\left(\frac{\partial}{\partial z} + \frac{i}{4Y}\right), \qquad p_{\bar z} = \frac1i\left(\frac{\partial}{\partial\bar z} - \frac{i}{4Y}\right), \\ p_\theta &= \frac1i\left(\frac{\partial}{\partial\theta} - \frac{\bar\theta}{4Y}\right), \qquad p_{\bar\theta} = \frac1i\left(\frac{\partial}{\partial\bar\theta} - \frac{\theta}{4Y}\right). \end{aligned} \tag{37}$$

Plugging the momenta into Eq.(35) gives

$$H = -\frac{1}{2m}{}^a g^b \partial_b \partial_a = -\frac{Y}{m}\left[(2Y - \theta\bar\theta)\partial_z\partial_{\bar z} + i\bar\theta\partial_z\partial_{\bar\theta} - i\theta\partial_z\partial - \theta\bar\theta\partial_\theta\partial_{\bar\theta}\right]. \tag{38}$$

In the first line in (38) one recognizes the same structure as in the bosonic case, where $H = -\frac{1}{2m}{}^a g^b \partial_b\partial_a = -\frac{1}{2m}y^2(\partial_x^2 + \partial_y^2)$. Since the Hamiltonian represents an observable, it should be, at least in a formal sense, hermitian. Therfore one is lead to the following
Def.: The complex conjugate of the partial derivatives on $\mathbf{C}_c \times \mathbf{C}_a$ are given by

$$(\bar\partial_z) \equiv \partial_{\bar z}, \qquad (\bar\partial_\theta) \equiv -\partial_{\bar\theta}. \tag{39}$$

It is quite surprising that the Hamiltonian can be factorized. Let us define

$$\Box = 2YD\bar D = 2Y(\partial_{\bar\theta}\partial_\theta + \theta\bar\theta\partial_z\partial_{\bar z} - \theta\partial_{\bar\theta}\partial_z - \bar\theta\partial_\theta\partial_z), \quad D = \partial_\theta + \theta\partial_z, \qquad \bar D = -\partial_{\bar\theta} + \bar\theta\partial_{\bar z} \tag{40}$$

[i.e. $D = D_\theta$, $\bar D = D_{\bar\theta}$, in the Vierbein notation of Eq.(1)]. Then we have the important relation:

$$\Delta_{SLB} = \Box^2. \tag{41}$$

Proof:

$$\begin{aligned}\Box^2 &= 4YD\bar D Y D\bar D \\ &= 4YD[(\bar DY)D\bar D + Y\bar D D\bar D] \\ &= 4Y[(D\bar DY)D\bar D - (\bar DY)D^2\bar D + (DY)\bar D D\bar D + YD\bar D D\bar D].\end{aligned} \tag{42}$$

One makes use of the relations

$$\begin{aligned} &D\bar D = -\bar D D, \qquad D^2 = \partial_z, \qquad \bar D^2 = -\partial_{\bar z}, \\ &DY = -\frac{i}{2}(\theta + i\bar\theta), \qquad \bar DY = \frac{1}{2}(\theta + i\bar\theta), \\ &D\bar DY = iD^2Y = i\partial_z Y = \frac{1}{2}\end{aligned} \tag{43}$$

and get

$$\Box^2 = -\frac{Y}{m}\left[(2Y - \theta\bar\theta)\partial_z\partial_{\bar z} + i\bar\theta\partial_z\partial_{\bar\theta} - i\theta\partial_z\partial - \theta\bar\theta\partial_\theta\partial_{\bar\theta}\right], \tag{44}$$

which completes the proof.

■

It is quite interesting to note that this possibility of taking the square roote of Δ_{SLB} crucially depends on the choice of the factor "2" in ds^2 of Eq.(6). Demanding $\Delta_{SLB} = \Box^2$ determines this factor unambiguously.
$\Box$ is a $SPL(2,\mathbf{R})$ invariant operator.
Proof: In Appendix A it is shown that one has for the $SPL(2,\mathbf{R})$ transformation (II.60) the following transformation properties

$$D = \frac{1}{\hat B(Z)}D', \qquad Y = |\hat B(Z)|^2 Y'. \tag{45}$$

Therefore

$$\Box = 2YD\bar D = 2|\hat B(Z)|^2 \frac{1}{\hat B(Z)}D'\frac{1}{\bar{\hat B}(Z)}\bar D' = 2Y'D'\bar D' = \Box'. \tag{46}$$

■

Generally I refer to the operator $\Box$ as the **Laplace-Dirac operator** on $\mathcal{SH}$.
With the invariant volume element on $\mathcal{SH}$, Δ_{SLB} and $\Box$ are hermitean with respect to the scalar product

$$(\Phi_1, \Phi_2) = \int dV(Z)\Phi_1\bar\Phi_2. \tag{47}$$

The operator $\Box$ is the zero-case of the more general operator $\Box_m$ which is defined by[1]

$$\Box_m = 2YD\bar D + m(i\theta - \bar\theta)\bar D. \tag{48}$$

This is the important operator for the fermionic string (see next chapter). In Ref.[10] also the operator $\hat\Box_m$ is introduced which is constructed by a linear isomorphism

$$\Box_m = Y^{\frac{m}{2}}(\hat\Box_m + \tfrac{m}{2})Y^{-\frac{m}{2}}. \tag{49}$$

Hence we have an unitary equivalence of $\Box_m$ and $\hat\Box_m + \frac{m}{2}$. Explicitly $\hat\Box_m$ reads:

$$\begin{aligned}\hat\Box_m &= 2YD\bar D + \frac{m}{2}(i\theta - \bar\theta)(\bar D + iD) \\ &= 2YD\bar D - i\frac{m}{2}(\theta_1 + \theta_2)(\partial_{\theta_1} - \partial_{\theta_2}) - i\frac{m}{2}\theta\bar\theta\partial_x.\end{aligned} \tag{50}$$

I denote this unitary equivalence by $\Box_m \cong \hat\Box + \frac{m}{2}$. Let us consider an even differentiable superfunction on $\mathcal{SH}$ (A, B even, $\chi, \tilde\chi$ odd)

$$\Phi(z, \bar z, \theta, \bar\theta) = A(z, \bar z) + \frac{1}{\sqrt{y}}\left[\theta\chi(z, \bar z) + \bar\theta\tilde\chi(z, \bar z)\right] + \frac{1}{y}\theta\bar\theta B(z, \bar z). \tag{51}$$

With the notation $-\Delta_m = -4y^2\partial_z\partial_{\bar z} + imy\partial_x = -y^2(\partial_x^2 + \partial_y^2) + imy\partial_x$ one gets the following equivalence relation [10]:

$$\hat\Box_m\Phi = s\Phi \Leftrightarrow \begin{cases} -\Delta_{-m}A = s(1-s)A, & B = \frac{s}{2}A \\ -\Delta_{-m-1}\chi = (\frac{1}{4} - s^2)\chi, & -\Delta_{-m+1}\tilde\chi = (\frac{1}{4} - s^2)\tilde\chi, \\ \tilde\chi(s - \frac{m}{2}) = -2y\partial_{\bar z}\chi - \frac{i}{2}(m-1)\chi, & \end{cases} \tag{52}$$

where s is an even supernumber. Thus, the solution of the Eigenvalue problem is formally the same as the classical bosonic one. However, the periodic boundary conditions [for e.g. $m = 0$: $\psi(\gamma Z) = \psi(z)$, $\gamma \in SPL(2,\mathbf{R})$] must be interpreted in the super language. By taking the body in all quantities, one recovers, of course, the old problem. The equivalence relation legitimates to set $s = \frac{1}{2} + ip$ ($p \in \mathbf{R}$, so called "small" Eigenvalues neglected). This reproduces the positivity of the operator $-\Delta_m$.

[1] I use a slightly different notation as in Baranov et al.[10,11] and Aoki [3]. In Refs.[3,10] a description is given, how such opertors can be constructed in a systematic approach.

An odd superfunction is constructed in taking the quantities A, B, χ and $\tilde{\chi}$ in Eq.(51) odd and even, respectively.
As is easily checked, $\Phi_1 = Y^s$ and $\Phi_2 = (\theta_1 + \theta_2)y^{-s}$ satisfy (52), i.e. Φ_1 and Φ_2 are an even and an odd solution of the Laplace-Dirac operator $\widehat{\Box}_m$, respectively, with Eigenvalue s:[1]

$$\widehat{\Box}_m \Phi_i = s\Phi_i, \qquad (i = 1, 2). \tag{53}$$

Let us note the squares of the Laplace-Dirac operators $\Box_m$ and $\widehat{\Box}_m$. They read

$$\begin{aligned}
\Box_m^2 &= \left[\Box_0 + m(i\theta - \bar\theta)\bar D\right]^2 \\
&= \Box_0^2 + m\Box_0(i\theta - \bar\theta)\bar D + m(i\theta - \bar\theta)\bar D\Box_0 + m^2(i\theta - \bar\theta)\bar D(i\theta - \bar\theta)\bar D \\
&= \Box_0^2 + 2mY(D\bar D + i\partial_{\bar z}) + m^2(i\theta - \bar\theta)\bar D; \qquad (55)\\
\widehat{\Box}_m^2 &= \left[\Box_0 + \frac{m}{2}(i\theta - \bar\theta)(\bar D + iD)\right]^2 \\
&= \Box_0^2 + \frac{m}{2}\Box_0(i\theta - \bar\theta)(\bar D + iD) + \frac{m}{2}(i\theta - \bar\theta)(\bar D + iD)\Box_0 \\
&\qquad + \frac{m^2}{4}(i\theta - \bar\theta)(\bar D + iD)(i\theta - \bar\theta)(\bar D + iD) \\
&= \Box_0^2 + imY\partial_z. \qquad (56)
\end{aligned}$$

Let us finally discuss quantization in the path integral formalism on a (m,n)-dimensional super Riemann manifold. I state just the most important results. A more detailed treatment is not discussed here. Let us consider the Eigenstate $|q>$ of the coordinate operator $\underset{\sim}{q}$ with Eigenvalue q and the properties

$$< q|q' > = [g(q)g(q')]^{-\frac{1}{4}}(q')\delta(q - q'), \qquad \mathbf{1} = \int dq\sqrt{g}|q>< q|. \tag{57}$$

Now define the **kernel function** $K(q'', q'; t'', t')$ describing the time evolution

$$\Psi(q''; t'') = \int dq\sqrt{g}K(q'', q'; t'', t')\Psi(q'; t') \tag{58}$$

by the matrix element

$$K(q'', q'; t'', t') = < q''|e^{-i(t''-t')H_{SW}}|q' > . \tag{59}$$

Here the super Weyl ordering prescription in the Hamiltonian (c.f. Eq.(35)) is chosen. I proceed in the usual manner, i.e. subdividing the time interval $T = t'' - t'$ into N-subintervals of equal length $\epsilon = T/N$ and let $t^{(k)} = t' + \epsilon k$, $q^{(k)} = q(t^{(k)})(k = 0, 1, \dots, N)$. First one gets

$$\begin{aligned}
K(q'', q; T) &= < q''|e^{-i(t''-t')H_{SW}}|q' > \\
&= \left(\prod_{j=1}^{N-1}\int\sqrt{g^{(j)}}\,dq^{(j)}\right) \times \prod_{j=1}^{N} < q^{(j)}|\exp\left[-i\epsilon H_{SW}\right]|q^{(j-1)} > . \qquad (60)
\end{aligned}$$

[1] Note that an Eigenfunction of the form $\phi = A + \frac{\theta\bar\theta}{y}B$ which is satisfying the relation (52) (Y^s is a special case of such an Eigenfunction) has under the assumption $s = \frac{1}{2} + ip\,(p \in \mathbf{R})$ with respect to the scalar-product (47) zero-norm. Even worse, in general the scalar-product (47) does not form a Hilbert space inner product in the sense that it is not positive definite - quite puzzeling in view of DeWitt's book [22]. The asymptotic behavior of the heat-kernel (see chapter VI and Appendix C) suggest that one would have to give up either positivity or diagonizability of self-adjoint operators (or both), to have a super reparametrization invariant notion of super Hilbert space [4].

The short-time matrix element can be evaluated $[q_M^{(j)} = \frac{1}{2}(q^{(j)} + q^{(j-1)}), \Delta q^{(j)} = q^{(j)} - q^{(j-1)}, dp = dp_z dp_y dp_\theta dp_{\bar\theta}]$:

$$\begin{aligned}
&< q^{(j)}|e^{-i\epsilon H_{SW}}|q^{(j-1)} > \\
&\qquad \simeq 1 - i\epsilon < q^{(j)}|H_{SW}|q^{(j-1)} > \simeq [g(q')g(q'')]^{-\frac{1}{4}}\int\frac{dp^{(j)}}{(2\pi)^m} \\
&\qquad \times \exp\left\{i\left[p^{(j)}\Delta q^{(j)} - \frac{\epsilon}{2m}\,{}^a g^b(q_M^{(j)})p_b^{(j)}p_a^{(j)} - \epsilon V(q_M^{(j)}) - \epsilon\Delta V(q_M^{(j)})\right]\right\}. \qquad (61)
\end{aligned}$$

Here use has been made of the matrix elements

$$< z|p_z > = \frac{1}{2\pi}\int dp_z e^{ip_z z}, \qquad < \theta|p_\theta > = \frac{1}{i}\int dp_\theta e^{ip_\theta\theta}, \tag{62}$$

where z denotes any of the commuting variables and θ any of the anticommuting variables. Equations (62) imply

$$\begin{aligned}
&\int dp_\theta dp_{\bar\theta} e^{ip_\theta(\theta''-\theta')}e^{ip_{\bar\theta}(\bar\theta''-\bar\theta')} \\
&\quad = \int dp_\theta dp_{\bar\theta}\left[1 + ip_\theta(\theta'' - \theta') + ip_{\bar\theta}(\bar\theta'' - \bar\theta') + p_\theta p_{\bar\theta}(\bar\theta'' - \bar\theta')(\theta'' - \theta')\right] \\
&\qquad + (\bar\theta'' - \bar\theta')(\theta'' - \theta') = \delta(\bar\theta'' - \bar\theta')\delta(\theta'' - \theta'). \qquad (63)
\end{aligned}$$

Now the integration formula for Gaussian integrals in superspace [22, p.46, p.304] is used:

$$\int dq\exp\left[\tfrac{i}{2}q^a{}_aM_b{}^bq - ip_a{}^aq\right] = \frac{(2\pi i)^{\frac{m}{2}}}{|\mathrm{sdet}M|^{\frac{1}{2}}}e^{\frac{i}{2}{}^aM^b{}_bp_ap}. \tag{64}$$

Here it is assume that $({}_aM_b)$ is of the block diagonal form

$$({}_aM_b) = \begin{pmatrix} A & C \\ -C^T & B \end{pmatrix},$$

where A is a matrix with Eigenvalues whose bodies are strictly positive and B a matrix of the block form

$$B = \mathrm{diag}\left[\begin{pmatrix} 0 & i\mu_1 \\ -i\mu_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & i\mu_n \\ -i\mu_n & 0 \end{pmatrix}\right] \tag{65}$$

with $\mu_i > 0\,(i = 1, \dots, n)$. Otherwise additional phases occur in (64) - see [22].[1] Modified for the inverse of $({}_aM_b)$, $({}^aM^b)$ the Gaussian integral reads (and note that I handle the variable q and p as a standard basis):

$$\int dp\exp\left[\tfrac{i}{2}p_a{}^aM^bp_b - iq^a{}_ap\right] = \frac{(2\pi i)^{\frac{m}{2}}}{|\mathrm{sdet}M|^{\frac{1}{2}}}e^{\frac{i}{2}q^a{}_ag_b{}^bq}. \tag{66}$$

[1] Phases due to negative Eigenvalues in the matrix A are not allowed in ordinary quantum mechanics, the underlying body of the discussed theory. Additional phases due to negative μ_i read as follows: For every $\mu_i < 0$ multilpy (64) by the factor i.

Therefore the p-integrations can be carried out. Finally one has to take the limit $N \to \infty$ and the path integral on a (m,n)-dimensional super Riemann manifold reads

$$K(q'',q';T) = [g(q')g(q'')]^{-\frac{1}{4}} \lim_{N\to\infty} \left(\frac{i^{m-2n}}{(2\pi)^m}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dq^{(j)}$$
$$\times \prod_{j=1}^{N} \sqrt{g(q_M^{(j)})} \exp\left\{ i\left[\frac{m}{2\epsilon}\Delta q^{(j),a}{}_a g_b(q_M^{(j)})\Delta q^{(j)} + \epsilon V(q_M^{(j)}) + \Delta V(q_M^{(j)})\right]\right\}. \quad (67)$$

A detailed proof of this representation will not be given here.

In case of $S\mathcal{H}$ one has $\Delta V = 0$ and finds explicitly for the path integral on $S\mathcal{H}$ $[Z = (z,\theta) = (x+iy, \theta_1 + i\theta_2)]$

$$K(Z'',Z';T) = 2\sqrt{Y'Y''} \lim_{N\to\infty} \left(\frac{1}{4\pi i}\right)^N \prod_{j=1}^{N-1} \int dx^{(j)}dy^{(j)}d\theta_1^{(j)}d\theta_2^{(j)} \prod_{j=1}^{N} \frac{1}{Y_M^{(j)}}$$
$$\times \exp\left\{ \frac{im}{2\epsilon Y_M^{(j)^2}} [\Delta z^{(j)}\Delta \bar z^{(j)} - i\bar\theta_M^{(j)}\Delta \bar z^{(j)}\Delta\theta^{(j)} - i\theta_M^{(j)}\Delta z^{(j)}\Delta\bar\theta^{(j)}\right.$$
$$\left. - (2Y_M^{(j)} + \theta_M^{(j)}\bar\theta_M^{(j)})\Delta\theta^{(j)}\Delta\bar\theta^{(j)}]\right\}. \quad (68)$$

In a rather tedious calculation it can be directly shown that this is the correct path integral on $S\mathcal{H}$. See Appendix B for some details.

The path integral (68) has not been calculated explicitly up to now. However, there exists a semiclassical solution by Uehara and Yasui [87] and a calculation of the heat kernel $\tilde K(T)$ by Aoki [3] by solving directly the equation

$$\left(\frac{\partial}{\partial t''} - \frac{1}{2m}\Box_m''^2\right)\tilde K(T) = \delta(q''-q'), \qquad (T = t''-t' > 0) \quad (69)$$

with the boundary condition

$$\tilde K(T) \to \frac{1}{2Y'}\delta(x''-x')\delta(y''-y')\delta(\theta''-\theta')\delta(\bar\theta''-\bar\theta'), \qquad (T\to 0^+). \quad (70)$$

A short summary of Aoki's results is given in Appendix C.

Let us consider the $SPL(2,\mathbf{R})$ transformation as given in Chapter II

$$z' = \frac{\delta\theta + az + b}{\gamma\theta + cz + d} \equiv \frac{A}{B}, \qquad \theta' = \frac{e\theta + \alpha z + \beta}{\gamma\theta + cz + d} \equiv \frac{\Gamma}{B}, \quad (1)$$

where

$$A(Z) = az + b - \theta\delta,\ B(Z) = cz + d - \theta\gamma,\ \Gamma(Z) = \alpha z + \beta + e\theta; \quad (2)$$
$$e = 1 + \frac{3}{2}\beta\alpha,\ \gamma = d\alpha - c\beta,\ \delta = b\alpha - a\beta. \quad (3)$$

The numbers a,b,c,d satisfy the relation $ad - bc = 1$ and are real even supernumbers. The numbers α and β are odd supernumbers with the property $\bar\alpha = i\alpha$, $\bar\beta = i\beta$. I also use the notation $Z' = (z',\theta') = \gamma(z,\theta) = \gamma Z$. A, B and Γ must be multiplied by $K = 1 + \frac{1}{2}\alpha\beta$ to give the correct normalization sdet$T = 1$ (see Eq.(II.64)). I denote these quantities by $\hat A$, $\hat B$ and $\hat\Gamma$, respectively. Let us introduce some important notions:

Def.1: Let $\Gamma \subset SPL(2,\mathbf{R})$ be a **discrete subgroup** and $U \subset S\mathcal{H}$ a **fundamental domain** of Γ which tesselates $S\mathcal{H}$.

Def.2: Let $\gamma \in \Gamma$. A function $f(Z)(Z \in S\mathcal{H})$ is called a **super automorphic function** of weight m iff it is satisfying the relation $f(\gamma Z) = j_\gamma^m(Z)f(Z)$, where j_γ^m is given by

$$j_\gamma^m = F_\gamma^m |F_\gamma|^{-m} = \left(\frac{F_\gamma}{\bar F_\gamma}\right)^{\frac{m}{2}}, \qquad F_\gamma = D\theta' = \frac{1}{\hat B(Z)}, \qquad (z',\theta') = \gamma(z,\theta). \quad (4)$$

The task is to construct the relevant operator for the super trace formula which maps super automorphic functions into super automorphic functions. Let us consider the integral operator L

$$L\phi(Z) = \int_{S\mathcal{H}} dV(W)k_m(Z,W)\phi(W). \quad (5)$$

L is called the **Selberg super integral operator** on $S\mathcal{H}$, where $k_m(Z,W)$ is the integral-kernel of an operator valued function of the operator $\Box_m$.
Now introduce the functions $\Phi(x)$ and $\Psi(x)$ sufficiently decreasing at ∞. Since the Laplace-Dirac operator $\Box$ is a $SPL(2,\mathbf{R})$ invariant operator its integral kernel (and the integral kernel of functions of $\Box$) must depend on $SPL(2,\mathbf{R})$ invariant quantities. Therefore one makes the Ansatz

$$k(Z,W) = k_0(Z,W) = \Phi[R(Z,W)] + r(Z,W)\Psi[R(Z,W)]. \quad (6)$$

The invariants $r(Z,W)$ and $R(Z,W)$ have been defined in Eqs.(III.22). For the heat-kernel of the operator $\Box^2$ the integral kernel k_{heat} can be explicitly calculated [3] and has the form of Eq.(6). Let be $m \in \mathbf{N}_0$. Introduce the quantity J^m:

$$J^m(Z,W) = \left(\frac{z - \bar w + i\theta\bar\nu}{\bar z - w + i\bar\theta\nu}\right)^{\frac{m}{2}}, \qquad (Z,W \in S\mathcal{H}), \quad (7)$$

and study the transformation properties of J^m for $J^m(Z,W) \to J^m(\gamma Z, \gamma W)$. Using Eq.(A.8)

$$\gamma(z - \bar{w} + i\theta\bar{\nu}) = (z - \bar{w} + i\theta\bar{\nu})\frac{1-\alpha\beta}{B(Z)\bar{B}(W)} = \frac{z - \bar{w} + i\theta\bar{\nu}}{\hat{B}(Z)\bar{\hat{B}}(W)}, \tag{8}$$

one gets the transformation rule for J^m:

$$\begin{aligned} J^m(\gamma Z, \gamma W) &= \left(\frac{z' - \bar{w}' + i\theta'\bar{\nu}'}{\bar{z}' - w' + i\bar{\theta}'\nu'}\right)^{\frac{m}{2}} \\ &= \left[\frac{\bar{B}(Z)}{B(Z)}\cdot\frac{z - \bar{w} + i\theta\bar{\nu}}{\bar{z} - w + i\bar{\theta}\nu}\cdot\frac{B(W)}{\bar{B}(W)}\right]^{\frac{m}{2}} = j_\gamma^m(Z)J^m(Z,W)j_\gamma^{-m}(W). \end{aligned} \tag{9}$$

Defining now

$$k_m(Z,W) = J^m(Z,W)k(Z,W) \tag{10}$$

and k_m has the required properties for the integral kernel of the operator L. Thus one formulates the following

Theorem 1: Let f be a super automorphic function with $f \in L^2(S\mathcal{H})$ and scalar product $[f_1, f_2 \in L^2(S\mathcal{H})]$

$$(f_1, f_2) = \int dV(Z) f_1(Z)\bar{f}_2(Z). \tag{11}$$

The Selberg super integal operator is then given by

$$Lf(z) = \int_{S\mathcal{H}} dV(Z)J^m(Z,W)\{\Phi[R(Z,W)] - r(Z,W)\Psi[R(Z,W)]\}f(W) \tag{12}$$

and maps super automorphic functions into super automorphic functions.
Together with the substitution $W = \gamma W'$ one easily shows

$$\begin{aligned} g(\gamma Z) &= \int_{S\mathcal{H}} dV(W)k_m(\gamma Z, W)f(W) = \int_{S\mathcal{H}} dV(W')k_m(\gamma Z, \gamma W')f(\gamma W') \\ &= \int_{S\mathcal{H}} dV(W')j_\gamma^m(Z)k_m(Z,W')j_\gamma^{-m}(W')j_\gamma^m(W')f(W') \\ &= j_\gamma^m(Z)\int_{S\mathcal{H}} dV(W)k_m(Z,W)f(W) = j_\gamma^m(Z)g(Z). \end{aligned} \tag{13}$$

■

Let f be a super automorphic function and $g = Lf$. Then:

$$\begin{aligned} g(Z) &= \int_{S\mathcal{H}} dV(W)k_m(Z,W)f(W) = \sum_{\{\gamma\}_p}\int_{\gamma U} dV(W)k_m(Z,W)f(W) \\ &= \sum_{\{\gamma\}_p}\int_U dV(W)k_m(Z,\gamma W)f(\gamma W) = \sum_{\{\gamma\}_p}\int_U dV(W)k_m(Z,\gamma W)j_\gamma^m(W)f(W) \\ &= \int_U dV(W)K(Z,W)f(W); \end{aligned} \tag{14}$$

where

$$K(Z,W) \equiv \sum_{\{\gamma\}_p} k_m(Z,\gamma W)j_\gamma^m(W) \tag{15}$$

is the **super automorphic kernel**.

Let us consider the supertrace of L. L represents an integral operator of an operator valued function h of the Dirac operator $\square_m$, i.e. $L \cong h(\square_m)$. Let us assume that the Hilbert space $L^2(S\mathcal{H})$ can be mapped onto $l_S^2 :=$ { the space of all sequences $\{a_n\} \subset \Lambda_\infty$ with the property that $\lim_{N\to\infty}\sum_{n=0}^N |a_n|^2$ exists }. This assumption is nothing but a generalization to the super case of the well-known theorem that every Hilbert space is isomorph to l^2. In l_S^2 $\square_m$ can be repesented as an infinite dimensional matrix. Let us denote $(\square_m)$ as this matrix representation. Let us furtheron consider the Eigenvalue problem

$$(\square_m)\Psi = \lambda\Psi, \tag{16}$$

where $\Psi \in l_S^2$ with appropriate boundary conditions. $(\square_m)$ diagonolized yields $(\square_m) \to \Lambda = \mathrm{diag}(\lambda_0^B, \lambda_1^B, \ldots, \lambda_0^F, \lambda_1^F, \ldots)$, where $\lambda_n^{B(F)}$ are the B(ose)- and F(ermi) Eigenvalues of $(\square_m)$ for the even and odd Eigenfunctions of $(\square_m)$, respectively. Clearly

$$\mathrm{str}\,\square_m = \mathrm{str}(\square_m) = \mathrm{str}\Lambda = \sum_{n=0}^\infty (\lambda_n^B - \lambda_n^F); \tag{17}$$

this expression must be in general regularized (e.g. by the zeta-function method). Let us consider further powers of $\square_m$: $\square_m^k$ $(k \in \mathbf{N})$. Then

$$\mathrm{str}\,\square_m^k = \mathrm{str}(\square_m)^k = \mathrm{str}\Lambda^k = \sum_{n=0}^\infty [(\lambda_n^B)^k - (\lambda_n^F)^k]. \tag{18}$$

Since every well-behaved function $h(z)$ can be approximated by powers of z, one finally gets for $\mathrm{str}(L)$ on the one hand,

$$\mathrm{str}(L) = \mathrm{str}[h(\square_m)] = \sum_{n=0}^\infty [h(\lambda_n^b) - h(\lambda_n^F)]; \tag{19}$$

on the other hand, one has for the transformation $W = \gamma Z = (N_\gamma z, \chi_\gamma\sqrt{N_\gamma}\theta)$:

$$j_\gamma^m = \frac{F_\gamma^m}{|F_\gamma|^m} = \left(\frac{D\chi_\gamma\sqrt{N_\gamma}\theta}{|D\chi_\gamma\sqrt{N_\gamma}\theta|}\right)^m = \chi_\gamma{}^m. \tag{20}$$

Thus we get for $\mathrm{str}(L)$:

$$\mathrm{str}(L) = \int_U dV(Z)K(Z,Z) = \sum_{\{\gamma\}_p}\int_U dV(Z)k_m(Z,\gamma Z)j_\gamma^m(Z) \equiv \sum_{\{\gamma\}_p}\chi_\gamma^m A(\gamma), \tag{21}$$

where $A(\gamma)$ is given by

$$\begin{aligned} A(\gamma) &= \int_U dV(Z)k_m(Z,\gamma Z) \\ &= \int_U \frac{dx\,dy\,d\theta\,d\bar{\theta}}{2Y} J^m(Z,\gamma Z)[\Phi(R) - r\Psi(R)] \\ &= \frac{1}{2}\int_1^{N_0} dy\int_{-\infty}^\infty dx \int \frac{d\theta d\bar{\theta}}{y + \frac{\theta\bar{\theta}}{2}} J^m(Z,\gamma Z)[\Phi(R) - r\Psi(R)]. \end{aligned} \tag{22}$$

Immediately the term corresponding to the identity transformation can be stated $[J^m(Z,Z) = (-1)^{\frac{m}{2}} = i^m]$

$$A_0^{(m)} \equiv A(I) = \frac{i^m}{2}\int_1^{N_{\gamma_0}} dy \int_{-\infty}^{\infty} dx \int \frac{d\theta d\bar\theta}{Y}\Phi(0) = i^m \pi(g-1)\Phi(0), \tag{23}$$

since $vol(RS_g) = 4\pi(g-1)$, where RS_g denotes a Riemann surface of genus g.
For the explicit evaluation of (22) we need the following
Theorem 2: Let L be the super Selberg operator and ϕ any Eigenfunction of $\widehat{\Box}_m$ in $\mathcal{SH}$ with $\widehat{\Box}_m\phi = s\phi$. Then

$$\int_{\mathcal{SH}} dV(Z)k_m(W,Z)\phi(Z) = h(s)\phi(W), \tag{24}$$

where the superfunction h depends only on s and the kernel k. The value of $h(s)$ is thus independent of the function ϕ.
Proof: In Ref.[10] the proof of this theorem is given for all function ϕ (even and odd), all $m \in \mathbf{Z}$ and all $W \in \mathcal{SH}$. Since we need theorem 2 only at a specific value, i.e. $W = Z_0 = (i,0)$, I restrict myself to that relatively easy case, which has also the advantage that the case of odd functions drops out. Thus for $W = Z_0 = (i,0)$:

$$\begin{aligned} J^m(i,Z) &= \left(\frac{i-\bar z}{-i-z}\right)^{\frac{m}{2}} = \left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{\frac{m}{2}}, \\ R(i,Z) &= \frac{|i-z|^2}{y+\frac{\theta\bar\theta}{2}} = \frac{x^2+(y-1)^2}{y}\left(1-\frac{\theta\bar\theta}{2y}\right) \equiv R_i\left(1-\frac{\theta\bar\theta}{2y}\right), \\ r(i,Z) &= \frac{\theta\bar\theta}{2y}, \\ Y^{s-1} &= \left(y+\frac{\theta\bar\theta}{2}\right)^{s-1} = y^{s-1} + \frac{s-1}{2}y^{s-2}\theta\bar\theta. \end{aligned} \tag{25}$$

Let ϕ an even superfunction as in Eq.(III.51) without linear in $\theta,\bar\theta$-terms, i.e. ϕ is of the form $\phi = A(1+\frac{s}{2y}\theta\bar\theta)$. Insertion yields:

$$\begin{aligned} L\phi(Z_0) &= \int dV(Z)k_m(Z_0,Z)\phi(Z) \\ &= \frac{1}{2}\int_0^\infty dy \int_{-\infty}^\infty dx \int \frac{d\theta d\bar\theta}{Y}J^m(i,Z)\{\Phi[R(i,Z)] - r(i,Z)\Psi[R(i,Z)]\}\left(1+\frac{s}{2y}\theta\bar\theta\right)A(z,\bar z) \\ &= \frac{1}{2}\int_0^\infty \frac{dy}{y}\int_{-\infty}^\infty dx \left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{\frac{m}{2}} A(x,y) \\ &\qquad \times \int d\theta\bar\theta\left[\Phi(R_i) + \frac{s-1}{2y}\theta\bar\theta\Phi(R_i) - \frac{\theta\bar\theta}{2y}R_i\Phi'(R_i) - \frac{\theta\bar\theta}{2y}\Psi(R_i)\right] \\ &= \frac{1}{4}\int_0^\infty \frac{dy}{y^2}\int_{-\infty}^\infty dx \left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{\frac{m}{2}} A(x,y)\underbrace{[-(s-1)\Phi(R_i) + R_i\Phi'(R_i) + \Psi(R_i)]}_{=\tilde\Phi(R_i)} \\ &= \frac{1}{4}\int_0^\infty \frac{dy}{y^2}\int_{-\infty}^\infty dx \left(\frac{x-i(y+1)}{x+i(y+1)}\right)^{\frac{m}{2}} A(x,y)\tilde\Phi(R_i). \end{aligned} \tag{26}$$

Since $\tilde\Phi$ depends only on $R_0(z,i)$, where $R_0 = \frac{|z-w|^2}{yv}$ is an $SL(2,\mathbf{R})$ invariant quantity, the last equation can be interpreted in terms of the Selberg trace formula for automorphic forms of weight m [49] with integral kernel $\tilde\Phi$. Now, an operator $\tilde L$ on the Poincaré upper half-plane whose kernel depends only on R_0 is in fact a function of the Laplace operator Δ_m. It follows that $\tilde L$ multiplies ϕ by

$$h(s) := \int_0^\infty y^{s-\frac{3}{2}}\tilde Q(y+y^{-1}-2)dy, \tag{27}$$

where $\tilde Q$ is given by

$$\tilde Q(y) = \int_0^\infty \tilde\Phi(x^2+y)dx. \tag{28}$$

This completes the proof.

∎

Let us turn to the calculation of $A(\gamma)$. The invariants r and R are given with the hyperbolic transformation $W = \gamma Z = (N_\gamma z, \chi_\gamma\sqrt{N_\gamma}\theta)$ as:

$$\begin{aligned} R(Z,\gamma Z) &= \frac{(N_\gamma-1)^2(x^2+y^2)}{N_\gamma y^2}\left(1-\frac{\theta\bar\theta}{2}\right), \\ r(Z,\gamma Z) &= (2-\chi_\gamma N_\gamma^{\frac{1}{2}} - \chi_\gamma N_\gamma^{-\frac{1}{2}})\frac{\theta\bar\theta}{2y}. \end{aligned} \tag{29}$$

For the J^m term one gets:

$$\begin{aligned} J^m(Z,\gamma Z) &= \left(\frac{z-N_\gamma\bar z + i\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{\bar z - N_\gamma z + i\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}\right)^{\frac{m}{2}} \\ &= \left[\left(\frac{x(N_\gamma-1)-iy(N_\gamma+1)}{x(N_\gamma-1)+iy(N_\gamma+1)} - \frac{i\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{x(N_\gamma-1)+iy(N_\gamma+1)}\right)\right. \\ &\qquad\qquad \left.\times\left(1-\frac{i\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{x(N_\gamma-1)+iy(N_\gamma+1)}\right)\right]^{\frac{m}{2}} \\ &= \left(\frac{x(N_\gamma-1)-iy(N_\gamma+1)}{x(N_\gamma-1)+iy(N_\gamma+1)}\right)^{\frac{m}{2}}\left(1-im\frac{x(N_\gamma-1)\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{x^2(N_\gamma-1)^2+y^2(N_\gamma+1)^2}\right) \end{aligned}$$

(Substitution $x = y\xi$)

$$= \left(\frac{\xi(N_\gamma-1)-i(N_\gamma+1)}{\xi(N_\gamma-1)+i(N_\gamma+1)}\right)^{\frac{m}{2}}\left(1-\frac{im\xi\chi_\gamma}{y}\frac{\sqrt{N_\gamma}(N_\gamma-1)\theta\bar\theta}{\xi^2(N_\gamma-1)^2+(N_\gamma+1)^2}\right)$$

(Substitution $\xi = \sqrt{\frac{N_\gamma}{N_\gamma-1}}\zeta$)

$$= \left(\frac{\zeta-2i\cosh\frac{\ln N_\gamma}{2}}{\zeta+2i\cosh\frac{\ln N_\gamma}{2}}\right)^{\frac{m}{2}}\left(1-\frac{im\zeta\chi_\gamma}{y}\frac{\theta\bar\theta}{\zeta^2+4\cosh^2\frac{\ln N_\gamma}{2}}\right).$$

Setting $\ln N_\gamma = u$ and finally

$$J^m(Z,\gamma Z) = \left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta+2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}\left(1-\frac{i\theta\bar\theta}{y}\frac{m\zeta\chi_\gamma}{\zeta^2+4\cosh^2\frac{u}{2}}\right) \tag{30}$$

$(\zeta = \sqrt{\frac{N_\gamma - 1}{N_\gamma}}\frac{y}{z},\ u = \ln N_\gamma)$.

We know that Y^s is an Eigenfunction of $\widehat{\Box}_m$ with Eigenvalue $\lambda = s$. Setting $W = Z_0 = (i, 0)$, then $Y_0^s = 1$. Theorem 2 gives therefore the multiplication by the function $h(s)$:

$$h(s) = \frac{1}{2}\int_0^\infty dy \int_{-\infty}^\infty dx \int \frac{d\theta d\bar\theta}{Y} J^m(i,Z)\Big\{\Phi\big[R(i,Z)\big] - r(i,Z)\Psi\big[R(i,Z)\big]\Big\}Y^s$$

$$= \frac{1}{2}\int_0^\infty dy \int_{-\infty}^\infty dx \int d\theta d\bar\theta\Big(y^{s-1} + \frac{s-1}{2}y^{s-2}\theta\bar\theta\Big)\Big(\frac{x - i(y+1)}{x + i(y+1)}\Big)^{\frac{m}{2}}$$
$$\times\Big\{\Phi\Big[\frac{x^2 + (y-1)^2}{y}\Big(1 - \frac{\theta\theta}{2y}\Big)\Big] - \frac{\theta\bar\theta}{2y}\Psi\Big[\frac{x^2+(y-1)^2}{y}\Big(1 - \frac{\theta\theta}{2y}\Big)\Big]\Big\}$$

$$= \frac{1}{4}\int_0^\infty dy \int_{-\infty}^\infty dx \Big(\frac{x - i(y+1)}{x + i(y+1)}\Big)^{\frac{m}{2}} y^{s-2}\Big[-(s-1)\Phi\Big(\frac{x^2+(y-1)^2}{y}\Big)$$
$$+ \frac{x^2+(y-1)^2}{y}\Phi'\Big(\frac{x^2+(y-1)^2}{y}\Big) + \Psi\Big(\frac{x^2+(y-1)^2}{y}\Big)\Big], \quad (31)$$

where the $(m/2)^{nth}$ power is to be a principle value (see [49], p.454). Performing in the y-integral a partial integration for $\mathrm{Re}(s) > 1$:

$$-(s-1)\int_0^\infty dy\, y^{s-2}\Big(\frac{x - i(y+1)}{x + i(y+1)}\Big)^{\frac{m}{2}}\Phi\Big(\frac{x^2+(y-1)^2}{y}\Big)$$
$$= \int_0^\infty dy\, y^{s-1}\Big(\frac{x - i(y+1)}{x + i(y+1)}\Big)^{\frac{m}{2}}$$
$$\times\Big[\Phi'\Big(\frac{x^2+(y-1)^2}{y}\Big)\frac{x^2+(y-1)^2}{y} - \frac{imx}{x^2+(y+1)^2}\Phi\Big(\frac{x^2+(y-1)^2}{y}\Big)\Big]. \quad (32)$$

Therefore for $h(s)$:

$$h(s) = \frac{1}{4}\int_0^\infty dy \int_{-\infty}^\infty dx\, y^{s-2}\Big(\frac{x - i(y+1)}{x + i(y+1)}\Big)^{\frac{m}{2}}$$
$$\times\Big[\Psi\Big(\frac{x^2+(y-1)^2}{y}\Big) + 2(y-1)\Phi'\Big(\frac{x^2+(y-1)^2}{y}\Big) - \frac{imxy}{x^2+(y+1)^2}\Phi\Big(\frac{x^2+(y-1)^2}{y}\Big)\Big]$$

(Substitutions $x = \sqrt{y}\xi$ and $y = e^u$)

$$= \frac{1}{4}\int_{-\infty}^\infty du \int_{-\infty}^\infty d\xi \frac{(\xi - 2i\cosh\frac{u}{2})^m}{(\xi^2 + 4\cosh^2\frac{u}{2})^{\frac{m}{2}}} e^{u(s-\frac{1}{2})}\Big[\Psi(\xi^2 + 4\sinh^2\tfrac{u}{2})$$
$$+ 2(e^u - 1)\Phi'(\xi^2 + 4\sinh^2\tfrac{u}{2}) - \frac{im\xi e^{\frac{u}{2}}}{\xi^2 + 4\cosh^2\frac{u}{2}}\Phi(\xi^2 + 4\sinh^2\tfrac{u}{2})\Big].$$

The ξ-integral is splitted into two parts in order to project out the even ξ-contributions: further the abbreviation $\alpha^m(\xi, u) = (\xi - 2i\cosh\frac{u}{2})^m$ is introduced (note that α^m is an even function in u):

$$h(s) = \frac{1}{4}\int_{-\infty}^\infty du e^{u(s-\frac{1}{2})}\int_0^\infty d\xi$$
$$\times\Big\{\frac{\Psi(\xi^2 + 4\sinh^2\frac{u}{2}) + 2(e^u - 1)\Phi'(\xi^2 + 4\sinh^2\frac{u}{2})}{(\xi^2 + 4\cosh^2\frac{u}{2})^{\frac{m}{2}}}\big[\alpha^m(\xi,u) + \alpha^m(-\xi,u)\big]$$
$$- \frac{im\xi e^{\frac{u}{2}}}{\xi^2 + 4\cosh^2\frac{u}{2}}\frac{\Phi(\xi^2 + 4\sinh^2\frac{u}{2})}{\xi^2 + 4\sinh^2\frac{u}{2}}\big[\alpha^m(\xi,u) - \alpha^m(-\xi,u)\big]\Big\}$$

(Substitutions $x = \xi^2 + 4\sinh^2\frac{u}{2}$)

$$= \frac{1}{8}\int_{-\infty}^\infty du e^{u(s-\frac{1}{2})}\int_{4\sinh^2\frac{u}{2}}^\infty \frac{dx}{(x+4)^{\frac{m}{2}}}$$
$$\times\Big[\frac{\Psi(x) + 2(e^u - 1)\Phi(x)}{\sqrt{x - 4\sinh^2\frac{u}{2}}}\big[\alpha^m(\xi(x),u) + \alpha^m(-\xi(x),u)\big]$$
$$- ime^{\frac{u}{2}}\Phi(x)\frac{\alpha^m(\xi(x),u) - \alpha^m(-\xi(x),u)}{x+4}\Big]. \quad (33)$$

Thus for appropriate h the operator $h(\widehat{\Box}_m)$ equals to an integral operator L of the form (5) whose integral kernel $k_m(Z,W)$ is related to h by the equations

$$h(s) = \int_{-\infty}^\infty du\, e^{u(s-\frac{1}{2})}g(u), \qquad (s = \tfrac{1}{2} + ip),$$
$$g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty dp\, e^{-iup}h(\tfrac{1}{2} + ip),$$
$$g(u) = \frac{1}{8}\int_{4\sinh^2\frac{u}{2}}^\infty \frac{dx}{(x+4)^{\frac{m}{2}}} \quad (34)$$
$$\times\Big[\frac{\Psi(x) + 2(e^u - 1)\Phi(x)}{\sqrt{x - 4\sinh^2\frac{u}{2}}}(\alpha_+^m + \alpha_-^m) - ime^{\frac{u}{2}}\Phi(x)\frac{\alpha_+^m - \alpha_-^m}{x+4}\Big],$$

where the abbreviation $\alpha_\pm^m = \alpha^m[\pm\xi(x), u]$ has been used. Since $\Box_m$ and $\widehat{\Box}_m + \frac{m}{2}$ are unitary equivalent, one can study traces of $\widehat{\Box}_m$ instead of $\Box_m$. But some care is needed. If one goes back to $\Box_m$, which is the relevant operator in the fermionic string, then $Y^{s-\frac{m}{2}}$ is an Eigenfunction of $\Box_m$ with Eigenvalue $\lambda = s$; therefore

$$\widehat{\Box}_m Y^s = sY \Rightarrow (\widehat{\Box}_m + \tfrac{m}{2})Y^s = (s + \tfrac{m}{2})Y^s$$
$$\Rightarrow \Box_m Y^{s-\frac{m}{2}} = (\widehat{\Box}_m + \tfrac{m}{2})Y^{s-\frac{m}{2}} = sY^s. \quad (35)$$

Considering now h as an operator valued function of $\Box_m$ one has $h(\Box_m) \cong h(\widehat{\Box}_m + \frac{m}{2})$. Therefore one has to replace in the calculation of $h(s)$ as a multiplier of the kernel of $h(\Box_m)$ Y^s by $Y^{s-\frac{m}{2}}$. Considering s as an Eigenvalue of $\widehat{\Box}_m$, this yields for the multiplier of the kernel of $h(\Box_m)$

$$h(s + \tfrac{m}{2}) = \int_{-\infty}^\infty du\, e^{u(s-\frac{1}{2})}g(u).$$
$$g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty dp\, e^{-iup}h(ip + \tfrac{m+1}{2}). \quad (36)$$

where $g(u)$ is explicitly given in terms of Φ and Ψ as in Eq.(34). Note that the contributions $\frac{m}{2}$ coming from $h(s+m/2)$ and $Y^{s-\frac{m}{2}}$ cancel. To distinguish between the functions h in Eqs.(34) and (36) I often denote h in Eq.(36) by $h_m(s) \equiv h(s+\frac{m+1}{2})$. Let us consider several combinations of $g(u)$ and $g(-u)$ for later use

$$g(u)+g(-u)$$
$$= \frac{1}{4}\int_{4\sinh^2\frac{u}{2}}^{\infty} \frac{dx}{(x+4)^{\frac{m}{2}}}\left[\frac{\Psi(x)+4\sinh^2\frac{u}{2}\Phi'(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m) - im\cosh\tfrac{u}{2}\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right] \tag{37}$$

$$g(u)e^{-\frac{u}{2}}+g(-u)e^{\frac{u}{2}}$$
$$= \frac{1}{4}\int_{4\sinh^2\frac{u}{2}}^{\infty} \frac{dx}{(x+4)^{\frac{m}{2}}}\left[\frac{\cosh\frac{u}{2}\Psi(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m) - im\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right] \tag{38}$$

$$g(u)-g(-u)$$
$$= \frac{1}{4}\sinh\tfrac{u}{2}\int_{4\sinh^2\frac{u}{2}}^{\infty} \frac{dx}{(x+4)^{\frac{m}{2}}}\left[\frac{4\cosh\frac{u}{2}\Phi'(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m) - im\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right] \tag{39}$$

$$g(u)e^{-\frac{u}{2}}-g(-u)e^{\frac{u}{2}} = \frac{1}{4}\sinh\tfrac{u}{2}\int_{4\sinh^2\frac{u}{2}}^{\infty} \frac{dx}{(x+4)^{\frac{m}{2}}}\frac{4\Phi'(x)-\Psi(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m) \tag{40}$$

I have now the relevant terms to calculate $A(\gamma)$:

$$A(\gamma) = \frac{1}{2}\int_1^{N_{\gamma 0}} dy \int_{-\infty}^{\infty} dx \int \frac{d\theta d\bar\theta}{2Y}$$
$$\times J^m(Z,\gamma Z)\{\Phi[R(Z,\gamma Z)] - r(Z,\gamma Z)\Psi[R(Z,\gamma Z)]\}$$
$$= \frac{1}{2}\int_1^{N_{\gamma 0}} dy \int_{-\infty}^{\infty} dx \int d\theta d\bar\theta \left(\frac{1}{y}-\frac{\theta\bar\theta}{y^2}\right)\left(\frac{x(N_\gamma-1)-iy(N_\gamma+1)}{x(N_\gamma-1)+iy(N_\gamma+1)}\right)^{\frac{m}{2}}$$
$$\times\left(1-im\frac{x(N_\gamma-1)\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{x^2(N_\gamma-1)^2+y^2(N_\gamma+1)^2}\right)$$
$$\times\left[\Phi(R_0)-\frac{\theta\bar\theta}{y}R_0\Phi'(R_0)-2(1-\chi_\gamma\cosh\tfrac{u}{2})\frac{\theta\bar\theta}{2y}\Psi(R_0)\right]$$

Performing the $\theta\bar\theta$-integration and the substitution $x = y\xi$

$$= \frac{1}{2}\int_1^{N_{\gamma 0}}\frac{dy}{y}\int_{-\infty}^{\infty} d\xi\left(\frac{\xi(N_\gamma-1)-i(N_\gamma+1)}{\xi(N_\gamma-1)+i(N_\gamma+1)}\right)^{\frac{m}{2}}$$
$$\times\left[R_0\Phi'(R_0)+(1-\chi_\gamma\cosh\tfrac{u}{2})\Psi(R_0)+\tfrac{1}{2}\Phi(R_0)+\frac{im\xi(N_\gamma-1)\chi_\gamma\sqrt{N_\gamma}\theta\bar\theta}{\xi^2(N_\gamma-1)^2+(N_\gamma+1)^2}\Phi(R_0)\right].$$

Here another substitution was performed, $\xi = \zeta\sqrt{N_\gamma}/(N_\gamma-1) = \zeta/(2\sinh\frac{u}{2})$, where

$N_\gamma \equiv e^u$ and $R_0 = \zeta^2+4\sinh^2\frac{u}{2}$. The y-integration yields

$$A(\gamma) = \frac{\ln N_{\gamma 0}}{4\sinh^2\frac{u}{2}}\int_{-\infty}^{\infty} d\zeta\left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta+2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}$$
$$\times\left[(\zeta^2+4\sinh^2\tfrac{u}{2})\Phi'(\zeta^2+4\sinh^2\tfrac{u}{2})+(1-\chi_\gamma\cosh\tfrac{u}{2})\Psi(\zeta^2+4\sinh^2\tfrac{u}{2})\right.$$
$$\left.+\tfrac{1}{2}\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})+\frac{im\zeta\chi_\gamma}{\zeta^2+4\cosh^2\frac{u}{2}}\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})\right]. \tag{41}$$

Let us consider the Φ and Φ'-terms in (41) and perform a partial integration in ζ

$$\int_{-\infty}^{\infty} d\zeta\left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta+2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}[\zeta^2\Phi'(\zeta^2+4\sinh^2\tfrac{u}{2})+\tfrac{1}{2}\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})]$$
$$= \int_{-\infty}^{\infty} d\zeta\left\{\left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta-2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}\zeta^2\Phi'(\zeta^2+4\sinh^2\tfrac{u}{2})\right.$$
$$\left.-\frac{1}{2}\zeta\frac{d}{d\zeta}\left[\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})\left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta+2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}\right]\right\}$$
$$= -im\cosh\tfrac{u}{2}\int_{-\infty}^{\infty} d\zeta\left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta+2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}\frac{\zeta\Phi(\zeta^2+4\sinh^2\frac{u}{2})}{\zeta^2+4\cosh^2\frac{u}{2}}. \tag{42}$$

Thus we get for $A(\gamma)$

$$A(\gamma) = \frac{\ln N_{\gamma 0}}{4\sinh^2\frac{u}{2}}\int_{-\infty}^{\infty} d\zeta\left(\frac{\zeta-2i\cosh\frac{u}{2}}{\zeta+2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}$$
$$\times\left[4\sinh^2\tfrac{u}{2}\Phi'(\zeta^2+4\sinh^2\tfrac{u}{2})+(1-\chi_\gamma\cosh\tfrac{u}{2})\Psi(\zeta^2+4\sinh^2\tfrac{u}{2})\right.$$
$$\left.+\frac{im\zeta}{\zeta^2+4\cosh^2\frac{u}{2}}(\chi_\gamma-\cosh\tfrac{u}{2})\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})\right].$$
$$= \frac{\ln N_{\gamma 0}}{4\sinh\frac{u}{2}}\int_0^{\infty}\frac{d\zeta}{(\zeta^2+4\sinh^2\frac{u}{2})^{\frac{m}{2}}}$$
$$\times\left\{\left[4\sinh^2\tfrac{u}{2}\Phi'(\zeta^2+4\sinh^2\tfrac{u}{2})+(1-\chi_\gamma\cosh\tfrac{u}{2})\Psi(\zeta^2+4\sinh^2\tfrac{u}{2})\right](\alpha_+^m+\alpha_-^m)\right.$$
$$\left.+\frac{im\zeta}{\zeta^2+4\cosh^2\frac{u}{2}}(\chi_\gamma-\cosh\tfrac{u}{2})\Phi(\zeta^2+4\sinh^2\tfrac{u}{2})(\alpha_+^m-\alpha_-^m)\right\}$$

(Substitution $x = \zeta^2+4\sinh^2\frac{u}{2}$, $dx = 2\zeta d\zeta = 2\sqrt{x-4\sinh^2\frac{u}{2}}d\zeta$)

$$= \frac{\ln N_{\gamma 0}}{8\sinh\frac{u}{2}}\int_{4\sinh^2\frac{u}{2}}^{\infty}\frac{dx}{(x+4\sinh^2\frac{u}{2})^{\frac{m}{2}}}$$
$$\times\left[\frac{(1-\chi_\gamma\cosh\frac{u}{2})\Psi(x)+4\sinh^2\frac{u}{2}\Phi'(x)}{\sqrt{x-4\sinh^2\frac{u}{2}}}(\alpha_+^m+\alpha_-^m)+im(\chi_\gamma-\cosh\tfrac{u}{2})\Phi(x)\frac{\alpha_+^m-\alpha_-^m}{x+4}\right]$$

[and finally by Eqs.(37) and (38)]

$$A(\gamma) = \frac{\ln N_{\gamma 0}}{N_\gamma^{\frac{1}{2}}-N_\gamma^{-\frac{1}{2}}}\left[g(u)+g(-u)-\chi_\gamma\left(g(u)e^{-\frac{u}{2}}+g(-u)e^{\frac{u}{2}}\right)\right]. \tag{43}$$

This is the result of Ref.[10]. Therefore the supertrace formula reads:

$$\mathrm{str}L = i^m(g-1)\pi\Phi_m(0) + \sum_{\{\gamma\}_p} \frac{{\chi_\gamma}^m \ln N_{\gamma_0}}{N_\gamma^{\frac{1}{2}} - N_\gamma^{-\frac{1}{2}}}\left[g(u)+g(-u)-\chi_\gamma\left(g(u)e^{-\frac{u}{2}}+g(-u)e^{\frac{u}{2}}\right)\right], \quad (44)$$

where $u = \ln N_\gamma = l_\gamma$ and $g(u)$ is given by

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{-iup} h(ip + \tfrac{m+1}{2}). \quad (45)$$

Furthermore in $\Phi(x)$ an index m is added to denote the dependence on m. Our final task is to eliminate $\Phi_m(0)$.
Let us first consider $m = 0$. By Eqs.(39):

$$g(u) - g(-u) = 2\sinh\tfrac{u}{2}\cosh\tfrac{u}{2}\int_{4\sinh^2\frac{u}{2}}^{\infty}\frac{\Phi_0'(x)dx}{\sqrt{x-4\sinh^2\frac{u}{2}}}. \quad (46)$$

Let us denote ($w = 4\sinh^2\frac{u}{2}$):

$$Q_0(w) = \frac{1}{\sinh u}[g(u)-g(-u)] = \int_w^\infty \frac{\Phi_0'(x)}{\sqrt{x-w}}dx. \quad (47)$$

Let us consider the integral

$$\begin{aligned}
-\frac{1}{\pi}\int_x^\infty \frac{dw}{\sqrt{w-x}}Q_0(w) &= \int_x^\infty \frac{dw}{\sqrt{w-x}}\int_w^\infty \frac{\Phi_0'(y)}{\sqrt{y-w}}dy \\
&= -\frac{1}{\pi}\int_x^\infty dy\,\Phi_0'(y)\int_w^y dw(w-x)^{-\frac{1}{2}}(x-w)^{-\frac{1}{2}} \\
&= -\frac{1}{\pi}\tfrac{1}{2}B(\tfrac{1}{2},\tfrac{1}{2})\int_x^\infty dy\Phi_0'(y) = \Phi_0(x),
\end{aligned} \quad (48)$$

where in the last step the integral [33, p.285]:

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}dx = (b-a)^{\nu+\mu-1}B(\nu,\mu) \quad (49)$$

was used and $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function. Thus the inversion formula reads

$$\Phi_0(x) = -\frac{1}{\pi}\int_x^\infty \frac{dw}{\sqrt{w-x}}Q_0(w). \quad (50)$$

Therefore:

$$\begin{aligned}
\Phi_0(0) &= -\frac{1}{\pi}\int_0^\infty \frac{dw}{\sqrt{w}}Q_0(w) = -\frac{1}{\pi}\int_0^\infty \frac{du}{\sinh\frac{u}{2}}[g(u)-g(-u)] \\
&= -\frac{1}{2\pi^2}\int_0^\infty \frac{du}{\sinh\frac{u}{2}}\int_{-\infty}^\infty dp\left[e^{-iup}h(ip+\tfrac{1}{2}) - e^{iup}h(ip+\tfrac{1}{2})\right] \\
&= \frac{i}{\pi^2}\int_{-\infty}^\infty dp\,h(ip+\tfrac{1}{2})\int_0^\infty \frac{\sin up}{\sinh\frac{u}{2}}du \\
&= \frac{i}{\pi}\int_{-\infty}^\infty h(ip+\tfrac{1}{2})\tanh\pi p\,dp,
\end{aligned}$$

where the integral [33, p.503]:

$$\int_0^\infty \frac{\sin(ax)}{\sinh(bx)}dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b} \quad (51)$$

has been used; therefore finally[1]

$$A_0^{(0)} = i(g-1)\int_{-\infty}^\infty h(ip+\tfrac{1}{2})\tanh\pi p\,dp. \quad (52)$$

It is possible to construct the inversion formulas for, e.g. the $m = 1$ and $m = 2$ cases by starting from Eq.(39). But this is rather tedious and cannot be easily generalized to all $m \in \mathbf{Z}$. Therefore I must develop a systematic approach to invert Eq.(39), i.e. to express $\Phi_m(x)$ by an integal (or integrals) over $g(u) - g(-u)$. The general inversion formula must then be evaluated for $\Phi_m(0)$.
Let us consider Eq.(39) by reinserting the variable $\xi = \sqrt{x - 4\sinh^2\frac{u}{2}}$:

$$\begin{aligned}
g(u) - g(-u) &= \frac{1}{2}\sinh\tfrac{u}{2}\int_{-\infty}^\infty d\xi\left(\frac{\xi - 2i\cosh\frac{u}{2}}{\xi + 2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}} \\
&\times\left[4\cosh\tfrac{u}{2}\Phi_m'(\xi^2+4\sinh^2\tfrac{u}{2}) - \frac{im\xi}{\xi^2+4\cosh^2\frac{u}{2}}\Phi_m(\xi^2+4\sinh^2\tfrac{u}{2})\right].
\end{aligned} \quad (53)$$

Let be $m \neq 0$. Performing a partial integration in the second term, where it is assumed that all the relevant terms are sufficiently decreasing at ∞:

$$\begin{aligned}
&\int_{-\infty}^\infty d\xi\left[\left(\frac{\xi - 2i\cosh\frac{u}{2}}{\xi + 2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}\frac{1}{\xi^2+4\cosh^2\frac{u}{2}}\right]\cdot\left[\xi\Phi_m(\xi^2+4\sinh^2\tfrac{u}{2})\right] \\
&= -\frac{1}{2im\cosh\frac{u}{2}}\int_{-\infty}^\infty d\xi\left(\frac{\xi - 2i\cosh\frac{u}{2}}{\xi + 2i\cosh\frac{u}{2}}\right)^{\frac{m}{2}}\left[\Phi_m(\xi^2+4\sinh^2\tfrac{u}{2}) + 2\xi^2\Phi_m'(\xi^2+4\sinh^2\tfrac{u}{2})\right].
\end{aligned} \quad (54)$$

With the abbreviation $w = 4\sinh^2\frac{u}{2}$ this gives in Eq.(53):

$$\begin{aligned}
&g(u) - g(-u) \\
&= \frac{1}{2}i^m\tanh\tfrac{u}{2}\int_{-\infty}^\infty d\xi\left(\frac{\sqrt{w+4}+i\xi}{\sqrt{w+4}-i\xi}\right)^{\frac{m}{2}}\left[(w+\xi^2+4)\Phi_m'(w+\xi^2)+\tfrac{1}{2}\Phi_m(w+\xi^2)\right].
\end{aligned} \quad (55)$$

Let us define (Q must not be confused with Q_0 in Eq.(47))

$$\left.\begin{aligned}
Q(w) &= 2\coth\tfrac{u}{2}[g(u) - g(-u)] \\
\tilde{\Phi}_m(x) &= i^m\left[(x+4)\Phi_m'(x) + \tfrac{1}{2}\Phi_m(x)\right]
\end{aligned}\right\} \quad (56)$$

and get the integral relation ($w = 4\sinh^2\frac{u}{2} \geq 0$):

$$Q(w) = \int_{-\infty}^\infty d\xi\left(\frac{\sqrt{w+4}+i\xi}{\sqrt{w+4}-i\xi}\right)^{\frac{m}{2}}\tilde{\Phi}_m(w+\xi^2). \quad (57)$$

[1] In Ref.[10] a factor of two is missing.

For this integral relation I can apply an inversion formula given by Hejhal [49, pp.454] which yields for $\tilde{\Phi}$:

$$\tilde{\Phi}_m(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} Q'(x+t^2)\left(\frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t}\right)^{\frac{m}{2}} dt. \tag{58}$$

Reinserting $\tilde{\Phi}$ I get a differential equation for Φ:

$$\Phi'_m(x) + \frac{1}{2(x+4)}\Phi_m(x) = \frac{-1}{i^m\pi(x+4)}\int_{-\infty}^{\infty} Q'(x+t^2)\left(\frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t}\right)^{\frac{m}{2}} dt \tag{59}$$

which can be easily solved to give the **inversion formula** for Φ:

$$i^m\Phi_m(x) = \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{\sqrt{y+4}}\int_{-\infty}^{\infty} Q'(y+t^2)\left(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\right)^{\frac{m}{2}} dt. \tag{60}$$

This is the main result of this section.
Note: 1) The integration constant in Eq.(60) is given by $\Phi_m(\infty) = 0$.
2) The inversion formula is valid for $m \in \mathbf{Z}$ [see below Eq.(61) for $m = 0$].
To get some confidence in the inversion formula let us consider Eq.(60) for some specific values of m.

1) $m = 0$:

$$\Phi_0(x) = \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{\sqrt{y+4}}\int_{-\infty}^{\infty} Q'(y+t^2)dt$$

($w = y^2 + t$)

$$= \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{\sqrt{y+4}}\int_y^{\infty}\frac{Q'(w)}{\sqrt{w-y}}dw$$

(Rearrangement of integrations)

$$= \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty} dw\, Q'(w)\int_x^{w}\frac{dy}{\sqrt{y+4}\sqrt{w-y}}$$

$\left[\text{Elementary integral: } \int\frac{dx}{\sqrt{(ax+b)(cx+d)}} = -\frac{2}{\sqrt{-ac}}\arctan\sqrt{-\frac{x+\frac{b}{a}}{x+\frac{d}{c}}}\ (ac<0)\right]$

$$= \frac{2}{\pi\sqrt{x+4}}\int_x^{\infty} Q'(w)\arctan\sqrt{\frac{w-x}{x+4}}dw$$

(Partial integration)

$$= -\frac{1}{\pi}\int_x^{\infty}\frac{Q(w)}{(w+4)\sqrt{w-x}}dw$$

($w = 4\sinh^2\frac{u}{2}$)

$$= -\frac{1}{\pi}\int_{2\operatorname{arsinh}\frac{\sqrt{x}}{2}}^{\infty}\frac{g(u)-g(-u)}{\sqrt{4\sinh^2\frac{u}{2}-x}}du. \tag{61}$$

This is equivalent with Eq.(50) and shows that the inversion formula is also valid for $m = 0$, i.e. the inversion formula is valid for all $m \in \mathbf{Z}$.

2) $m = 1$:

$$i\Phi_1(x) = \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{\sqrt{y+4}}\int_{-\infty}^{\infty} Q'(y+t^2)\left(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\right)^{\frac{1}{2}} dt$$

$$= \frac{2}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{y+4}\int_0^{\infty} Q'(y+t^2)\sqrt{y+4+t^2}dt$$

($w = y^2 - t$)

$$= \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{y+4}\int_y^{\infty} Q'(w)\sqrt{\frac{w+4}{w-y}}dw$$

(Rearrangement of integrations)

$$= \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty} dw\, Q'(w)\sqrt{w+4}\int_x^{w}\frac{dy}{(y+4)\sqrt{w-y}}$$

$\left(\text{Set } z = y+4 \text{ and elementary integral: } \int\frac{dx}{x\sqrt{ax+b}} = -\frac{2}{\sqrt{b}}\operatorname{artanh}\sqrt{\frac{ax+b}{b}}\right)$

$$= \frac{2}{\pi\sqrt{x+4}}\int_x^{\infty} Q'(w)\operatorname{artanh}\sqrt{\frac{w-x}{w+4}}dw$$

(Partial integration)

$$= -\frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{Q(w)}{\sqrt{(w-x)(w+4)}}dw$$

($w = 4\sinh^2\frac{u}{2}$)

$$= -\frac{4}{\pi\sqrt{x+4}}\int_{2\operatorname{arsinh}\frac{\sqrt{x}}{2}}^{\infty}\frac{g(u)-g(-u)}{\sqrt{4\sinh^2\frac{u}{2}-x}}\cosh\tfrac{u}{2}du. \tag{62}$$

In particular for $x = 0$:

$$i\Phi_1(0) = -\frac{1}{\pi}\int_0^{\infty}\coth\tfrac{u}{2}[g(u)-g(-u)]du$$

$$= \frac{i}{\pi^2}\int_{-\infty}^{\infty} dp\, h_1(ip+\tfrac{1}{2})\int_0^{\infty}\coth\tfrac{u}{2}\sin up\, du$$

$$= \frac{i}{\pi}\int_{-\infty}^{\infty} h_1(ip+\tfrac{1}{2})\coth\pi p\, dp, \tag{63}$$

where in the last step the integral [33, p.504]

$$\int_0^{\infty}\sin ax\frac{\cosh\beta x}{\sinh\gamma x}dx = \frac{\pi}{2\gamma}\frac{\sinh\frac{\pi a}{\gamma}}{\cosh\frac{\pi a}{\gamma}+\cos\frac{\pi\beta}{\gamma}} \tag{64}$$

was used. This gives finally for $A_0^{(1)}$ by Eq.(23)

$$A_0^{(1)} = i(g-1)\int_{-\infty}^{\infty}\coth\pi p\, h(ip+1)dp. \tag{65}$$

3) $m = 2$:

$$i^2\Phi_2(x) = \frac{1}{\pi\sqrt{x+4}}\int_x^\infty \frac{dy}{\sqrt{y+4}}\int_{-\infty}^\infty Q'(y+t^2)\Big(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\Big)dt$$
$$= \frac{2}{\pi\sqrt{x+4}}\int_0^\infty \frac{dy}{\sqrt{y+4}}\int_0^\infty Q'(y+t^2)dt$$
$$+ \frac{4}{\pi\sqrt{x+4}}\int_0^\infty \frac{dy}{\sqrt{y+4}^3}\int_0^\infty t^2Q'(y+t^2)dt. \quad (66)$$

The first integral is up to a factor Eq.(61). For the second I get:

$$\frac{4}{\pi\sqrt{x+4}}\int_0^\infty \frac{dy}{\sqrt{y+4}^3}\int_0^\infty t^2Q'(y+t^2)dt$$
$$= \frac{2}{\pi\sqrt{x+4}}\int_x^\infty \frac{dy}{(y+4)^{\frac{3}{2}}}\int_0^\infty t\Big[\frac{d}{dt}Q(y+t^2)\Big]dt \qquad \text{(Partial integration)}$$
$$= -\frac{2}{\pi\sqrt{x+4}}\int_x^\infty \frac{dy}{(y+4)^{\frac{3}{2}}}\int_0^\infty Q(y+t^2)dt \qquad (w = y+t^2 \text{ and rearrangement})$$
$$= -\frac{1}{\pi\sqrt{x+4}}dw\,Q(w)\int_x^w \frac{dy}{\sqrt{w-y}\sqrt{y+4}^3}$$

$\left(\text{Elementary integral: } \int \frac{dx}{\sqrt{ax+b}\sqrt{cx+d}^3} = \frac{2}{ad-bc}\sqrt{\frac{ax+b}{cx+d}}\right)$

$$= -\frac{2}{\pi(x+4)}\int_x^\infty \frac{\sqrt{w-x}}{w+4}Q(w)dw \qquad (w = 4\sinh^2\tfrac{u}{2})$$
$$= -\frac{4}{\pi(x+4)}\int_{2\operatorname{arsinh}\frac{\sqrt{x}}{2}}^\infty [g(u)-g(-u)]\sqrt{4\sinh^2\tfrac{u}{2}-x}\,du. \quad (67)$$

This gives for $m = 2$ the inversion formula

$$i^2\Phi_2(x) = -\frac{1}{\pi}\int_{2\operatorname{arsinh}\frac{\sqrt{x}}{2}}^\infty \frac{g(u)-g(-u)}{\sqrt{4\sinh^2\frac{u}{2}-x}}du.$$
$$-\frac{4}{\pi(x+4)}\int_{2\operatorname{arsinh}\frac{\sqrt{x}}{2}}^\infty [g(u)-g(-u)]\sqrt{4\sinh^2\tfrac{u}{2}-x}\,du. \quad (68)$$

In particular for $x = 0$:

$$i^2\Phi_2(0) = \frac{i}{\pi}\int_{-\infty}^\infty h_2(ip+\tfrac{1}{2})\tanh\pi p\,dp - \frac{2}{\pi}\int_{-\infty}^\infty g(u)\sinh\tfrac{u}{2}\,du \quad (69)$$

and therefore finally

$$A_0^{(2)} = i(g-1)\int_{-\infty}^\infty h_2(ip+\tfrac{1}{2})\tanh\pi p\,dp + (1-g)\big[h_2(1)-h_2(0)\big]. \quad (70)$$

4) $m = 3$: Similarly as for $m = 2$ it is straightforward to show that

$$\left.\begin{aligned} i^3\Phi_3(0) &= \frac{i}{\pi}\int_{-\infty}^\infty h_3(ip+\tfrac{1}{2})\coth\pi p\,dp - \frac{2}{\pi}\int_{-\infty}^\infty g(u)\sinh u\,du \\ A_0^{(3)} &= i(g-1)\int_{-\infty}^\infty h_3(ip+\tfrac{1}{2})\coth\pi p\,dp - (1-g)\big[h_3(\tfrac{3}{2})-h_3(-\tfrac{1}{2})\big]. \end{aligned}\right\} \quad (71)$$

Let us for a moment turn to test functions h for the operator $\hat{\Box}_m$, i.e. let us consider the functions $h(s) \equiv h_0(s)$. By Eqs.(34) relating h and g, $\hat{g}$ and $\hat{Q}$ do not depend on m ("hatted" quantities belonging to $\hat{\Box}_m$). The equations for $\hat{\Phi}_m$ ($m = 0,1,2,3$) suggest the following general structure for $\hat{\Phi}_m$:

$$i^m\hat{\Phi}_m(0) = \frac{i}{\pi}\int_{-\infty}^\infty h(ip+\tfrac{1}{2})\tanh\pi p\,dp$$
$$-\frac{2}{\pi}\sum_{k=1}^{m/2}\int_{-\infty}^\infty \hat{g}(u)\sinh(k-\tfrac{1}{2})u\,du \qquad (m \text{ even}) \quad (72)$$
$$i^m\hat{\Phi}_m(0) = \frac{i}{\pi}\int_{-\infty}^\infty h(ip+\tfrac{1}{2})\coth\pi p\,dp$$
$$-\frac{2}{\pi}\sum_{k=1}^{(m-1)/2}\int_{-\infty}^\infty \hat{g}(u)\sinh ku\,du \qquad (m \text{ odd}). \quad (73)$$

In particularly, it remains to show that for all m (even and odd):

$$i^{m+2}\hat{\Phi}_{m+2}(0) - i^m\hat{\Phi}_m(0) = -\frac{2}{\pi}\int_{-\infty}^\infty \hat{g}(u)\sinh\tfrac{m+1}{2}u\,du. \quad (74)$$

Having proved Eq.(74) once, one can go back to the operator $\Box_m$ and all related quantities. Equation (74) is now proved by induction for $m \to m+2$.
1) Since each step forward is by two units in the induction we have to distinguish between the even and odd cases. Eqs.(61) and (69), respectively (63) and (71) show that Eq.(74) is correct for $m = 0$ and $m = 1$, respectively.
2) Let us consider Eq.(74) and insert for $i^m\hat{\Phi}_m(0)$ and $i^{m+2}\hat{\Phi}_{m+2}(0)$:

$$i^{m+2}\hat{\Phi}_{m+2}(0) - i^m\hat{\Phi}_m(0) = \frac{1}{2\pi}\int_x^\infty \frac{dy}{\sqrt{y+4}}\hat{Q}'(y+t^2)$$
$$\times\left[\Big(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\Big)^{\frac{m+2}{2}} - \Big(\frac{\sqrt{y+4+t^2}-t}{\sqrt{y+4+t^2}+t}\Big)^{\frac{m}{2}}\right]$$
$$= -\frac{1}{\pi}\int_0^\infty \frac{dy}{(y+4)^{\frac{m+3}{2}}}\int_{-\infty}^\infty t\hat{Q}'(y+t^2)\big(\sqrt{y+4+t^2}-t\big)^{m+1}dt$$

(Partial integration)

$$= -\frac{m+1}{2\pi}\int_0^\infty \frac{dy}{(y+4)^{\frac{m+3}{2}}}\int_{-\infty}^\infty \hat{Q}(y+t^2)\frac{\big(\sqrt{y+4+t^2}-t\big)^{m+1}}{\sqrt{y+4+t^2}}dt$$
$$= -\frac{m+1}{2\pi}\int_0^\infty \frac{dy}{(y+4)^{\frac{m+3}{2}}}$$
$$\times\int_0^\infty \frac{\hat{Q}(y+t^2)}{\sqrt{y+4+t^2}}\Big[\big(\sqrt{y+4+t^2}-t\big)^{m+1} + \big(\sqrt{y+4+t^2}+t\big)^{m+1}\Big]$$

$(w = 4\sinh^2\frac{u}{2})$

$$= -\frac{m+1}{4\pi}\int_0^\infty \frac{dy}{(y+4)^{\frac{m+3}{2}}}$$
$$\times\int_y^\infty \frac{Q(w)}{\sqrt{w+4}\sqrt{w-y}}\Big[\big(\sqrt{w+4}-\sqrt{w-y}\big)^{m+1} + \big(\sqrt{w+4}+\sqrt{w-y}\big)^{m+1}\Big].$$

Again with a rearrangement of integrations:

$$i^{m+2}\hat{\Phi}_{m+2}(0) - i^m\hat{\Phi}_m(0) = -\frac{m+1}{4\pi}\int_0^\infty dw\,\frac{\hat{Q}(w)}{\sqrt{w+4}}$$
$$\times\int_0^w \frac{dy}{\sqrt{w-y}}\left[\frac{\left(\sqrt{w+4}-\sqrt{w-y}\right)^{m+1}}{(y+4)^{\frac{m+3}{2}}} + \frac{\left(\sqrt{w+4}+\sqrt{w-y}\right)^{m+1}}{(y+4)^{\frac{m+3}{2}}}\right]$$

[Substitution $t^2 = 1-(y+4)/(w+4)$]

$$= -\frac{m+1}{2\pi}\int_0^\infty dw\,\frac{\hat{Q}(w)}{w+4}\int_0^{\sqrt{\frac{w}{w+4}}}\left[\frac{(1-t)^{\frac{m-1}{2}}}{(1+t)^{\frac{m+3}{2}}} + \frac{(1+t)^{\frac{m-1}{2}}}{(1-t)^{\frac{m+3}{2}}}\right]dt. \quad (75)$$

Let us consider the t-integration and replace $m \to m+2$:

$$(m+3)\int_0^{\sqrt{\frac{w}{w+4}}}\left[\frac{(1-t)^{\frac{m+1}{2}}}{(1+t)^{\frac{m+5}{2}}} + \frac{(1+t)^{\frac{m+1}{2}}}{(1-t)^{\frac{m+5}{2}}}\right]dt$$

(Partial integration:)

$$= \underbrace{-2\left[\frac{(1-t)^{\frac{m+1}{2}}}{(1+t)^{\frac{m+3}{2}}} - \frac{(1+t)^{\frac{m+1}{2}}}{(1-t)^{\frac{m+3}{2}}}\right]\Bigg|_0^{\sqrt{\frac{w}{w+4}}}}_{=4\cosh\frac{u}{2}\sinh\frac{m+2}{2}u\quad (w=4\sinh^2\frac{u}{2}\text{ reinserted})}$$
$$-(m+1)\int_0^{\sqrt{\frac{w}{w+4}}}\left[\frac{(1-t)^{\frac{m-1}{2}}}{(1+t)^{\frac{m+3}{2}}} + \frac{(1+t)^{\frac{m-1}{2}}}{(1-t)^{\frac{m+3}{2}}}\right]dt. \quad (76)$$

Thus repeating the calculations of Eq.(75) for $m \to m+2$ and taking into account the result of Eq.(76) yield together with $w = 4\sinh^2\frac{u}{2}$ and Eq.(74):

$$i^{m+4}\hat{\Phi}_{m+4}(0) - i^{m+2}\hat{\Phi}_{m+2}(0)$$
$$= -\frac{8}{\pi}\int_0^\infty\left[\hat{g}(u)-\hat{g}(-u)\right]\cosh\tfrac{u}{2}\sinh\tfrac{m+2}{2}du - \left[i^{m+2}\hat{\Phi}_{m+2}(0) - i^m\hat{\Phi}_m(0)\right]$$
$$= -\frac{2}{\pi}\int_{-\infty}^\infty \hat{g}(u)\sinh\tfrac{m+3}{2}du. \quad (77)$$

This proves the induction!

∎

I summarize. I have formulated the Selberg Supertrace formula on super Riemannian surfaces for operator valued functions of the Laplace-Dirac operator $\Box_m$. Let h be a testfunction with the properties (following Baranov et al.[10]):

i) $h(\frac{1}{2}+ip) \in C^\infty(\mathbf{R})$,

ii) $h(\frac{1}{2}+ip)$ need not be an even function in p,

iii) $h(\frac{1}{2}+ip) \propto O(\frac{1}{p^2})(p\to\pm\infty)$.

iv) $h(\frac{1}{2}+ip)$ is holomorphic in the strip $|\mathrm{Im}(p)| \leq 1+\frac{m}{2}+\epsilon$, $\epsilon > 0$ to guarantee absolute convergence in the sums of Eq.(82) below (see [49] p.30]).

Its Fourier transform g is given by:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty dp e^{-iup}h(ip+\tfrac{m+1}{2}). \quad (78)$$

The term $A_0^{(m)}$ corresponding to the identity transformation reads

$$A_0^{(m)} = i(g-1)\int_{-\infty}^\infty h(ip+\tfrac{m+1}{2})\tanh\pi p\,dp$$
$$-(1-g)\sum_{k=1}^{m/2}\left[h(\tfrac{m}{2}+k) - h(\tfrac{m}{2}-k-1)\right] \qquad (m\text{ even}) \quad (79)$$

$$A_0^{(m)} = i(g-1)\int_{-\infty}^\infty h(ip+\tfrac{m+1}{2})\coth\pi p\,dp$$
$$+(1-g)\sum_{k=1}^{(m-1)/2}\left[h(\tfrac{m+1}{2}+k) - h(\tfrac{m-1}{2}-k)\right] \qquad (m\text{ odd}). \quad (80)$$

The last two equations can be combined and stated in a compact form yielding[1]

$$A_0^{(m)} = (1-g)\int_0^\infty \frac{g(u)-g(-u)}{\sinh\frac{u}{2}}T_m(\cosh\tfrac{u}{2})du, \qquad (m\in\mathbf{Z}), \quad (81)$$

where $T_m(\cosh\frac{u}{2}) = \cosh\frac{m}{2}u$ denotes the m^{th} Chebyshev-polynomial in $\cosh\frac{u}{2}$. Thus the supertrace formula reads (l_γ primitive geodesic):

$$\sum_{n=0}^\infty\left[h_m(p_n^B) - h_m(p_n^F)\right] = (1-g)\int_0^\infty\frac{g(u)-g(-u)}{\sinh\frac{u}{2}}\cosh\tfrac{m}{2}du$$
$$+\sum_{\{\gamma\}_p}\sum_{k=1}^\infty\frac{l_\gamma\chi_\gamma{}^{km}}{e^{\frac{kl_\gamma}{2}}-e^{-\frac{kl_\gamma}{2}}}\left[g(kl_\gamma)+g(-kl_\gamma)-\chi_\gamma{}^k\left(g(kl_\gamma)e^{-\frac{kl_\gamma}{2}}+g(-kl_\gamma)e^{\frac{kl_\gamma}{2}}\right)\right] \quad (82)$$

Equation (82) completes the work of Refs.[10,11] by explicit statement of the inversion formula (60) and the $A_0^{(m)}$-term, respectively.

[1] Note the similarity with the $A_{0,Selberg}^{(m)}$-term in Eq.(I.25).

1. The Selberg Zeta-Function Z

The Selberg zeta-functions were originally introduced by Selberg [82] in order to study spectra of Laplacians on compact Riemann surfaces of genus g. The Selberg zeta-function is defined by

$$Z(s) := \prod_{\{\gamma\}_p} \prod_{k=0}^{\infty} \left[1 - e^{-(s+k)l_\gamma}\right], \qquad (\mathrm{Re}(s) > 1). \tag{1}$$

Here l_γ is the length of a primitive $\gamma \in \Gamma$ and $\{\gamma\}_p$ denotes primitive conjugacy classes. The analytic properties of $Z(s)$ can be studied by the Selberg trace formula with the help of the regularized resolvent function $h(p) = 1/[p^2 - (s-\frac{1}{2})^2] - 1/[p^2 - (\sigma - \frac{1}{2})^2]$. This is an even function with the property $h(p) \to O(p^{-4})(p \to \pm\infty)$. Inserting h into the Selberg trace formula (I.22) one finds for $\mathrm{Re}(s) > 1$ [49,84]:

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)} = B + \frac{1}{s(s-1)} + 2(g-1)[\Psi(s) - \Psi(1)] + \sum_{n=1}^{\infty}\left[\frac{1}{E_n + s(s-1)} - \frac{1}{E_n}\right], \tag{2}$$

where the $E_n = \frac{1}{4} + p_n^2$ are the Eigenvalues of the Laplacian $-\Delta$ on the Poincaré upper half-plane in a compact domain as already defined in chapter I and $B = \frac{1}{2}\frac{Z''(1)}{Z'(1)} - 1$. Thus, $Z(s)$ is an entire function of s of order 2 and Eq.(2) extends meromorphically to all $s \in \mathbf{C}$. Furthermore one can state: $Z(s)$ has

- : at $s = -k\,(k \in \mathbf{N})$ zeros with multiplicity $(2g-2)(2k+1)$,
- : at $s = 0$ a zero with multiplicity $2g - 1$ and
- : at $s = 1$ a zero of multiplicity one.

These zeros are called the "trivial" ones. Furthermore:

- : at $s = \frac{1}{2} \pm ip_n$ $(p_n \in \mathbf{R})$ "nontrivial" zeros with the same multiplicity as the corresponding Eigenvalue,
- : at $0 < s < 1$ "nontrivial" zeros are located which correspond to the so called "small Eigenvalues" with the same multiplicity as the corresponding Eigenvalue.

These Eigenvalues are the Eigenvalues of the Laplacian $-\Delta$ defined on compact domains on the Poincaré upper haf-plane with periodic boundary conditions for its $4g$ boundaries. These compact domains correspond to a compact Riemann surface of genus g. Setting $s \to 1 - s$ in Eq.(2) and subtracting it from (2) one gets by direct integration the functional equation for the Selberg zeta-function

$$Z(s) = Z(1-s)\exp\left[4\pi(g-1)\int_0^{s-\frac{1}{2}} y \tan \pi y \, dy\right]. \tag{3}$$

One could hope of finding an analytic continuation of $Z(s)$ in, say, $0 \le \mathrm{Re}(s) \le 1$, where it should be possible to determine on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ the nontrivial zeros and therefore the Eigenvalues $E_n = \frac{1}{4} + p_n^2$ of the corresponding Laplacian (small Eigenvalues, of course, in a similar manner; for the regular oktagon $(g = 2)$ it is has

recently been shown, that there are no such Eigenvalues [7]). Unfortunately, up to now no such analytic continuation has been constructed.

There is an increasing amount of literature concerning the Selberg zeta-function. It has been of interest, because determinants of Laplacians can be expressed by combinations of the zeta-function and its derivatives. Define $D_\Delta(z) = \det'(-\Delta + z)$, where the prime denotes the omission of zero modes. Then the Selberg zeta-function and D_Δ are connected by the relation [84]:

$$Z(s) = s(s-1)D_\Delta[s(s-1)]\left[(2\pi)^{1-s}e^{C+s(s-1)}G(s)G(s+1)\right]^{2(g-1)}, \tag{4}$$

where $C = \frac{1}{4} - \ln\sqrt{2\pi} - 2\zeta'(-1)$ and $G(z)$ denotes the Barnes G-function defined by

$$\begin{aligned} G(z+1) &= (2\pi)^{\frac{z}{2}} e^{-\frac{z}{4} - \frac{1+\gamma_E}{2}z^2} \prod_{n=1}^{\infty}\left[\left(1 + \frac{z}{n}\right)^n e^{-z + \frac{z^2}{2n}}\right] \\ &= (2\pi)^{\frac{z}{2}} e^{-\frac{z}{4} - \frac{1+\gamma_E}{2}z^2} \prod_{n=1}^{\infty}\left[\frac{\Gamma(n)}{\Gamma(z+n)} e^{z\Psi(n) + \frac{1}{2}z^2\Psi'(n)}\right] \end{aligned} \tag{5}$$

[$\Psi(z) = \Gamma'(z)/\Gamma(z)$ the digamma-function, $\gamma_E = -\Psi(1)$ Euler's constant]. $G(z)$ has the important properties $G(z+1) = \Gamma(z)G(z)$ and $G(1) = 1$. By taking the limit $s \to 1$ in Eq.(4) one gets

$$\det{}'(-\Delta) = Z'(1)e^{(g-1)[\ln 2\pi + 4\zeta'(-1) - \frac{1}{2}]}, \tag{6}$$

where $\zeta(z)$ denotes the Riemann zeta-function. The properties of the Selberg zeta-function can also be used to show that the bosonic string theorie diverges - see [45] and end of chapter VI. Steiner [84] has computed the important relation

$$\begin{aligned} Z(s) = s(s-1)Z'(1)e^{\gamma_\Delta s(s-1)}&\left[(2\pi)^{1-s}e^{s(s-1)}G(s)G(s+1)\right]^{2(g-1)} \\ &\times \prod_{n=1}^{\infty}\left(1 + \frac{s(s-1)}{E_n}\right)\exp\left(-\frac{s(s-1)}{E_n}\right), \end{aligned} \tag{7}$$

where $\gamma_\Delta = 2(g-1)\gamma_E - \frac{Z''(1)}{2Z'(1)} - 1$. Equations (4) and (7) are connected by the formula

$$D_\Delta(z) = Z'(1)e^{\gamma_\Delta z - 2(g-1)C}\prod_{n=1}^{\infty}\left(1 + \frac{z}{E_n}\right)\exp\left(-\frac{z}{E_n}\right). \tag{8}$$

From Eq.(7) one can deduce

$$Z(s) = Z(s+1)\frac{s-1}{s+1}e^{-2s\gamma_\Delta}\left[\frac{se^{2s}\Gamma^2(s)}{2\pi}\right]^{2(1-g)}\prod_{n=1}^{\infty}\frac{E_n + s(s-1)}{E_n + s(s+1)}\exp\left(\frac{2s}{E_n}\right). \tag{9}$$

Finally the functional relation (3) can be rewritten yielding

$$Z(s) = Z(1-s)\left[(2\pi)^{1-2s}\frac{G(s)G(s+1)}{G(1-s)G(2-s)}\right]^{2(g-1)}. \tag{10}$$

Introducing the function $R(s) = Z(s)/Z(s+1)$ one finds the functional relation for the function $R(s)$ which reads

$$R(s)R(-s) = \frac{Z(s)Z(-s)}{Z(1+s)Z(1-s)} = (2\sin\pi s)^{4(g-1)}. \tag{11}$$

More information about $Z(s)$ can be found, e.g. in [24,49,62,84].

The purpose of the following sections is to discuss the analytic properties of the Selberg super zeta-functions Z_0 and Z_1 (definitions see below). It turns out that it is possible to proceed in a fashion similar to that used for the ordinary Selberg zeta-function, however, with some restrictions. Thus the program is:

- : to determine the "trivial" zeros,
- : to determine the "nontrivial" zeros,
- : to find functional equations and,
- : to find functional equations combining the two Selberg super zeta-functions.

Since in the supertrace formula terms with and without the character χ_γ appear, I have to choose appropriate test functions which separate this combination. It turns out that I must choose odd test functions which are allowed in the supertrace formula, in contrast to the ordinary trace formula. Similarities to, as well as differences from the original pure bosonic case will occur. I also derive functional relations and relations between the two possible zeta-functions. In a paper by Baranov and Schwarz [11], such a relation was already stated, unfortunately however, with a missing factor.

2. The Selberg Super Zeta-Function Z_1

The Selberg super zeta-functions are defined by

$$Z_q(s) := \prod_{\{\gamma\}_p}\prod_{k=0}^{\infty}\left[1 - \chi_\gamma{}^q e^{-(s+k)l_\gamma}\right], \qquad (\mathrm{Re}(s) > 1), \tag{12}$$

where q can take on the values $q = 0, 1$, respectively. χ_γ describes the spin structure and l_γ is the length of a primitive geodesic, as already defined.[1] The γ product is taken over all primitive conjugacy classes $\gamma \in \Gamma$. The Selberg super R-functions are defined by

$$R_q(s) := \frac{Z_q(s)}{Z_q(s+1)} = \prod_{\{\gamma\}_p}\left[1 - \chi_\gamma{}^q e^{-sl_\gamma}\right], \qquad (\mathrm{Re}(s) > 1). \tag{13}$$

To study the analytic properties of Z_0 and Z_1 let us consider the Selberg supertrace formula for $m = 0$, i.e.[2]

$$\sum_{n=0}^{\infty}\left[h(p_n^B) - h(p_n^F)\right] = i(g-1)\int_{-\infty}^{\infty} h(ip+\tfrac{1}{2})\tanh\pi p\,dp$$
$$+\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{e^{\frac{kl_\gamma}{2}} - e^{-\frac{kl_\gamma}{2}}}\left[g(kl_\gamma)+g(-kl_\gamma)-\chi_\gamma{}^k\left(g(kl_\gamma)e^{-\frac{kl_\gamma}{2}}+g(-kl_\gamma)e^{\frac{kl_\gamma}{2}}\right)\right]. \tag{14}$$

[1] If not otherwise noted, I do not distinguish between the these two numbers l_γ in Eq.(1) and Eq.(12).

[2] Througout this chapter I denote by $\lambda^{B(F)} = \frac{1}{2} + p_n^{B(F)}$ $(n \in \mathbf{N})$ the Bose and Fermi Eigenvalues of $\Box$, respectively.

To get informations for Z_1 or R_1, respectively, one has to choose a test function $h(p)$ so that the first two terms in the square bracket in the supertrace formula cancel, i.e. $g(u) = -g(-u)$. I choose the function ($\mathrm{Re}(s) > 1$, $\mathrm{Re}(\sigma) > 1$):

$$h_s(p) = 2(\lambda - \tfrac{1}{2})\left[\frac{1}{s^2 - (\lambda-\frac{1}{2})^2} - \frac{1}{\sigma^2 - (\lambda-\frac{1}{2})^2}\right]\Bigg|_{\lambda=\frac{1}{2}+ip} = 2ip\left(\frac{1}{s^2+p^2} - \frac{1}{\sigma^2+p^2}\right). \tag{15}$$

The second term plays the role of a regulator in order that all the involved terms in the supertrace formula are convergent. Thus for $g(u)$:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iup}h_s(p)dp = \frac{2}{\pi}\int_0^{\infty} p\sin up\left(\frac{1}{s^2+p^2} - \frac{1}{\sigma^2+p^2}\right)dp, \tag{16}$$

and therefore $g(u)$ is an odd function as required. Using [33, p.406]:

$$\int_0^{\infty}\frac{x\sin ax}{\beta^2+x^2}dx = \frac{\pi}{2}e^{-a\beta} \tag{17}$$

I get ($u > 0$) $g(u) = (e^{-su} - e^{-\sigma u})$ and for $u \in \mathbf{R}$

$$g(u) = \mathrm{sign}(u)(e^{-s|u|} - e^{-\sigma|u|}). \tag{18}$$

finally for $G(u,\chi)$

$$G(u,\chi_\gamma) = 2\chi_\gamma(e^{-s|u|} - e^{-\sigma|u|})\sinh\tfrac{u}{2}. \tag{19}$$

Thus only the χ_γ-term remains in the supertrace formula which allows to study the properties of Z_1 alone. Inserting $G(u,\chi)$ into the length term yields

$$\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{2\sinh\frac{kl_\gamma}{2}}G(kl_\gamma,\chi_\gamma) = \sum_{\{\gamma\}_p}\sum_{k=1}^{\infty} l_\gamma\chi_\gamma{}^k(e^{-skl_\gamma} - e^{-\sigma kl_\gamma})$$
$$= \sum_{\{\gamma\}_p}\left[\frac{l_\gamma\chi_\gamma e^{-sl_\gamma}}{1-\chi_\gamma e^{-sl_\gamma}} - \frac{l_\gamma\chi_\gamma e^{-\sigma l_\gamma}}{1-\chi_\gamma e^{-\sigma l_\gamma}}\right] = \frac{R_1'(s)}{R_1(s)} - \frac{R_1'(\sigma)}{R_1(\sigma)}. \tag{20}$$

In the last step the property of the logarithmic derivative of the Selberg super R-functions has been used, i.e. for $\mathrm{Re}(s) > 1$:

$$\frac{d}{ds}\ln R_q(s) = \frac{d}{ds}\ln\prod_{\{\gamma\}_p}\left[1 - \chi_\gamma{}^q e^{-sl_\gamma}\right] = \sum_{\{\gamma\}_p}\frac{l_\gamma\chi_\gamma{}^q e^{-sl_\gamma}}{1-\chi_\gamma{}^q e^{-sl_\gamma}}. \tag{21}$$

The A_0 term gives

$$A_0 = i(g-1)\int_{-\infty}^{\infty} h_s(p)\tanh\pi p\,dp$$
$$= \frac{4}{\pi}(1-g)\int_0^{\infty}\frac{du}{\sinh\frac{u}{2}}\left[\int_0^{\infty}\frac{p\sin up}{p^2+s^2}dp - \int_0^{\infty}\frac{p\sin up}{p^2+\sigma^2}dp\right]$$
$$= 4(1-g)\int_0^{\infty} e^{-\frac{s+\sigma}{2}u}\frac{\sinh\frac{\sigma-s}{2}}{\sinh\frac{u}{2}}du, \tag{22}$$

where the integrals (17) and (IV.51) have been used. Using now [33, p.356]

$$\int_0^\infty e^{-\mu x}\frac{\sinh \beta x}{\sinh bx}dx = \frac{1}{2b}\left[\Psi\left(\frac{1}{2}+\frac{\mu+\beta}{2b}\right) - \Psi\left(\frac{1}{2}+\frac{\mu-\beta}{2b}\right)\right], \tag{23}$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$, $z \in \mathbf{C}$, I get finally for A_0

$$A_0 = 4(g-1)[\Psi(s+\tfrac{1}{2}) - \Psi(\sigma+\tfrac{1}{2})]. \tag{24}$$

Let us denote by $\Delta n_0^{(0)} = n_0^B - n_0^F$ the difference between the number of even and odd zero modes of the Dirac operator $\Box$. Thus the supertrace formula for the function h_s reads

$$\sum_{n=1}^{\infty}[h_s(p_n^B) - h_s(p_n^F)] - \Delta n_0^{(0)}\left[\frac{1}{(s-\frac{1}{2})(s+\frac{1}{2})} - \frac{1}{(\sigma-\frac{1}{2})(\sigma+\frac{1}{2})}\right]$$
$$= 4(g-1)\left[\Psi(s+\tfrac{1}{2}) - \Psi(\sigma+\tfrac{1}{2})\right] + \frac{R_1'(s)}{R_1(s)} - \frac{R_1'(\sigma)}{R_1(\sigma)}. \tag{25}$$

First I discuss the trivial structure of zeros and poles of R_1 and Z_1. I read off the analytic properties of the R_1-function:

- : For $s = \frac{1}{2}$: there is a pole, zero or a regular point depending on whether $\Delta n_0^{(0)} > 0$, $\Delta n_0^{(0)} < 0$ or $\Delta n_0 = 0$, respectively.
- : For $s = -\frac{1}{2}$: there is a zero of multiplicity $4(g-1) + \Delta n_0^{(0)}$ (assuming that $|\Delta n_0| < 4(g-1)$).
- : For $s = -\frac{1}{2} - k$ $(k \in \mathbf{N})$: there are zeros with multiplicity $4(g-1)$.

In the discussion has been used that $\operatorname{Res}\Psi(z)|_{z=-n} = -1$ $(k \in \mathbf{N}_0)$. Therefore the analytic properties of Z_1 read:

- : For $s = \frac{1}{2}$: there is a pole, zero or a regular point depending on whether $\Delta n_0^{(0)} > 0$, $\Delta n_0^{(0)} < 0$ or $\Delta n_0 = 0$, respectively.
- : For $s = -\frac{1}{2} - k$ $(k \in \mathbf{N}_0)$: there are zeros with multiplicity $4(k+1)(g-1)$.

Second let us turn to the nontrivial zeros and poles of these two functions (first so called "small Eigenvalues" not considered). Since

$$h_s(p) = 2ip\left[\frac{1}{(s+ip)(s-ip)} - \frac{1}{(\sigma+ip)(\sigma-ip)}\right] \tag{26}$$

we have

$$\operatorname{Res}\left[h_s(p_n^B)\right]\Big|_{s=ip_n^B} = 1, \qquad \operatorname{Res}\left[h_s(p_n^B)\right]\Big|_{s=-ip_n^B} = -1 \tag{27}$$

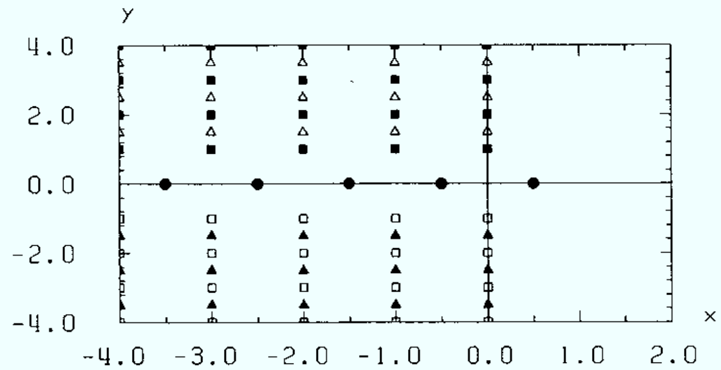
with signs of the residua reversed for the Fermi Eigenvalues. Thus $R_1(s)$ has

- : for $s = ip_n^{B(F)}$: there are zeros (poles) with the same multiplicity as the corresponding Eigenvalue of $\Box$.
- : for $s = -ip_n^{B(F)}$: reversed situation for poles and zeros.

Note the crucial dependence on the signs.
Since $R_1(1 \pm ip)$ is regular we can conclude by $Z_1(1 \pm ip) = R_1(1 \pm ip) \cdot Z_1(2 \pm ip)$ that $Z_1(s)$ is regular on the line $\mathrm{Re}(s) = 1$. Furthermore this gives by $Z_1(ip) = R_1(ip) \cdot Z_1(1+ip)$ that $Z_1(s)$ has on the line $\mathrm{Re}(s) = 0$ the same properties as $R_1(s)$, i.e. zeros (poles) for $s = ip_n^{B(F)}$ and poles (zeros) for $s = -ip_n^{B(F)}$. Repeating this procedure for $Z_1(ip-k) = R_1(ip-k) \cdot Z_1(ip-k+1)$ $(k \in \mathbf{N})$, we get an infinite number of critical lines for Z_1 located at $\mathrm{Re}(s) = -k$ $(k \in \mathbf{N}_0)$. Therefore the analytic properties of Z_1 for the nontrivial zeros and poles $(k \in \mathbf{N}_0)$ read:

- : For $s = ip_n^{B(F)} - k$: there are zeros (p_N^B) and poles (p_n^F) with the same multiplicity as the corresponding Eigenvalue of $\Box$.
- : For $s = -ip_n^{B(F)} - k$: there are poles (p_N^B) and zeros (p_n^F) with the same multiplicity as the corresponding Eigenvalue of $\Box$.

Figure 3: Zeros and poles of the zeta-function Z_1



Finally let us discuss the case of so called small Eigenvalues $(0 \le \lambda \le \frac{1}{2})$, which are also unknown and likely do not exist for small g [7]. We can see from Eq.(25) that for R_1 they are located at $-\frac{1}{2} \le s \le \frac{1}{2}$:

- : for $s = \pm(\lambda_n^B - \frac{1}{2})$ there are zeros (poles) and
- : for $s = \mp(\lambda_n^F - \frac{1}{2})$ there are poles (zeros) with the same multiplicity as the corresponding Eigenvalue of $\Box$, respectively.

By the same considerations as for the other nontrivial zeros and poles we get the structure for the Z_1-function $(k \in \mathbf{N}_0)$:

- : for $s = \lambda_n^{B(F)} - \frac{1}{2} - k$ there are zeros (poles) and
- : for $s = -(\lambda_n^{B(F)} - \frac{1}{2}) - k$ there are poles (zeros) with the same multiplicity as the corresponding Eigenvalue of $\Box$, respectively.

All these Eigenvalues are of course, even numbers, i.e. elements of $\mathbf{C}_c$. Therefore the supertrace formula can be extended meromorphically to all $s \in \Lambda_\infty$ and R_1 and Z_1 are meromorphic functions in Λ_∞.

In figure 3 the analytic properties of the Z_1-function are displayed. The trivial zeros are indicated by filled dots, the position of the bosonic zeros and poles by filled and empty squares, respectively, and the position of the fermionic zeros and poles by filled and empty triangles, respectively. The small Eigenvalues are not considered. The x- and y-axis are taken at the body of Λ_∞, i.e. $(\Lambda_\infty)_{Body} = \mathbf{C}$. The y-axis is taken in arbitrary units.

Let us consider Eq.(25) in the limit $\sigma \to \frac{1}{2}$ and get

$$\lim_{\sigma\to\frac{1}{2}} \left[\frac{R_1'(\sigma)}{R_1(\sigma)} + \frac{\Delta n_0^{(0)}}{\sigma^2 - \frac{1}{4}} - 4(g-1)\Psi(\sigma + \tfrac{1}{2})\right] = A_1 - \gamma_E, \tag{28}$$

where $\Psi(1) = -\gamma_E = -0.57721\ldots$ is the Euler number and A_1 is given by

$$\left.\begin{aligned} A_1 &= \frac{R_1'(\frac{1}{2})}{R_1(\frac{1}{2})}, && (\Delta n_0^{(0)} = 0),\\ &= \frac{R_1^{(1-\Delta n_0^{(0)})}(\frac{1}{2})}{(1-\Delta n_0^{(0)})R_1^{(-\Delta n_0^{(0)})}(\frac{1}{2})}, && (\Delta n_0^{(0)} < 0),\\ &= \frac{\oint(\sigma-\frac{1}{2})^{\Delta n_0^{(0)}-2}R_1(\sigma)\,d\sigma}{\oint(\sigma-\frac{1}{2})^{\Delta n_0^{(0)}-1}R_1(\sigma)\,d\sigma} && (\Delta n_0^{(0)} > 0). \end{aligned}\right\} \tag{29}$$

Therefore

$$\sum_{n=1}^{\infty}\left[\frac{2ip_n^B}{s^2+(p_n^B)^2} - \frac{2ip_n^B}{\frac{1}{4}+(p_n^B)^2} - \frac{2ip_n^F}{s^2+(p_n^F)^2} + \frac{2ip_n^F}{\frac{1}{4}+(p_n^F)^2}\right] + 4(g-1)\gamma_E + A_1$$
$$= \frac{\Delta n_0^{(0)}}{(s-\frac{1}{2})(s+\frac{1}{2})} + 4(g-1)\Psi(s+\tfrac{1}{2}) + \frac{R_1'(s)}{R_1(s)}. \tag{30}$$

h_s has the symmetry $h_s = h_{-s}$. Writing down Eq.(30) for $s \to -s$ and subtracting it from Eq.(30) gives with $\Psi(\frac{1}{2}+s) = \Psi(\frac{1}{2}-s) + \pi\tan\pi s$ [55, p.14] the functional equation in differential form for the R_1-function

$$\frac{d}{ds}\ln R_1(s)R_1(-s) = -4(g-1)\pi\tan\pi s. \tag{31}$$

Of course, every information about the nontrivial zeros is lost. This equation can be integrated yielding

$$R_1(s)R_1(-s) = \tilde{A}_1(\cos\pi s)^{4(g-1)} \tag{32}$$

where $\tilde{A}_1$ is a constant given e.g. by $\tilde{A}_1 = R_1(s_0)R_1(-s_0)(\cos\pi s_0)^{4(1-g)}$ with some $s_0 \in \mathbf{C}$, which is however, independent of s_0. I have, e.g. (no small Eigenvalue $\lambda = \frac{1}{2}$ assumed) for $s_0 = 0$: $\tilde{A}_1 = R_1^2(0)$.

3. The Selberg Super Zeta-Function Z_0

In this section I derive the analytic properties of the Selberg super zeta-function Z_0 and present a functional equation connecting the two Selberg super zeta-functions Z_0 and Z_1.[1] I consider the test function ($\mathrm{Re}(s) > \frac{3}{2}$):

$$h_s(p) = \frac{1}{\lambda(1-\lambda) - s(1-s)}\Big|_{\lambda=ip+\frac{1}{2}} = \frac{1}{p^2 + (s-\frac{1}{2})^2}. \tag{33}$$

This gives at once $A_0 = 0$ because h_s is an even function in p. Furthermore for $g(u)$:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iup}}{(s-\frac{1}{2})^2+p^2}dp = \frac{1}{2s-1}e^{-(s-\frac{1}{2})|u|}, \tag{34}$$

([55, p.431]). This gives for $G(u,\chi)$:

$$G(kl_\gamma,\chi_\gamma) = \frac{e^{-(s-\frac{1}{2})kl_\gamma}}{s-\frac{1}{2}}(1-\chi_\gamma{}^k\cosh\tfrac{kl_\gamma}{2}). \tag{35}$$

Therefore the right hand side of the supertrace formula reads

$$\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{2\sinh\frac{kl_\gamma}{2}}\frac{e^{-(s-\frac{1}{2})kl_\gamma}}{s-\frac{1}{2}}(1-\chi_\gamma{}^k\cosh\tfrac{kl_\gamma}{2})$$
$$= \frac{1}{2s-1}\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{1-e^{-kl_\gamma}}\left[2e^{-skl_\gamma} - \chi_\gamma{}^k e^{-(s-\frac{1}{2})kl_\gamma} - \chi_\gamma{}^k e^{-(s+\frac{1}{2})kl_\gamma}\right]$$
$$= \frac{1}{2s-1}\left[2\frac{d\ln Z_0(s)}{ds} - \frac{d\ln Z_1(s-\frac{1}{2})}{ds} - \frac{d\ln Z_1(s+\frac{1}{2})}{ds}\right]$$
$$= \frac{1}{2s-1}\frac{d}{ds}\ln\left[\frac{Z_0^2(s)}{Z_1(s-\frac{1}{2})Z_1(s+\frac{1}{2})}\right]. \tag{36}$$

Here the properties of the logarithmic derivative of the super zeta-functions have been used.

$$\frac{d}{ds}\ln Z_q(s) = \frac{d}{ds}\ln\prod_{\{\gamma\}_p}\prod_{k=0}^{\infty}\left[1-\chi_\gamma{}^q e^{-(s+k)l_\gamma}\right]$$
$$= \sum_{\{\gamma\}_p}\sum_{k=0}^{\infty}\frac{l_\gamma\chi_\gamma{}^q e^{-(s+k)l_\gamma}}{1-\chi_\gamma{}^q e^{-(s+k)l_\gamma}} = \sum_{\{\gamma\}_p}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}l_\gamma\chi_\gamma{}^{qn}e^{-(s+k)nl_\gamma}$$
$$= \sum_{\{\gamma\}_p}\sum_{n=1}^{\infty}\frac{l_\gamma\chi_\gamma{}^{qn}e^{-snl_\gamma}}{1-e^{-nl_\gamma}}. \tag{37}$$

Thus the supertrace formula for the test function h_s reads

$$\sum_{n=1}^{\infty}\left[h_s(p_n^B) - h_s(p_n^F)\right] - \frac{\Delta n_0^{(0)}}{s(1-s)} = \frac{1}{2s-1}\frac{d}{ds}\ln\left[\frac{Z_0^2(s)}{Z_1(s-\frac{1}{2})Z_1(s+\frac{1}{2})}\right]. \tag{38}$$

[1] It is also possible to derive the analytic properties of Z_0 similarly as the reasoning for Z_1 as in section 2. However, this approach here seems to be more straightforward to me.

Due to our knowledge of the analytic properties of the Z_1-function we can deduce the analytic properties of the Z_0-function:

- : $s = -k\,(k \in \mathbf{N}_0)$: There are trivial zeros with multiplicity $(g-1)(4k+2)$.

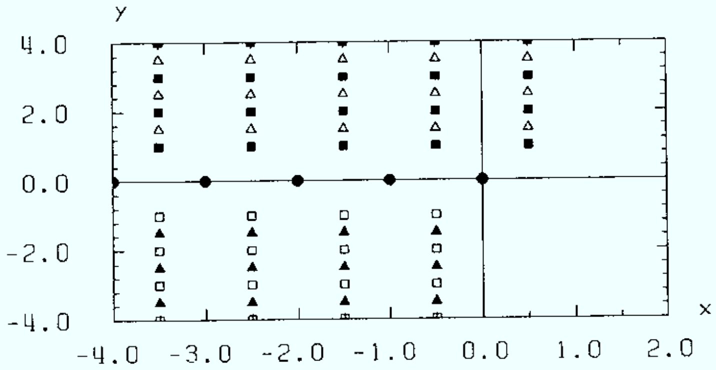
Since both sides of Eq.(38) must be regular for $s = \frac{1}{2} \pm ip_n - k\,(k \in \mathbf{N}_0)$ we get further

- : $s = \frac{1}{2} + ip_n^{B(F)} - k$: There are zeros (p_n^B) and poles (p_n^F),
- : $s = -\frac{1}{2} - ip_n^{B(F)} - k$: there are poles (p_n^B) and zeros (p_n^F),

with the same multiplicity as the corresponding Eigenvalue, respectively. Similarly, as for Z_1, we get an infinite number of critical lines, which makes it very unlikely that a functional equation for Z_0exists like Eq.(10) for the ordinary Selberg zeta-function. By the same considerations as for Z_1 one gets the structure for the Z_0-function for the "small Eigenvalues" $(k \in \mathbf{N}_0)$:

- : for $s = \lambda_n^{B(F)} - k$ there are zeros (poles) and
- : for $s = 1 - \lambda_n^{B(F)} - k$ there are poles (zeros) with the same multiplicity as the corresponding Eigenvalue of $\square$, respectively.

Figure 4: Zeros and poles of the zeta-function Z_0



A functional equation for R_0 can be derived, as can be seen in the next section. Of course Eq.(38) and Z_0 can be extended meromorphically to all $s \in \Lambda_\infty$. In figure 4 the analytic structure of Z_0 is scetched. The trivial zeros are indicated by filled dots, the position of the bosonic zeros and poles by filled and empty squares, respectively, and the position of the fermionic zeros and poles by filled and empty triangles, respectively. The small Eigenvalues are not considered. The x- and y-axis are again taken at the body of Λ_∞, i.e. $(\Lambda_\infty)_{Body} = \mathbf{C}$. The y-axis is taken in arbitrary units.

The test function h_s is invariant under the change $s \to 1-s$. Performing this substitution in Eq.(38) and subtracting it from (38) yields the functional equation

$$\frac{d}{ds}\ln\left[\frac{Z_0^2(s)}{Z_1(s-\frac{1}{2})Z_1(s+\frac{1}{2})}\right] = \frac{d}{ds}\ln\left[\frac{Z_0^2(1-s)}{Z_1(\frac{1}{2}-s)Z_1(\frac{3}{2}-s)}\right]. \tag{39}$$

We consider the functional equation (31) for the R_1-function and perform the substitution $s \to \frac{1}{2} - s$. By expressing the R_1-function by the quotient of the Z_1-functions, this yields

$$\frac{d}{ds}\ln\left[\frac{Z_1(\frac{1}{2}-s)Z_1(s-\frac{1}{2})}{Z_1(\frac{3}{2}-s)Z_1(s+\frac{1}{2})}\right] = 4\pi(g-1)\cot\pi s. \tag{40}$$

Thus we find by combining Eqs.(39) and (40) the functional equation in differential form connecting Z_0 and Z_1:

$$\frac{d}{ds}\ln\left[\frac{Z_1(\frac{1}{2}-s)Z_0(s)}{Z_1(\frac{1}{2}+s)Z_0(1-s)}\right] = 2\pi(g-1)\cot\pi s. \tag{41}$$

The functional equation can be integrated yielding[1]

$$\frac{Z_1(\frac{1}{2}-s)Z_0(s)}{Z_1(\frac{1}{2}+s)Z_0(1-s)} = C_0(\sin\pi s)^{2(g-1)}, \tag{42}$$

where C_0 is, e.g. given by $Z_1(\frac{1}{2}-s_0)Z_0(s_0)/[Z_1(\frac{1}{2}+s_0)Z_0(1-s_0)(\sin\pi s_0)^{2(1-g)}]$ with some $s_0 \in \mathbf{C}$ which is, however independent of s_0. This gives, e.g. for $s_0 = \frac{1}{2}$: $C_0 = Z_1(0)/Z_1(1) = R_1(0) = \sqrt{\tilde{A}_1}$.

4. The Super Zeta-Function Z_S

To get around the difficulties of the combination of the Z_0 and Z_1 functions for general test functions h in the Selberg supertrace formula let us (following Matsumoto, Uehara and Yasui [61]) define the super zeta-function Z_S:

$$\begin{aligned} Z_S(s) &:= \prod_{\{\gamma\}_p}\prod_{n=0}^{\infty} \operatorname{sdet}\left[\mathbf{1} - \operatorname{diag}(1, e^{-l_\gamma}, \chi_\gamma e^{-\frac{l_\gamma}{2}}, \chi_\gamma e^{-\frac{l_\gamma}{2}})e^{-(s+n)l_\gamma}\right] \\ &= \prod_{\{\gamma\}_p}\prod_{n=0}^{\infty} \frac{[1-e^{-(s+n)l_\gamma}][1-e^{-(s+n+1)l_\gamma}]}{[1-\chi_\gamma e^{-(s+n+\frac{1}{2})l_\gamma}]^2} = \frac{Z_0(s)Z_0(s+1)}{Z_1^2(s+\frac{1}{2})}. \end{aligned} \tag{43}$$

Let us consider the resolvent of $\square_0^2$: $R_s(\square_0^2) = (s^2 - \square_0^2)^{-1}$ $(\mathrm{Re}(s) > 1)$. Therefore

$$h(p) = \left.\frac{1}{s^2 - \lambda^2}\right|_{\lambda = \frac{1}{2}+ip} = \frac{1}{(s^2 - \frac{1}{4}) - ip + p^2}. \tag{44}$$

The Fourier transform of $h(p)$ yields:

$$g(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(p)e^{-iup}dp = g_1(u) + g_2(u), \tag{45}$$

where

$$\begin{aligned} g_1(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos up}{(s^2-\frac{1}{4}) - ip + p^2}dp = g_1(-u) \\ g_2(u) &= \frac{-i}{2\pi}\int_{-\infty}^{\infty}\frac{\sin up}{(s^2-\frac{1}{4}) - ip + p^2}dp = -g_2(-u). \end{aligned} \tag{46}$$

[1] In Ref.[11] the $(\sin\pi s)^{2(g-1)}$-dependence is missing.

Using the integrals [33, p.407]:

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{(b+cx)\sin ax}{p+qx+x^2}dx &= \left[\frac{cq-b}{\sqrt{p-q^2}}\sin aq + c\cos aq\right]\pi e^{-a\sqrt{p-q^2}}\\ \int_{-\infty}^{\infty}\frac{(b+cx)\cos ax}{p+qx+x^2}dx &= \left[\frac{b-cq}{\sqrt{p-q^2}}\cos aq + c\sin aq\right]\pi e^{-a\sqrt{p-q^2}}\end{aligned} \tag{47}$$

we get for $u>0$:

$$\begin{aligned} g_1(u) &= \frac{1}{2s}\cosh\tfrac{u}{2}e^{-us}\\ g_2(u) &= \frac{1}{2s}\sinh\tfrac{u}{2}e^{-us}.\end{aligned} \tag{48}$$

Therefore ($u\in\mathbf{R}$):

$$g(u) = \frac{1}{2s}e^{\frac{u}{2}-s|u|}, \tag{49}$$

which gives for $G(u,\chi)$

$$G(u,\chi_\gamma) = \frac{1}{s}e^{-us}(\cosh\tfrac{u}{2}-\chi_\gamma), \tag{50}$$

and the right hand side of the supertrace formula reads

$$\frac{1}{2s}\sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma}{e^{\frac{kl_\gamma}{2}}-e^{-\frac{kl_\gamma}{2}}}e^{-skl_\gamma}\left(e^{\frac{kl_\gamma}{2}}+e^{-\frac{kl_\gamma}{2}}-2\chi_\gamma{}^k\right) = \frac{1}{2s}\frac{Z_S'(s)}{Z_S(s)}. \tag{51}$$

For the A_0 we get

$$\begin{aligned} A_0 &= i(g-1)\int_{-\infty}^{\infty}h(p)\tanh\pi p\,dp\\ &= i\frac{g-1}{\pi}\int_0^\infty\frac{du}{\sinh\frac{u}{2}}\int_{-\infty}^{\infty}\frac{\sin up}{(s^2-\frac14)-ip+p^2}dp\\ &= \frac{1-g}{s}\int_0^\infty\frac{du}{\sinh\frac{u}{2}}\sinh\tfrac{u}{2}e^{-us} = \frac{1-g}{s^2},\end{aligned} \tag{52}$$

where the integral (47a) has been used. Therefore we have for the resolvent-kernel-function the supertrace formula

$$\sum_{n=1}^{\infty}\left[\frac{1}{s^2-(\lambda_n^B)^2}-\frac{1}{s^2-(\lambda_n^F)^2}\right]+\frac{\Delta n_0^{(0)}+g-1}{s^2} = \frac{1}{2s}\frac{Z_S'(s)}{Z_S(s)}. \tag{53}$$

Therefore Eqs. (53) and Z_S can be extended meromorphically to all $s\in\Lambda_\infty$. The very simple analytic structure of Z_S can be read off:

- : $s=0$ there is a zero with multiplicity $2(g-1+\Delta n_0^{(0)})$,
- : $s=\pm(\frac12+ip_n^B)$ there are zeros (poles) and
- : $s=\pm(\frac12+ip_n^F)$ there are poles (zeros),

with the same multiplicity as the corresponding Eigenvalue of $\square$, respectively. A very simple functional relation can be deduced from Eq.(53), reading

$$\frac{d\ln Z_S(s)}{ds} = \frac{d\ln Z_S(-s)}{ds}. \tag{54}$$

In terms of Z_0 and Z_1 Eq.(54) gives:[1]

$$\frac{d}{ds}\ln\frac{Z_0(s)Z_0(s+1)}{Z_1^2(s+\frac12)} = \frac{d}{ds}\ln\frac{Z_0(-s)Z_0(1-s)}{Z_1^2(\frac12-s)}. \tag{55}$$

Equation (54) or (55), respectively, integrated gives $Z_S(s)=Z_S(-s)$, thus $Z_S(s)$ is an even function in s. Combining Eqs.(31), (41) and (55) the functional equation for the R_0 function is deduced, which reads:

$$\frac{d}{ds}\ln R_0(s)R_0(-s) = 4\pi(g-1)\cot\pi s. \tag{56}$$

Equation (56) can be integrated to give

$$R_0(s)R_0(-s) = B_0(\sin\pi s)^{4(g-1)}. \tag{57}$$

where the constant B_0 is e.g. given by $B_0 = R_0(s_0)R_0(-s_0)(\sin\pi s_0)^{4(1-g)}$ with some $s_0\in\mathbf{C}$, where B_0 is independent of s_0, e.g. for $s=\pm\frac12$: $B_0 = Z_0(-\frac12)/Z_0(\frac32) = R_0(-\frac12)$. Of course, any information about the nontrivial zeros and poles is lost. Note that the same relation holds also for the ordinary Selberg zeta-function; however, in this case the integration constant is given by $B=2^{4(g-1)}$, see Eq.(11).

From Eqs.(32), (42) and (57) many relations linking Z_0 and Z_1 for particular arguments can be deduced, e.g.

$$B_0 = \frac{Z_0(-\frac12)}{Z_0(\frac32)} = \frac{Z_1(-1)Z_1(1)}{Z_1(2)Z_1(0)} = \frac{Z_1^2(0)}{Z_1^2(1)} = C_0^2 = \tilde{A}_1. \tag{58}$$

However, I do not see any valuable consequence as, e.g. determining from these relations the constants $\tilde{A}_1$, B_0 and C_0 like for the Selberg zeta-function. It is also not obvious to me to derive from these relations a functional relation for Z_0 or Z_1, respectively, like Eq.(10) for the ordinary Selberg zeta-function. On the contrary: I believe that such relations do not exist for Z_0 and Z_1, because we have an infinite number of critical lines for these two funtions. The immediate consequence of such relations, if they would exist, would be that we could solve the Eigenvalue problem for the operator $\square_0$ by just looking at the poles (for λ_n^B) and zeros (for λ_n^F) at, e.g. the critical line $\mathrm{Re}(s)=-\frac12$ for $Z_0(s)$. The values at the critical line $\mathrm{Re}(s)=-\frac12$ for $Z_0(s)$ would be related to the line $\mathrm{Re}(s)=\frac32$, where Z_0 could be easily calculated by Eq.(12) once a sufficient large enough set of geodesics $\{l_\gamma\}$ would be known. This is, however, very unlikely (but not a proof).

[1] In comparision to Ref.[61] one has to take the limit $a=1$ in the formulas.

1. Resolvent and Heat-Kernel

Since $\Box_m^2$ is not a positive definite operator I calculate the superdeterminant of $c^2-\Box_m^2$ for $\mathrm{Re}(c)>m$ and analytically continue in c. Similar considerations have been done by Aoki [3] by means of the supertrace of the heat kernel of $\Box_m^2$. Fitted with the knowledge of the analytical properties of the Selberg super zeta-functions I can avoid the indirect reasoning of Aoki to get the superdeterminants in compact form. For this purpose the functional relations for Z_0 and Z_1 of the previous chapter are used. These functional relations have not been available in [3]; without proof Aoki has used the functional relation of the Selberg zeta-function, assuming that it is also valid in the super case. As discussed at the end of the previous section this seems to be very unlikely that such a functional relation exist. Furthermore, statements of the spectrum of the operators $\widehat{\Box}_m$ and its relation to the spectrum of $\Box$ can be made (and similarly for $\Box_m$ which I do not consider explicitly).

Let be $m\in\mathbf{N}_0$. The superdeterminant is defined using the ζ-function regularization as[1]

$$\begin{aligned}\mathrm{sdet}(c^2-\Box_m^2)&=\exp\left[-\frac{\partial}{\partial s}\zeta_m(s;c)\Big|_{s=0}\right]\\ \zeta_m(s;c)&=\mathrm{str}\big[(c^2-\Box_m^2)^{-s}\big]\\ &=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\mathrm{str}\{\exp[-t(c^2-\Box_m^2)]\},\end{aligned}\tag{1}$$

where use has been made of the integral [33, p.317]:

$$\int_0^\infty x^{\nu-1}e^{-\mu x}dx=\mu^{-\nu}\Gamma(\nu).\tag{2}$$

So one has two possibilities to study $\mathrm{sdet}(c^2-\Box_m^2)$:
1) by computing $\mathrm{str}[(c^2-\Box_m^2)^{-s}]$ (generalized resolvent) or
2) by computing $\mathrm{str}\{\exp[-t(c^2-\Box_m^2)]\}$ (heat-kernel).
I prefer the second method. It has the advantage that integration constants are explicitly given (see below). The function h corresponding to the heat-kernel of $c^2-\Box_m^2$ reads

$$h_{hk}(s)=e^{t[(s+\frac{m}{2})^2-c^2]}.\tag{3}$$

Therefore for $g(u)$

$$\begin{aligned}g(u)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iup}h_{hk}(ip+\tfrac12)dp\\ &=\frac{1}{2\pi}e^{t[(\frac{m+1}{2})^2-c^2]}\int_{-\infty}^{\infty}e^{-p^2t+i[(m+1)t-u]p}dp\\ &=\frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{u^2}{4t}-c^2t+(m+1)\frac{u}{2}\right].\end{aligned}\tag{4}$$

[1] This method of regularization of determinants by zeta-functions was introduced by Ray and Singer [73] in differential geometry and Hawking [47] in field theory.

This gives

$$\begin{aligned}G(u,\chi)&=\frac{1}{\sqrt{\pi t}}e^{-u^2/4t-c^2t}\big[\cosh(m+1)\tfrac{u}{2}-\chi\cosh m\tfrac{u}{2}\big]\\ g(u)-g(-u)&=\frac{1}{\sqrt{\pi t}}e^{-u^2/4t-c^2t}\sinh(m+1)\tfrac{u}{2}.\end{aligned}\tag{5}$$

Splitting now the calculation of $\zeta_m(s;c)$ into two terms corresponding to the identity transformation and the length term, respectively, gives:

$$\zeta_m(s;c)=\zeta_m^I(s;c)+\zeta_m^\Gamma(s;c).\tag{6}$$

I first calculate ζ_m^I:

$$\begin{aligned}\zeta_m^I(s;c)&=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}A_0^{(m)}(t)dt\\ A_0^{(m)}(t)&=\frac{1-g}{\sqrt{\pi t}}e^{-c^2t}\int_0^\infty e^{-\frac{u^2}{4t}}\frac{\sinh(m+1)\frac{u}{2}}{\sinh\frac{u}{2}}\cosh m\tfrac{u}{2}\,du\\ &=(1-g)e^{-c^2t}\sum_{k=0}^{m}e^{k^2t}.\end{aligned}\tag{7}$$

Equation (7) can be proved by induction:
First step: $m=0$:

$$A_0^{(0)}(t)=\frac{1-g}{\sqrt{\pi t}}\int_0^\infty e^{-u^2/4t}du=(1-g)e^{-c^2t}.\tag{8}$$

Second step: $m\to m+1$:

$$\begin{aligned}A_0^{(m+1)}(t)&=\frac{1-g}{\sqrt{\pi t}}e^{-c^2t}\int_0^\infty\frac{e^{-u^2/4t}}{\sinh\frac{u}{2}}\sinh(m+2)\tfrac{u}{2}\cosh(m+1)\tfrac{u}{2}\,du\\ &=\frac{1-g}{\sqrt{\pi t}}e^{-c^2t}\int_0^\infty\frac{e^{-u^2/4t}}{\sinh\frac{u}{2}}\big[\cosh(m+1)u\sinh\tfrac{u}{2}+\sinh(m+1)\tfrac{u}{2}\cosh m\tfrac{u}{2}\big]du\\ &=A_0^{(m)}+\frac{1-g}{\sqrt{\pi t}}e^{-c^2t}\int_0^\infty e^{-u^2/4t}\cosh(m+1)u\,du\\ &=(1-g)\left[e^{-c^2t}\sum_{k=0}^{m}e^{k^2t}+e^{-c^2t+(m+1)^2t}\right]=(1-g)e^{-c^2t}\sum_{k=0}^{m+1}e^{k^2t},\end{aligned}\tag{9}$$

where the integral [33, p.317]:

$$\int_0^\infty e^{-\beta x^2}\cosh\alpha x\,dx=\frac12\sqrt{\frac{\pi}{\beta}}e^{\alpha^2/4\beta}\tag{10}$$

has been used.

■

Similarly:

$$\begin{aligned}A_0^{(-1)}&=0\\ A_0^{(-m)}(t)&=(g-1)e^{-c^2t}\sum_{k=0}^{m-2}e^{k^2t}\qquad(m=2,3,\ldots).\end{aligned}\tag{11}$$

This gives for ζ_m^I:

$$\zeta_m^I(s;c) = \frac{1-g}{\Gamma(s)}\sum_{k=0}^{m}\int_0^\infty t^{s-1}e^{-(c^2-k^2)t}\,dt = (1-g)\sum_{k=0}^{m}(c^2-k^2)^{-s}. \tag{12}$$

For later use it is easily calculated

$$\frac{\partial}{\partial s}\zeta_m^I(s;c)\bigg|_{s=0} = (g-1)\sum_{k=0}^{m}\ln(c^2-k^2). \tag{13}$$

Let us calculate ζ_m^Γ in two alternative ways. The first is appropriate to the analysis of the spectrum, the second to the calculation of the superdeterminants.
1) The supertrace formula for the heat-kernel now reads:

$$\sum_{n=1}^{\infty}\left\{e^{t[(\lambda_{n,m}^B)^2-c^2]} - e^{[t(\lambda_{n,m}^F)^2-c^2]}\right\} = (1-g)e^{-c^2t}\sum_{k=0}^{m}e^{k^2t}$$
$$+\frac{e^{-c^2t}}{\sqrt{4\pi t}}\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma\chi_\gamma{}^{km}}{\sinh\frac{kl_\gamma}{2}}e^{-\frac{k^2l_\gamma{}^2}{4t}}\left[\cosh(m+1)\frac{kl_\gamma}{2} - \chi_\gamma{}^k\cosh m\frac{kl_\gamma}{2}\right] \tag{14}$$

and the A_0 term appropriately replaced for negative integers. With the help of Eqs.(2), (7) and the integral [33, p.340]:

$$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}-\gamma x}dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}K_\nu(2\sqrt{\beta\gamma}) \tag{15}$$

this gives for the supertrace formula of the generalized resolvent kernel:

$$\sum_{n=0}^{\infty}\left[\frac{1}{[c^2-(\lambda_{n,m}^B)^2]^s} - \frac{1}{[c^2-(\lambda_{n,m}^F)^2]^s}\right]$$
$$= (1-g)\sum_{k=0}^{m}\frac{1}{(c^2-k^2)^s} + \frac{2}{\Gamma(s)}\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma\chi_\gamma{}^{km}}{e^{\frac{kl_\gamma}{2}}-e^{-\frac{kl_\gamma}{2}}}\left(\frac{kl_\gamma}{2c}\right)^{s-\frac{1}{2}}$$
$$\times K_{s-\frac{1}{2}}(ckl_\gamma)\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right) - \chi_\gamma{}^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]. \tag{16}$$

This gives explicitly for $s=1$ (m even):

$$\sum_{n=0}^{\infty}\left[\frac{1}{c^2-(\lambda_{n,m}^B)^2} - \frac{1}{c^2-(\lambda_{n,m}^F)^2}\right] + (g-1)\sum_{k=0}^{m}\frac{1}{c^2-k^2}$$
$$= \frac{1}{2c\sqrt{\pi}}\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma}{e^{\frac{kl_\gamma}{2}}-e^{-\frac{kl_\gamma}{2}}}e^{-kcl_\gamma}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right) - \chi_\gamma{}^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]$$
$$= \frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_0(\frac{m}{2}+c+1)Z_0(c-\frac{m}{2})}{Z_1(c+\frac{m+1}{2})Z_1(c+\frac{1-m}{2})}\right], \tag{17}$$

where Eq.(V.37) has been used (logaritmic derivative of the super zeta-functions). For $s=1$ and m odd:

$$\sum_{n=0}^{\infty}\left[\frac{1}{[c^2-(\lambda_n^B)^2]} - \frac{1}{[c^2-(\lambda_n^F)^2]}\right] + (g-1)\sum_{k=0}^{m}\frac{1}{c^2-k^2}$$
$$= \frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_1(\frac{m}{2}+c+1)Z_1(c-\frac{m}{2})}{Z_0(\frac{m+1}{2}+c)Z_0(c+\frac{1-m}{2})}\right]. \tag{18}$$

2) Let first m be an even number. Let us consider the representation ($\mathrm{Re}(s)<1$):

$$t^{s-1} = \frac{2}{\Gamma(1-s)}\int_0^\infty \frac{\lambda+c}{[\lambda(\lambda+2c)]^s}e^{-\lambda(\lambda+2c)t}d\lambda; \tag{19}$$

This integral representation follows with the help of [33, p.318] ($\mathrm{Re}(\nu)>-1$):

$$\int_u^\infty (x-u)^\nu e^{-\mu x}dx = \mu^{-\nu-1}\Gamma(\nu+1)e^{-u\mu}. \tag{20}$$

Therefore we get for $\zeta_m^\Gamma(c;s)$ with the help of Eq.(14) and the representation $K_{\pm\frac{1}{2}}(z) = \sqrt{\pi/2z}e^{-z}$:

$$\zeta_m^\Gamma(s;c) = \frac{\sin\pi s}{\pi}\int_0^\infty\frac{d\lambda}{[\lambda(\lambda+2c)]^s}$$
$$\times 2\sum_{\gamma\in\Gamma}\sum_{k=1}^{\infty}\frac{l_\gamma}{1-e^{-kl_\gamma}}e^{-kl_\gamma(\lambda+c+\frac{1}{2})}\left[\cosh\left(\frac{m+1}{2}kl_\gamma\right) - \chi_\gamma{}^k\cosh\left(\frac{m}{2}kl_\gamma\right)\right]$$
$$= \frac{\sin\pi s}{\pi}\int_0^\infty\frac{d\lambda}{[\lambda(\lambda+2c)]^s}\frac{d}{d\lambda}\ln\left[\frac{Z_0(\frac{m}{2}+\lambda+c+1)Z_0(\lambda+c-\frac{m}{2})}{Z_1(\frac{m+1}{2}+\lambda+c)Z_1(\frac{m-1}{2}+\lambda+c)}\right], \tag{21}$$

where Eq.(V.37) has been used again. Let be $f(s) = \sin(\pi s)[\lambda(\lambda+2c)]^{-s}]$. Then $f'(s)|_{s=0} = \pi$ and we get for $\zeta'(0;c)$ ($\mathrm{Re}(s)>m$):

$$\zeta'(0;c) = (g-1)\sum_{k=0}^{m}\ln(c^2-k^2)$$
$$+\int_0^\infty d\lambda\frac{d}{d\lambda}\ln\left[\frac{Z_0(\lambda+\frac{m}{2}+c+1)Z_0(\lambda-\frac{m}{2}+c)}{Z_1(\lambda+\frac{m+1}{2}+c)Z_1(\lambda+\frac{1-m}{2}+c)}\right]$$
$$= (g-1)\sum_{k=0}^{m}\ln(c^2-k^2) - \ln\left[\frac{Z_0(c+1+\frac{m}{2})Z_0(c-\frac{m}{2})}{Z_1(c+\frac{m+1}{2})Z_1(c+\frac{1-m}{2})}\right]. \tag{22}$$

Here it has been have used that $\lim_{s\to\infty}Z_q(s)=1$, which follows at once from the Euler product representation of the Selberg super zeta-functions. Therefore ($m = 0, 2, \ldots$):

$$\mathrm{sdet}(c^2-\Box_m^2) = \frac{Z_0(c+\frac{m}{2}+1)Z_0(c-\frac{m}{2})}{Z_1(c-\frac{m+1}{2})Z_1(c+\frac{1-m}{2})c^{2(g-1)}}\prod_{k=1}^{m}(c^2-k^2)^{1-g}. \tag{23}$$

Similarly ($m = 2, 4, \ldots$):

$$\operatorname{sdet}(c^2 - \Box^2_{-m}) = \frac{Z_0(c - \frac{m}{2} + 1) Z_0(c + \frac{m}{2})}{Z_1(c + \frac{m+1}{2}) Z_1(c + \frac{1-m}{2})} \prod_{k=0}^{m-2} (c^2 - k^2)^{g-1}. \tag{24}$$

For m an odd number the roles of Z_0 and Z_1 are just reversed and it follows immediately ($m = 1, 3, \ldots$):

$$\operatorname{sdet}(c^2 - \Box^2_m) = \frac{Z_1(c + 1 + \frac{m}{2}) Z_1(c - \frac{m}{2})}{Z_0(c + \frac{m+1}{2}) Z_0(c + \frac{1-m}{2}) c^{2g-2}} \prod_{k=0}^{m-2} (c^2 - k^2)^{1-g}. \tag{25}$$

Similarly ($m = 1, 3, \ldots$):

$$\operatorname{sdet}(c^2 - \Box^2_{-m}) = \frac{Z_1(c + 1 - \frac{m}{2}) Z_1(c + \frac{m}{2})}{Z_0(c + \frac{m+1}{2}) Z_0(c + \frac{1-m}{2})} \prod_{k=1}^{m-2} (c^2 - k^2)^{g-1}. \tag{26}$$

Equations (23-26) are the starting points for the calculation of determinants. Because the super zeta-functions are meromorphic functions in Λ_∞, the same holds for the superdeterminants.

Let us denote by $\hat\Theta(t) := \operatorname{str}'[\exp(t\Box_0^2)]$. Thus:

$$\operatorname{sdet}'(c^2 - \Box_0^2) = \exp\left\{ -\frac{\partial}{\partial s}\left[\frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} e^{-tc^2} \hat\Theta(t)\Big|_{s=0}\right]\right\}. \tag{27}$$

Some statements about $\hat\Theta$ can be made and an equation expressing $\hat\Theta$ by the zeta-function Z_S can be derived. Let us consider the supertrace formula for the resolvent kernel:

$$\operatorname{str}'(c^2 - \Box_0^2)^{-1} = \int_0^\infty e^{-tc^2} \hat\Theta(t) dt = \frac{1}{2c} \frac{Z_S'(c)}{Z_S(c)} - \frac{g - 1 + \Delta n_0^{(0)}}{c^2}. \tag{28}$$

This equation can be inverted by the theory of Laplace transformations yielding (see e.g. [25], pp.129):

$$\begin{aligned} \hat\Theta(t) &= \frac{1}{2\pi} \int_{b-i\infty}^{b+i\infty} \left[\frac{1}{2c} \frac{Z_S'(c)}{Z_S(c)} - \frac{g - 1 + \Delta n_0^{(0)}}{c^2}\right] dc^2 \\ &= -\frac{1}{\sqrt{4\pi t}} \int_0^\infty u e^{-u^2/4t} (\mathcal{L}^{-1} \ln Z_S)(u)\, du - (g - 1 + \Delta n_0^{(0)}). \end{aligned} \tag{29}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transformation. In particular this gives

$$\hat\Theta(0) = -(g - 1 + \Delta n_0^{(0)}); \tag{30}$$

this result is consistent with equation (14). Equation (14) gives also that for $t \to \infty$ the supertrace for the heat-kernel for $\Box_m^2$ diverges according to ($m \in \mathbf{N}$)

$$\Theta_m(t) = \operatorname{str}'[\exp(t\Box_m^2)] \simeq (1 - g) e^{m^2 t}, \qquad (t \to \infty) \tag{31}$$

(and similar for negative integers), a result found by Aoki [3].

2. Discussion of the Spectrum

The operator $\hat\Box$ is simpler to study than the operator $\Box_m$, because the trivial contribution $\frac{m}{2}$ to the Eigenvalues has been subtracted (see Eq.(III.49) for the unitary equivalence between $\Box_m$ and $\hat\Box_m + \frac{m}{2}$).

For applying the supertrace formula for the operator $\hat\Box_m$ the formulas must be changed appropriately - see Eqs.(IV.34):

$$\begin{aligned} A_0^{(m)} &= (1 - g) \int_0^\infty \frac{g(u) - g(-u)}{\sinh \frac{u}{2}} T_m(\cosh \tfrac{u}{2}) du \\ G(u, \chi) &= g(u) + g(-u) - \chi\left[g(u) e^{-\frac{u}{2}} + g(-u) e^{\frac{u}{2}}\right], \end{aligned} \tag{32}$$

where

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-iup} h(ip + \tfrac{1}{2}) dp. \tag{33}$$

Thus the function h has taken on the argument $ip + \frac{1}{2}$. This gives immediately:

$$\begin{aligned} g(u) &= \frac{1}{\sqrt{4\pi t}} e^{-u^2/4t - c^2 t + \frac{u}{2}} \\ G(u, \chi) &= \frac{1}{\sqrt{\pi t}} e^{-u^2/4t - c^2 t} (\cosh \tfrac{u}{2} - \chi) \\ A_0^{(m)} &= \frac{1 - g}{\sqrt{\pi t}} e^{-c^2 t} \int_0^\infty e^{-u^2/4t} \cosh m \tfrac{u}{2}\, du \\ &= (1 - g) e^{-(c^2 - \frac{m^2}{4})t}, \end{aligned} \tag{34}$$

where in the last expression the integral (9) has been used. Therefore we get for the supertrace formula for the resolvent of $\hat\Box_m^2$ for m even ("hatted" quantities belonging to $\hat\Box$):

$$\sum_{n=1}^\infty \left[\frac{1}{c^2 - (\hat\lambda_{n,m}^B)^2} - \frac{1}{c^2 - (\hat\lambda_{n,m}^F)^2}\right] = \frac{1 - g}{(c - \frac{m}{2})(c + \frac{m}{2})} + \frac{1}{2c} \frac{d}{dc} \ln\left[\frac{Z_0(c+1) Z_0(c)}{Z_1^2(c + \frac{1}{2})}\right]; \tag{35}$$

and similarly for m odd:

$$\sum_{n=1}^\infty \left[\frac{1}{c^2 - (\hat\lambda_{n,m}^B)^2} - \frac{1}{c^2 - (\hat\lambda_{n,m}^F)^2}\right] = \frac{1 - g}{(c - \frac{m}{2})(c + \frac{m}{2})} + \frac{1}{2c} \frac{d}{dc} \ln\left[\frac{Z_1(c+1) Z_1(c)}{Z_0^2(c + \frac{1}{2})}\right]. \tag{36}$$

Analysing for the particular values $c = \epsilon$ and $c = \pm\frac{m}{2} + \epsilon$ this gives for m even ($|\epsilon| \ll 1$):

$$\frac{1}{2c} \frac{d}{dc} \ln\left[\frac{Z_0(c+1) Z_0(c)}{Z_1^2(c + \frac{1}{2})}\right] - \frac{g - 1}{(c - \frac{m}{2})(c + \frac{m}{2})} \propto \begin{cases} \dfrac{\Delta \hat n_0^{(0)}}{\epsilon^2} & (c = \epsilon,\ m = 0), \\ \pm \dfrac{g - 1 + \Delta \hat n_0^{(0)}}{\epsilon^2} & (c = \epsilon,\ m \neq 0), \\ \mp \dfrac{1 - g}{m\epsilon} & (c = \pm\frac{m}{2} - \epsilon,\ m \neq 0) \end{cases} \tag{37}$$

and regularly otherwise up to the nontrivial zeros and poles of Z_0 and Z_1. For m odd for $c = \pm\frac{1}{2} + \epsilon$ and $c = \pm\frac{m}{2} + \epsilon$ ($|\epsilon| \ll 1$):

$$\frac{1}{2c}\frac{d}{dc}\ln\left[\frac{Z_1(c+1)Z_1(c)}{Z_0^2(c+\frac{1}{2})}\right] - \frac{g-1}{(c-\frac{m}{2})(c+\frac{m}{2})} \propto \begin{cases} \mp\dfrac{g-1+\Delta\hat{n}_0^{(0)}}{\epsilon} & (c = \pm\frac{1}{2}+\epsilon,\ m=1), \\ \mp\dfrac{\Delta\hat{n}_0^{(0)}}{\epsilon} & (c=\pm\frac{1}{2}+\epsilon,\ m\neq 1), \\ \pm\dfrac{1-g}{m\epsilon} & (c=\pm\frac{m}{2}+\epsilon, m\neq 1) \end{cases} \tag{38}$$

and regularly otherwise up to the nontrivial zeros and poles of Z_0 and Z_1. Let us discuss two scenarios for $\Delta\hat{n}_0^{(0)}$:

1) $\Delta\hat{n}_0^{(0)} = 0$: This yields for the various trivial modes of $\widehat{\Box}_m$ for m even:

<u>m even:</u>

$\Delta\hat{n}_0^{(0)} = 0$

$\Delta\hat{n}_0^{(m)} = g-1 \qquad (m\neq 0),$

$\Delta\hat{n}_{\pm\frac{m}{2}}^{(m)} = 1-g \qquad (m\neq 0);$

<u>m odd:</u>

$\Delta\hat{n}_0^{(1)} = g-1$

$\Delta\hat{n}_{\pm\frac{1}{2}}^{(m)} = 0 \qquad (m\neq 1),$

$\Delta\hat{n}_{\pm\frac{m}{2}}^{(m)} = 1-g \qquad (m\neq 1).$

2) $\Delta\hat{n}_0^{(0)} = 1-g$: In a similar way:

<u>m even:</u>

$\Delta\hat{n}_0^{(0)} = 1-g$

$\Delta\hat{n}_0^{(m)} = 0 \qquad (m\neq 0),$

$\Delta\hat{n}_{\pm\frac{m}{2}}^{(m)} = 1-g; \qquad (m\neq 0);$

<u>m odd:</u>

$\Delta\hat{n}_0^{(1)} = 0$

$\Delta\hat{n}_{\pm\frac{1}{2}}^{(m)} = g-1 \qquad (m\neq 1),$

$\Delta\hat{n}_{\pm\frac{m}{2}}^{(m)} = 1-g \qquad (m\neq 1).$

That trivial-modes or trivial Eigenvalues (as the trivial-modes of $\Box_m$) appear can be understood in the view of the corresponding results for the classical Laplacian $-\Delta_m$ as discussed, e.g. by Hejhal [49, p.408]. Let $\{\lambda_n^{(m)}\}$ be the set of all Eigenvalues of the Laplace-operator $-\Delta_m = -y^2(\partial_x^2+\partial_y^2) + imy\partial_x$ and $m \geq 2$. Then ($n\in\mathbf{N}$):

$$\{\lambda_n^{(m)}\} = \{\tfrac{m}{2}(1-\tfrac{m}{2})\}_{k=1}^{d} \cup \{\lambda_n^{(m-2)} | \lambda_n^{(m-2)} \neq \tfrac{m}{2}(1-\tfrac{m}{2})\}, \tag{39}$$

where $d = \delta + (g-1)(m-1)$ and δ takes on the values 0 and 1, depending on m. There are several methods of obtaining this result. E.g. one can first consider the Selberg trace formula for the (regularized) resolvent-kernel function and deduces this statement from the analytical properties of the Selberg zeta-function (nontrivial Eigenvalues) and the poles occuring in the A_0 term (trivial-modes); second, one can consider commutation relations of the differential operators ∇_z^k acting on tensorfields which give simple recursion formulas for the Laplacian Δ_m depending on the curvature R of the space in question [16].

Equations (35) and (36) give also the relation of the Eigenvalues $\hat{\lambda}_{n,m}$ of $\widehat{\Box}_m$ and $\hat{\lambda}_n$ of $\widehat{\Box}_0$. Due to the analytic structure of the super zeta-functions for the nontrivial Eigenvalues this gives:

$$\hat{\lambda}_{n,m} = \hat{\lambda}_n, \qquad (n\in\mathbf{N}, m\in\mathbf{N}). \tag{40}$$

This simple result corresponds again to the classical one noted in Eq.(39).

3. Determinants and the Fermionic String Integrand

The starting points for the calculation of determinants of the operator $\Box_m^2$ are Eqs.(23-26) which all can be analytically continued to $c=0$ (including a omission of zero-modes if necessarry). Let us first consider Eq.(23) for $m=0$. Performing the limit $c\to\epsilon$ for $|\epsilon|\ll 1$:

$$\mathrm{sdet}(-\Box_0^2) = \frac{1}{(2g-2)!}\cdot\frac{Z_0(1)Z_0^{(2g-2)}(0)}{[\tilde{Z}_1(\frac{1}{2})]^2}\epsilon^{2\Delta n_0^{(0)}}. \tag{41}$$

Here by $\tilde{Z}_1(\frac{1}{2})$ the appropriate derivative or residuum of Z_1 at $s=\frac{1}{2}$ is denoted, depending whether $\Delta n_0^{(0)} \leq 0$ or $\Delta n_0^{(0)} > 0$, respectively. To make this quantity well-defined I subtract from $\mathrm{sdet}(-\Box_0^2)$ the zero-mode which I denote by priming the sdet. Using further the functional relation (V.42) for Z_0 and Z_1 I get finally:

$$\mathrm{sdet}'(-\Box_0^2) = \left[\pi^{g-1}\frac{Z_0(1)}{\tilde{Z}_1(\frac{1}{2})}\right]^2\frac{Z_1(0)}{Z_1(1)}. \tag{42}$$

For calculating the superdeterminant for m even and $m\geq 2$ a subtraction of zero- or trivial-modes is not necessary. Proceeding similarly as for $m=0$ we get for $m=2,4,\ldots$:

$$\mathrm{sdet}(-\Box_m^2) = \left[\left(\frac{\pi}{m!}\right)^{g-1}\frac{Z_0(1+\frac{m}{2})}{Z_1(\frac{m+1}{2})}\right]^2\frac{Z_1(0)}{Z_1(1)}. \tag{43}$$

Similarly ($m=2,4,\ldots$):

$$\mathrm{sdet}(-\Box_{-m}^2) = \left[\left(\frac{(m-2)!}{\pi}\right)^{g-1}\frac{Z_0(\frac{m}{2})}{Z_1(\frac{m+1}{2})}\right]^2\frac{Z_1(1)}{Z_1(0)}. \tag{44}$$

For $m=1,3,\ldots$:

$$\mathrm{sdet}(-\Box_m^2) = \left[\left(\frac{\pi}{i\,m!}\right)^{g-1}\frac{Z_1(1+\frac{m}{2})}{Z_0(\frac{m+1}{2})}\right]^2\frac{Z_1(0)}{Z_1(1)}; \tag{45}$$

and $m=3,5,\ldots$:

$$\mathrm{sdet}(-\Box_{-m}^2) = \left[\left(\frac{(m-2)!i}{\pi}\right)^{g-1}\frac{Z_1(\frac{m}{2})}{Z_0(\frac{m+1}{2})}\right]^2\frac{Z_1(1)}{Z_1(0)}. \tag{46}$$

Note the differences to Ref.[3] which are due to the additional super zeta-functions. The case of $\Box_{-1}^2$ must be treated separately because of the appearence of zero-modes which must be subtracted. Therefore denoting the omission of zero-modes by priming the super determinant I get

$$\mathrm{sdet}'(-\Box^2_{-1}) = \left[\pi^{1-g}\frac{\bar{Z}_1(\frac{1}{2})}{Z_0(1)}\right]^2 \frac{Z_1(1)}{Z_1(0)}. \qquad (47)$$

From the introduction we know that the relevant string integrand is given by $\mathrm{sdet}'(-\Box^2_0)$ and $\mathrm{sdet}(-\Box^2_2)$. Equations (42) and (43) yield:

$$\begin{aligned}
&\left[\mathrm{sdet}'(-\Box^2_0)\right]^{-\frac{5}{2}}\left[\mathrm{sdet}(-\Box^2_2)\right]^{\frac{1}{2}} \\
&= \left(\pi^{g-1}\frac{Z_0(1)}{\bar{Z}_1(\frac{1}{2})}\right)^{-5}\left(\frac{\pi}{2}\right)^{g-1}\frac{Z_0(2)}{Z_1(\frac{3}{2})}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2,
\end{aligned}$$

or alternatively

$$= \left(\pi^{g-1}\frac{Z_0(1)}{\bar{Z}_1(\frac{1}{2})}\right)^{-5}\left(\frac{\pi}{2}\right)^{g-1}\frac{Z_0(2)}{Z_1(\frac{3}{2})}\frac{Z_0(\frac{3}{2})}{Z_0(-\frac{1}{2})}, \qquad (48)$$

and we can conclude that this expression is well defined. Furthermore for Z_g of Eq.(I.38):

$$\begin{aligned}
Z_g &= \int_{s\mathcal{M}_g} d(SWP)\left[\mathrm{sdet}'(-\Box^2_0)\right]^{-\frac{5}{2}}\left[\mathrm{sdet}'(-\Box^2_2)\right]^{\frac{1}{2}} \\
&= \left(\frac{1}{2\pi^4}\right)^{g-1}\int_{s\mathcal{M}_g} d(SWP)\left(\frac{Z_0(1)}{\bar{Z}_1(\frac{1}{2})}\right)^{-5}\frac{Z_0(2)}{Z_1(\frac{3}{2})}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2. \qquad (49)
\end{aligned}$$

This is the main result of this section. Note the appearence of the various ratios of the Selberg super zeta-functions. The main difference to Aoki [3] who first calculated super determinants of Laplace-Dirac operators lies in the additional factor $[Z_1(1)/Z_1(0)]^2$ in the superdeterminants. This factor is unambigously given by the functional equations which have been used to derive Eq.(48) and it changes the supermoduli dependence of the integrand. In the result of Ref.[3] this factor did not appear because the functional relations of the Selberg super zeta-functions were not available.

Unfortunately no statement, like the analysis of Gross and Periwal [45], about the growing properties of this expression for increasing genus g can be made because an expansion in the super moduli gives unknown signs in the soul-contributions of the Selberg super zeta-functions. Gross and Periwal found for the relevant terms in the genus-expansion of the Polyakov partition function Z for the bosonic string

$$\left.\begin{aligned}
Z &= \sum_{g=0}^{\infty} Z_g \\
Z_g &= \int_{\mathcal{M}_g} d(WP)\left[\det(\Delta_0^-)\right]^{-13}\det(\Delta_1^-) \\
\det{}'(\Delta_0^-) &= Z'(1)e^{(g-1)[\ln 2\pi - 4\zeta'(-1)] - \frac{1}{2}}, \\
\det(\Delta_1^-) &= Z(2)e^{(g-1)[3\ln 2\pi - 4\zeta'(-1) - \frac{9}{2}]}, \\
Z'(1) &= C_\epsilon B_\epsilon^{4\pi(g-1)}\exp\left\{-\int_\epsilon^\infty \frac{dt}{t}\left[\Theta(t) - 1\right]\right\} \\
Z(2) &= Z(1+\delta) = Z'(1)K(\delta,\epsilon)^{4\pi(g-1)}M(\delta,\epsilon),
\end{aligned}\right\} \qquad (50)$$

where $d(WP)$ is the Weil-Peterson measure, $\mathcal{M}_g$ the moduli space, Z the usual Selberg zeta-function, Θ the trace of the heat-kernel on compact domains on the Poincaré upper half-plane and C_ϵ, B_ϵ, $K(\delta,\epsilon)$ and $M(\delta,\epsilon)$ are constants independent of the genus g. From Eq.(50) one finds that $Z'(1)^{-1}$ and $Z(2)$ have an exponential growth for increasing genus; together with the properties of the Weil-Peterson measure, i.e. that its volume increases factorial-like for increasing genus one concludes that the pertubation series in g for the partition function Z diverges.

Nevertheless we can discuss the behaviour of the fermionic string integrand for the case of degenerate super Riemann surfaces. For this purpose let us consider such a surface, i.e. a pinching process takes place and at least the lenght of one geodesic vanishes. Let l_0 be the geodesics of $\gamma_0 \in \Gamma$ with $(l_0)_{Body} < (l_\gamma)_{Body}$ for all $\gamma \in \Gamma$ with $\gamma \neq \gamma_0$. Let us introduce the partial zeta-functions $\mathcal{Z}_q(s) = \mathcal{Z}_q(s, l_0)$ with

$$\mathcal{Z}_q(s) := \prod_{n=0}^{\infty}\left[1 - \chi^q_{\gamma_0}e^{-(s+n)l_0}\right], \qquad (q = 0,1,\ \mathrm{Re}(s) > 1). \qquad (51)$$

For the entire zeta-functions one has

$$Z_q(s) = \left[\mathcal{Z}_q(s)\right]^{g(l_0)}\prod_{\substack{\gamma\in\Gamma \\ \gamma\neq\gamma_0}}\prod_{k=0}^{\infty}\left[1 - \chi^q_{\gamma_0}e^{-(s+k)l_\gamma}\right], \qquad (\mathrm{Re}(s) > 1). \qquad (52)$$

where $g(l_0)$ denotes the multiplicity of l_0. A discussion for the bosonic string is due to Wolpert [95] who showed that for $l_0 \to 0$ one has (set $q = 0$ in Eq.(51) and interpret all quantities in terms of the bosonic case)

$$\mathcal{Z}(s) \propto l_0^{\frac{1}{2}-s}\exp\left(-\frac{\pi^2}{6l_0} + O(l_0)\right), \qquad (l_0 \to 0). \qquad (53)$$

This asymptotic behaviour has the immediate consequence that the bosonic string has a divergence due to geodesics of zero-length. This result can be generalised to the fermionic string! To see this let us start by taking the logarithm of partial zeta-function:

$$\begin{aligned}
-\ln \mathcal{Z}_q(s) &= -\sum_{n=0}^{\infty}\ln(1 - \chi^q_{\gamma_0}e^{-(s+n)l_0}) \\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\chi^k_{\gamma_0}e^{-k(s+n)l_0} = \sum_{k=1}^{\infty}\frac{\chi^k_{\gamma_0}e^{-ksl_0}}{k(1 - e^{-kl_0})} \\
&= \frac{1}{l_0}\sum_{k=1}^{\infty}\frac{\chi^k_{\gamma_0}e^{-ksl_0}}{k^2} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{\chi^k_{\gamma_0}e^{-ksl_0}}{k} + \frac{l_0}{12}\sum_{k=1}^{\infty}\chi^k_{\gamma_0}e^{-ksl_0} + \ldots, \qquad (54)
\end{aligned}$$

where the denominator was expanded as

$$\frac{1}{1 - e^{-kl_0}} = \frac{1}{kl_0} + \frac{1}{2} + \frac{kl_0}{12} + O(k^2 l_0^2). \qquad (55)$$

For the various sums one gets

$$\begin{aligned}
\frac{1}{l_0}\sum_{k=1}^{\infty}\frac{\chi^k_{\gamma_0}e^{-ksl_0}}{k^2} &= \frac{1}{l_0}z\Phi(z,2,1), \qquad (z = \chi_{\gamma_0}e^{-sl_0}) \\
&= \frac{1}{l_0}\left(\frac{\pi^2}{6} + \ln\chi_{\gamma_0} - sl_0 - (sl_0 - \ln\chi_{\gamma_0})\ln(sl_0 - \ln\chi_{\gamma_0})\right) + O(l_0). \qquad (56)
\end{aligned}$$

where $\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}$ is Lerch's transcendent [55, p.32]. With the expansion

$$\Phi(z,m,a) = z^{-a}\left\{ \sum_{\substack{n=0\\ n\neq m-1}}^{\infty} \zeta(m-n,a)\frac{(\ln z)^n}{n!} + \frac{(\ln z)^{m-1}}{(m-1)!}\left[\Psi(m) - \Psi(a) - \ln\left(\ln\frac{1}{z}\right)\right]\right\} \tag{57}$$

(note $\zeta(s,1) = \zeta(s)$, $\zeta(2) = \pi^2/6$ and $\Psi(2) - \Psi(1) = 1$). Furthermore

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^{\infty} \frac{\chi_{\gamma_0}^k e^{-ksl_0}}{k} &= -\frac{1}{2}\ln(1-\chi_{\gamma_0}e^{-sl_0}) \\ &= -\frac{1}{2}\ln(sl_0) + O(l_0), \qquad (\chi_{\gamma_0} = 1), \end{aligned} \tag{58}$$

$$\begin{aligned} \frac{l_0}{12}\sum_{k=1}^{\infty} \chi_{\gamma_0}^k e^{-skl_0} &= \frac{l_0}{12}\frac{\chi_{\gamma_0}e^{-sl_0}}{1-\chi_{\gamma_0}e^{-sl_0}} \\ &= \frac{1}{sl_0} - \frac{1}{2} + O(l_0), \qquad (\chi_{\gamma_0} = 1). \end{aligned} \tag{59}$$

The last two expansions are valid for $(sl_0)_{Body} > 0$. Furthermore no singularities appear for $\chi_{\gamma_0} = -1$. This gives in the case of $\chi_{\gamma_0} = 1$ the expansion for $(-\ln \mathcal{Z}_q)$:

$$-\ln \mathcal{Z}_0(s) = \frac{\pi^2}{6l_0} + (s - \tfrac{1}{2})\ln l_0 + const. + O(l_0), \tag{60}$$

which is equivalent with Eq.(54). For $\chi_{\gamma_0} = -1$ things are changed and we get:

$$\begin{aligned} &-\ln \mathcal{Z}_1(s)|_{\chi_{\gamma_0}=-1} \\ &= \frac{1}{l_0}\left(\frac{\pi^2}{6} + i\pi\right) - \frac{i\pi}{l_0}\ln\left(l_0 - \frac{i\pi}{s}\right) + s\ln\left(l_0 - \frac{i\pi}{s}\right) + const. + O(l_0). \end{aligned} \tag{61}$$

Therefore we have to descriminate between $\chi_{\gamma_0} = 1$ and $\chi_{\gamma_0} = -1$. Let us first assume that the character χ_{γ_0} corresponding to the smallest geodesic is positive or that this can be achieved by an appropriate redefinition of the $4g$ generators γ_i, γ_i^{-1} $(i = 1, \ldots, 2g)$. In the relevant combinations for $\mathrm{sdet}(-\Box_m^2)$ this gives:

$$\begin{aligned} \frac{\mathcal{Z}_0(s-\frac{m}{2})\mathcal{Z}_0(s+1+\frac{m}{2})}{\mathcal{Z}_1(s+\frac{m+1}{2})\mathcal{Z}_1(s+\frac{1-m}{2})} &\propto const., \qquad (m \in \mathbf{Z}, \text{even}, l_0 \to 0), \\ \frac{\mathcal{Z}_1(s-\frac{m}{2})\mathcal{Z}_1(s+1+\frac{m}{2})}{\mathcal{Z}_0(s+\frac{m+1}{2})\mathcal{Z}_0(s+\frac{1-m}{2})} &\propto const., \qquad (m \in \mathbf{Z}, \text{odd}, l_0 \to 0), \end{aligned} \tag{62}$$

where the *const.* may depend on s. Therefore in this case the determinants are proportional to a constant ($m \in \mathbf{Z}$) and thus the fermionic string integrand is finite.

In the case of $\chi_{\gamma_0} = -1$ things are changed and we get in the limit $l_0 \to 0$, e.g. for $m = 0$.

$$\begin{aligned} \frac{\mathcal{Z}_0(s)\mathcal{Z}_0(s+1+)}{\mathcal{Z}_1^2(s+\frac{1}{2})} &\propto e^{\frac{2i\pi}{l_0}} l_0^{-2s}\left(l_0 - \frac{i\pi}{s+\frac{1}{2}}\right)^{2s-1-\frac{2i\pi}{l_0}}, \\ \frac{\mathcal{Z}_1(s)\mathcal{Z}_1(s+1)}{\mathcal{Z}_0^2(s+\frac{1}{2})} &\propto e^{\frac{-2i\pi}{l_0}} l_0^{2s}\left(l_0 - \frac{i\pi}{s}\right)^{\frac{i\pi}{l_0}-s}\left(l_0 - \frac{i\pi}{s+1}\right)^{\frac{i\pi}{l_0}-s-1}. \end{aligned} \tag{63}$$

where again the *const.* may depend on s. In this case the fermionic string integrand diverges for $l_0 \to 0$ as in the bosonic case.

Finally let us consider the product of the superdeterminants of $-\Box_0^2$ and $-\Box_{-1}^2$:

$$\mathrm{sdet}'(-\Box_0^2)\cdot \mathrm{sdet}'(-\Box_{-1}^2) = 1 \tag{64}$$

which follows directly from Eqs.(42) and (47). Generalizing this interesting result we find (omitting zero-modes if necessary):

$$\mathrm{sdet}'(-\Box_{-m}^2)\cdot \mathrm{sdet}'(-\Box_{m-1}^2) = (-1)^{g-1}(m-1)^{2-2g}, \qquad (m \in \mathbf{Z}). \tag{65}$$

Let be $f_m = \prod_{k=0}^{m}(-k^2)^{g-1}$. Redefining the superdeterminant by $\mathrm{sdet}'(-\Box_m^2) := \mathrm{sdet}'(-\Box_m^2)/f_m$ $(m \geq 0)$ and $\mathrm{sdet}'(-\Box_{-m}^2) := \mathrm{sdet}'(-\Box_{-m}^2)\cdot f_{m-2}$ $(m \geq 1)$ this gives the relation

$$\mathrm{sdet}'(-\Box_{-m}^2)\cdot \mathrm{sdet}'(-\Box_{m-1}^2) = 1, \qquad (m \in \mathbf{Z}). \tag{66}$$

An equation like this was already stated by Baranov and Schwarz [11] by more general considerations. I close with this result, which nicely confirms my own considerations.

VII. Summary

In this paper I have discussed the Selberg supertrace formula on super Riemann surfaces and some of its most important consequences. In chapter II a short survey of super analysis and super manifolds was given. Fitted with that knowledge it was possible to study in chapter III physics on the super Poincaré upper half-plane, i.e. classical and quantum motion. Starting from the Vielbein approach a well-behaving nonsingular metric on $S\mathcal{H}$ was constructed. From this metric the classical Lagrangian and Hamiltonian could be easily constructed. Also, the construction of the quantum Hamiltonian has been straightforward and was defined by the super Laplace-Beltrami operator Δ_{SLB} on $S\mathcal{H}$. The path integral on $S\mathcal{H}$ was written down and proved in Appendix B to be the correct one. Surprisingly enough this super Laplace-Beltrami operator could be factorized into the square of the Laplace-Dirac operator: $\Delta_{SLB} = \square^2 = (2YD\bar{D})^2$. Also the more general operators $\widehat{\square}_m$ and $\square_m$ were introduced which are connected by a linear isomorphism.

With chapter IV the main part of this paper started. The Selberg super operator L on $S\mathcal{H}$ was defined and it was found that the operator L multiplies an arbitrary Eigenfunction of $\square_m$ by the function h, where h is only defined by the Eigenvalue s of this Eigenfunction with respect to $\square_m$ and the integral kernel of L. The calculation yielded that for appropriate h the operator $h(\square_m)$ equals to an integral operator L whose integral kernel $k_m(Z,W)$ is related to h by the equations

$$
\begin{aligned}
h_m(s) &= \int_{-\infty}^{\infty} du\, e^{u(s-\frac{1}{2})} g(u), \qquad (s = \tfrac{1}{2} + ip), \\
g(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-iup} h_m(\tfrac{1}{2} + ip), \\
g(u) &= \frac{1}{8}\int_{4\sinh^2\frac{u}{2}}^{\infty} \frac{dx}{(x+4)^{\frac{m}{2}}} \\
&\quad\times \left[\frac{\Psi(x) + 2(e^u - 1)\Phi(x)}{\sqrt{x - 4\sinh^2\frac{u}{2}}}(\alpha_+^m + \alpha_-^m) - im e^{\frac{u}{2}}\Phi(x)\frac{\alpha_+^m - \alpha_-^m}{x+4}\right].
\end{aligned}
\tag{1}
$$

It was found that the Selberg supertrace formula reads

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left[h_m(p_n^B) - h_m(p_n^F)\right] &= (1-g)\int_0^{\infty}\frac{g(u) - g(-u)}{\sinh\frac{u}{2}}\cosh\frac{m}{2}\,du \\
&+ \sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l_\gamma \chi_\gamma^{km}}{e^{\frac{kl_\gamma}{2}} - e^{-\frac{kl_\gamma}{2}}}\left[g(kl_\gamma) + g(-kl_\gamma) - \chi_\gamma^k\left(g(kl_\gamma)e^{-\frac{kl_\gamma}{2}} + g(-kl_\gamma)e^{\frac{kl_\gamma}{2}}\right)\right].
\end{aligned}
\tag{2}
$$

The inversion formula which is needed in the supertrace formula to calculate the term $A_0^{(m)} = i^m\pi(g-1)\Phi_m(0)$ which corresponds to the identity transformation was found to be given by

$$
i^m\Phi_m(x) = \frac{1}{\pi\sqrt{x+4}}\int_x^{\infty}\frac{dy}{\sqrt{y+4}}\int_{-\infty}^{\infty}Q'(y+t^2)\left(\frac{\sqrt{y+4+t^2} - t}{\sqrt{y+4+t^2}+t}\right)^{\frac{m}{2}}dt. \tag{3}
$$

where $Q(u) = 2\coth\frac{u}{2}[g(u) - g(-u)]$. $A_0^{(m)}$ and the inversion formula for $\Phi_m(0)$ completed the work of Baranov et al. [10.11] by explicitly stating the $A_0^{(m)}$-term and the inversion formula, respectively.

Chapter V was devoted to the discussion of the analytic properties of the two Selberg super zeta-functions Z_0 and Z_1:

$$
\begin{aligned}
Z_0(s) &= \prod_{\gamma\in\Gamma}\prod_{k=0}^{\infty}\left[1 - e^{-(s+k)l_\gamma}\right], \qquad (\mathrm{Re}(s) > 1), \\
Z_1(s) &= \prod_{\gamma\in\Gamma}\prod_{k=0}^{\infty}\left[1 - \chi_\gamma e^{-(s+k)l_\gamma}\right], \qquad (\mathrm{Re}(s) > 1).
\end{aligned}
\tag{4}
$$

By considering specific test functions for the analytic properties of Z_1 ($k \in \mathbf{N}_0$) could be derived:

- : $s = \frac{1}{2}$: There are a pole, zero with multiplicity $|\Delta n_0^{(0)}|$, respectively, or a regular point depending on whether $\Delta n_0^{(0)} > 0$, $\Delta n_0^{(0)} < 0$ or $\Delta n_0^{(0)} = 0$.
- : $s = -\frac{1}{2} - k$: There are zeros with multiplicity $4(k+1)(g-1)$.
- : $s = ip_n^{B,F} - k$: There are zeros (p_N^B) and poles (p_n^F);
- : $s = -ip_n^{B,F} - k$: There are poles (p_N^B) and zeros (p_n^F)

with the same multiplicity as the corresponding Eigenvalue of $\square$, respectively. The crucial importance of $\Delta n_0^{(0)} = \#even(zero-modes) - \#odd(zero-modes)$ of the operator $\square$ has become clear. A functional relation for R_1 and Z_1, respectively, was derived:

$$
R_1(s)R_1(-s) = \frac{Z_1(s)Z_1(-s)}{Z_1(s+1)Z_1(1-s)} = \frac{Z_1^2(0)}{Z_1(1)}(\cos\pi s)^{4(g-1)}. \tag{5}
$$

Similarly for Z_0 ($k \in \mathbf{N}_0$):

- : $s = -k$: There are trivial zeros with multiplicity $(g-1)(4k+2)$.
- : $s = \frac{1}{2} + ip_n^{B,F} - k$: There are zeros (p_n^B) and poles (p_n^F);
- : $s = -\frac{1}{2} - ip_n^{B,F} - k$: There are poles (p_n^B) and zeros (p_n^F),

of course, with the same multiplicity as the corresponding Eigenvalues of $\square$, respectively. It was found that there is no zero of $Z_0(s)$ at $s = 1$. This is quite different in comparision to the usual Selberg zeta-function.
Also a functional equation for R_0 and Z_0, respectively, was derived reading

$$
R_0(s)R_0(-s) = \frac{Z_0(s)Z_0(-s)}{Z_0(s-1)Z_0(1-s)} = \frac{Z_0(-\frac{1}{2})}{Z_0(\frac{3}{2})}(\sin\pi s)^{4(g-1)}. \tag{6}
$$

For both functions we get an infinite set of critical lines located for Z_0 at $\mathrm{Re}(s) = \frac{1}{2} - k$ ($k \in \mathbf{N}_0$) and for Z_1 at $\mathrm{Re}(s) = -k$ ($k \in \mathbf{N}_0$).
Unfortunately no functional equation for Z_0 or Z_1 as for the ordinary Selberg zeta-function could be found. However, I have argued the unlikelihood that such a relation exists, based on the existence of the infinite number of critical lines.
This appearence of an infinite number of critical lines for the two functions Z_0 and Z_1 is surprising, because there is not any classical analogy for this feature. However, in view of the functional relations for $R_0(Z_0)$ and $R_1(Z_1)$ this is a consistent result. The functional relations are not of any use for the determination of the spectrum of the Laplace-Dirac operator $\square$. This in turn is the same situation as in the classical

case. There is up to now no way into the critical domain of the (super) zeta-functions in the complex plane, where the nontrivial zeros (and/or poles) are located.

By an appropriate test function h a functional relation connecting Z_0 and Z_1 could be derived:

$$\frac{Z_1(\frac{1}{2}-s)Z_0(s)}{Z_1(\frac{1}{2}+s)Z_0(1-s)} = \frac{Z_1(0)}{Z_1(1)}(\sin \pi s)^{2(g-1)}. \tag{7}$$

A relation like this was already stated by Baranov and Schwarz [11], but without the characteristic $(\sin \pi s)^{2g-2}$-dependence.

Having discussed the properties of Z_0 and Z_1 I treated in the final chapter the spectrum and superdeterminants of the Laplacian-Dirac operators $\hat{\Box}_m$ and $\Box_m$, respectively. Denoting by $\Delta\hat{n}_\lambda^{(m)}$ the difference of the even and odd trivial-modes λ of the operator $\hat{\Box}_m$ two scenarios for $\Delta\hat{n}_0^{(0)}$, i.e. $\Delta\hat{n}_0^{(0)} = 0$ and $\Delta\hat{n}_0^{(0)} = 1-g$, respectively, were discussed.
For the nontrivial Eigenvalues of $\hat{\Box}_m$ I found that they are determined by the nontrivial Eigenvalues of $\hat{\Box}_0$ as

$$\hat{\lambda}_{n,m} = \hat{\lambda}_n, \qquad (n \in \mathbf{N}, m \in \mathbf{N}). \tag{8}$$

The calculation of the determinants was performed with the well-known zeta-regularization method. The result for e.g. $\mathrm{sdet}'(-\Box_0^2)$ was given by

$$\mathrm{sdet}'(-\Box_0^2) = \left[\pi^{g-1}\frac{Z_0(1)}{\hat{Z}_1(\frac{1}{2})}\right]^2 \frac{Z_1(0)}{Z_1(1)}. \tag{9}$$

and similarly for $\mathrm{sdet}'(-\Box_m^2)$ $(m \in \mathbf{Z})$. These representations showed clearly that the superdeterminants are well-defined quantities. Since the superdeterminants were well-defined, it could be shown that the fermionic string integrand in the Polyakov approach is well-defined. The remaining integral over the super moduli space reads

$$\begin{aligned} Z_g &= \int_{s\mathcal{M}_g} d(SWP)\left[\mathrm{sdet}'(-\Box_0^2)\right]^{-\frac{5}{2}}\left[\mathrm{sdet}'(-\Box_2^2)\right]^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi^4}\right)^{g-1}\int_{s\mathcal{M}_g} d(SWP)\left(\frac{Z_0(1)}{\hat{Z}_1(\frac{1}{2})}\right)^{-5}\frac{Z_0(2)}{Z_1(\frac{3}{2})}\left(\frac{Z_1(1)}{Z_1(0)}\right)^2. \end{aligned} \tag{10}$$

Unfortunately no statement about the growing properties of this expression for increasing genus g like the analysis of Gross and Periwal [45] could be made.
However, it could discussed what happens for the fermionic string integrand if a pinching takes place. Here a divergence as well as a convergence can happen, depending on the spin structure.

An interesting feature of the determinants is that there is a typical factor of $Z_1(0)/Z_1(1)$. This factor does not appear in the work of Aoki [3], who started from quite analogous expressions but used the functional relation of the ordinary Selberg zeta-function [49] instead of the functional relations for the Selberg super zeta-functions. But this factor is an unambigous consequence of the functional relations which I derived in chapter V. I do not see any way to simplify this charateristic factor any further by exploiting all these functional relations. Therefore this factor

gives an additional contribution in the super moduli dependence of the superdeterminants and thus also for the fermionic string integrand.

An interesting relation for the determinants was deduced reading

$$\mathrm{sdet}'(-\Box_{-m}^2)\cdot\mathrm{sdet}'(-\Box_{m-1}^2) = (-1)^{g-1}(m-1)^{2-2g}, \qquad (m \in \mathbf{Z}). \tag{11}$$

These results which are all direct consequences of the Selberg super trace formula demonstrate in an impressing way the power of the trace formula.

In Appendix E I have given a review of path integration on the Poincaré upper half-plane $\mathcal{H}$. It is a short summary of recent results in this field obtained by F.Steiner and myself [38-43].

The fact that the fermionic string theory is, formulated in the super analysis formulation, well-defined, is a step forward in the understanding of the whole string theory. However, one must keep in mind that the fermionic string is as well as the bosonic string nothing but a toy-model. To incorporate supersymmetry or to get the standard-model gauge symmetries, the superstring or the heterotic string theory is needed (The higher-loop partition function for the latter has been constructed by Moore, Nelson and Polchinski [63]). Whereas the incorporation of the superstring can be done by the GSO-projection, it is not obvious to formulate a Selberg trace formula for the heterotic string case and to study its consequences. Again new surprising features may occur.
I think that we must face the possibility that we do not know up to now enough mathematics to understand this new physics, and once again physics may be too hard for the physicists.

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1. The Volume Element $dV(Z)$

Let us consider a $SPL(2,\mathbf{R})$ transformation which is following Chapter II given by

$$z' = \frac{\delta\theta + az + b}{\gamma\theta + cz + d} \equiv \frac{A}{B}, \qquad \theta' = \frac{e\theta + \alpha z + \beta}{\gamma\theta + cz + d} \equiv \frac{\Gamma}{B}, \tag{1}$$

where

$$A(Z) = az + b - \theta\delta,\ B(Z) = cz + d - \theta\gamma,\ \Gamma(Z) = \alpha z + \beta + e\theta; \tag{2}$$

$$\epsilon = 1 + \frac{3}{2}\beta\alpha,\ \gamma = d\alpha - c\beta,\ \delta = b\alpha - a\beta. \tag{3}$$

The numbers a, b, c, d satisfy the relation $ad - bc = 1$ and are real even supernumbers. The numbers α and β are odd supernumbers with property $\bar\alpha = i\alpha$, $\bar\beta = i\beta$. A, B and Γ must be multiplied by $K = 1 + \frac{1}{2}\alpha\beta$ to give the correct normalization sdet$T = 1$ (see Eq.(II.64)). I denote these quantities by $\hat A$, $\hat B$ and $\hat\Gamma$, respectively. Throughout this appendix I assume that $\gamma \in \Gamma$ is a $SPL(2\mathbf{R})$ transformation (but do not confuse with $\gamma = d\alpha - c\beta$; the correct distinction between these two quantities is in general clear by context).
Let us introduce the number

$$Z_{Z\bar W} := z - \bar w + i\theta\bar\nu, \qquad \Rightarrow \qquad Y = \frac{1}{2i} Z_{Z\bar Z} = \frac{1}{2i}(z - \bar z + i\theta\bar\theta). \tag{4}$$

$Z_{Z\bar W}$ is transformed under the action of $\gamma \in SPL(2,\mathbf{R})$ into

$$Z_{Z\bar W} \mapsto \gamma Z_{Z\bar W} = Z'_{Z\bar W} = \frac{A(Z)\bar B(W) - \bar A(W)B(Z) + i\Gamma(Z)\bar\Gamma(W)}{B(Z)\bar B(W)}. \tag{5}$$

Let us first consider the first two terms in the numerator of Eq.(5):

$$\begin{aligned}
&A(Z)\bar B(W) - \bar A(W)B(Z)\\
&= (az + b - \theta\delta)\overline{(cw + d - \nu\gamma)} - \overline{(az + b - \theta\delta)}(cw + d - \nu\gamma)\\
&= \underbrace{(az + b)(c\bar w + d) - (a\bar w + b)(cz + d)}_{= z - \bar w}\\
&\qquad \underbrace{+ i\bar\nu(az + b)(d\alpha - c\beta) - i\bar\nu(b\alpha - a\beta)(cz + d)}_{= i\bar\nu(\alpha z + \beta)}\\
&\qquad \underbrace{- \theta(c\bar w + d)(b\alpha - a\beta) + \theta(d\alpha - c\beta)(a\bar w + b)}_{= \theta(\alpha\bar w + \beta)}\\
&\qquad \underbrace{+ i\theta(b\alpha - a\beta)(d\alpha - c\beta)\bar\nu - (b\alpha - a\beta)\bar\nu\theta(d\alpha - c\beta)}_{= 2i\alpha\beta\theta\bar\nu}\\
&= z - \bar w + i\bar\nu(\alpha z - \beta) + \theta(\alpha\bar w + \beta) - 2i\alpha\beta\theta\bar\nu.
\end{aligned} \tag{6}$$

For the third term:

$$\begin{aligned}
i\Gamma(Z)\bar\Gamma(W) &= i(\alpha z + \beta + \epsilon\theta)\overline{(\alpha w + \beta + e\nu)}\\
&= \alpha\beta(z - \bar w) - i\bar\nu(\alpha z - \beta) - \theta(\alpha\bar w + \beta) + i(1 - 3\alpha\beta)\theta\bar\nu.
\end{aligned} \tag{7}$$

Thus combining Eqs.(6) and (7)

$$Z'_{Z\bar W} = Z_{Z\bar W}\frac{1}{\hat B(Z)\bar{\hat B}(W)}. \tag{8}$$

Similarly

$$z' - w' - \theta'\nu' = \frac{z - w - \theta\nu}{\hat B(Z)\hat B(W)}; \tag{9}$$

in particular:

$$Y' = Y\frac{1 - \alpha\beta}{|B(Z)|^2} = \frac{Y}{|\hat B(Z)|^2}. \tag{10}$$

Let us consider $D\theta'$:

$$D\theta' = D\frac{\Gamma(Z)}{B(Z)} = \frac{[D\Gamma(Z)]B(Z) + \Gamma(Z)[DB(Z)]}{B^2(Z)}.$$

In detail:

$$\begin{aligned}
&[D\Gamma(Z)]B(Z) + \Gamma(Z)[DB(Z)] =\\
&\qquad [\epsilon B(Z) + \theta\alpha B(Z) + \gamma\Gamma + \Gamma\theta c](cz + d)\left(1 - \frac{\alpha\beta}{2} - \frac{\theta\gamma}{cz + d}\right)
\end{aligned}$$

and therefore

$$\frac{[D\Gamma(Z)]B(Z) + \Gamma(Z)[DB(Z)]}{B(Z)} = 1 - \frac{\alpha\beta}{2}.$$

Thus

$$D\theta' = \frac{1 - \frac{\alpha\beta}{2}}{B(Z)} = \frac{1}{\hat B(Z)} \equiv F_\gamma(Z). \tag{11}$$

With Eq.(II.52) we have the transformation properties of the operator D:

$$D = \frac{1 - \frac{\alpha\beta}{2}}{B(Z)}D' = \frac{1}{\hat B(Z)}D'. \tag{12}$$

Let us consider now the Jacobian J in the volume element for the transformation of Eq.(1): $dz\,d\bar z\,d\theta\,d\bar\theta = J\,dz'd\bar z'd\theta'd\bar\theta'$. It is given by:

$$\begin{aligned}
J &= \operatorname{sdet}\begin{pmatrix}
(\partial_z z') & (\partial_z \bar z') & (\partial_z \theta') & (\partial_z \bar\theta')\\
(\partial_{\bar z} z') & (\partial_{\bar z} \bar z') & (\partial_{\bar z} \theta') & (\partial_{\bar z} \bar\theta')\\
(\partial_\theta z') & (\partial_\theta \bar z') & (\partial_\theta \theta') & (\partial_\theta \bar\theta')\\
(\partial_{\bar\theta} z') & (\partial_{\bar\theta} \bar z') & (\partial_{\bar\theta} \theta') & (\partial_{\bar\theta} \bar\theta')
\end{pmatrix}\\
&= \operatorname{sdet}\begin{pmatrix}
(\partial_z z') & 0 & (\partial_z \theta') & 0\\
0 & (\partial_{\bar z} \bar z') & 0 & (\partial_{\bar z} \bar\theta')\\
(\partial_\theta z') & 0 & (\partial_\theta \theta') & 0\\
0 & (\partial_{\bar\theta} \bar z') & 0 & (\partial_{\bar\theta} \bar\theta')
\end{pmatrix}\\
&= \frac{1}{(\partial_\theta\theta')(\partial_{\bar\theta}\bar\theta')}\det\Bigg[\begin{pmatrix}(\partial_z z') & 0\\ 0 & (\partial_{\bar z}\bar z')\end{pmatrix}\\
&\qquad - \begin{pmatrix}(\partial_z\theta') & 0\\ 0 & (\partial_{\bar z}\bar\theta')\end{pmatrix}\cdot\begin{pmatrix}(\partial_\theta\theta')^{-1} & 0\\ 0 & (\partial_{\bar\theta}\bar\theta')^{-1}\end{pmatrix}\cdot\begin{pmatrix}(\partial_\theta z') & 0\\ 0 & (\partial_{\bar\theta}\bar z')\end{pmatrix}\Bigg]\\
&= \frac{(\partial_z z')(\partial_\theta\theta') - (\partial_z\theta')(\partial_\theta z')}{(\partial_\theta\theta')}\cdot\frac{(\partial_{\bar z}\bar z')(\partial_{\bar\theta}\bar\theta') - (\partial_{\bar z}\bar\theta')(\partial_{\bar\theta}\bar z')}{(\partial_{\bar\theta}\bar\theta')} \equiv \tilde J\cdot\bar{\tilde J}
\end{aligned} \tag{13}$$

and we see that the Jacobian factorizes. Let us calculate $\tilde{J}$. It is given by

$$\tilde{J} = \frac{[(\partial_z A)B - A(\partial_z B)][(\partial_\theta \Gamma)B + \Gamma(\partial_\theta B)] - [(\partial_z \Gamma)B - \Gamma(\partial_z B)][(\partial_\theta A)B - A(\partial_\theta B)]}{[(\partial_\theta \Gamma)B + \Gamma(\partial_\theta B)]^2}. \tag{14}$$

For the denominator:

$$[(\partial_\theta \Gamma)B + \Gamma(\partial_\theta B)]^2 = (\epsilon B - \Gamma\gamma)^2 = B(cz + d)\left[1 - \alpha\beta - 3\theta\frac{d\alpha - c\beta}{cz + d}\right] \tag{15}$$

and therefore

$$\frac{1}{[(\partial_\theta \Gamma)B + \Gamma(\partial_\theta B)]^2} = \frac{1}{B(cz + d)}\left[1 + \alpha\beta + 3\theta\frac{d\alpha - c\beta}{cz + d}\right]. \tag{16}$$

For the numerator:

$$\begin{aligned}
&[(\partial_z A)B - A(\partial_z B)][(\partial_\theta \Gamma)B + \Gamma(\partial_\theta B)] - [(\partial_z \Gamma)B - \Gamma(\partial_z B)][(\partial_\theta A)B - A(\partial_\theta B)] \\
&= (aB - Ac)(\epsilon B - \Gamma\gamma) - (\alpha B - c\Gamma)(A\gamma - \delta B) \\
&= \underbrace{e(aB^2 - ABc - ab\theta\gamma + 2Ac\theta\gamma + c\theta\delta B)}_{(cz+d)\left[1 - \frac{3}{2}\alpha\beta - 3\theta\frac{d\alpha - c\beta}{cz+d}\right]} \\
&\qquad + \underbrace{(\alpha z + d)(2Ac\gamma - aB\gamma - c\delta B)}_{=\alpha\beta(cz+d)} + \underbrace{\alpha\delta B^2 - \alpha\gamma AB}_{=-\alpha\beta(cz+d)} \\
&\qquad\qquad = (cz + d)\left[1 - \frac{3}{2}\alpha\beta - 3\theta\frac{d\alpha - c\beta}{cz + d}\right].
\end{aligned} \tag{17}$$

Thus:

$$\tilde{J} = \frac{1 - \frac{1}{2}\alpha\beta}{B(Z)} = \frac{1}{\hat{B}(Z)}, \qquad \Rightarrow \qquad J = \frac{1 - \alpha\beta}{|B(Z)|^2} = \frac{1}{|\hat{B}(Z)|^2}. \tag{18}$$

Combining Eqs.(10) and (18), the invariance of $dV(Z)$ is proven, i.e.

$$\frac{dz\,d\bar{z}\,d\theta\,d\bar{\theta}}{2Y} = \frac{dz'\,d\bar{z}'\,d\theta'\,d\bar{\theta}'}{2Y'}. \tag{19}$$

2. The Invariance of $R(Z,W)$

Let us consider the two-point quantities $r(Z,W)$ and $R(Z,W)$. They are given by

$$R(Z,W) = \frac{|z - w - \theta\nu|^2}{YV} = \frac{|z - w - \theta\nu|^2}{(y + \theta\bar{\theta}/2)(v + \nu\bar{\nu}/2)}, \tag{20}$$

$$\begin{aligned}
r(Z,W) = \frac{2\theta\bar{\theta} + i(\nu - i\bar{\nu})(\theta + i\bar{\theta})}{4Y} + \frac{2\nu\bar{\nu} + i(\theta - i\bar{\theta})(\nu + i\bar{\nu})}{4V} \\
+ \frac{(\nu + i\bar{\nu})(\theta + i\bar{\theta})\mathrm{Re}(z - w - \theta\nu)}{4YV}.
\end{aligned} \tag{21}$$

Furthermore we have $\cosh d(Z,W) = 1 + \frac{1}{2}R(Z,W) - 2r(Z,W)$. Of course, the invariance of d follows from the invariance of r and R.
Let us consider R. From Eqs.(8) and (9) we see at once that

$$R(\gamma Z, \gamma W) = R(Z,W) \tag{22}$$

and the invariance of R with respect to the action of $SPL(2\mathbf{R})$ is shown. The invariance of $r(Z,W)$ is similar but much more involved and will not be given here.

3. The Vierbein E^A

Let us consider the quantity $E^z = (dz + \theta d\theta)/Y$ under the action of the transformation (1). First with Eq.(10):

$$Y' = \gamma Y = \frac{Y}{|\hat{B}(Z)|^2}. \tag{23}$$

Next this gives

$$\begin{aligned}
dz' + \theta' d\theta' &= \gamma(dz + \theta d\theta) \\
&= \frac{1}{B}\left(\frac{\partial A}{\partial z}dz - \frac{\partial A}{\partial \theta}d\theta\right) - \frac{1}{B^2}\left(\frac{\partial B}{\partial z}dz - \frac{\partial B}{\partial \theta}d\theta\right) \\
&\qquad + \frac{\Gamma}{B}\left[\frac{1}{B}\left(\frac{\partial \Gamma}{\partial z}dz - \frac{\partial \Gamma}{\partial \theta}d\theta\right) - \frac{\Gamma}{B^2}\left(\frac{\partial B}{\partial z}dz - \frac{\partial B}{\partial \theta}d\theta\right)\right] \\
&= \frac{1}{B^2}[(Ba - Ac + \Gamma\alpha)dz - (B\delta - A\gamma + \Gamma\epsilon)d\theta] = \frac{dz + \theta d\theta}{\hat{B}^2(Z)}.
\end{aligned} \tag{24}$$

Thus combining Eqs.(23) and (24) we get

$$E'^z = \gamma E^z = e^{-i\phi}E^z \tag{25}$$

with some phase factor given by $\phi = \mathrm{Im}(B)/\mathrm{Re}(B)$. Therefore E^z (and $E^{\bar{z}}$) are up to phase factors invariant under the action of $SPL(2,\mathbf{R})$. For the combination $E^z E^{\bar{z}}$ this phase factor drops out. The calculation for E^θ (and $E^{\bar{\theta}}$) is similar.

Appendix B: Proof of the Path Integral on $S\mathcal{H}$

In this Appendix I want to show that from the short time kernel of the path integral on $S\mathcal{H}$ [I denote $Z = (z,\theta)$, $Y_M = y_M + \theta_M\bar{\theta}_M/2$, where $y_M = (y' + y'')/2$, $\theta_M = (\theta' + \theta'')/2$, $\Delta z = (z'' - z')$, $\Delta\theta = (\theta'' - \theta')$ and h.c.]:

$$\begin{aligned}
K(Z', Z''; \epsilon) &= \frac{\sqrt{Y'Y''}}{2\pi i Y_M} \\
&\quad \times \exp\left\{\frac{im}{2\epsilon Y_M^2}\left[\Delta z\Delta\bar{z} - i\bar{\theta}_M\Delta\bar{z}\Delta\theta - i\theta_M\Delta z\Delta\bar{\theta} - (2Y_M - \theta_M\bar{\theta}_M)\Delta\theta\Delta\bar{\theta}\right]\right\}
\end{aligned} \tag{1}$$

and the time evolution equation

$$\Psi(Z; t + \epsilon) = \int \frac{dx'dy'd\theta'd\bar{\theta}'}{2Y'}K(Z, Z'; \epsilon)\Psi(Z'; t) \tag{2}$$

the "super"-Schrödinger equation can be derived, i.e. ($z = x + iy$, $\theta = \theta_1 + i\theta_2$):

$$\begin{aligned}
i\frac{\partial\Psi(Z;t)}{\partial t} &= -\frac{Y}{m}\left[(2Y - \theta\bar{\theta})\partial_z\partial_{\bar{z}} + i\bar{\theta}\partial_z\partial_{\bar{\theta}} - i\theta\partial_{\bar{z}}\partial_\theta + \partial_{\bar{\theta}}\partial_\theta\right]\Psi(Z,t) \\
&= -\frac{1}{2m}\left[y(y + i\theta_2\theta_1)(\partial_x^2 + \partial_y^2) + i(y + i\theta_2\theta_1)\partial_{\theta_2}\partial_{\theta_1}\right. \\
&\qquad \left. + y(\theta_1\partial_y\partial_{\theta_1} + \theta_2\partial_y\partial_{\theta_2} + \theta_2\partial_x\partial_{\theta_1} - \theta_1\partial_x\partial_{\theta_2})\right]\Psi(Z,t).
\end{aligned} \tag{3}$$

One has to perform a Taylor expansion in Eq.(2):

$$\Psi(Z;t)+\epsilon\frac{\partial}{\partial t}\Psi(z;t)=B_0\Psi(z;t)+B_a\partial_a\Psi(Z;t)+B_{ab}\partial_b\partial_a\Psi(Z;t)+\dots, \quad (4)$$

where the coefficients B_a and B_{ab} in terms of the real coordinates x, y, θ_1, θ_2 up to $O(\epsilon)$ are given by

$$\begin{aligned}B_0 &= \int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)\\ &=\frac{\sqrt{y+i\theta_2\theta_1}}{4\pi i}\int\frac{dx'dy'd\theta_1'd\theta_2'}{\sqrt{y'+i\theta_2'\theta_1'}(y_M+i\theta_{M\,2}\theta_{M\,1})}\\ &\times\exp\left\{\frac{im}{2\epsilon Y_M^2}\left[\Delta z\Delta\bar z-i\bar\theta_M\Delta\bar z\Delta\theta-i\theta_M\Delta z\Delta\bar\theta-(2Y_M+\theta_M\bar\theta_M)\Delta\theta\Delta\bar\theta\right]\right\}\simeq 1. \quad (5)\end{aligned}$$

$$\left.\begin{aligned}B_x &= \int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(x-x')\simeq 0\\ B_y &= \int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(y-y')\simeq 0\\ B_{\theta_1} &= \int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(\theta_1-\theta_1')\simeq 0\\ B_{\theta_2} &= \int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(\theta_2-\theta_2')\simeq 0.\end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned}B_{xx} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(x-x')^2\simeq\frac{i\epsilon}{2m}y(y+i\theta_2\theta_1)\\ B_{yy} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(y-y')^2\simeq\frac{i\epsilon}{2m}y(y+i\theta_2\theta_1)\\ B_{\theta_1\theta_2} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(\theta_1-\theta_1')(\theta_2-\theta_2')\simeq\frac{i\epsilon}{2m}i(y+i\theta_2\theta_1)\\ B_{x\theta_1} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(x-x')(\theta_1-\theta_1')\simeq\frac{i\epsilon}{4m}y\theta_2\\ B_{x\theta_2} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(x-x')(\theta_2-\theta_2')\simeq\frac{i\epsilon}{4m}(-y\theta_1)\\ B_{y\theta_1} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(y-y')(\theta_1-\theta_1')\simeq\frac{i\epsilon}{4m}y\theta_1\\ B_{y\theta_2} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(y-y')(\theta_2-\theta_2')\simeq\frac{i\epsilon}{4m}y\theta_2\\ B_{xy} &= \frac12\int\frac{dx'dy'd\theta_1'd\theta_2'}{2(y'+i\theta_2'\theta_1')}K(Z,Z';\epsilon)(x-x')(y-y')\simeq 0.\end{aligned}\right\} \quad (7)$$

I just present some details of the calculation of B_0; the other coefficients are calculated in a similar way.[1] First let us consider the Lagrangian in the midpoint formulation on the lattice:

$$L=\frac{m}{2\epsilon^2 Y_M^2}\left[\Delta z\Delta\bar z-i\bar\theta_M\Delta\bar z\Delta\theta-i\theta_M\Delta z\Delta\bar\theta-(2Y_M+\theta_M\bar\theta_M)\Delta\theta\Delta\bar\theta\right]. \quad (8)$$

[1] The calculations were done with the help of the algebraic computer program REDUCE.

The exponential expanded in the Grassmann variables yields $[\Delta x = x - x',\ \Delta y = y - y']$

$$\begin{aligned}e^{i\epsilon L} &= \exp\left[-\frac{m}{2i\epsilon}\frac{(x-x')^2+(y-y')^2}{y_M^2}\right]\\ &\times\left\{1-\frac{m^2}{4\epsilon^2y_M^4}\bar\theta\theta\bar\theta'\theta'(\Delta^2x+\Delta^2y)\right.\\ &\quad+\frac{im}{\epsilon y_M}\theta\bar\theta'\left[1+\frac{i\Delta x}{2y_M}-\frac{\Delta^2x+\Delta^2y}{8y_M^2}\right]+\frac{im}{\epsilon y_M}\theta'\bar\theta\left[1-\frac{i\Delta x}{2y_M}-\frac{\Delta^2x+\Delta^2y}{8y_M^2}\right]\\ &\quad\left.+\frac{im}{\epsilon y_M}\theta\bar\theta\left[1-\frac{\Delta y}{2y_M}+\frac{\Delta^2x+\Delta^2y}{8y_M^2}\right]+\frac{im}{\epsilon y_M}\theta'\bar\theta'\left[1+\frac{\Delta y}{2y_M}+\frac{\Delta^2x+\Delta^2y}{8y_M^2}\right]\right\}. \quad (9)\end{aligned}$$

Furthermore we have

$$\begin{aligned}\frac{1}{Y_M} &= \frac{1}{y_M}\left[1+\frac{\bar\theta\theta'-\theta'\bar\theta'-\theta\bar\theta'-\theta\bar\theta}{8y_M}\right]\\ \sqrt{\frac{Y''}{Y'}} &= \sqrt{\frac{y''}{y'}}\left[1+\frac{\bar\theta'\theta'}{4y'}-\frac{\bar\theta\theta}{4y}-\frac{\bar\theta\theta\bar\theta'\theta'}{16yy'}\right].\end{aligned} \quad (10)$$

In the usual way y' and y_M have to be expanded according to

$$\frac{1}{y_M}\simeq\frac1y\left[1+\frac{\Delta y}{2y}+\frac{\Delta^2y}{4y^2}\right],\qquad\frac{1}{\sqrt{y}}\simeq\frac{1}{\sqrt{y}}\left[1+\frac{\Delta y}{2y}+\frac38\frac{\Delta^2y}{y^2}\right]. \quad (11)$$

In multiplication the various terms and inserting into the Taylor expansions we have to take into account terms up to the sixth and eighth order in Δx and Δy, respectively. Explicitly, after integrating out the Grassmann variables

$$\begin{aligned}B_0 &\simeq \frac{m}{2\pi i\epsilon y^2}\underbrace{\int\int dx'dy'\exp\left[-\frac{m}{2i\epsilon}\frac{\Delta^2x+\Delta^2y}{y^2}\right]}_{\simeq\frac{2\pi i\epsilon}{m}y^2(\epsilon\to 0)}\\ &\times\left\{1-\frac{3\pi i\epsilon}{8m}+\frac{\Delta^2x+22\Delta^2y}{8y^2}\right.\\ &\qquad+\frac{11im}{8\epsilon y^4}(\Delta^2x\Delta^2y+\Delta^4y)-\frac{m^2}{8\epsilon^2y^6}(\Delta^4x\Delta^2y+2\Delta^2x\Delta^4y+\Delta^4y)\\ &+\theta_2\theta_1\left[\frac{\epsilon}{8my}-\frac{65}{16}\frac{m}{\epsilon y^5}(\Delta^4y+\Delta^2x\Delta^2y)+\frac{17}{16}\frac{im^2}{\epsilon^2y^7}(\Delta^6y+2\Delta^2x\Delta^4y-\Delta^4x\Delta^2y)\right.\\ &\qquad\left.\left.-\frac{i}{y}-\frac{21}{8}\frac{\Delta^2y}{y^3}-\frac{m}{16\epsilon^3y^9}(\Delta^8y-\Delta^6x\Delta^2y+3\Delta^4x\Delta^2y+3\Delta^2x\Delta^4y)\right]\right\}. \quad (12)\end{aligned}$$

One makes use of the following identities

$$\begin{aligned}\Delta^2x &= \Delta^2y\doteq\frac{i\epsilon}{m}y^2, & \Delta^6x &= \Delta^6y\doteq 15\left(\frac{i\epsilon}{m}y^2\right)^3,\\ \Delta^4x &= \Delta^4y\doteq 3\left(\frac{i\epsilon}{m}y^2\right)^2, & \Delta^8y &\doteq 105\left(\frac{i\epsilon}{m}y^2\right)^4;\end{aligned} \quad (13)$$

respecting that in the required limit the integral over x and y gives $2\pi i\epsilon y^2/m$ (as denoted), the result of Eq.(5) is proven.

■

Appendix C: The Heat Kernel on $S\mathcal{H}$

In this Appendix I want to give a short summary of the results of Aoki [3] who first calculated the heat kernel on $S\mathcal{H}$. Let us first describe his definition of the Laplacian he uses. It reads

$$\Delta_m := \Delta_0 + m(z - \bar{z} - \theta\bar{\theta})[\hat{D}\hat{\bar{D}} + \tfrac{1}{2}(\partial_z + \partial_{\bar{z}})] - \tfrac{m^2}{2}(\theta - \bar{\theta})(\hat{D} + \hat{\bar{D}}) + \tfrac{m^2}{4}, \tag{1}$$

where $m \in \mathbf{Z}$, $\hat{D} = \partial_\theta + \theta\partial_z$, $\hat{\bar{D}} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$ and

$$\Delta_0 = \left[(z - \bar{z} - \theta\bar{\theta})^2 + \theta\bar{\theta}(z - \bar{z} - \theta\bar{\theta})\right]\partial_z\partial_{\bar{z}} + (z - \bar{z} - \theta\bar{\theta})(\partial_\theta\partial_{\bar{\theta}} + \theta\partial_\theta\partial_{\bar{z}} - \bar{\theta}\partial_{\bar{\theta}}\partial_z). \tag{2}$$

Now recall the Laplacians $\square_m$ and $\widehat{\square}_m$ used by Baranov et al. [10]. Thus we find the following equivalence

$$(\widehat{\square}_m + \tfrac{m}{2})^2 = \Delta_m, \tag{3}$$

where

$$\begin{aligned}\widehat{\square}_m &= (z - \bar{z} - \theta\bar{\theta})\hat{D}\hat{\bar{D}} - \tfrac{m}{2}(\theta - \bar{\theta})(\hat{D} + \hat{\bar{D}})\\ \square_m &= (z - \bar{z} - \theta\bar{\theta})^{\frac{m}{2}}(\widehat{\square}_m + \tfrac{m}{2})(z - \bar{z} - \theta\bar{\theta})^{-\frac{m}{2}}.\end{aligned} \tag{4}$$

These equations establish the unitary equivalences of Δ_m and $\square_m^2$, i.e. $\Delta_m \cong \square_m^2$. This gives, of course, also $\square_m \cong \widehat{\square}_m + \frac{m}{2}$ in my notation.

Aoki studies the heat kernel $\hat{G}_M^t$ which is defined by $[Z = (z,\theta)$, $W = (w,\nu) = (u + iv, \nu)]$:

$$\begin{aligned}(\partial_t + \Delta_m)\hat{G}_m^t(Z,W) &= 0 \qquad (t > 0)\\ \hat{G}_m^t(Z,W) &\to \frac{z - \bar{z} - \theta\bar{\theta}}{-4}\delta(x - u)\delta(y - v)(\theta - \nu)(\bar{\theta} - \bar{\nu}) \qquad (t \to 0^+).\end{aligned} \tag{5}$$

Now consider some redefinitions, i.e. define G_m^t by

$$\hat{G}_m^t(Z,W) = \left(\frac{z - \bar{w} - \theta\bar{\nu}}{w - \bar{z} - \bar{\theta}\nu}\right)^{\frac{m}{2}} G_m^t(Z,W), \tag{6}$$

then

$$\begin{aligned}(\partial_t + \tilde{\Delta})G_m^t(Z,W) &= 0 \qquad (t > 0)\\ \frac{1}{\sinh d(Z,W)}G_m^t(Z,W) &\to -\frac{1}{2\pi}\delta[\cosh d(Z,W) - 1]i\Delta(Z,W)\bar{\Delta}(Z,W) \qquad (t \to 0^+).\end{aligned} \tag{7}$$

Here $\tilde{\Delta}_m$ is a transformed Laplacian and d, Δ and $\bar{\Delta}$ are $SPL(2,\mathbf{R})$ invariant quantities given by:

$$\begin{aligned}\tilde{\Delta}_m &= \left(\frac{z_{1\bar{2}}}{z_{2\bar{1}}}\right)^{-\frac{m}{2}}\Delta_m\left(\frac{z_{1\bar{2}}}{z_{2\bar{1}}}\right)^{\frac{m}{2}}\\ \cosh d(Z,W) &= 1 - 2\frac{z_{12}z_{\bar{1}\bar{2}}}{z_{1\bar{1}}z_{2\bar{2}}}\\ \Delta(Z,W) &= \frac{\theta z_{1\bar{2}} + \nu z_{\bar{2}1} + \bar{\nu}z_{12} + \theta\nu\bar{\nu}}{\sqrt{z_{12}z_{2\bar{2}}z_{\bar{2}1}}}\\ z_{ab} &= z_a - z_b - \theta_a\theta_b, \qquad (a,b = 1,2), \qquad Z_1 = Z,\ Z_2 = W.\end{aligned} \tag{8}$$

Now expand

$$G_m^t(Z,W) = g_m^t(r) + i\Delta\bar{\Delta}h_m^t(d). \tag{9}$$

This leads to coupled differential equations for g_m^t and h_m^t:

$$\left\{\frac{\partial}{\partial t} - \left[\frac{\partial^2}{\partial d^2} + \left(\frac{1}{\sinh d} - m\tanh\frac{d}{2}\right)\frac{\partial}{\partial d} + \frac{m^2}{4}\left(1 + \frac{2}{\cosh d + 1}\right)\right]\right\}g_m^t(d) = \frac{2(m+1)}{\sinh d}h_m^t(d) \tag{10}$$

$$\left\{\frac{\partial}{\partial t} - \left[\frac{\partial^2}{\partial d^2} + \left(-\frac{1}{\sinh d} - m\tanh\frac{d}{2}\right)\frac{\partial}{\partial d} + \frac{\cosh d}{\sinh^2 d} - \frac{m(1+\frac{m}{2})}{\cosh r + 1} + \frac{m^2}{4}\right]\right\}h_m^t(d) - m\frac{\sinh d}{\cosh d + 1}\left(\frac{\partial^2}{\partial d^2} + \frac{1}{\sinh d}\frac{\partial}{\partial d} - \frac{m/2}{\cosh d} - \frac{m^2}{4}\right)g_m^t(d) = 0 \tag{11}$$

with the boundary conditions:

$$\begin{aligned}g_m^t(d) &\to 0 \qquad (t \to 0^+)\\ \frac{1}{\sinh d}h_m^t(d) &\to -\frac{1}{2\pi}\delta(\cosh d - 1) \qquad (t \to 0^+).\end{aligned} \tag{12}$$

The solutions for g_m^t and h_m^t read

$$g_m^t(d) = -\frac{1}{2\pi^{\frac{3}{2}}t^{\frac{1}{2}}}\int_d^\infty \frac{\sinh\frac{m+1}{2}u\,e^{-u^2/4t}}{\sqrt{2(\cosh u - \cosh d)}}T_m\left(\frac{\cosh\frac{u}{2}}{\cosh\frac{d}{2}}\right)du \tag{13}$$

$$h_m^t(d) = -\frac{\sinh d}{4\pi^{\frac{3}{2}}t^{\frac{3}{2}}}\int_d^\infty \frac{\cosh(\frac{m}{2}u)e^{-u^2/4t}}{\sqrt{2(\cosh u - \cosh d)}}T_{m+1}\left(\frac{\cosh\frac{u}{2}}{\cosh\frac{d}{2}}\right)du - \frac{m}{2}\tanh\frac{d}{2}\,g_m^t(d), \tag{14}$$

where T_k ($k \in \mathbf{N}_0$) are Chebyshev-polynomials. One can also expand $\hat{G}_m^t$ as

$$\hat{G}_m^t(Z,W) = \hat{g}_M^t(z,w) + \frac{\theta\bar{\theta}}{z - \bar{z}}\hat{F}_m^t(z,w) + \text{terms involving}\,\nu\bar{\nu}$$

with the relation

$$[g_m^t(d)]_{Body} = \left(\frac{z - \bar{w}}{w - \bar{z}}\right)^{-\frac{m}{2}}\hat{g}_m^t(z,w).$$

As is shown on Ref.[3] the following Laplace representation can be deduced for $\hat{g}_m^t$:

$$\hat{g}_m^t(z,w) = -(m+1)\int_{c+iR}\frac{ds}{2\pi i}\int_{-\infty}^\infty dk\int_0^\infty dp\,\frac{e^{st}\Psi_{p,k}(x,y)\bar{\Psi}_{p,k}(u,v)}{[s - (\frac{m+1}{2} + ip)^2][s - (\frac{m+1}{2} - ip)^2]}. \tag{15}$$

where the functions $\Psi_{p,k}$ are the normalized Eigenfunctions of the Maass operator $D_{-n} = y^2(\partial_x^2 + \partial_y^2) + 2im\partial_x$ and are given by [3,18,39] ($\mu \equiv -\text{sign}(k)\frac{m}{2}$, $p > 0$, $k \in \mathbf{R}$):

$$\Psi_{p,k}(x,y) = |\Gamma(\tfrac{1}{2} - \mu + ip)|\sqrt{\frac{p\sinh 2\pi p}{4\pi^3|k|}}e^{ikx}W_{\mu,ip}(2|k|y), \tag{16}$$

where the $W_{\mu,\nu}(z)$ are Whittacker functions. The representation (15) shows a cut in the complex s-plane located along the critical line $\text{Re}\,s = \frac{m+1}{2}$ which gives the spectrum of $\sqrt{\Delta_m} \cong \widehat{\square}_m + \frac{m}{2} \cong \square_m$ reading $\lambda = \frac{m+1}{2} \pm ip$ ($p > 0$).

Appendix D: The Path Integral on the Poincaré Upper Half-Plane $\mathcal{H}$

In this Appendix I present three diferent path integral treatments for a particle moving freely on the Poincaré upper half-plane $\mathcal{H} \equiv \{z = x+iy | y > 0\}$, endowed with the hyperbolic geometry. The Poincaré upper half-plane is analytically equivalent to three further Riemannian spaces: the pseudosphere Λ^2, the Poincaré disc D and the hyperbolic strip S (as already noted in the introduction).

For a review of classical and quantum mechanical motion (in bounded and unbounded domains) in these four Riemannian spaces, see e.g. Balazs/Voros [8].

I do not consider motion in bounded domains. For an attempt to calculate wave-functions and energy levels see Aurich, Sieber and Steiner [7]. To construct the path integral on $\mathcal{H}$, I follow the canonical approach as described in previous papers [41,42]. I use the prescription, which I called "product form"-definition. Let us summarize in short the most important facts of this prescription (for details see [38]).

Let us start with the generic case [i.e. the classical Lagrangian is given by $\mathcal{L}_{Cl}(q,\dot q) = \frac{m}{2} g_{ab}\dot q^a \dot q^b - V(q)$, the classical Hamiltonian by $\mathcal{H}_{Cl}(q,\dot q) = \frac{m}{2} g^{ab} p_a p_b + V(q)$] and write the metric tensor g_{ab} in the form (which under reasonable assumptions is always possible, e.g. positive definite scalar product):

$$g_{ab}(q) = \sum_{c=1}^{d} h_{ac}(q) h_{bc}(q) \tag{1}$$

(d=dimension of the Riemannian manifold). The quantum Hamiltonian is constructed in the usual way by the Laplace-Beltrami operator Δ_{LB} (I set $\hbar = 1$; in the following sums over repeated indices are implicitely understood):

$$H = -\frac{1}{2m}\Delta_{LB} + V(q) = -\frac{1}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a}\sqrt{g}g^{ab}\frac{\partial}{\partial q^b} + V(q) \tag{2}$$

(g=determinant of the metric tensor g_{ab}). Let us introduce momentum operators $p_a = -i(\partial_a + \Gamma_a/2)$, where $\Gamma_a = \partial_a \ln\sqrt{g}$. Rewriting the Hamiltonian (2) in terms of the momentum operators p_a let us choose a **product ordering**-definition:

$$H = \frac{1}{2m} h^{ac}(q) p_a p_b h^{bc}(q) + V(q) + \Delta V(q) \tag{3}$$

with the well-defined quantum correction ΔV given by ($h := \det(h_{ab}) = \sqrt{g}$):

$$\Delta V = \frac{1}{8m}\Bigg[4h^{ac}h^{bc}{}_{,ab} + 2h^{ac}h^{bc}\frac{h_{,ab}}{h} + 2h^{ac}\left(h^{bc}{}_{,b}\frac{h_{,a}}{h} - h^{bc}{}_{,a}\frac{h_{,b}}{h}\right) - h^{ac}h^{bc}\frac{h_{,a}h_{,b}}{h^2}\Bigg]. \tag{4}$$

Let us assume that g_{ab} is proportional to the unit tensor, i.e. $g_{ab} = f^2\delta_{ab}$. Then ΔV simplifies into

$$\Delta V = \hbar^2 \frac{d-2}{8mf^4}\sum_{a=1}^{d}\left[(4-d)f_{,a}^2 + 2f\cdot f_{,aa}\right]. \tag{5}$$

This implies that if the dimension of the space is $d = 2$, then the quantum correction ΔV vanishes.

For details consult Ref. [38]. Using the Trotter formula $e^{-it(A+B)} = s\text{-}\lim_{N\to\infty}\left(e^{-itA/N}e^{-itB/N}\right)^N$ (e.g. [75]) and the short-time approximation for the matrix element $< q''|e^{-i\epsilon H}|q' >$ one obtains in the usual manner the **Lagrangian path integral in the "product form"-definition** [$q^{(j)} = q(t^{(j)})$, $f^{(j)} = f(t^{(j)})$, $t^{(j)} = t' + j\epsilon$, $\epsilon = T/N = (t''-t')/N$, $N\to\infty$, $\Delta q^{(j)} = q^{(j)} - q^{(j-1)}$]:

$$K(q'',q';T) = \int \sqrt{g}\,\mathcal{D}q(t)\exp\left\{i\int_{t'}^{t''}\left[\frac{m}{2}h_{ac}h_{bc}\dot q^a\dot q^b - V(q) - \Delta V(q)\right]dt\right\}$$

$$:= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{Nd}{2}}\prod_{j=1}^{N-1}\int\sqrt{g(q^{(j)})}\,dq^{(j)}$$

$$\times\exp\left\{i\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{ac}(q^{(j-1)})h_{bc}(q^{(j)})\Delta q^{a,(j)}\Delta q^{b,(j)} - \epsilon V(q^{(j)}) - \epsilon\Delta V(q^{(j)})\right]\right\}. \tag{6}$$

The expression in square brackets is nothing but the classical Lagrangian with an additional quantum correction potential ΔV: $\mathcal{L}_{eff} = \mathcal{L}_{Cl} - \Delta V$. Clearly, one has to prove that with the short time kernel of this path integral the time-dependent Schrödinger equation

$$\left[-\frac{1}{2m}\Delta_{LB} + V(q)\right]\psi(q;t) = \frac{1}{i}\frac{\partial}{\partial t}\psi(q;t) \tag{7}$$

can be derived via the time evolution equation

$$\psi(q'';t'') = \int\sqrt{g(q')}K(q'',q';T)\psi(q';t')dq'. \tag{8}$$

This is in fact the case - see [38].

In $\mathcal{H}$ the metric is given by $g_{ab} = \delta_{ab}/y^2$ ($x\in\mathbf{R}, y > 0$). The classical Lagrangian and the Hamiltonian read, respectively:

$$\mathcal{L}_{Cl}(x,\dot x,y,\dot y) = \frac{m}{2y^2}(\dot x^2 + \dot y^2), \qquad \mathcal{H}_{Cl}(x,p_x,y,p_y) = \frac{y^2}{2m}(p_x^2 + p_y^2), \tag{9}$$

and the quantum Hamiltonian is given by

$$H = -\frac{y^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{10}$$

The scalar product for functions $f_1, f_2 \in L^2(\mathcal{H})$ reads,

$$(f_1,f_2)_{\mathcal{H}} = \int_{-\infty}^{\infty}dx\int_0^{\infty}\frac{dy}{y^2}f_1(x,y)\overline{f_2}(x,y). \tag{11}$$

States $\Psi \in D(H)\cap L^2(\mathcal{H})$ must satify the boundary condition $\lim_{y\to 0}\Psi(x,y) = 0$ ($x\in\mathbf{R}$). Following the "product-form" prescription, I get for the hermitian momenta:

$$\left.\begin{aligned}\Gamma_x &= 0, & p_x &= \frac{1}{i}\frac{\partial}{\partial x}\\ \Gamma_y &= -\frac{2}{y}, & p_y &= \frac{1}{i}\left(\frac{\partial}{\partial y} - \frac{2}{y}\right),\end{aligned}\right\} \tag{12}$$

and the Hamiltonian rewritten in the product ordering yields:

$$H = \frac{1}{2m} y(p_x^2 + p_y^2) y. \tag{13}$$

Here it has been used that for this special two-dimensional metric: $\Delta V = 0$. Thus we can infer that the path integral on the Poincaré upper half-plane in the "product form"-definition reads:

$$K^{\mathcal{H}}(x'', y'', x', y'; T) = \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dx^{(j)} \int_0^{\infty} \frac{dy^{(j)}}{y^{(j)2}} \cdot \exp\left[\frac{im}{2\epsilon} \sum_{j=1}^N \frac{\Delta^2 x^{(j)} + \Delta^2 y^{(j)}}{y^{(j)} y^{(j-1)}}\right]. \tag{14}$$

I want to emphasize that in the "product form"-definition for the path integral on $\mathcal{H}$, there is no additional quantum potential or curvature term. Via the time evolution Eq.(8) it is an easy computation to show that Eq.(14) is indeed the right path integral on $\mathcal{H}$ (see [41] for details).

I now present three alternative ways to calculate the path integral (14).

1) To make the path integral manageable I perform a time-transformation (see [42]):

$$s(t) \equiv \int_{t'}^{t} \frac{1}{f(y(\sigma))} d\sigma, \qquad s'' = s(t''), \qquad s(t') = 0 \tag{15}$$

with $f(y) = 1/y^2$. The variables x and y are transformed into

$$\begin{aligned} x(t) &\to \xi(s) \quad \text{with} \quad \xi(s(t)) = x(t) \\ y(t) &\to \eta(s) \quad \text{with} \quad \eta(s(t)) = y(t) \end{aligned} \tag{16}$$

with $\xi(0) = x'$, $\xi(s'') = x''$, $\eta(0) = y'$ and $\eta(s'') = y''$. Let us assume that the constraint

$$\int_0^{s''} \frac{ds}{\eta^2(s)} = T \tag{17}$$

has for all admissible paths a unique solution $s'' > 0$. Of course, since T is fixed, the "time" s'' will be path-dependent. To incorporate the constraint (17) the identity

$$1 = \frac{1}{y''^2} \int_0^{\infty} ds'' \delta\left(\int_0^{s''} \frac{ds}{\eta^2(s)} - T\right) = \frac{1}{y''^2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iTE} \int_0^{\infty} ds'' \exp\left(i \int_0^{s''} ds \frac{E}{\eta^2(s)}\right) \tag{18}$$

has to be used in the path integral (14). The only difference to the prescription given in [42] is that we have now only a time- and not a space-time-transformation. This has the consequence that the additional factor in equation (IV.6) of [42] is absent in the present case. Defining the energy-dependent Feynman kernel $G(E)$ via the Fourier transformation[1]

$$K^{\mathcal{H}}(x'', y'', x', y'; T) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-iTE} G^{\mathcal{H}}(x'', y'', x', y'; E) dE \tag{19}$$

[1] To work with well-defined mathematical formulas let us assume that E has a small positive imaginary part $i\epsilon$, and write $E + i\epsilon$ (with real E) instead of E whenever necessary. Also, square roots will be positive. See e.g. [39,46] for details.

I obtain the transformation formula

$$G^{\mathcal{H}}(x'', y'', x', y'; E) = i \int_0^{\infty} \tilde{K}(\xi'', \eta'', \xi', \eta'; s'') ds'', \tag{20}$$

where the transformed path integral is given by

$$\begin{aligned} \tilde{K}(\xi'', \eta'', \xi', \eta'; s'') &= \int D\xi(s) \mu_\lambda[\eta] D\eta(s) \exp\left[\frac{im}{2} \int_0^{s''} (\dot{\xi}^2 + \dot{\eta}^2) ds\right] \\ &= \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta}\right)^N \int_{-\infty}^{\infty} \int_0^{\infty} d\xi_{(1)} d\eta_{(1)} \cdots \int_{-\infty}^{\infty} \int_0^{\infty} d\xi_{(N-1)} d\eta_{(N-1)} \\ &\quad \times \mu_\lambda[\eta_{(j)}] \exp\left\{\frac{im}{2\delta} \sum_{j=1}^N \left[(\xi_{(j)} - \xi_{(j-1)})^2 + (\eta_{(j)} - \eta_{(j-1)})^2\right]\right\} \end{aligned} \tag{21}$$

with $\delta = s''/N$ and $\lambda = \sqrt{1/4 - 2mE}$. The functional measure is given by

$$\mu_\lambda[\eta] \to \prod_{j=1}^N \left[\sqrt{\frac{2\pi m}{i\delta} \eta_{(j)} \eta_{(j-1)}} \exp\left(-\frac{m}{i\delta} \eta_{(j)} \eta_{(j-1)}\right) I_\lambda\left(\frac{m}{i\delta} \eta_{(j)} \eta_{(j-1)}\right)\right]. \tag{22}$$

I_λ denotes a modified Bessel function. Following the general theory [42], it has been used:

$$g_{ab} = \delta_{ab}, \quad \sqrt{g} = 1, \quad \Gamma_\xi = 0, \quad \Gamma_\eta = 0, \quad \Delta V = 0. \tag{23}$$

The path integral in (21) factorizes into a path integral for a free particle in $\xi \in \mathbf{R}$, and into a radial path integral with "angular momentum" λ in the variable $\eta \in \mathbf{R}^+$. Using the well-known path integral identity

$$\int \mu_\lambda[r] Dr(t) \exp\left(\frac{im}{2} \int_{t'}^{t''} \dot{r}^2 dt\right) = \sqrt{r'r''} \frac{m}{iT} \exp\left(\frac{im}{2T}(r'^2 + r''^2)\right) I_\lambda\left(\frac{m}{iT} r'r''\right) \tag{24}$$

(see Peak and Inomata [69] and Steiner and Grosche [41]), we can immediately write down the solution of (21):

$$\tilde{K}(\xi'', \eta'', \xi', \eta'; s'') = \sqrt{\frac{\eta'\eta''}{2\pi}} \left(\frac{m}{is''}\right)^{3/2} \exp\left\{-\frac{m}{2is''}[(\xi'' - \xi')^2 + \eta''^2 + \eta'^2]\right\} I_\lambda\left(\frac{m}{is''} \eta'\eta''\right). \tag{25}$$

Inserting (25) into Eq.(20), the s''-integration can be carried out by first performing a Feynman-Wick rotation ($s'' \to -i\tau$, $\tau \in \mathbf{R}^+$), and then introducing the integration variable $z = my'y''/\tau$ and the Poincaré distance $\cosh d(z'', z') \equiv [(x'' - x')^2 + y'^2 + y''^2]/2y'y''$. I obtain (for the integral see p.712 of reference [33]):

$$G^{\mathcal{H}}(x'', y'', x', y'; E) = \frac{m}{\sqrt{2\pi}} \int_0^{\infty} e^{-z\cosh d} I_{ip}(z) \frac{dz}{\sqrt{z}} = \frac{m}{\pi} Q_{-\frac{1}{2}-ip}(\cosh d), \tag{26}$$

where I have introduced the momentum $p \equiv \sqrt{2mE - 1/4}$. Eq.(26) gives a closed expression for the energy-dependent Green's function (resolvent kernel) in terms of the

Legendre function of the second kind $\mathcal{Q}_\nu$[1]. This result agrees with the one obtained by solving directly the Schrödinger equation (see e.g. [46]). Using the integrals (see [33], pp.819, 732):

$$\left.\begin{aligned}\mathcal{Q}_{\nu-\frac12}\left(\frac{a^2+b^2+c^2}{2ab}\right) &= \int_0^\infty dp'\,\frac{p'\tanh\pi p'}{\nu^2+p'^2}\mathcal{P}_{ip'-\frac12}\left(\frac{a^2+b^2+c^2}{2ab}\right)\\ \mathcal{P}_{\nu-\frac12}\left(\frac{a^2+b^2+c^2}{2ab}\right) &= \frac{4\sqrt{ab}}{\pi^2}\cos\nu\pi\int_0^\infty dk\,K_\nu(ak)K_\nu(bk)\cos ck,\end{aligned}\right. \tag{27}$$

Eq.(26) can be rewritten as

$$\begin{aligned}&G^{\mathcal{H}}(x'',y'',x',y';E)\\ &= \frac{1}{\pi^3}\int_{-\infty}^\infty dk\int_0^\infty dp'\,\frac{p'\sinh\pi p'}{(p'^2+\frac14)/2m-E}\sqrt{y'y''}\,K_{ip'}(|k|y')\,K_{ip'}(|k|y'')\,e^{ik(x''-x')}\end{aligned} \tag{28}$$

(K_ν denotes a modified Bessel function). The representation (28) shows clearly that $G(E)$ has a cut on the positive real axis in the complex energy plane with a branch point at $E=1/8m$. We thus infer that the quantum mechanical motion on the Poincaré upper half-plane $\mathcal{H}$ has a continuous energy spectrum. From (28) the normalized wave functions and the energy spectrum can be read off:

$$\left.\begin{aligned}\psi_{p,k}(x,y) &= \sqrt{\frac{p\sinh\pi p}{\pi^3}}\,e^{ikx}\sqrt{y}\,K_{ip}(|k|y)\quad(x\in\mathbf{R},y>0)\\ E_p &= \frac{1}{2m}\left(p^2+\frac14\right)\end{aligned}\right\} \tag{29}$$

with $p>0$ and $k\in\mathbf{R}\setminus\{0\}$. These are the correct wave functions. The spectrum has a largest lower bound $E_0=1/8m$. A state with $p=0$ and $E_0=1/8m$ does not exist, because $\psi_{0,k}$ vanishes identically. One also has to exclude the case $k=0$, which is obvious from the asymptotic behaviour of the K_ν function for $z\to0$: $K_{ip}(z)\to\frac12\left[\Gamma(ip)\left(\frac{2}{z}\right)^{ip}+\Gamma(-ip)\left(\frac{2}{z}\right)^{-ip}\right]$. It is nevertheless possible to define a function $\phi_p(y):=y^{ip+1/2}$ which is an Eigenstate of H, $H\phi_p=E_p\phi_p$, but this function is not normalizable in $\mathcal{H}$. ϕ_p is only normalizable in a bounded domain. Let us discuss in short the "zero-momentum" energy shift $E_0=\frac{1}{8m}$. Let us consider the classical Hamiltonian $\mathcal{H}_{Cl}$ and insert (introducing $\hbar$) the Heisenberg uncertainty relations $xp_x\geq\hbar/2$ and $yp_y\geq\hbar/2$. This gives for the energy of quantum motion on $\mathcal{H}$ the lower bound:

$$E_{\mathcal{H}}\geq\frac{\hbar^2}{8m}\left(1+\frac{y^2}{x^2}\right)>\frac{\hbar^2}{8m}. \tag{30}$$

The value $E_0=\inf E_{\mathcal{H}}=\frac{\hbar^2}{8m}$ can never be taken on because $\{z|y=0\}\notin\mathcal{H}$. E_0 is the largest lower bound on $\mathcal{H}$.

On the pseudosphere Λ^2, the Poincaré disc D and the hyperbolic strip S the

[1] I use $\mathcal{P}^\mu_\nu(z)$, $\mathcal{Q}^\mu_\nu(z)$ for $z\in\mathbf{C}\setminus[-1,1]$ and $P^\mu_\nu(x)$, $Q^\mu_\nu(x)$ for $x\in(-1,1)$ for the Legendre functions of the first and second kind, respectively.

wave functions are given by $(p>0,l\in\mathbf{Z},k\in\mathbf{R})$:

$$\left.\begin{aligned}\Psi^{\Lambda^2}_{p,l}(\tau,\phi) &= \sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma(ip+l+1/2)\mathcal{P}^{-l}_{ip-\frac12}(\cosh\tau)\,e^{il\phi}\\ \Psi^D_{p,l}(r,\psi) &= \sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma(\tfrac12+ip+l)\mathcal{P}^{-l}_{ip-\frac12}\left(\frac{1+r^2}{1-r^2}\right)e^{il\psi}\\ \Psi^S_{p,k}(X,Y) &= \sqrt{\frac{p\sinh\pi p}{4\pi(\cosh^2\pi k+\sinh^2\pi p)}}\sqrt{\cos Y}\,P^{ip}_{ik-\frac12}(\sin Y)e^{ikX}.\end{aligned}\right\} \tag{31}$$

The corresponding path integral expressions on $\mathcal{H}$, Λ^2 D and S are equivalent to each other. This has been discussed in detail in Refs.[40,43]; see also the end of this section.

Finally, we perform a Fourier transformation in (28) to get the time-dependent Feynman kernel

$$\begin{aligned}&K^{\mathcal{H}}(x'',y'',x',y';T)\\ &= \frac{1}{\pi^3}\int_{-\infty}^\infty dk\int_0^\infty dp\,p\sinh\pi p\,e^{-iT\frac{p^2+1/4}{2m}}\sqrt{y'y''}K_{ip}(|k|y')\,K_{ip}(|k|y'')\,e^{ik(x''-x')}.\end{aligned} \tag{32}$$

The $\psi_{p,k}$ form an orthonormal basis

$$\int_{-\infty}^\infty dx\int_0^\infty\frac{dy}{y^2}\bar\psi_{p,k}(x,y)\psi_{p',k'}(x,y)=\delta(k-k')\delta(p-p'). \tag{33}$$

Proof: Inserting $\psi_{p,k}$ from Eq.(29) and performing the x-integration yields:

$$N=\delta(k-k')\frac{2\sqrt{pp'\sinh\pi p\sinh p'}}{\pi^2}\int_0^\infty\frac1y K_{ip}(y)K_{ip'}(y)\,dy. \tag{34}$$

Using the integral ([33], p.693):

$$\begin{aligned}&\int_0^\infty y^{-\lambda}K_\mu(ay)K_\nu(by)\,dy=\frac{a^{\lambda-\nu-1}b^\nu}{2^{2+\lambda}\Gamma(1-\lambda)}\\ &\times\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)\\ &\qquad\times F\left(\frac{1-\lambda+\mu+\nu}{2},\frac{1-\lambda-\mu+\nu}{2};1-\lambda;1-\frac{b^2}{a^2}\right).\end{aligned} \tag{35}$$

Let $a=b=1$, $\lambda=1-2\epsilon$, $\mu=ip$ and $\nu=ip+2iq$, $q=(p'-p)/2$, then

$$\int_0^\infty y^{2\epsilon-1}K_{ip}(y)K_{ip+2iq}(y)\,dy=\frac{\Gamma(\epsilon+ip+iq)\Gamma(\epsilon+iq)\Gamma(\epsilon-iq)\Gamma(\epsilon-ip-iq)}{\Gamma(2\epsilon)2^{3-2\epsilon}}. \tag{36}$$

The "good" terms yield in the limit $\epsilon\to0$:

$$\lim_{\epsilon\to0}\frac{\Gamma(\epsilon+ip+iq)\Gamma(\epsilon-ip-iq)}{2^{3-2\epsilon}}=\frac18|\Gamma(ip-iq)|^2=\frac{\pi}{8(p+q)\sinh\pi(p-q)}, \tag{37}$$

where I have used a well-know property of the Γ-function. The remaining terms yield

$$\lim_{\epsilon\to 0}\frac{\Gamma(\epsilon+iq)\Gamma(\epsilon-iq)}{\Gamma(2\epsilon)} = 2\pi\lim_{\epsilon\to 0}\frac{\epsilon}{\pi(\epsilon^2+q^2)} = 4\pi\delta(p'-p), \tag{38}$$

and Eq.(33) is proved.
Vice versa, the $\psi_{p,k}$ form a complete set, i.e.

$$\int_{-\infty}^{\infty} dk \int_0^{\infty} dp\, \psi_{p,k}(x'',y'',)\bar\psi_{p,k}(x',y') = y'y''\delta(x''-x')\delta(y''-y') \tag{39}$$

(the factor $y'y'' = (g'g'')^{-\frac14}$ has to be included due to the Riemannian structure of $\mathcal{H}$, see e.g. [67]).

Proof: Consider the integral ([33] p.772):

$$\int_0^{\infty} dx\, K_{ix}(a)K_{ix}(b)\cosh[(\pi-\phi)x] = K_0(\sqrt{a^2+b^2-2ab\cos\phi}). \tag{40}$$

Differentiation with respect to ϕ gives on the left hand side:

$$-\frac{\partial}{\partial\phi}\int_0^{\infty} dx\, K_{ix}(a)K_{ix}(b)\cosh[(\pi-\phi)x] = \int_0^{\infty} dx\, x\sinh[(\pi-\phi)x]K_{ix}(a)K_{ix}(b), \tag{41}$$

while the right hand side yields:

$$-\frac{\partial}{\partial\phi}K_0(\sqrt{a^2+b^2-2ab\cos\phi}) = \frac{ab\sin\phi}{\sqrt{a^2+b^2-2ab\cos\phi}}K_1(\sqrt{a^2+b^2-2ab\cos\phi}). \tag{42}$$

Here some properties of the K_ν-function have been used (see e.g. [33], p.510). Therefore in the limit $\phi\to 0$ and for $y'\neq y''$:

$$\int_0^{\infty} dp\, p\sinh\pi p\, K_{ip}(|k|y')K_{ip}(|k|y'') = 0. \tag{43}$$

It remains to consider the case $y'\simeq y''$. Let us set $y=y'$, $y''=y+\delta$ with $|\delta|\ll 1$ and $\cos\varphi\simeq 1-\phi^2/2$ for $|\phi|\ll 1$. Using $K_0\simeq -\ln(z/2)$ $(z\to 0)$ we get for the right hand side of Eq.(40):

$$\frac{\pi}{2}K_0(|k|\sqrt{y'^2+y''^2-2y'y''\cos\phi}) \simeq \frac{\pi}{2}\left[\ln\frac{|k|}{2}+\frac12\ln(\delta^2+y^2\phi^2)\right] \quad (|\delta|,|\phi|\ll 1) \tag{44}$$

and in the limit $\phi\to 0$:

$$\int_0^{\infty} dp\, p\sinh\pi p K_{ip}(|k|y')K_{ip}(|k|y'') = \frac{\pi^2}{2}\sqrt{y'y''}\,\delta(y'-y''). \tag{45}$$

Together with the well-known equation $\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x''-x')} = \delta(x''-x')$ the completeness relation (39) is proven.

2) In the second approach in calculating the path integral (14) - following the idea of Kubo [54] - let us start by integrating $K(T)$ over $x''\equiv x^{(N)}$. This gives:

$$\int_{-\infty}^{\infty} K^{\mathcal{H}}(x'',y'',x',y';T)dx'' = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1}\int_0^{\infty}\frac{dy^{(j)}}{y^{(j)2}}$$
$$\times\exp\left[\frac{im}{2\epsilon}\sum_{j=1}^N\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right]\cdot\prod_{j=1}^N\int_{-\infty}^{\infty}dx^{(j)}\exp\left(-\frac{m}{2i\epsilon}\frac{\Delta^2 x^{(j)}}{y^{(j)}y^{(j-1)}}\right)$$
$$=\sqrt{y'y''}\lim_{n\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac N2}\prod_{j=1}^{N-1}\int_0^{\infty}\frac{dy^{(j)}}{y^{(j)}}\exp\left[\frac{im}{2\epsilon}\sum_{j=1}^N\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right]\equiv\tilde K(y'',y';T). \tag{46}$$

Let us look for a transformation $z=z(y)$ so that

$$\dot z^2 = \frac{\dot y^2}{y^2}. \tag{47}$$

A simple calculation yields $z=\ln y$, respectively the inverse transformation $y=e^z$. $z(y)$ has the property $(0,\infty)\mapsto\mathbf{R}$. Thus $dy/y=dz$, and the kinetic term in the exponential in the path integral (46) gives in a Taylor expansion:

$$\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}} = e^{\Delta z^{(j)}-2z^{(j)}}\left[e^{z^{(j)}}-e^{z^{(j)}-\Delta z^{(j)}}\right]^2 \simeq \Delta^2 z^{(j)} + \frac{\Delta^4 z^{(j)}}{12}. \tag{48}$$

Exploiting the path integral identity $\Delta^4 z \doteq 3\left(\frac{im}{\epsilon}\right)^2$ I get:[1]

$$\exp\left(\frac{im}{2\epsilon}\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right)\doteq\exp\left(\frac{im}{2\epsilon}\Delta^2 z^{(j)}-\frac{i\epsilon}{8m}\right). \tag{49}$$

Therefore the path integral (46) gives essentially a path integral of a free particle in $\mathbf{R}$ in the z-coordinate. We get with $\hat K(z'',z';T)\equiv\tilde K(y'',y';T)$:

$$\hat K(z'',z';T)$$
$$=\exp\left[\frac{z'+z''}{2}-\frac{iT}{8m}\right]\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac N2}\prod_{j=1}^{N-1}\int_{-\infty}^{\infty}dz^{(j)}\exp\left(\frac{im}{2\epsilon}\sum_{j=1}^N\Delta^2 z^{(j)}\right)$$
$$=\exp\left[\frac{z'+z''}{2}-\frac{iT}{8m}\right]\left(\frac{m}{2\pi iT}\right)^{\frac12}\exp\left[\frac{im}{2T}(z''-z')^2\right]$$
$$=e^{-iT/8m}\sqrt{y'y''}\left(\frac{m}{2\pi iT}\right)^{\frac12}\exp\left[-\frac{m}{2iT}\ln^2\left(\frac{y''}{y'}\right)\right]=\tilde K(y'',y';T). \tag{50}$$

Let us note that $l=\ln(y''/y')$ is nothing but the hyperbolic distance d in $\mathcal{H}$ if $(x''-x')=0$. Introducing now

$$k := \frac{\xi^2+y'^2+y''^2}{2y'y''} \tag{51}$$

[1] I use the symbol $\doteq$ (following DeWitt [21]) to denote "equivalence as far as use in the path integral is concerned". A discussion concerning these identities can be, e.g. found in Feynman and Hibbs [27]; see also [42] and references therein.

I obtain

$$\tilde{K}(y'',y';T)=\int_l^\infty K(k,y'',y';T)\frac{dk}{\sqrt{k-l}}=\left(\frac{m}{2\pi iT}\right)^{\frac{1}{2}}\exp\left(-\frac{m}{2iT}\operatorname{arcosh}^2 l-\frac{iT}{8m}\right). \tag{52}$$

This integral equation can be solved exactly and the result reads (see [54]):

$$K^{\mathcal{H}}(d;T)=\sqrt{2}\left(\frac{m}{2\pi iT}\right)^{\frac{3}{2}}\int_d^\infty\frac{udu}{\sqrt{\cosh u-\cosh d}}\exp\left[-\frac{m}{2iT}u^2-\frac{iT}{8m}\right]. \tag{53}$$

Fourier transformation to get the Green's function $G(E)=\int_0^\infty e^{iTE}K(T)dT$ yields:

$$G^{\mathcal{H}}(d;E)=\sqrt{2}\left(\frac{m}{2\pi i}\right)^{\frac{3}{2}}\int_d^\infty\frac{udu}{\sqrt{\cosh u-\cosh d}}du \times\int_0^\infty T^{-\frac{3}{2}}\exp\left[-\frac{m}{2iT}u^2-\left(\frac{i}{8m}-iE\right)T\right]dT$$

$$=\frac{m}{\pi\sqrt{2}}\int_d^\infty\frac{\exp\left(-iu\sqrt{2mE-\frac{1}{4}}\right)}{\sqrt{\cosh u-\cosh d}}du=\frac{m}{\pi}\mathcal{Q}_{-\frac{1}{2}-i\sqrt{2mE-\frac{1}{4}}}(\cosh d). \tag{54}$$

This is the previous result. Note that $K^{\mathcal{H}}$ and $G^{\mathcal{H}}$ are only a function of the hyperbolic distance d. Therefore Eqs.(53) and (54) gives the result in all the four spaces $\mathcal{H}$, D, Λ^2 and S.

3) In the third approach in calculating the path integral (14) let us start by performing a Fourier expansion of $K^{\mathcal{H}}(T)$:

$$\begin{aligned}K^{\mathcal{H}}(x'',y'',x',y';T)&=\int_{-\infty}^\infty K_k(y'',y';T)\,e^{ik(x''-x')}dk\\ K_k(y'',y';T)&=\frac{1}{2\pi}\int_{-\infty}^\infty K^{\mathcal{H}}(x'',y'',x',y';T)e^{-ik(x''-x')}dx''.\end{aligned} \tag{55}$$

This gives if Eq.(14) is inserted into (55) for $K_k^{\mathcal{H}}(T)$:

$$\begin{aligned}K_k(y'',y';T)&=\frac{1}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N\prod_{j=1}^{N-1}\int_0^\infty\frac{dy^{(j)}}{y^{(j)2}}\exp\left[\frac{im}{2\epsilon}\sum_{j=1}^N\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right]\\
&\qquad\times\prod_{j=1}^N\int_{-\infty}^\infty dx^{(j)}\exp\left[-\frac{m}{2i\epsilon}\frac{\Delta^2 x^{(j)}}{y^{(j)}y^{(j-1)}}-ik\Delta x^{(j)}\right]\\
&=\frac{\sqrt{y'y''}}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}}\prod_{j=1}^{N-1}\int_0^\infty\frac{dy^{(j)}}{y^{(j)}}\exp\left\{i\sum_{j=1}^N\left[\frac{m}{2\epsilon}\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}-\epsilon\frac{k^2y^{(j)}y^{(j-1)}}{2m}\right]\right\}\\
&\equiv\frac{\sqrt{y'y''}}{2\pi}\int\frac{Dy(t)}{y}\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}\frac{\dot{y}^2}{y^2}-\frac{k^2y^2}{2m}\right)dt\right].\end{aligned} \tag{56}$$

Performing the transformation $z=\ln y$, $y=e^z$ in (56) and repeating the same procedure as in Eq.(46) I get with $\hat{K}_k(z'',z';T)\equiv K_k(y'',y';T)$:

$$K_k(z'',z';T)=\frac{1}{2\pi}e^{(z'-z'')/2-iT/8m}\int Dz(t)\exp\left[i\int_{t'}^{t''}\left(\dot{z}^2-\frac{k^2}{2m}e^{2z}\right)dt\right]. \tag{57}$$

This path integral is nothing but a path integral for the potential of Liouville quantum mechanics with the potential $V(z)=\frac{k^2}{2m}e^{2z}$. This path integral was calculated by Steiner and Grosche [41] and the result for $\hat{K}_k$ therefore reads,

$$\hat{K}_k(z'',z';T)=\frac{1}{\pi^3}e^{(z'+z'')/2}\int_0^\infty dpp\sinh\pi p\exp\left[-\frac{iT}{2m}\left(p^2+\frac{1}{4}\right)\right]K_{ip}(|k|e^{z'})K_{ip}(|k|e^{z''}). \tag{58}$$

Inserting $y=e^z$, this gives finally for the Feynman kernel on $\mathcal{H}$:

$$\begin{aligned}K^{\mathcal{H}}(x'',y'',x',y';T)&=\frac{1}{\pi^3}\int_{-\infty}^\infty dk\int_0^\infty dp\,p\sinh\pi p\\
&\qquad\times\exp\left[-\frac{iT}{2m}\left(p^2+\frac{1}{4}\right)\right]\sqrt{y'y''}K_{ip}(|k|y')K_{ip}(|k|y'')e^{ik(x''-x')}\end{aligned} \tag{59}$$

with the correct energy-spectrum and wave-functions as in (29).
Using now the integral repesentation ([33], p.732):

$$\int_0^\infty K_\nu(ax)K_\nu(bx)\cos cx\,dx=\frac{\pi^2}{4\sqrt{ab}\cos\nu\pi}\mathcal{P}_{-\frac{1}{2}+\nu}\left(\frac{a^2+b^2+c^2}{2ab}\right), \tag{60}$$

and the addition theorem for the associated Legendre functions: ([33], p.1014)

$$\mathcal{P}_\nu(zz'-\sqrt{z^2-1}\sqrt{z'^2-1}\cos\phi)=\sum_{l=-\infty}^\infty(-1)^l e^{il\phi}\frac{\Gamma(\nu-l+1)}{\Gamma(\nu+l+1)}\mathcal{P}_\nu^l(z)\mathcal{P}_\nu^l(z'), \tag{61}$$

the identity:

$$\begin{aligned}&\frac{1}{\pi^3}\int_{-\infty}^\infty dk\int_0^\infty dp\,p\sinh\pi p\,e^{-\frac{iT}{2m}(p^2+\frac{1}{4})}\sqrt{y'y''}K_{ip}(|k|y')\,K_{ip}(|k|y'')\,e^{ik(x''-x')}\\
&=\frac{1}{2\pi^2}\sum_{l=-\infty}^\infty\int_0^\infty dpp\sinh\pi p\,e^{-\frac{iT}{2m}(p^2+\frac{1}{4})}|\Gamma(\tfrac{1}{2}+ip-l)|^2\\
&\qquad\times e^{il(\phi''-\phi')}\mathcal{P}^l_{-\frac{1}{2}+ip}(\cosh\tau')\mathcal{P}^l_{-\frac{1}{2}+ip}(\cosh\tau'').\end{aligned} \tag{62}$$

can be derived. Here use has been made of Eq.(I.14). The right hand side of Eq.(62) represents the Feynman kernel on Λ^2 and thus this shows the equivalence of the Feynman-kernels on $\mathcal{H}$ and Λ^2, i.e. $K^{\mathcal{H}}(T)\cong K^{\Lambda^2}(T)$. Inserting on the right hand side of Eq.(62) the variables of the Disc D I get the Feynamn kernel on D:

$$\begin{aligned}K^D(r'',r',\psi'',\psi';T)&=\frac{1}{2\pi^2}\int_0^\infty dp\sum_{l=-\infty}^\infty p\sinh\pi p\exp\left[-\frac{iT}{2m}\left(p^2-\frac{1}{4}\right)\right]\\
&\qquad\times|\Gamma(\tfrac{1}{2}+ip+l)|^2\mathcal{P}^{-l}_{ip-\frac{1}{2}}\left(\frac{1+r'^2}{1-r'^2}\right)\mathcal{P}^{-l}_{ip-\frac{1}{2}}\left(\frac{1+r''^2}{1-r''^2}\right)e^{il(\psi''-\psi')}.\end{aligned} \tag{63}$$

There is no obvious simple manipulation in, e.g. Eq.(63) to achieve the Feynman kernel on the hyperbolic strip S. One has to calculate K^S directly. This has been done in Ref.[40] and the result reads:

$$\begin{aligned}K^S(X'',X',Y'',Y';T)&=\frac{1}{4\pi}\int_{-\infty}^\infty dk\int_{-\infty}^\infty dp\frac{p\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\\
&\qquad\times\sqrt{\cos Y'\cos Y''}P^{ip}_{ik-\frac{1}{2}}(\sin Y'')P^{-ip}_{ik-\frac{1}{2}}(\sin Y')e^{ik(X''-X')}e^{-\frac{iT}{2m}(p^2+\frac{1}{4})}.\end{aligned} \tag{64}$$

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